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CONSTRUCTIVE APPROXIMATION

Dual Wavelet Frames and Riesz Bases in Sobolev Spaces

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Abstract This paper generalizes the mixed extension principle in $L_2(\mathbb{R}^d)$ of (Ron and Shen in J. Fourier Anal. Appl. 3:617–637, 1997) to a pair of dual Sobolev spaces $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$. In terms of masks for $\phi, \psi^1, \dots, \psi^L \in H^s(\mathbb{R}^d)$ and $\tilde{\phi}, \tilde{\psi}^1, \dots, \tilde{\psi}^L \in H^{-s}(\mathbb{R}^d)$, simple sufficient conditions are given to ensure that $(X^s(\phi; \psi^1, \dots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^L))$ forms a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, where

$$X^s\bigl(\phi; \psi^1, \dots, \psi^L\bigr) := \bigl\{\phi(\cdot - k) : k \in \mathbb{Z}^d\bigr\} \cup \bigl\{2^{j(d/2-s)}\psi^\ell(2^j \cdot -k) : j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d,\ \ell = 1,\ \dots, L\bigr\}.$$

For $s > 0$, the key of this general mixed extension principle is the regularity of ϕ, $\psi^1, \dots, \psi^L$, and the vanishing moments of $\tilde{\psi}^1, \dots, \tilde{\psi}^L$, while allowing $\tilde{\phi}$, $\tilde{\psi}^1, \dots, \tilde{\psi}^L$ to be tempered distributions not in $L_2(\mathbb{R}^d)$ and $\psi^1, \dots, \psi^L$ to have no vanishing moments. So, the systems $X^s(\phi; \psi^1, \dots, \psi^L)$ and $X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^L)$ may not be able to be normalized into a frame of $L_2(\mathbb{R}^d)$. As an example, we show that $\{2^{j(1/2-s)}B_m(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is a wavelet frame in $H^s(\mathbb{R})$ for any $0 < s < m - 1/2$, where B_m is the B-spline of order m. This simple construction is also applied to multivariate box splines to obtain wavelet frames with short supports,

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noting that it is hard to construct nonseparable multivariate wavelet frames with small supports. Applying this general mixed extension principle, we obtain and characterize dual Riesz bases $(X^s(\phi;\psi^1,\dots,\psi^L), X^{-s}(\tilde{\phi};\tilde{\psi}^1,\dots,\tilde{\psi}^L))$ in Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. For example, all interpolatory wavelet systems in (Donoho, Interpolating wavelet transform. Preprint, 1997) generated by an interpolatory refinable function $\phi \in H^s(\mathbb{R})$ with $s > 1/2$ are Riesz bases of the Sobolev space $H^s(\mathbb{R})$. This general mixed extension principle also naturally leads to a characterization of the Sobolev norm of a function in terms of weighted norm of its wavelet coefficient sequence (decomposition sequence) without requiring that dual wavelet frames should be in $L_2(\mathbb{R}^d)$, which is quite different from other approaches in the literature.

Keywords Dual wavelet frames · Wavelet frames · Riesz bases · Sobolev spaces

Mathematics Subject Classification (2000) 42C40 · 41A15 · 41A63 · 42B35

1 Introduction

This paper gives a systematic study of dual wavelet frames derived from refinable functions in a Sobolev space $H^s(\mathbb{R}^d)$ and its dual space $H^{-s}(\mathbb{R}^d)$. This is then applied to obtain corresponding parallel results on dual Riesz wavelet bases in a pair of dual Sobolev spaces. We set up the framework, state some main results, and provide examples to show how the theory can be applied to obtain and understand some interesting wavelet systems. Dual wavelet frames are investigated in Sect. 2, and Sect. 3 is devoted to the study of dual Riesz wavelet bases.

1.1 Dual Wavelet Frame Systems

For a real number s, we denote by $H^s(\mathbb{R}^d)$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \big|\hat{f}(\xi)\big|^2 \big(1+\|\xi\|^2\big)^s \, d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. For $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ is defined as $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $\xi \in \mathbb{R}^d$, where $x\cdot\xi$ is the inner product of the two vectors x and ξ in $\mathbb{R}^d$. Note that $H^s(\mathbb{R}^d)$ is a Hilbert space under the inner product:

$$\langle f, g\rangle_{H^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\big(1+\|\xi\|^2\big)^s\,d\xi, \quad f,g \in H^s\big(\mathbb{R}^d\big).$$

Moreover, for each $g \in H^{-s}(\mathbb{R}^d)$,

$$\langle f, g\rangle := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi, \quad f \in H^s\big(\mathbb{R}^d\big)$$

defines a linear functional on $H^s(\mathbb{R}^d)$. The spaces $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ form a pair of dual spaces.

Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For given $\phi, \psi^1, \dots, \psi^L \in H^s(\mathbb{R}^d)$, a properly normalized wavelet system in $H^s(\mathbb{R}^d)$ is defined as:

$$X^s(\phi; \psi^1, \dots, \psi^L) := \{\phi_{0,k} : k \in \mathbb{Z}^d\} \cup \{\psi^{\ell,s}_{j,k} : j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d,\ \ell = 1, \dots, L\}$$
$$\text{with } \phi_{0,k} := \phi(\cdot - k) \text{ and } \psi^{\ell,s}_{j,k} := 2^{j(d/2-s)}\psi^\ell(2^j \cdot -k). \tag{1.1}$$

We say that $X^s(\phi; \psi^1, \dots, \psi^L)$ is a *wavelet frame* in $H^s(\mathbb{R}^d)$ if there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1\|f\|^2_{H^s(\mathbb{R}^d)} &\leqslant \sum_{k\in\mathbb{Z}^d} \left|\langle f, \phi_{0,k}\rangle_{H^s(\mathbb{R}^d)}\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle f, \psi^{\ell,s}_{j,k}\rangle_{H^s(\mathbb{R}^d)}\right|^2 \\ &\leqslant C_2\|f\|^2_{H^s(\mathbb{R}^d)}, \quad f \in H^s(\mathbb{R}^d). \end{aligned} \tag{1.2}$$

We say that $X^s(\phi; \psi^1, \dots, \psi^L)$ is *a Bessel wavelet sequence* in $H^s(\mathbb{R}^d)$ if the right-side inequality of (1.2) holds. For a real number s, let $\phi, \psi^1, \dots, \psi^L$ be functions (or distributions) in $H^s(\mathbb{R}^d)$ and let $\tilde\phi, \tilde\psi^1, \dots, \tilde\psi^L$ belong to $H^{-s}(\mathbb{R}^d)$. We say that $(X^s(\phi; \psi^1, \dots, \psi^L), X^{-s}(\tilde\phi; \tilde\psi^1, \dots, \tilde\psi^L))$ is *a pair of dual wavelet frames in* $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ when the following two conditions are satisfied:

(1) $X^s(\phi; \psi^1, \dots, \psi^L)$ is a wavelet frame in $H^s(\mathbb{R}^d)$ and $X^{-s}(\tilde\phi; \tilde\psi^1, \dots, \tilde\psi^L)$ is a wavelet frame in $H^{-s}(\mathbb{R}^d)$.
(2) The identity

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}^d} \langle f, \tilde\phi_{0,k}\rangle\langle \phi_{0,k}, g\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \langle f, \tilde\psi^{\ell,-s}_{j,k}\rangle\langle \psi^{\ell,s}_{j,k}, g\rangle, \tag{1.3}$$

holds for all $f \in H^s(\mathbb{R}^d)$ and $g \in H^{-s}(\mathbb{R}^d)$.

If $(X^s(\phi; \psi^1, \dots, \psi^L), X^{-s}(\tilde\phi; \tilde\psi^1, \dots, \tilde\psi^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, then we have the wavelet representations in the Sobolev spaces $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ as follows:

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}^d} \langle f, \tilde\phi_{0,k}\rangle\phi_{0,k} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \langle f, \tilde\psi^{\ell,-s}_{j,k}\rangle\psi^{\ell,s}_{j,k}, \quad f \in H^s(\mathbb{R}^d), \\ g &= \sum_{k\in\mathbb{Z}^d} \langle g, \phi_{0,k}\rangle\tilde\phi_{0,k} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \langle g, \psi^{\ell,s}_{j,k}\rangle\tilde\psi^{\ell,-s}_{j,k}, \quad g \in H^{-s}(\mathbb{R}^d), \end{aligned} \tag{1.4}$$

with the series converging unconditionally in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively.

Wavelet frames, especially wavelet frames in $L_2(\mathbb{R}^d)$, are well-studied in the literature. The interested reader should consult [11, 20, 21, 27, 30, 38, 49–51] and the references therein for more details. Recently, wavelet frames are used to derive efficient algorithms for image restorations (see, e.g., [5–10, 22]). Some of those applications indicate that it is desirable to have a wavelet frame system in some Sobolev space with its dual system in the corresponding dual Sobolev space. This is one of the motivations of our adventures here.

For a tempered distribution f defined on $\mathbb{R}^d$, we denote

$$\begin{aligned}\nu_2(f) &:= \sup\{s \in \mathbb{R} : f \in H^s(\mathbb{R}^d)\} \\ &= \sup\Big\{s \in \mathbb{R} : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + \|\xi\|^2)^s \, d\xi < \infty\Big\}.\end{aligned} \tag{1.5}$$

If $f \notin H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$, then we simply set $\nu_2(f) := -\infty$.

Now we have the following result on pairs of dual wavelet frames, which is the mixed extension principle for a pair of dual wavelet frames in Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, and which is a special case of Theorem 2.4, for trigonometric polynomial masks. We denote by δ the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$ below and throughout this paper.

Theorem 1.1 *Let $\hat{a}, \widehat{b^1}, \ldots, \widehat{b^L}$ and $\hat{\tilde{a}}, \widehat{\tilde{b}^1}, \ldots, \widehat{\tilde{b}^L}$ be 2π-periodic trigonometric polynomials in d-variables with $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. Define ϕ and $\tilde{\phi}$ by*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi) \quad \text{and} \quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j}\xi), \quad \xi \in \mathbb{R}^d \tag{1.6}$$

and the wavelet functions $\psi^1, \ldots, \psi^L, \tilde{\psi}^1, \ldots, \tilde{\psi}^L$ by

$$\widehat{\psi^\ell}(2\xi) := \widehat{b^\ell}(\xi)\hat{\phi}(\xi) \quad \text{and} \quad \widehat{\tilde{\psi}^\ell}(2\xi) := \widehat{\tilde{b}^\ell}(\xi)\hat{\tilde{\phi}}(\xi), \quad \xi \in \mathbb{R}^d, \ \ell = 1, \ldots, L. \tag{1.7}$$

Suppose that

(1) *the following identity holds*

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi + \gamma\pi) + \sum_{\ell=1}^{L} \overline{\widehat{b^\ell}(\xi)}\widehat{\tilde{b}^\ell}(\xi + \gamma\pi) = \delta_\gamma, \quad \gamma \in \{0, 1\}^d; \tag{1.8}$$

(2) *for a real number $s \in \mathbb{R}$ satisfying*

$$\nu_2(\phi) > s \quad \text{and} \quad \nu_2(\tilde{\phi}) > -s, \tag{1.9}$$

there exist nonnegative numbers α and $\tilde{\alpha}$, with $\alpha > -s$ and $\tilde{\alpha} > s$, such that the following conditions on vanishing moments hold:

$$\widehat{b^\ell}(\xi) = O(\|\xi\|^\alpha) \quad \text{and} \quad \widehat{\tilde{b}^\ell}(\xi) = O(\|\xi\|^{\tilde{\alpha}}), \quad \xi \to 0, \ \ell = 1, \ldots, L. \tag{1.10}$$

Then $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ which implies that (1.4) holds. Furthermore, there are positive *constants C_1 and C_2 such that*

$$C_1\|g\|^2_{H^{-s}(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^{\ell,s}_{j,k}\rangle\right|^2 \leqslant C_2\|g\|^2_{H^{-s}(\mathbb{R}^d)},$$
$$g\in H^{-s}(\mathbb{R}^d) \tag{1.11}$$

and

$$C_2^{-1}\|f\|^2_{H^{s}(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d} \left|\langle f, \tilde\phi_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle f, \tilde\psi^{\ell,-s}_{j,k}\rangle\right|^2 \leqslant C_1^{-1}\|f\|^2_{H^{s}(\mathbb{R}^d)},$$
$$f\in H^{s}(\mathbb{R}^d). \tag{1.12}$$

When $s=0$, Theorem 1.1 was obtained in [50] as the mixed extension principle in $L_2(\mathbb{R}^d)$ which is a generalization of the unitary extension principle of [49]. The mixed extension principle was stated in [50] under the assumptions that both systems are Bessel in $L_2(\mathbb{R}^d)$ whose masks satisfy (1.8). When all masks are trigonometric polynomials, the above result asserts that the Bessel property is satisfied if Item (2) of Theorem 1.1 holds with $s=0$ and $\alpha=\tilde\alpha=1$. In fact, for a compactly supported refinable function $\phi\in L_2(\mathbb{R}^d)$ whose refinement mask is a trigonometric polynomial, by [30, Theorem 2.2] $\phi\in H^{\alpha}(\mathbb{R})$ for some $\alpha>0$. Hence (1.9) holds with $s=0$ for both ϕ and $\tilde\phi$. Furthermore, it was shown in [30, Theorem 2.3] that both $X^0(\phi;\psi^1,\dots,\psi^L)$ and $X^0(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^L)$ are Bessel in $L_2(\mathbb{R}^d)$ whenever (1.10) holds with $\alpha=\tilde\alpha=1$. Therefore, for trigonometric polynomial masks, if (1.8) and (1.10) hold, then $(X^0(\phi;\psi^1,\dots,\psi^L), X^0(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^L))$ is a pair of dual wavelet frames in $L_2(\mathbb{R}^d)$, as already given in [11, 20, 21], but without the assumptions of Bessel properties of either system. Since it is not easy to check the Bessel property, Theorem 1.1 gives a new input even for this well-understood case. For the refinable function ϕ defined in (1.6) with a trigonometric polynomial mask $\hat a$, the quantity $\nu_2(\phi)$ can be computed via its mask (see, e.g., [31, Algorithm 2.1], [32, Theorem 4.1], and [29, 34, 36, 43, 51]).

More importantly, Theorem 1.1 also gives the mixed extension principle for a pair of dual systems $X^s(\phi;\psi^1,\dots,\psi^L)$ and $X^{-s}(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^L)$ in the pair of spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ for $s\neq 0$. Without loss of generality, assuming that $s>0$, then Theorem 1.1 says that the regularity of ϕ, the vanishing moments of $\tilde\psi^\ell$, $\ell=1,\dots,L$, and (1.8) make both systems form a pair of dual frames in the pair of spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. It is not required that $\tilde\phi\in L_2(\mathbb{R}^d)$ or ψ^ℓ, $\ell=1,\dots,L$, have any order of vanishing moments. In other words, it could be that neither of the systems is a frame in $L_2(\mathbb{R}^d)$; e.g., one of the system's wavelets may not have the required vanishing moments and the other system's wavelets may not be in $L_2(\mathbb{R}^d)$. It gives great flexibility in constructions by dropping the requirement that both wavelet frames should be in $L_2(\mathbb{R}^d)$. For example, as stated in the next result, for a refinable function $\phi\in H^s(\mathbb{R}^d)$ with a trigonometric polynomial refinement mask, $\{2^{j(d/2-s)}\phi(2^j\cdot -k) : j\in\mathbb{N}_0, k\in\mathbb{Z}^d\}$ is a frame in $H^s(\mathbb{R}^d)$. This is related to [20, Corollary 3.4] for dimension one and is of interest in numerical analysis as shown in [4, 17, 18]. Note that this system cannot form a frame in $L_2(\mathbb{R}^d)$, since its wavelets do not have any vanishing moments.

Inequalities (1.11) and (1.12) yield a characterization of the Sobolev norm of a function in either $H^s(\mathbb{R})$ or $H^{-s}(\mathbb{R})$ via the weighted ℓ_2-norm of its wavelet coefficient sequence in (1.4). Characterization of the Sobolev norm and more general Besov norm of a function in terms of its weighted wavelet coefficient sequence has already been given in [2, 3, 41] using a pair of dual wavelet frames in $L_2(\mathbb{R}^d)$, under the assumption that both wavelet frames must have a certain required order of regularity and vanishing moments simultaneously. Our approach is different from those in the literature where a pair of dual frame systems in $L_2(\mathbb{R}^d)$ is first constructed, then it is shown that the weighted ℓ_2-norm of the wavelet coefficients of a function in $H^s(\mathbb{R}^d)$ is equivalent to its Sobolev norm (see, e.g., [2, 3, 16, 17, 26, 41]). This approach in the literature requires that wavelet systems have positive regularity and vanishing moments. On the other hand, Theorem 1.1 completely separates the vanishing moments and regularity of two competing requirements for two systems. One can require the analysis system to have the desired order of vanishing moments to achieve the sparsity, while requiring the synthesis system to have the desired order of regularity for representing functions in $H^s(\mathbb{R}^d)$.

As we will see in the next result, it is much easier to derive a frame system in $H^s(\mathbb{R}^d)$ with its dual frame in $H^{-s}(\mathbb{R}^d)$ than a pair of dual wavelet frames in $L_2(\mathbb{R}^d)$. To state the result, we first introduce some notation: For a smooth function $f : \mathbb{R}^d \mapsto \mathbb{C}$ and $j = 1, \dots, d$, we denote by $\partial_j f$ the partial derivative of f with respect to the jth coordinate. For $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we denote $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$ and $|\beta| := \beta_1 + \cdots + \beta_d$.

We say that a mask $\hat{a}$ satisfies the *sum rules* of order m if $\partial^\beta \hat{a}(\gamma\pi) = 0$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| < m$ and for all $\gamma \in \{0, 1\}^d \backslash \{0\}$. In particular, we denote by $sr(\hat{a})$ the largest nonnegative integer m such that $\hat{a}$ satisfies the sum rules of order m. As an application of Theorem 1.1, we have the following result:

Theorem 1.2 *Let $\hat{a}$ be a 2π-periodic trigonometric polynomial in d-variables with $\hat{a}(0) = 1$. Define ϕ by $\hat{\phi}(\xi) := \prod_{j=1}^\infty \hat{a}(2^{-j}\xi)$ for $\xi \in \mathbb{R}^d$. Assume that $\phi \in L_2(\mathbb{R}^d)$ (which implies $\nu_2(\phi) > 0$ by* [30, *Theorem* 2.2] *or* [32, *Corollary* 4.2]). *Then for any real number s such that $0 < s < \min(\nu_2(\phi), sr(\hat{a}))$,*

$$\left\{2^{j(d/2-s)}\phi(2^j \cdot -k) : j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d\right\} \tag{1.13}$$

is a wavelet frame in $H^s(\mathbb{R}^d)$.

Box splines are important examples of refinable functions in several variables. For a given $d \times m$ (direction) matrix Ξ of full rank with integer entries and $m \geqslant d$, the Fourier transform of its associated box spline M_Ξ is given by

$$\widehat{M_\Xi}(\xi) := \prod_{k \in \Xi} \frac{1 - e^{-ik\cdot\xi}}{ik \cdot \xi}, \quad \xi \in \mathbb{R}^d, \tag{1.14}$$

where $k \in \Xi$ means that k is a column vector of Ξ and k goes through all the columns of Ξ once and only once. The box spline M_Ξ is refinable, and its refinement mask is given by

$$\widehat{a_\Xi}(\xi) = \prod_{k\in\Xi} \frac{1+e^{-ik\cdot\xi}}{2}, \quad \xi \in \mathbb{R}^d. \tag{1.15}$$

The box spline M_Ξ belongs to $C^{m(\Xi)-1}$, where $m(\Xi)+1$ is the minimum number of columns that can be discarded from Ξ to obtain a matrix of rank $< d$. In other words, we have $\nu_2(M_\Xi) = m(\Xi) + 1/2$. When Ξ is a $1 \times m$ row vector with all its components being 1, the box spline M_Ξ is the well-known B-spline of order m and has the mask $2^{-m}(1+e^{-i\xi})^m$. The reader should consult [1] for more details on box splines. Theorem 1.2 can be applied to the box splines to obtain frame systems with short supports in Sobolev spaces.

Example 1.3 Let M_Ξ be the box spline defined in (1.14) with a direction matrix Ξ. Then $\nu_2(M_\Xi) = m(\Xi) + 1/2$ and $\widehat{M_\Xi}(2\xi) = \widehat{a_\Xi}(\xi)\widehat{M_\Xi}(\xi)$, where the mask $\widehat{a_\Xi}$ is defined in (1.15). Now by Theorem 1.2, we see that for any s such that $0 < s < \min(m(\Xi)+1/2, sr(\widehat{a_\Xi}))$,

$$\left\{2^{j(d/2-s)} M_\Xi(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}^d\right\} \tag{1.16}$$

is a wavelet frame in $H^s(\mathbb{R}^d)$. We mention two particularly interesting examples. The first example is the univariate B-spline. Let $\Xi = [1,\dots,1]$ be a $1\times m$ direction matrix, M_Ξ (also denoted by B_m in the abstract) is the B-spline of order m whose mask is given by $\widehat{a_\Xi}(\xi) = 2^{-m}(1+e^{-i\xi})^m$. Since $sr(\widehat{a_\Xi}) = m$ and $\nu_2(M_\Xi) = m - 1/2$, we have $\min(\nu_2(M_\Xi), sr(\widehat{a_\Xi})) = m - 1/2$, and the wavelet system in (1.16) is a wavelet frame in $H^s(\mathbb{R})$ for all $0 < s < m - 1/2$.

The second example is a special case of box splines in $d > 1$ variables. Let Ξ be the direction matrix of order $d \times m(d+1)$ whose column vectors consist of m copies of the following $d+1$ column vectors: $(1,0,\dots,0)^T, (0,1,0,\dots,0)^T, \dots, (0,\dots,0,1)^T$ and $(1,1,\dots,1)^T$. Then we have $\nu_2(M_\Xi) = 2m - 1/2$ and $sr(\widehat{a_\Xi}) = 2m$. Therefore, we have $\min(\nu_2(M_\Xi), sr(\widehat{a_\Xi})) = 2m - 1/2$ and the wavelet system in (1.16) is a wavelet frame in $H^s(\mathbb{R}^d)$ for all $0 < s < 2m - 1/2$.

1.2 Dual Riesz Wavelet Bases

A pair of dual wavelet frame systems $X^s(\phi; \psi^1,\dots,\psi^L)$ and $X^{-s}(\tilde\phi; \tilde\psi^1,\dots,\tilde\psi^L)$ becomes a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if and only if the following biorthogonality condition holds: for all $j, j \in \mathbb{N}_0$, $k, k' \in \mathbb{Z}^d$ and $\ell, \ell' = 1,\dots,L$,

$$\begin{aligned} &\langle \phi_{0,k}, \tilde\phi_{0,k'}\rangle = \delta_{k-k'}, \qquad \left\langle \psi^{\ell,s}_{j,k}, \tilde\psi^{\ell',-s}_{j',k'}\right\rangle = \delta_{j-j'}\delta_{k-k'}\delta_{\ell-\ell'}, \\ &\left\langle \phi_{0,k}, \tilde\psi^{\ell',-s}_{j',k'}\right\rangle = 0, \qquad \left\langle \psi^{\ell,s}_{j,k}, \tilde\phi_{0,k'}\right\rangle = 0, \end{aligned} \tag{1.17}$$

where δ denotes the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$.

It is well known that when $s = 0$ and $L = 2^d - 1$, (1.17) can be reduced to condition $\langle \phi_{0,k}, \tilde\phi_{0,k'}\rangle = \delta_{k-k'}$ as shown in [15, 50]. Furthermore, this can be characterized in terms of masks $\hat{a}$ and $\hat{\tilde{a}}$ (see, e.g., [34, Theorem 3.1] and [28, 29, 52]). To state the result for the more general case $s \neq 0$, we need to recall the definition of a quantity $\nu_2(\hat{a})$ from [29] (also see [31, 32, 34]). For a 2π-periodic function $\hat{a}$ in d-variables such that $sr(\hat{a}) = m$, that is, $\hat{a}$ satisfies the sum rules of order m but not $m+1$, we define a quantity $\nu_2(\hat{a})$ to be ([29, Page 61] and [31])

$$\nu_2(\hat{a}) := -d/2 - \log_2 \max\Big\{\limsup_{n\to\infty} \big\|\nabla^\beta a^n\big\|_{\ell_2(\mathbb{Z}^d)}^{1/n} : \beta \in \mathbb{N}_0^d, |\beta| < m\Big\}, \tag{1.18}$$

where $\|\nabla^\beta a^n\|_{\ell_2(\mathbb{Z}^d)}^2 := \sum_{k\in\mathbb{Z}^d} |[\nabla^\beta a^n]_k|^2$ and the sequence $\{[\nabla^\beta a^n]_k\}_{k\in\mathbb{Z}^d}$ is determined by: for $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \big[\nabla^\beta a^n\big]_k e^{-ik\cdot\xi} &= \widehat{\nabla^\beta a^n}(\xi) \\ &:= \big(1 - e^{-i\xi_1}\big)^{\beta_1} \cdots \big(1 - e^{-i\xi_d}\big)^{\beta_d} \hat{a}\big(2^{n-1}\xi\big) \cdots \hat{a}(2\xi)\hat{a}(\xi).\end{aligned} \tag{1.19}$$

It is known that $\nu_2(\hat{a}) \leqslant \nu_2(\phi)$ ([29, 32]) where ϕ is the compactly supported refinable function associated with the 2π-periodic trigonometric polynomial $\hat{a}$. The equality holds when the underlying refinable function ϕ and its shifts form a Riesz system. Details can be found in [28, 29, 31, 32, 34, 36, 43, 48, 51].

When $\hat{a}$ is a 2π-periodic trigonometric polynomial, it is known in the literature ([19], [31, Algorithm 2.1], and [34, Sect. 4]) that the quantity $\nu_2(\hat{a})$ can be easily computed by calculating the spectral radius of a finite matrix. For the convenience of the reader, let us recall the procedure from [19] for the univariate case here. We write $\hat{a}(\xi) = (1 + e^{-i\xi})^m \hat{c}(\xi)$ for some nonnegative integer m and some 2π-periodic trigonometric polynomial $\hat{c}(\xi)$ with $\hat{c}(\pi) \neq 0$. We write $|\hat{c}(\xi)|^2 = \sum_{k=-K}^{K} c_k e^{-ik\xi}$, where K is some nonnegative integer. Denote by $\rho(\hat{a})$ the spectral radius of the square matrix $(c_{2j-k})_{-K\leqslant j,k\leqslant K}$. Then $\nu_2(\hat{a}) = -1/2 - \log_2 \sqrt{\rho(\hat{a})}$. Note that if $\hat{a} = c$ is a nonzero constant, then by the definition of $\nu_2(\hat{a})$ in (1.18) we have $\rho(\hat{a}) = |c|^2$ and $\nu_2(\hat{a}) = -1/2 - \log_2 |c|$.

As an application of Theorem 3.1 that is stated in a more general multivariate setting, we have the following result on pairs of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$:

Corollary 1.4 *Let $\hat{a}$ and $\hat{\tilde{a}}$ be 2π-periodic trigonometric polynomials with $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. Define ϕ and $\tilde{\phi}$ by*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \hat{a}\big(2^{-j}\xi\big) \quad \textit{and} \quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}\big(2^{-j}\xi\big), \quad \xi \in \mathbb{R} \tag{1.20}$$

and the wavelet functions ψ and $\tilde{\psi}$ by

$$\hat{\psi}(2\xi) := \overline{\eta} e^{-i\xi} \overline{\hat{\tilde{a}}(\xi+\pi)} \hat{\phi}(\xi) \quad \textit{and} \quad \hat{\tilde{\psi}}(2\xi) := \eta^{-1} e^{-i\xi} \overline{\hat{a}(\xi+\pi)} \hat{\tilde{\phi}}(\xi), \quad \xi \in \mathbb{R}, \tag{1.21}$$

where η is any nonzero complex constant. Let $s \in \mathbb{R}$ be a real number. Then $(X^s(\phi;\psi), X^{-s}(\tilde{\phi};\tilde{\psi}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ if and only if

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) + \overline{\hat{a}(\xi+\pi)}\hat{\tilde{a}}(\xi+\pi) = 1, \quad \xi \in \mathbb{R} \tag{1.22}$$

and

$$\nu_2(\hat{a}) > s \quad \text{and} \quad \nu_2(\hat{\tilde{a}}) > -s. \tag{1.23}$$

When $s = 0$, the above result is well-known in the wavelet analysis on pairs of biorthogonal wavelet bases in $L_2(\mathbb{R})$ (e.g., [15, 19] and [34, Theorem 3.1]). When $s \neq 0$, say $s > 0$, then $\tilde{\phi}$ could be a distribution, rather than a function in $L_2(\mathbb{R})$; hence $\tilde{\psi} \notin L_2(\mathbb{R})$. Furthermore, $\hat{\tilde{a}}$ may not have a zero at π. In this case, ψ has no vanishing moments. As a result, neither $X^s(\phi;\psi)$ nor $X^{-s}(\tilde{\phi};\tilde{\psi})$ is a Riesz system in $L_2(\mathbb{R})$; however, $(X^s(\phi;\psi), X^{-s}(\tilde{\phi};\tilde{\psi}))$ forms a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$.

This is a different approach from those in the literature. A widely used approach introduced by [46] in the literature starts with a Riesz wavelet basis $X^0(\phi;\psi)$ in $L_2(\mathbb{R})$ with a certain required order of regularity, then shows that $X^s(\phi;\psi)$, as a properly renormalized system of $X^0(\phi;\psi)$ in $H^s(\mathbb{R})$, is also a Riesz wavelet basis in $H^s(\mathbb{R})$ for some range of s (also see [13, 16, 17, 19]). This implies that the wavelet must have a certain required order of regularity and vanishing moments simultaneously. When orthonormal wavelet systems in $L_2(\mathbb{R})$ are used, it is natural to demand that the wavelet have both regularity and vanishing moments, since there is only one wavelet that plays both roles. This approach is also applied to obtain Riesz bases in $H^s(\mathbb{R})$ from other biorthogonal Riesz wavelets in $L_2(\mathbb{R})$, but it requires the pair to be a Riesz basis of $L_2(\mathbb{R})$. There are no complete and systematic discussions on dual Riesz bases in dual spaces for general $s \neq 0$ in the setting of this paper as far as we know. Hence, there is no complete study of the Riesz property and its dual basis of the class of interpolatory wavelet systems introduced by [25], since the interpolatory wavelet system is not a Riesz basis of $L_2(\mathbb{R})$ due to the fact that the wavelet function $\psi = \phi(2\cdot -1)$ has no vanishing moment and its dual wavelet is a distribution. Our approach is to have two dual Riesz wavelet bases in dual spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$, respectively, such that one of them has the required order of regularity and the other one has the required order of vanishing moments, instead of demanding both systems to be Riesz bases in $L_2(\mathbb{R})$. This is exactly the case for the interpolatory wavelets [25].

For $m \in \mathbb{N}$, consider the mask

$$A_m(\xi) := \cos^{2m}(\xi/2) P_{m,m}\big(\sin^2(\xi/2)\big) \quad \text{with } P_{m,m}(x) := \sum_{j=0}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^j.$$

It is well known ([19]) that $A_m(\xi) + A_m(\xi+\pi) = 1$. Furthermore, the corresponding refinable function ϕ_m is interpolatory, i.e., it is continuous and satisfies $\phi_m(k) = \delta_k$ for all $k \in \mathbb{Z}$. It has been proved in [19] (also see [24]) that $\lim_{m\to\infty} \nu_2(A_m) = +\infty$. The interpolatory wavelets introduced in [25] are constructed from such interpolatory refinable functions.

Example 1.5 For a positive integer m, let

$$\widehat{a_m}(\xi) = A_m(\xi),$$

and its corresponding interpolatory refinable function be ϕ_m. Let $\tilde{\phi}_m := \delta$. Then $\tilde{\phi}_m$ is refinable and $\hat{\tilde{\phi}}_m = 1$ with its refinement mask $\widehat{\tilde{a}_m} = 1$. Recall that $\nu(\widehat{a_1}) = 3/2$, $\nu(\widehat{a_m}) > \nu(\widehat{a_1})$ for all $m > 1$, and $\lim_{m\to\infty} \nu_2(\widehat{a_m}) = +\infty$. Note also that $\nu(\widehat{\tilde{a}_m}) = -1/2$, for all m. Let $\widehat{b_m}(\xi) = \frac{1}{2}e^{-i\xi}$ and $\widehat{\tilde{b}_m}(\xi) = 2e^{-i\xi}\widehat{a_m}(\xi+\pi) = 2e^{-i\xi}(1 - \widehat{a_m}(\xi))$. Then $\psi_m = \phi_m(2\cdot -1)$. The pair of systems $(X^s(\phi_m;\psi_m), X^{-s}(\tilde{\phi}_m;\tilde{\psi}_m))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ for any $s \in (1/2, \nu_2(\widehat{a_m}))$ by Corollary 1.4.

For $f \in H^s(\mathbb{R})$ with $1/2 < s < \nu_2(\widehat{a_m})$, let $\phi = \phi_m$ and $\psi = \psi_m$, and $\tilde{\phi} = \tilde{\phi}_m$ and $\tilde{\psi} = \tilde{\psi}_m$ be the corresponding dual distributions. Let $\phi^s_{j,k} = 2^{j(1/2-s)}\phi(2^j\cdot -k)$ and $\tilde{\phi}^{-s}_{j,k} = 2^{j(1/2+s)}\tilde{\phi}(2^j\cdot -k)$, $j > 0$ and $\phi_{0,k} = \phi(\cdot - k)$ and $\tilde{\phi}_{0,k} = \tilde{\phi}(\cdot - k)$. The interpolation operator at the jth dyadic level is defined as

$$\mathcal{P}_j f := \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}^{-s}_{j,k}\rangle \phi^s_{j,k}.$$

Using the fact $\langle f, 2^j\tilde{\phi}(2^j\cdot)\rangle = f(0)$, one obtains

$$\mathcal{P}_j f = \sum_{k\in\mathbb{Z}} 2^{j(s-1/2)} f\big(k/2^j\big)\phi^s_{j,k} = \sum_{k\in\mathbb{Z}} f\big(k/2^j\big)\phi\big(2^j\cdot -k\big).$$

Therefore,

$$\begin{aligned} f &= \mathcal{P}_0 f + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} (f - \mathcal{P}_j f)\big((2k+1)/2^{j+1}\big)\psi\big(2^j\cdot -1-k\big) \\ &= \mathcal{P}_0 f + \sum_{j=0}^{\infty} 2^{j(s-1/2)} \sum_{k\in\mathbb{Z}} (f - \mathcal{P}_j f)\big((2k+1)/2^{j+1}\big)\psi^s_{j,k} \\ &= \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{0,k}\rangle\phi_{0,k} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^{-s}_{j,k}\rangle\psi^s_{j,k}. \end{aligned}$$

Since $X^s(\phi;\psi)$ is a Riesz basis of $H^s(\mathbb{R})$, the ℓ_2-norm of the sequence

$$\big\{f(k), 2^{j(s-1/2)}(f - \mathcal{P}_j f)\big((2k+1)/2^{j+1}\big)\big\}_{k\in\mathbb{Z}, j\geqslant 0}$$

is equivalent to the Sobolev norm of f in $H^s(\mathbb{R})$. This coincides with the characterization of the regularity of a given function in terms of its interpolatory wavelet coefficients in [25] (restricted to a Sobolev space). However, the characterization in [25] itself only gives the "frame" property of the wavelet system in $H^s(\mathbb{R})$ which does not imply the Riesz property of the system. Here, we have shown that the interpolatory wavelet system is a Riesz basis in $H^s(\mathbb{R})$ if and only if $1/2 < s < \nu_2(\widehat{a_m}) = \nu_2(\phi_m)$. Note that $\nu_2(\widehat{a_m}) \geqslant \nu_2(\widehat{a_1}) = 3/2$.

A special example of this class of interpolatory wavelets which is used in numerical analysis is the hierarchical basis. The hierarchical (wavelet) basis is generated by piecewise linear polynomial $\phi = \max(0, 1 - |x|)$ and $\psi = \phi(2 \cdot -1)$. It is known that $X^1(\phi; \psi)$ indeed is a Riesz basis in the Sobolev space $H^1(\mathbb{R})$, (see [16–18] and many references therein). The above example shows that $X^s(\phi; \psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$ for all $1/2 < s < 3/2$. For this fact, also see [45].

It is well known (e.g., see [15, 19, 28, 42]) that the interpolatory refinable function can also be used to construct biorthogonal wavelets in $L_2(\mathbb{R})$. The main idea is to factorize the mask A_m into a pair of dual masks. In order to make the construction work, it is important to make sure that both corresponding refinable functions are in $L_2(\mathbb{R})$. This idea of [15] is also used to construct biorthogonal wavelets from pseudo-splines by [23] and multivariate biorthogonal wavelets by [27, 28, 42]. However, since we allow one of ϕ or $\tilde{\phi}$ to be a distribution when we consider pairs of dual Riesz wavelet systems in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, it gives us more freedom in the choice of factorization. The following example shows that the mask A_m can be factorized in a very arbitrary way to get dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ by applying Corollary 1.4.

Example 1.6 For a positive integer m and $0 \leqslant \ell \leqslant 2m$, denote

$$\hat{a}(\xi) := 2^{-\ell}\big(1 + e^{-i\xi}\big)^{\ell} \quad \text{and}$$
$$\hat{\hat{a}}(\xi) := 2^{\ell-2m} e^{i(m-\ell)\xi}\big(1 + e^{-i\xi}\big)^{2m-\ell} P_{m,m}\big(\sin^2(\xi/2)\big).$$

Then it is obvious that $\overline{\hat{a}(\xi)}\hat{\hat{a}}(\xi) = A_m(\xi)$, and so (1.22) is satisfied. It is well known that $\nu_2(\hat{a}) = \ell - 1/2$ and $\nu_2(\hat{\hat{a}}) = \nu_2(A_m) - \ell$. For any real number s such that $s \in (\ell - \nu_2(A_m), \ell - 1/2)$, it is easy to check that $\nu_2(\hat{a}) > s$ and $\nu_2(\hat{\hat{a}}) > -s$. Define $\phi, \tilde{\phi}$ and $\psi, \tilde{\psi}$ as in (1.20) and (1.21). Therefore, by Corollary 1.4, $(X^s(\phi; \psi), X^{-s}(\tilde{\phi}; \tilde{\psi}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$. Since $\lim_{m\to\infty} \nu_2(A_m) = +\infty$ and $\nu_2(A_m) \geqslant 3/2$ (see, e.g., [19]), the interval $(\ell - \nu_2(A_m), \ell - 1/2)$ is not empty.

The second corollary of Theorem 3.1 gives a sufficient condition on the Riesz wavelet bases in $H^s(\mathbb{R})$. In order to state our result on dual Riesz wavelet bases in Sobolev spaces, let us introduce a notion $\mu_2(\hat{a})$, which is similar and closely related to the quantity $\nu_2(\hat{a})$ in (1.18) ([34, 35]). For a 2π-periodic function $\hat{a}$, we define $\mu_2(\hat{a})$ to be the supremum of all $\nu_2(\hat{\hat{a}})$ such that $|\hat{a}(\xi)| \leqslant |\hat{\hat{a}}(\xi)|$ for almost every $\xi \in \mathbb{R}^d$ and $\hat{\hat{a}}$ is a 2π-periodic trigonometric polynomial with $\hat{\hat{a}}(0) = 1$. If $\hat{a}$ is a 2π-periodic trigonometric polynomial with $\hat{a}(0) = 1$, it is evident that $\mu_2(\hat{a}) = \nu_2(\hat{a})$ (see [34]).

Corollary 1.7 *Let $\hat{a}$ and $\hat{b}$ be 2π-periodic functions with exponential decay such that $\hat{a}(0) = 1$ and $\hat{a}(\pi)\hat{b}(0) = 0$. Define ϕ and ψ by*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \hat{a}\big(2^{-j}\xi\big) \quad \textit{and} \quad \hat{\psi}(2\xi) = \hat{b}(\xi)\hat{\phi}(\xi), \quad \xi \in \mathbb{R}. \tag{1.24}$$

Assume that

$$d(\xi) := \hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}, \tag{1.25}$$

and define $\hat{\tilde{a}}(\xi) := \overline{\hat{b}(\xi+\pi)}/\overline{d(\xi)}$. *Suppose that*

$$\mu_2(\hat{a}) > s \quad \textit{and} \quad \mu_2(\hat{\tilde{a}}) > -s \tag{1.26}$$

hold for some real number $s \in \mathbb{R}$. *Then* $X^s(\phi;\psi)$ *is a Riesz wavelet basis in* $H^s(\mathbb{R})$.

A similar result to Corollary 1.7 for the case $s \geqslant 0$ is obtained in [45] under a different set of conditions. It is required in [45] that the compactly supported refinable function ϕ is in $L_2(\mathbb{R})$ with a 2π-periodic trigonometric polynomial mask and its shifts are stable. We allow ϕ to have an exponentially decaying mask and $s \in \mathbb{R}$ in Corollary 1.7 instead of $s \geqslant 0$ in [45]. The stability condition on ϕ is not explicitly assumed in Corollary 1.7, although it is in fact implicitly implied there by the mask conditions. Hence, there is no need for us to check the stability of ϕ, which could be a highly nontrivial task. Moreover, it has been shown in [34, Theorem 3.2] (also see [37]), Corollary 1.7 is sharp for $s = 0$, that is, the condition in (1.25) and (1.26) is both necessary and sufficient for $X^0(\phi;\psi)$ to be a Riesz wavelet basis in $L_2(\mathbb{R})$. Riesz wavelets in $L_2(\mathbb{R}^d)$ have been studied in a few other papers (see, e.g., [12, 14, 16, 23, 24, 33–35, 39, 40, 44, 47]).

The next example shows that by applying Corollary 1.7, one can have a simple construction of Riesz wavelet bases in $H^s(\mathbb{R})$ from splines with very short support. The short-supported Riesz wavelet bases in $L_2(\mathbb{R})$ derived from splines were studied in [39]. The analysis here is simpler by applying Corollary 1.7, since checking the Bessel property of the dual wavelets becomes straightforward due to Theorem 2.3. Recall that the most difficult part in [39] is to prove the Bessel property in $L_2(\mathbb{R})$ of the dual system. This example also shows that the frame system given in the abstract has a subsystem that forms a Riesz basis in $H^s(\mathbb{R})$ for $(m-1)/2 < s < m-1/2$ (or $m/2 < s < m-1/2$) when m is even (or odd).

Example 1.8 For any positive even integer $m \geqslant 2$, we denote

$$\hat{a}(\xi) := 2^{-m}\left|1+e^{-i\xi}\right|^m = \cos^m(\xi/2) \quad \text{and} \quad \hat{b}(\xi) := e^{-i\xi}.$$

Let ϕ and ψ be defined in (1.24). Note that $\psi = 2\phi(2\cdot -1)$. Then ϕ is the B-spline of order m and $\nu_2(\hat{a}) = m - 1/2$. Now we show that $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$ for all $m/2 - 1/2 < s < m - 1/2$. Note that

$$d(\xi) := -e^{-i\xi}\left[\cos^m(\xi/2) + \sin^m(\xi/2)\right] \neq 0 \quad \forall \xi \in \mathbb{R} \quad \text{and}$$
$$\hat{\tilde{a}}(\xi) := \frac{1}{\cos^m(\xi/2) + \sin^m(\xi/2)}.$$

It is easy to check that $\cos^m(\xi/2) + \sin^m(\xi/2) \geqslant 2^{1-m/2}$ for all $\xi \in \mathbb{R}$. Therefore, $|\hat{\tilde{a}}(\xi)| \leqslant 2^{m/2-1}$ for all $\xi \in \mathbb{R}$. By [34, Theorem 4.1 and Lemma 4.3], we have

$\nu_2(\hat{\tilde{a}}) \geqslant 1/2 - m/2$. By Corollary 1.7, $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$ for all $m/2 - 1/2 < s < m - 1/2$.

For an odd integer $m \geqslant 2$, define

$$\hat{a}(\xi) := 2^{-m} e^{i\xi(m-1)/2}\big(1+e^{-i\xi}\big)^m \quad \text{and} \quad \hat{b}(\xi) := 1.$$

Let ϕ and ψ be defined in (1.24). Note that $\psi = 2\phi(2\cdot)$. Then ϕ is the B-spline of order m and by [34, Theorem 4.1], we have $\nu_2(\hat{a}) = m - 1/2$. Now we show that $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$ for all $m/2 < s < m - 1/2$. Note that

$$d(\xi) := e^{-i\xi/2}\big[\cos^m(\xi/2) - i\sin^m(\xi/2)\big] \neq 0 \quad \forall \xi \in \mathbb{R} \quad \text{and}$$
$$\big|\hat{\tilde{a}}(\xi)\big| = \frac{1}{\sqrt{\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)}}.$$

It is easy to check that $\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2) \geqslant 2^{1-m}$ for all $\xi \in \mathbb{R}$. Therefore, $|\hat{\tilde{a}}(\xi)| \leqslant 2^{m/2-1/2}$ for all $\xi \in \mathbb{R}$. By [34, Theorem 4.1 and Lemma 4.3], we have $\nu_2(\hat{\tilde{a}}) \geqslant -m/2$. By Corollary 1.7, $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$ for all $m/2 < s < m - 1/2$.

2 Dual Wavelet Frames in Sobolev Spaces

In this section, we study a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ derived from a pair of refinable functions. First, we relate the dual wavelet frame property to the characterization of a Sobolev space by using the weighted norm of a wavelet frame coefficient sequence (decomposition sequence). Secondly, we give sufficient conditions for a Bessel wavelet sequence in $H^s(\mathbb{R}^d)$. Then we state the mixed extension principle in Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. We shall present the proofs to Theorems 1.1 and 1.2 at the end of this section.

2.1 Characterization of Sobolev Spaces

First, we give the following result on wavelet frames in Sobolev spaces, which roughly says that the frame or Bessel property of a system $X^s(\phi;\psi^1,\dots,\psi^L)$ in $H^s(\mathbb{R}^d)$ can be checked by using all the elements in $H^{-s}(\mathbb{R}^d)$ instead of those in $H^s(\mathbb{R}^d)$ as stated in the definition.

Proposition 2.1 *Let $s \in \mathbb{R}$ and $\phi, \psi^1, \dots, \psi^L \in H^s(\mathbb{R}^d)$. Then $X^s(\phi;\psi^1,\dots,\psi^L)$ is a wavelet frame in $H^s(\mathbb{R}^d)$ (i.e., (1.2) holds with two positive constants C_1 and C_2), if and only if*

$$C_1\|g\|^2_{H^{-s}(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d} \big|\langle g, \phi_{0,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \big|\langle g, \psi^{\ell,s}_{j,k}\rangle\big|^2 \leqslant C_2\|g\|^2_{H^{-s}(\mathbb{R}^d)},$$
$$g \in H^{-s}\big(\mathbb{R}^d\big). \tag{2.1}$$

Proof Define an operator $\theta_s : H^s(\mathbb{R}^d) \mapsto H^{-s}(\mathbb{R}^d)$ by $\widehat{\theta_s(f)}(\xi) := \hat{f}(\xi)(1 + \|\xi\|^2)^s, \xi \in \mathbb{R}^d$. Then

$$\begin{aligned}\|\theta_s(f)\|^2_{H^{-s}(\mathbb{R}^d)} &:= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big|\widehat{\theta_s(f)}(\xi)\big|^2\big(1+\|\xi\|^2\big)^{-s}\,d\xi\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big|\hat{f}(\xi)\big|^2\big(1+\|\xi\|^2\big)^{s}\,d\xi = \|f\|^2_{H^s(\mathbb{R}^d)}.\end{aligned}$$

From this, it is easy to see that θ_s is an isometric and onto mapping between $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$. On the other hand, for $f, h \in H^s(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle f, h\rangle_{H^s(\mathbb{R}^d)} &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{h}(\xi)}\big(1+\|\xi\|^2\big)^{s}\,d\xi\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\xi)\big(1+\|\xi\|^2\big)^{s}\overline{\hat{h}(\xi)}\,d\xi\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\theta_s(f)}(\xi)\overline{\hat{h}(\xi)}\,d\xi = \big\langle\theta_s(f), h\big\rangle.\end{aligned}$$

This implies that (1.2) is equivalent to (2.1) by taking $g = \theta_s(f)$ in (2.1). $\square$

Let $X^s(\phi; \psi^1, \dots, \psi^L)$ be a frame in $H^s(\mathbb{R}^d)$. Basically, (2.1) in Proposition 2.1 says that for any $g \in H^{-s}(\mathbb{R}^d)$, its Sobolev norm $\|g\|_{H^{-s}(\mathbb{R}^d)}$ is equivalent to the ℓ_2-norm of its "wavelet coefficients" $\langle g, \phi_{0,k}\rangle$ and $\langle g, \psi^{\ell,s}_{j,k}\rangle$, $j \in \mathbb{N}_0, k \in \mathbb{Z}^d$ and $\ell = 1, \dots, L$. More precisely, noting that $\psi^{\ell,s}_{j,k} = 2^{-js}\psi^{\ell,0}_{j,k}$, we see that (2.1) is equivalent to

$$\begin{aligned}C_1\|g\|^2_{H^{-s}(\mathbb{R}^d)} &\leqslant \sum_{k\in\mathbb{Z}^d}\big|\langle g, \phi_{0,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}2^{-2js}\big|\big\langle g, \psi^{\ell,0}_{j,k}\big\rangle\big|^2\\ &\leqslant C_2\|g\|^2_{H^{-s}(\mathbb{R}^d)}, \quad g \in H^{-s}\big(\mathbb{R}^d\big).\end{aligned}\tag{2.2}$$

If $(X^s(\phi; \psi^1, \dots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, then we indeed have the wavelet representations for functions in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ in (1.4). Now, the Sobolev norm of any $g \in H^{-s}(\mathbb{R}^d)$ is indeed characterized by the weighted ℓ_2-norm of its wavelet coefficient sequence through (2.2). Similarly, we have

$$\begin{aligned}&C_2^{-1}\|f\|^2_{H^s(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d}\big|\langle f, \tilde{\phi}_{0,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}2^{2js}\big|\big\langle f, \tilde{\psi}^{\ell,0}_{j,k}\big\rangle\big|^2 \leqslant C_1^{-1}\|f\|^2_{H^s(\mathbb{R}^d)},\\ &\quad f \in H^s\big(\mathbb{R}^d\big).\end{aligned}\tag{2.3}$$

Altogether, we have:

Proposition 2.2 *Assume that* $(X^s(\phi;\psi^1,\dots,\psi^L), X^{-s}(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^L))$ *is a pair of dual wavelet frames in* $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. *Then* (1.11) *and* (1.12), *or equivalently*, (2.2) *and* (2.3), *respectively, hold.*

The frame property of the system $X^s(\phi;\psi^1,\dots,\psi^L)$ in $H^s(\mathbb{R}^d)$ implies that for $f\in H^s(\mathbb{R}^d)$ the inequality (1.2) holds. However, for the stability of the wavelet decomposition and reconstruction algorithm suggested in [21], one needs (2.3). The above result says that they are guaranteed by the fact that $X^s(\phi;\psi^1,\dots,\psi^L)$ and $X^{-s}(\tilde\phi;\tilde\psi^1,\dots,\tilde\psi^L)$ are a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

2.2 Bessel Properties in Sobolev Spaces

The following result gives sufficient conditions for a wavelet sequence to be Bessel in a Sobolev space. As stated in Theorem 2.3, the Bessel property of a wavelet sequence in $H^s(\mathbb{R}^d)$ is mainly determined by the regularity of the wavelet ψ (or the regularity of the refinable function from which ψ is derived) if $s>0$, and by the vanishing moments of the wavelet ψ if $s\leqslant 0$.

For two functions $f,g:\mathbb{R}^d\mapsto\mathbb{C}$, we define

$$[f,g]_t(\xi):=\sum_{k\in\mathbb{Z}^d} f(\xi+2\pi k)\overline{g(\xi+2\pi k)}\big(1+\|\xi+2\pi k\|^2\big)^t,\qquad \xi\in\mathbb{R}^d \text{ and } t\in\mathbb{R}. \tag{2.4}$$

A function $\phi\in H^s(\mathbb{R}^d)$ is stable if and only if there are $0<C_1\leqslant C_2<\infty$ such that inequalities

$$C_1\leqslant[\hat\phi,\hat\phi]_s\leqslant C_2$$

hold almost everywhere. The stability of ϕ is equivalent to the fact that ϕ and its integer shifts form a Riesz basis in $H^s(\mathbb{R}^d)$.

Theorem 2.3 *Let* $\phi\in H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, *be a function satisfying* $[\hat\phi,\hat\phi]_t\in L_\infty(\mathbb{R}^d)$ *for some* $t>s$. *Define* $\hat\psi(2\xi):=\hat b(\xi)\hat\phi(\xi)$, $\xi\in\mathbb{R}^d$, *where* $\hat b$ *is a* 2π*-periodic measurable function in d-variables. Assume that there exists a nonnegative number* $\alpha>-s$ (*we may choose* $\alpha=0$ *if* $s>0$) *and a positive constant* C *such that*

$$\big|\hat b(\xi)\big|\leqslant C\min\big(1,\|\xi\|^\alpha\big),\qquad a.e.\ \xi\in\mathbb{R}^d. \tag{2.5}$$

Then $X^s(\phi;\psi)$ *is a Bessel wavelet sequence in* $H^s(\mathbb{R}^d)$.

Proof By the proof of Proposition 2.1, to show the Bessel property of $X^s(\phi;\psi)$, it is enough to prove that there exists a positive constant C_1 such that

$$\sum_{k\in\mathbb{Z}^d}\big|\langle g,\phi_{0,k}\rangle\big|^2+\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|\langle g,\psi^s_{j,k}\rangle\big|^2\leqslant C_1\|g\|^2_{H^{-s}(\mathbb{R}^d)}\quad \forall g\in H^{-s}\big(\mathbb{R}^d\big), \tag{2.6}$$

where $\phi_{0,k}:=\phi(\cdot-k)$ and $\psi^s_{j,k}:=2^{j(d/2-s)}\psi(2^j\cdot-k)$.

First, by the Plancherel's theorem and Parseval's identity, it is not difficult to verify that

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g,\phi(\cdot-k)\rangle\right|^2=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left|[\hat{g},\hat{\phi}]_0(\xi)\right|^2\,d\xi. \tag{2.7}$$

By the Cauchy–Schwarz inequality, we have $|[\hat{g},\hat{\phi}]_0(\xi)|^2\leqslant[\hat{g},\hat{g}]_{-s}(\xi)[\hat{\phi},\hat{\phi}]_s(\xi)$. Since $t>s$ and $[\hat{\phi},\hat{\phi}]_t\in L_\infty(\mathbb{R}^d)$, it is straightforward to see that $[\hat{\phi},\hat{\phi}]_s(\xi)\leqslant[\hat{\phi},\hat{\phi}]_t(\xi)$. Therefore, $[\hat{\phi},\hat{\phi}]_s\in L_\infty(\mathbb{R}^d)$. Consequently, we deduce from (2.7) that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\left|\langle g,\phi_{0,k}\rangle\right|^2&\leqslant\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}[\hat{g},\hat{g}]_{-s}(\xi)[\hat{\phi},\hat{\phi}]_s(\xi)\,d\xi\\&\leqslant\left\|[\hat{\phi},\hat{\phi}]_s\right\|_{L_\infty(\mathbb{R}^d)}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}[\hat{g},\hat{g}]_{-s}(\xi)\,d\xi\\&=\left\|[\hat{\phi},\hat{\phi}]_s\right\|_{L_\infty(\mathbb{R}^d)}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\hat{g}(\xi)\right|^2\left(1+\|\xi\|^2\right)^{-s}d\xi\\&=\left\|[\hat{\phi},\hat{\phi}]_s\right\|_{L_\infty(\mathbb{R}^d)}\|g\|^2_{H^{-s}(\mathbb{R}^d)}.\end{aligned}$$

Similarly, we have

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g,\psi^s_{j,k}\rangle\right|^2=\frac{2^{j(d-2s)}}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left|\left[\hat{g}\left(2^j\cdot\right),\hat{\psi}\right]_0(\xi)\right|^2d\xi.$$

Now by the definition of ψ, we have $\hat{\psi}(\xi)=\hat{b}(\xi/2)\hat{\phi}(\xi/2)$ and it follows from the above identity that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\left|\langle g,\psi^s_{j,k}\rangle\right|^2\\&\quad=\frac{2^{-2js}2^{jd}}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left|\sum_{\gamma\in\{0,1\}^d}\hat{b}(\xi/2+\gamma\pi)\left[\hat{g}(2^{j+1}\cdot),\hat{\phi}\right]_0(\xi/2+\gamma\pi)\right|^2d\xi\\&\quad\leqslant\frac{2^{-2js}2^{(j+2)d}}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left|\hat{b}(\xi)\right|^2\left|\left[\hat{g}\left(2^{j+1}\cdot\right),\hat{\phi}\right]_0(\xi)\right|^2d\xi\\&\quad\leqslant\frac{2^{-2js}2^{(j+2)d}}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left|\hat{b}(\xi)\right|^2\left[\hat{g}\left(2^{j+1}\cdot\right),\hat{g}\left(2^{j+1}\cdot\right)\right]_{-t}(\xi)[\hat{\phi},\hat{\phi}]_t(\xi)\,d\xi\\&\quad\leqslant(2\pi)^{-d}\left\|[\hat{\phi},\hat{\phi}]_t\right\|_{L_\infty(\mathbb{R}^d)}2^{-2js}2^{(j+2)d}\\&\qquad\times\int_{[-\pi,\pi]^d}\left|\hat{b}(\xi)\right|^2\left[\hat{g}\left(2^{j+1}\cdot\right),\hat{g}\left(2^{j+1}\cdot\right)\right]_{-t}(\xi)\,d\xi\\&\quad=(2\pi)^{-d}\left\|[\hat{\phi},\hat{\phi}]_t\right\|_{L_\infty(\mathbb{R}^d)}2^{-2js}2^{(j+2)d}\int_{\mathbb{R}^d}\left|\hat{b}(\xi)\right|^2\left|\hat{g}\left(2^{j+1}\xi\right)\right|^2\left(1+\|\xi\|^2\right)^{-t}d\xi\end{aligned}$$

$$= \pi^{-d} \big\| [\hat{\phi}, \hat{\phi}]_t \big\|_{L_\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \big|\hat{g}(\xi)\big|^2 2^{-2js} \big(1 + \|2^{-j-1}\xi\|^2\big)^{-t} \big|\hat{b}\big(2^{-j-1}\xi\big)\big|^2 \, d\xi.$$

Hence, we conclude that

$$\sum_{k \in \mathbb{Z}^d} \big|\langle g, \psi^s_{j,k}\rangle\big|^2 \leqslant \frac{\|[\hat{\phi}, \hat{\phi}]_t\|_{L_\infty(\mathbb{R}^d)}}{\pi^d} \int_{\mathbb{R}^d} \big|\hat{g}(\xi)\big|^2 \big(1 + \|\xi\|^2\big)^{-s} \times \frac{2^{-2js}(1 + \|\xi\|^2)^s |\hat{b}(2^{-j-1}\xi)|^2}{(1 + \|2^{-j-1}\xi\|^2)^t} \, d\xi. \tag{2.8}$$

Denote

$$B_{s,t}(\xi) := \sum_{j=0}^{\infty} \frac{2^{-2js}(1 + \|\xi\|^2)^s |\hat{b}(2^{-j-1}\xi)|^2}{(1 + \|2^{-j-1}\xi\|^2)^t}, \quad \xi \in \mathbb{R}^d. \tag{2.9}$$

Assume that $B_{s,t} \in L_\infty(\mathbb{R}^d)$, which will be proven in the second part of the proof. Then it follows from (2.8) that

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \big|\langle g, \psi^s_{j,k}\rangle\big|^2 &\leqslant \frac{\|[\hat{\phi}, \hat{\phi}]_t\|_{L_\infty(\mathbb{R}^d)}}{\pi^d} \int_{\mathbb{R}^d} \big|\hat{g}(\xi)\big|^2 \big(1 + \|\xi\|^2\big)^{-s} \\
&\quad \times \sum_{j=0}^{\infty} \frac{2^{-2js}(1 + \|\xi\|^2)^s |\hat{b}(2^{-j-1}\xi)|^2}{(1 + \|2^{-j-1}\xi\|^2)^t} \, d\xi \\
&= \frac{\|[\hat{\phi}, \hat{\phi}]_t\|_{L_\infty(\mathbb{R}^d)}}{\pi^d} \int_{\mathbb{R}^d} \big|\hat{g}(\xi)\big|^2 \big(1 + \|\xi\|^2\big)^{-s} B_{s,t}(\xi) \, d\xi \\
&\leqslant 2^d \big\| [\hat{\phi}, \hat{\phi}]_t \big\|_{L_\infty(\mathbb{R}^d)} \|B_{s,t}\|_{L_\infty(\mathbb{R}^d)} \\
&\quad \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big|\hat{g}(\xi)\big|^2 \big(1 + \|\xi\|^2\big)^{-s} \, d\xi \\
&= 2^d \big\| [\hat{\phi}, \hat{\phi}]_t \big\|_{L_\infty(\mathbb{R}^d)} \|B_{s,t}\|_{L_\infty(\mathbb{R}^d)} \|g\|^2_{H^{-s}(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, (2.6) holds with $C_1 = \|[\hat{\phi}, \hat{\phi}]_s\|_{L_\infty(\mathbb{R}^d)} + 2^d \|[\hat{\phi}, \hat{\phi}]_t\|_{L_\infty(\mathbb{R}^d)} \times \|B_{s,t}\|_{L_\infty(\mathbb{R}^d)} < \infty$.

To complete the proof, we show that $B_{s,t} \in L_\infty(\mathbb{R}^d)$. In order to do so, let us consider the two cases $s > 0$ and $s \leqslant 0$ separately. We also note that $\hat{b} \in L_\infty(\mathbb{R}^d)$ in both cases by (2.5).

Suppose $s > 0$. For $\|\xi\| \leqslant 1$, by $t > s > 0$, we have $(1 + \|\xi\|^2)^s \leqslant 2^s$ and $(1 + \|2^{-j-1}\xi\|^2)^t \geqslant 1$. Therefore, for $\|\xi\| \leqslant 1$, by the definition of $B_{s,t}$ in (2.9), we deduce that

$$B_{s,t}(\xi) = \sum_{j=0}^{\infty} \frac{2^{-2js}(1 + \|\xi\|^2)^s |\hat{b}\big(2^{-j-1}\xi\big)|^2}{(1 + \|2^{-j-1}\xi\|^2)^t} \leqslant 2^s \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \sum_{j=0}^{\infty} 2^{-2js}$$

$$= \frac{2^s \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)}}{1-2^{-2s}} < \infty.$$

For $\|\xi\| > 1$, there exists a nonnegative integer J such that $2^J < \|\xi\| \leqslant 2^{J+1}$. Write $B_{s,t}(\xi) = B^1_{s,t}(\xi) + B^2_{s,t}(\xi)$, where

$$\begin{aligned} B^1_{s,t}(\xi) &:= \sum_{j=0}^{J} \frac{2^{-2js}(1+\|\xi\|^2)^s |\hat{b}(2^{-j-1}\xi)|^2}{(1+\|2^{-j-1}\xi\|^2)^t}, \\ B^2_{s,t}(\xi) &:= \sum_{j=J+1}^{\infty} \frac{2^{-2js}(1+\|\xi\|^2)^s |\hat{b}(2^{-j-1}\xi)|^2}{(1+\|2^{-j-1}\xi\|^2)^t}. \end{aligned} \tag{2.10}$$

Now by $t > s > 0$ and $2^J < \|\xi\| \leqslant 2^{J+1}$ with $J \geqslant 0$, we have

$$\begin{aligned} B^1_{s,t}(\xi) &\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \sum_{j=0}^{J} \frac{2^{-2js}(1+2^{2(J+1)})^s}{2^{2(J-j-1)t}} \\ &= \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2t}\bigl(2^{-2J}+2^2\bigr)^s \sum_{j=0}^{J} 2^{2(J-j)(s-t)} \\ &\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2t} 5^s \sum_{j=0}^{\infty} 2^{2j(s-t)} = \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2t} 5^s / \bigl(1-2^{s-t}\bigr) < \infty. \end{aligned}$$

On the other hand, since $t > s > 0$ and $J \geqslant 0$, we have

$$\begin{aligned} B^2_{s,t}(\xi) &\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \sum_{j=J+1}^{\infty} \frac{2^{-2js}(1+\|\xi\|^2)^s}{(1+\|2^{-j-1}\xi\|^2)^t} \\ &\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \sum_{j=J+1}^{\infty} 2^{-2js}\bigl(1+2^{2(J+1)}\bigr)^s \\ &= \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \bigl(2^{-2J}+2^2\bigr)^s \sum_{j=J+1}^{\infty} 2^{-2(j-J)s} \leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 5^s \sum_{j=1}^{\infty} 2^{-2js} \\ &= \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 5^s 2^{-2s} / \bigl(1-2^{-2s}\bigr) < \infty. \end{aligned}$$

Hence, we conclude that

$$\begin{aligned} B_{s,t}(\xi) \leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} \max\bigl(&2^s/\bigl(1-2^{-2s}\bigr), \\ &2^{2t}5^s/\bigl(1-2^{s-t}\bigr) + 2^{-2s}5^s/\bigl(1-2^{-2s}\bigr)\bigr) < \infty \quad \forall \xi \in \mathbb{R}^d, \end{aligned}$$

that is, for the case $s > 0$, we have proved that $B_{s,t} \in L_\infty(\mathbb{R}^d)$.

Suppose that $s \leqslant 0$. For $\|\xi\| \leqslant 1$ and $j \in \mathbb{N}_0$, since $t > s$ and $s \leqslant 0$, by (2.5) and $\alpha + s > 0$, we have $(1 + \|2^{-j-1}\xi\|^2)^{-t} \leqslant \max(2^{-t}, 1) \leqslant 2^{|t|}$ and

$$\begin{aligned}
B_{s,t}(\xi) &= \sum_{j=0}^{\infty} \frac{2^{-2js}(1 + \|2^{-j-1}\xi\|^2)^{-t}|\hat{b}(2^{-j-1}\xi)|^2}{(1 + \|\xi\|^2)^{-s}} \\
&\leqslant 2^{|t|} \sum_{j=0}^{\infty} \left|\hat{b}\left(2^{-j-1}\xi\right)\right|^2 2^{-2js} \\
&\leqslant 2^{|t|} C^2 \sum_{j=0}^{\infty} \left\|2^{-j-1}\xi\right\|^{2\alpha} 2^{-2js} \\
&\leqslant 2^{|t|} C^2 2^{-2\alpha} \sum_{j=0}^{\infty} 2^{-2j(\alpha+s)} \\
&= 2^{|t|} C^2 2^{-2\alpha} / \left(1 - 2^{-2(\alpha+s)}\right) < \infty.
\end{aligned}$$

For $\|\xi\| > 1$, there exists a nonnegative integer J such that $2^J < \|\xi\| \leqslant 2^{J+1}$. Then for $j = 0, \ldots, J$, if $t \geqslant 0$, we have

$$\begin{aligned}
\left(1 + \left\|2^{-j-1}\xi\right\|^2\right)^{-t} &\leqslant \left(1 + 2^{2(J-j-1)}\right)^{-t} = 2^{-2(J-j)t}\left(2^{-2(J-j)} + 2^{-2}\right)^{-t} \\
&\leqslant 2^{2|t|} 2^{-2(J-j)t}
\end{aligned}$$

and if $t < 0$,

$$\left(1+\left\|2^{-j-1}\xi\right\|^2\right)^{-t} \leqslant \left(1+2^{2(J-j)}\right)^{-t} = 2^{-2(J-j)t}\left(2^{-2(J-j)}+1\right)^{-t} \leqslant 2^{2|t|} 2^{-2(J-j)t}.$$

We write $B_{s,t}(\xi) = B^1_{s,t}(\xi) + B^2_{s,t}(\xi)$, where $B^1_{s,t}$ and $B^2_{s,t}$ are defined in (2.10). Then by $2^J < \|\xi\| \leqslant 2^{J+1}$ and $J \geqslant 0$, it follows from $s \leqslant 0$ and $t > s$ that

$$\begin{aligned}
B^1_{s,t}(\xi) &= \sum_{j=0}^{J} \frac{2^{-2js}(1 + \|2^{-j-1}\xi\|^2)^{-t}|\hat{b}(2^{-j-1}\xi)|^2}{(1 + \|\xi\|^2)^{-s}} \\
&\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2|t|} \sum_{j=0}^{J} \frac{2^{-2js}2^{-2(J-j)t}}{(1 + 2^{2J})^{-s}} \\
&= \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2|t|} \left(1 + 2^{-2J}\right)^s \sum_{j=0}^{J} 2^{-2(J-j)(t-s)} \\
&\leqslant \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2|t|} \sum_{j=0}^{\infty} 2^{-2j(t-s)} \\
&= \|\hat{b}\|^2_{L_\infty(\mathbb{R}^d)} 2^{2|t|} / \left(1 - 2^{-2(t-s)}\right) < \infty.
\end{aligned}$$

Similarly, by $2^J < \|\xi\| \leqslant 2^{J+1}$, we have $(1+\|2^{-j-1}\xi\|^2)^{-t} \leqslant 2^{|t|}$ for all $j \geqslant J+1$ and

$$\begin{aligned} B_{s,t}^2(\xi) &= \sum_{j=J+1}^{\infty} \frac{2^{-2js}(1+\|2^{-j-1}\xi\|^2)^{-t}|\hat{b}(2^{-j-1}\xi)|^2}{(1+\|\xi\|^2)^{-s}} \\ &\leqslant 2^{|t|} \sum_{j=J+1}^{\infty} \frac{2^{-2js}|\hat{b}(2^{-j-1}\xi)|^2}{(1+2^{2J})^{-s}}. \end{aligned}$$

By (2.5) and $\alpha > -s \geqslant 0$, we have

$$\begin{aligned} B_{s,t}^2(\xi) &\leqslant 2^{|t|}C^2 \sum_{j=J+1}^{\infty} \frac{2^{-2js}\|2^{-j-1}\xi\|^{2\alpha}}{(1+2^{2J})^{-s}} \\ &\leqslant 2^{|t|}C^2 \sum_{j=J+1}^{\infty} \frac{2^{-2js}2^{2(J-j)\alpha}}{(1+2^{2J})^{-s}} \\ &= 2^{|t|}C^2\big(1+2^{-2J}\big)^s \sum_{j=J+1}^{\infty} 2^{2(J-j)(\alpha+s)} \\ &\leqslant 2^{|t|}C^2 \sum_{j=1}^{\infty} 2^{-2j(\alpha+s)} \\ &= 2^{|t|}C^2 2^{-2(\alpha+s)}/\big(1-2^{-(\alpha+s)}\big) < \infty. \end{aligned}$$

Therefore, for the case $s \leqslant 0$, we conclude that $B_{s,t} \in L_\infty(\mathbb{R}^d)$. The proof is now complete. $\square$

For Bessel wavelet sequences in $L_2(\mathbb{R})$, see [27, Proposition 2.6], [30, Theorem 2.3], and [16, 34, 37, 39, 40, 50].

2.3 Mixed Extension Principle in Sobolev Spaces

In this section, we present the mixed extension principle for a pair of dual Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ from which Theorem 1.1 follows.

For a tempered distribution f defined on $\mathbb{R}^d$, we define

$$\mu_2(f) := \sup\big\{s \in \mathbb{R} : [\hat{f}, \hat{f}]_s \in L_\infty\big(\mathbb{R}^d\big)\big\}. \tag{2.11}$$

The mixed extension principle on pairs of dual wavelet frames in Sobolev spaces is stated for general refinement masks. It is more general than Theorem 1.1, since for any compactly supported function f, we shall see later from Proposition 2.6 that $\nu_2(f) = \mu_2(f)$.

Theorem 2.4 *Let $\hat{a}, \widehat{b^1}, \dots, \widehat{b^L}$ and $\hat{\tilde{a}}, \widehat{\tilde{b}^1}, \dots, \widehat{\tilde{b}^L}$ be 2π-periodic functions in $L_\infty(\mathbb{R}^d)$ such that $\hat{a}(\xi) - 1 = O(\|\xi\|^\beta)$ and $\hat{\tilde{a}}(\xi) - 1 = O(\|\xi\|^\beta)$ as $\xi \to 0$ for* *some* $\beta > 0$. *Define* $\phi, \tilde{\phi}$ *as in* (1.6) *and* $\psi^1, \dots, \psi^L$ *and* $\tilde{\psi}^1, \dots, \tilde{\psi}^L$ *as in* (1.7). *Assume that for a real number* $s \in \mathbb{R}$,

(i) (1.8) *and* (1.10) *are satisfied*;
(ii) $\mu_2(\phi) > s$ *and* $\mu_2(\tilde{\phi}) > -s$.

Then $(X^s(\phi; \psi^1, \dots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^L))$ *is a pair of dual wavelet frames in* $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. *Furthermore, the frame decompositions in* (1.4) *and the two inequalities* (1.11) *and* (1.12) *hold.*

Proof Since $\mu_2(\phi) > s$, we can take t such that $\mu_2(\phi) > t > s$. By the definition of $\mu_2(\phi)$ in (2.11), we see that $[\hat{\phi}, \hat{\phi}]_t \in L_\infty(\mathbb{R}^d)$. By (1.10) and Theorem 2.3, we conclude that $X^s(\phi; \psi^1, \dots, \psi^L)$ is a Bessel wavelet sequence in $H^s(\mathbb{R}^d)$. Similarly, we can deduce that $X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^L)$ is a Bessel wavelet sequence in $H^{-s}(\mathbb{R}^d)$, i.e., there exists a positive constant C_1 such that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \left|\langle f, \tilde{\phi}_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle f, \tilde{\psi}^{\ell,-s}_{j,k}\rangle\right|^2 \leqslant C_1 \|f\|^2_{H^s(\mathbb{R}^d)}, \quad f \in H^s(\mathbb{R}^d),\\
&\sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^{\ell,s}_{j,k}\rangle\right|^2 \leqslant C_1 \|g\|^2_{H^{-s}(\mathbb{R}^d)}, \quad g \in H^{-s}(\mathbb{R}^d).
\end{aligned} \tag{2.12}$$

By Proposition 2.1, in order to show the frame property of the systems, we need to prove the left-side inequalities in (1.11) and (1.12) for the systems to be frames. Let $\mathcal{B}(\mathbb{R}^d)$ denote the set of all tempered distributions f such that $\hat{f}$ is compactly supported and $\hat{f} \in L_\infty(\mathbb{R}^d)$. Note that $\mathcal{B}(\mathbb{R}^d) \subset H^\nu(\mathbb{R}^d)$ for all $\nu \in \mathbb{R}$. For any $h \in H^s(\mathbb{R}^d)$ and $\tilde{h} \in H^{-s}(\mathbb{R}^d)$, by Plancherel's Theorem and Parseval's identity, it is not difficult to verify that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \langle f, \tilde{h}(\cdot - k)\rangle \langle h(\cdot - k), g\rangle = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} [\hat{f}, \hat{\tilde{h}}]_0(\xi)[\hat{h}, \hat{g}]_0(\xi)\, d\xi,\\
&\quad f, g \in \mathcal{B}(\mathbb{R}^d).
\end{aligned} \tag{2.13}$$

Now by the definition of $\phi, \tilde{\phi}, \psi^1, \dots, \psi^L, \tilde{\psi}^1, \dots, \tilde{\psi}^L$, it follows from (2.13) and (1.8) that (e.g., see [20, Theorem 2.2])

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\psi}^\ell_{j,k}\rangle \langle \psi^\ell_{j,k}, g\rangle = \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\phi}_{j+1,k}\rangle \langle \phi_{j+1,k}, g\rangle - \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\phi}_{j,k}\rangle \langle \phi_{j,k}, g\rangle,\\
&\quad f, g \in \mathcal{B}(\mathbb{R}^d),
\end{aligned} \tag{2.14}$$

where $\psi^\ell_{j,k} := 2^{jd/2}\psi^\ell(2^j \cdot -k)$ and $\phi_{j,k} := 2^{jd/2}\phi(2^j \cdot -k)$. Note that $\psi^{\ell,s}_{j,k} = 2^{-js}\psi^\ell_{j,k}$ and $\tilde{\psi}^{\ell,-s}_{j,k} = 2^{js}\tilde{\psi}^\ell_{j,k}$. Therefore, $\langle f, \tilde{\psi}^\ell_{j,k}\rangle \langle \psi^\ell_{j,k}, g\rangle = \langle f, \tilde{\psi}^{\ell,-s}_{j,k}\rangle \langle \psi^{\ell,s}_{j,k}, g\rangle$.

Now it follows from (2.14) that

$$\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{n-1}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{j,k}^{\ell,-s}\rangle\langle\psi_{j,k}^{\ell,s},g\rangle=\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{n,k}\rangle\langle\phi_{n,k},g\rangle,$$
$$f,g\in\mathcal{B}(\mathbb{R}^d).\tag{2.15}$$

At the end of the proof, we will show that

$$\langle f,g\rangle=\lim_{n\to\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{n,k}\rangle\langle\phi_{n,k},g\rangle,\qquad f,g\in\mathcal{B}(\mathbb{R}^d).\tag{2.16}$$

Then by (2.15) and (2.16), we conclude that

$$\langle f,g\rangle=\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{j,k}^{\ell,-s}\rangle\langle\psi_{j,k}^{\ell,s},g\rangle,\qquad f,g\in\mathcal{B}(\mathbb{R}^d)\tag{2.17}$$

with the series converging absolutely by (2.12).

By the Cauchy–Schwarz inequality, it follows from (2.17) that

$$\begin{aligned}\left|\langle f,g\rangle\right|^2&\leqslant\left[\sum_{k\in\mathbb{Z}^d}\left|\langle f,\tilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle\right|+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\tilde{\psi}_{j,k}^{\ell,-s}\rangle\langle\psi_{j,k}^{\ell,s},g\rangle\right|\right]^2\\&\leqslant\left(\sum_{k\in\mathbb{Z}^d}\left|\langle f,\tilde{\phi}_{0,k}\rangle\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\tilde{\psi}_{j,k}^{\ell,-s}\rangle\right|^2\right)\\&\quad\times\left(\sum_{k\in\mathbb{Z}^d}\left|\langle\phi_{0,k},g\rangle\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle\psi_{j,k}^{\ell,s},g\rangle\right|^2\right).\end{aligned}$$

By (2.12), we deduce from the above inequality that

$$\left|\langle f,g\rangle\right|^2\leqslant C_1\|f\|_{H^s(\mathbb{R}^d)}^2\left(\sum_{k\in\mathbb{Z}^d}\left|\langle g,\phi_{0,k}\rangle\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle g,\psi_{j,k}^{\ell,s}\rangle\right|^2\right),$$
$$f,g\in\mathcal{B}(\mathbb{R}^d).\tag{2.18}$$

Hence, we have

$$\frac{1}{C_1}\sup_{f\in\mathcal{B}(\mathbb{R}^d)\backslash\{0\}}\frac{|\langle f,g\rangle|^2}{\|f\|_{H^s(\mathbb{R}^d)}^2}\leqslant\sum_{k\in\mathbb{Z}^d}\left|\langle g,\phi_{0,k}\rangle\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle g,\psi_{j,k}^{\ell,s}\rangle\right|^2,$$
$$g\in\mathcal{B}(\mathbb{R}^d).$$

Note that $\mathcal{B}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$. Moreover, $H^s(\mathbb{R}^d)$ is the dual space of $H^{-s}(\mathbb{R}^d)$. So, it follows from the above inequality and the second inequality of (2.12) that for any $g \in \mathcal{B}(\mathbb{R}^d)$,

$$C_1^{-1}\|g\|^2_{H^{-s}(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d}\left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle g, \psi^{\ell,s}_{j,k}\rangle\right|^2 \leqslant C_1\|g\|^2_{H^{-s}(\mathbb{R}^d)}. \tag{2.19}$$

Since $\mathcal{B}(\mathbb{R}^d)$ is dense in $H^{-s}(\mathbb{R}^d)$, we conclude that (2.19) holds for all $g \in H^{-s}(\mathbb{R}^d)$. Now by Proposition 2.1, $X^s(\phi; \psi^1, \ldots, \psi^L)$ is a wavelet frame in $H^s(\mathbb{R}^d)$. Similarly, we can prove that $X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ is a wavelet frame in $H^{-s}(\mathbb{R}^d)$. Now by (2.15) and (2.16), we see that (2.17) holds for all $f \in H^s(\mathbb{R}^d)$ and $g \in H^{-s}(\mathbb{R}^d)$. Hence, $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

To complete the proof, we show (2.16). By (2.13), we have

$$\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\phi}_{n,k}\rangle\langle \phi_{n,k}, g\rangle = \frac{2^{nd}}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left[\hat{f}\left(2^n\cdot\right), \hat{\tilde{\phi}}\right]_0(\xi)\left[\hat{\phi}, \hat{g}\left(2^n\cdot\right)\right]_0(\xi)\, d\xi,$$
$$f, g \in \mathcal{B}\left(\mathbb{R}^d\right),$$

that is, we have

$$\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\phi}_{n,k}\rangle\langle \phi_{n,k}, g\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{g}(\xi)}\left[\hat{f}\left(2^n\cdot\right), \hat{\tilde{\phi}}\right]_0\left(2^{-n}\xi\right) d\xi,$$
$$f, g \in \mathcal{B}\left(\mathbb{R}^d\right). \tag{2.20}$$

Since $f, g \in \mathcal{B}(\mathbb{R}^d)$, by the definition of $\mathcal{B}(\mathbb{R}^d)$, there exists a positive number N such that $\hat{f}(\xi) = \hat{g}(\xi) = 0$ for all $\xi \notin [-N, N]^d$. For $n > \log_2(N/\pi)$, it is easy to check that $\overline{\hat{g}(\xi)}\hat{f}(\xi + 2\pi k 2^n) = 0$ for all $k \in \mathbb{Z}^d\backslash\{0\}$ and $\xi \in \mathbb{R}^d$. Therefore,

$$\begin{aligned}
&\hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{g}(\xi)}\left[\hat{f}\left(2^n\cdot\right), \hat{\tilde{\phi}}\right]_0\left(2^{-n}\xi\right)\\
&\quad = \hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{g}(\xi)}\sum_{k\in\mathbb{Z}^d}\hat{f}\left(\xi + 2\pi 2^n k\right)\overline{\hat{\tilde{\phi}}\left(2^{-n}\xi + 2\pi k\right)}\\
&\quad = \hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{g}(\xi)}\hat{f}(\xi)\overline{\hat{\tilde{\phi}}\left(2^{-n}\xi\right)} = \hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{\tilde{\phi}}\left(2^{-n}\xi\right)}\hat{f}(\xi)\overline{\hat{g}(\xi)}.
\end{aligned} \tag{2.21}$$

Since $[\hat{\phi}, \hat{\phi}]_s \in L_\infty(\mathbb{R}^d)$ and $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}]_{-s} \in L_\infty(\mathbb{R}^d)$, we observe that $|\hat{\tilde{\phi}}(\xi)\overline{\hat{\phi}(\xi)}| \leqslant [|\hat{\tilde{\phi}}|, |\hat{\phi}|]_0(\xi) \leqslant C_2$ for almost every $\xi \in \mathbb{R}^d$, where $C_2 := \|[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}]_{-s}\|^{1/2}_{L_\infty(\mathbb{R}^d)}\|[\hat{\phi}, \hat{\phi}]_s\|^{1/2}_{L_\infty(\mathbb{R}^d)} < \infty$. So, by (2.21), for $n > \log_2(N/\pi)$, we have

$$\begin{aligned}
&\left|\hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{g}(\xi)}\left[\hat{f}\left(2^n\cdot\right), \hat{\tilde{\phi}}\right]_0\left(2^{-n}\xi\right)\right|\\
&\quad = \left|\hat{\phi}\left(2^{-n}\xi\right)\overline{\hat{\tilde{\phi}}\left(2^{-n}\xi\right)}\hat{f}(\xi)\overline{\hat{g}(\xi)}\right| \leqslant C_2\left|\hat{f}(\xi)\overline{\hat{g}(\xi)}\right|, \quad \text{a.e. } \xi \in \mathbb{R}^d.
\end{aligned}$$

Since $f, g \in \mathcal{B}(\mathbb{R}^d)$, we have $\hat{f}\overline{\hat{g}} \in L_1(\mathbb{R}^d)$. By the Lebesgue dominated convergence theorem and $\lim_{n\to\infty} \hat{\phi}(2^{-n}\xi) = \lim_{n\to\infty} \hat{\tilde{\phi}}(2^{-n}\xi) = 1$, it follows from (2.20) that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\phi}_{n,k}\rangle\langle \phi_{n,k}, g\rangle &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lim_{n\to\infty} \hat{\phi}\big(2^{-n}\xi\big)\overline{\hat{g}(\xi)}\big[\hat{f}\big(2^n\cdot\big), \hat{\tilde{\phi}}\big]_0\big(2^{-n}\xi\big)\, d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)} \lim_{n\to\infty} \hat{\phi}\big(2^{-n}\xi\big)\overline{\hat{\tilde{\phi}}\big(2^{-n}\xi\big)}\, d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \\
&= \langle f, g\rangle,
\end{aligned}$$

which verifies (2.16). □

As a direct consequence of the proof of Theorem 2.4, the OEP (Oblique Extension Principle) in [11, 21] (also see [20] for a comprehensive study of dual wavelet frames obtained by OEP) can easily be generalized from $L_2(\mathbb{R}^d)$ to Sobolev spaces.

Corollary 2.5 *Let $\hat{a}, \widehat{b^1}, \ldots, \widehat{b^L}$ and $\hat{\tilde{a}}, \widehat{\tilde{b}^1}, \ldots, \widehat{\tilde{b}^L}$ be 2π-periodic trigonometric polynomials with $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. Define ϕ and $\tilde{\phi}$ as in (1.6) and $\psi^1, \ldots, \psi^L$, $\tilde{\psi}^1, \ldots, \tilde{\psi}^L$ as in (1.7). Suppose that there is a 2π-periodic trigonometric polynomial Θ such that $\Theta(0) = 1$ and*

$$\begin{aligned}
&\Theta(2\xi)\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) + \sum_{\ell=1}^{L} \overline{\widehat{b^\ell}(\xi)}\widehat{\tilde{b}^\ell}(\xi) = \Theta(\xi), \\
&\Theta(2\xi)\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi+\gamma\pi) + \sum_{\ell=1}^{L} \overline{\widehat{b^\ell}(\xi)}\widehat{\tilde{b}^\ell}(\xi+\gamma\pi) = 0, \qquad \gamma \in \{0,1\}^d\backslash\{0\}.
\end{aligned} \tag{2.22}$$

For a real number $s \in \mathbb{R}$, if (1.9) is satisfied and (1.10) holds for some nonnegative numbers α and $\tilde{\alpha}$ such that $\alpha > -s$ and $\tilde{\alpha} > s$, then $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\eta; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, where $\hat{\eta}(\xi) := \Theta(\xi)\hat{\tilde{\phi}}(\xi)$.

Proof Since ϕ and η are compactly supported, by (1.9) and Proposition 2.6, we have $\mu_2(\phi) > s$ and $\mu_2(\eta) > -s$. By the same proof of Theorem 2.4, it follows from Proposition 2.3 and (1.10) that $X^s(\phi; \psi^1, \ldots, \psi^L)$ is a wavelet frame in $H^s(\mathbb{R}^d)$ and $X^{-s}(\eta; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ is a wavelet frame in $H^{-s}(\mathbb{R}^d)$. Now by (2.22) and the definition of ϕ and $\tilde{\phi}$ in (1.6), we deduce (see [20, Theorem 2.2]) that

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \big\langle f, \tilde{\psi}_{j,k}^{\ell,-s}\big\rangle\big\langle \psi_{j,k}^{\ell,s}, g\big\rangle = \sum_{k\in\mathbb{Z}^d} \langle f, \eta_{j+1,k}\rangle\langle \phi_{j+1,k}, g\rangle - \sum_{k\in\mathbb{Z}^d} \langle f, \eta_{j,k}\rangle\langle \phi_{j,k}, g\rangle,$$
$$f, g \in \mathcal{B}\big(\mathbb{R}^d\big).$$

Now by $\lim_{n\to\infty}\hat{\phi}(2^{-n}\xi)=\lim_{n\to\infty}\hat{\eta}(2^{-n}\xi)=1$, the same proof of Theorem 2.4 yields

$$\langle f,g\rangle=\sum_{k\in\mathbb{Z}^d}\langle f,\eta_{0,k}\rangle\langle\phi_{0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{j,k}^{\ell,-s}\rangle\langle\psi_{j,k}^{\ell,s},g\rangle,\quad f,g\in\mathcal{B}(\mathbb{R}^d).$$

The rest of the proof is the same as that of Theorem 2.4. □

2.4 Proofs of Theorems 1.1 and 1.2

In this section, we shall prove Theorems 1.1 and 1.2. For this, we need the following result that asserts that $\mu_2(f)=\nu_2(f)$ when f is compactly supported. With this, the proof of Theorem 1.1 follows directly from Theorem 2.4. We further remark that the regularity exponent $\nu_2(\phi)$ of a compactly supported refinable function ϕ can be computed from its refinement masks. The details can be found in [28, 29, 31, 32, 34, 36, 43, 48, 51].

Proposition 2.6 *Let $f\in H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, be a compactly supported tempered distribution. Then $[\hat{f},\hat{f}]_s\in L_\infty(\mathbb{R}^d)$. Consequently, for any compactly supported tempered distribution f, we have $\mu_2(f)=\nu_2(f)$, where the quantities $\mu_2(f)$ and $\nu_2(f)$ are defined in* (2.11) *and* (1.5), *respectively.*

Proof Since f is compactly supported, we can take a compactly supported $C^\infty(\mathbb{R}^d)$ function η such that η takes value $(2\pi)^{-d}$ on the support of f. Therefore, $f=(2\pi)^d f\eta$ that leads to $\hat{f}=\hat{f}*\hat{\eta}$. Since η is a compactly supported function in $C^\infty(\mathbb{R}^d)$, there exists a positive constant C_1 such that $|\hat{\eta}(\xi)|\leqslant C_1(1+\|\xi\|^2)^{-d-|s|/2}$ for all $\xi\in\mathbb{R}^d$. Therefore, we have

$$\begin{aligned}
|\hat{f}(\xi)|^2&=|[\hat{f}*\hat{\eta}](\xi)|^2=\left|\int_{\mathbb{R}^d}\hat{f}(\zeta)\hat{\eta}(\xi-\zeta)\,d\zeta\right|^2\\
&\leqslant C_1^2\left|\int_{\mathbb{R}^d}|\hat{f}(\zeta)|\left(1+\|\xi-\zeta\|^2\right)^{-d-|s|/2}d\zeta\right|^2\\
&\leqslant C_1^2\int_{\mathbb{R}^d}\left(1+\|\xi-\zeta\|^2\right)^{-d}d\zeta\int_{\mathbb{R}^d}|\hat{f}(\zeta)|^2\left(1+\|\xi-\zeta\|^2\right)^{-d-|s|}d\zeta\\
&=\frac{C_2}{(2\pi)^d}\int_{\mathbb{R}^d}|\hat{f}(\zeta)|^2\left(1+\|\xi-\zeta\|^2\right)^{-d-|s|}d\zeta,
\end{aligned}$$

where $C_2:=(2\pi)^d C_1^2\int_{\mathbb{R}^d}(1+\|\zeta\|^2)^{-d}\,d\zeta<\infty$. Now we have the following estimate

$$\begin{aligned}
[\hat{f},\hat{f}]_s(\xi)&=\sum_{k\in\mathbb{Z}^d}|\hat{f}(\xi+2\pi k)|^2\left(1+\|\xi+2\pi k\|^2\right)^s\\
&\leqslant\frac{C_2}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}|\hat{f}(\zeta)|^2\left(1+\|\xi+2\pi k-\zeta\|^2\right)^{-d-|s|}
\end{aligned}$$

$$\times \left(1 + \|\xi + 2\pi k\|^2\right)^s d\zeta$$

$$= C_2 \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\hat{f}(\zeta)\right|^2 \left(1 + \|\zeta\|^2\right)^s A(\xi, \zeta)\, d\zeta,$$

where

$$A(\xi, \zeta) := \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + \|\xi + 2\pi k - \zeta\|^2)^d} \left(\frac{(1 + \|\xi + 2\pi k\|^2)^s}{(1 + \|\zeta\|^2)^s (1 + \|\xi + 2\pi k - \zeta\|^2)^{|s|}} \right).$$

Denote

$$B(x, y) := \frac{1 + \|x\|^2}{(1 + \|y\|^2)(1 + \|x - y\|^2)}, \quad x, y \in \mathbb{R}^d.$$

Note that $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Now

$$2\left(1 + \|y\|^2\right)\left(1 + \|x - y\|^2\right) - \left(1 + \|x\|^2\right) = 1 + \|x - 2y\|^2 + 2\|y\|^2 \|x - y\|^2 > 0$$
$$\forall x, y \in \mathbb{R}^d.$$

Therefore, it follows from the above inequality that $B(x, y) \leqslant 2$ for all $x, y \in \mathbb{R}^d$. Note that

$$\frac{(1 + \|\xi + 2\pi k\|^2)^s}{(1 + \|\zeta\|^2)^s (1 + \|\xi + 2\pi k - \zeta\|^2)^{|s|}} = \begin{cases} [B(\xi + 2\pi k, \zeta)]^s, & \text{if} \quad s \geqslant 0, \\ [B(\zeta, \xi + 2\pi k)]^{-s}, & \text{if} \quad s < 0. \end{cases}$$

Now we can estimate $A(\xi, \zeta)$ as follows:

$$\begin{aligned} A(\xi, \zeta) &= \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + \|\xi + 2\pi k - \zeta\|^2)^d} \left(\frac{(1 + \|\xi + 2\pi k\|^2)^s}{(1 + \|\zeta\|^2)^s (1 + \|\xi + 2\pi k - \zeta\|^2)^{|s|}} \right) \\ &\leqslant \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + \|\xi + 2\pi k - \zeta\|^2)^d} \left[\max\left(B(\xi + 2\pi k, \zeta), B(\zeta, \xi + 2\pi k)\right) \right]^{|s|} \\ &\leqslant 2^{|s|} \sup_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + \|x + 2\pi k\|^2)^d} =: C_3 < \infty. \end{aligned}$$

Consequently, we conclude that

$$[\hat{f}, \hat{f}]_s(\xi) \leqslant C_2 C_3 \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\hat{f}(\zeta)\right|^2 \left(1 + \|\zeta\|^2\right)^s d\zeta = C_2 C_3 \|f\|^2_{H^s(\mathbb{R}^d)} < \infty.$$

Therefore, $[\hat{f}, \hat{f}]_s \in L_\infty(\mathbb{R}^d)$.

Note that $[\hat{f}, \hat{f}]_s \in L_\infty(\mathbb{R}^d)$ implies $f \in H^s(\mathbb{R}^d)$. Hence, we always have $\mu_2(f) \leqslant \nu_2(f)$. Conversely, if f has compact support, by what has been proved, then $f \in H^s(\mathbb{R}^d)$ implies $[\hat{f}, \hat{f}]_s \in L_\infty(\mathbb{R}^s)$. Therefore, for any compactly supported tempered distribution f, we also have $\nu_2(f) \leqslant \mu_2(f)$. This implies that for a compactly supported tempered distribution f, we always have the equality $\mu_2(f) = \nu_2(f)$. ☐

With Proposition 2.6, the proof of Theorem 1.1 follows immediately.

Proof of Theorem 1.1 Since ϕ and $\tilde{\phi}$ are compactly supported, by Proposition 2.6, we have $\mu_2(\phi) = \nu_2(\phi)$ and $\mu_2(\tilde{\phi}) = \nu_2(\tilde{\phi})$. Now Theorem 1.1 is a special case of Theorem 2.4. □

Next, we present the proof of Theorem 1.2.

Proof of Theorem 1.2 Since ϕ is a compactly supported function in $L_2(\mathbb{R}^d)$ with a 2π-periodic trigonometric polynomial mask, by [30, Theorem 2.2] or [32, Corollary 4.2], we have $\nu_2(\phi) > 0$.

Denote $\theta(\xi_1, \dots, \xi_d) := 2^{-2d} \prod_{j=1}^{d} (1 + e^{-i\xi_j})^2$. Let $m := sr(\hat{a})$. That is, $\partial^\beta \hat{a}(\gamma\pi) = 0$ for $|\beta| < m$ and $\gamma \in \{0, 1\}^d \backslash \{0\}$. By $\theta(0)\hat{a}(0) \neq 0$ and [29, Lemma 3.3], there exists a 2π-periodic trigonometric polynomial $\hat{c}$ in d-variables such that

$$\hat{c}(0) = 1, \qquad \partial^\beta \hat{c}(0) = \partial^\beta \left[\frac{1}{\theta(\cdot)\overline{\hat{a}(\cdot)}} \right](0), \quad \beta \in \mathbb{N}_0^d,\ |\beta| < m, \tag{2.23}$$

and $\|1 - \hat{c}\|_{L_\infty(\mathbb{R}^d)} < 1/2$. Now we define $\hat{\tilde{a}}(\xi) := \theta(\xi)\hat{c}(\xi)$. Note that $\|1 - \hat{c}\|_{L_\infty(\mathbb{R}^d)} < 1/2$ implies that $\|\hat{c}\|_{L_\infty(\mathbb{R}^d)} \leqslant 3/2 < 2$. Hence $|\hat{\tilde{a}}(\xi)| = |\theta(\xi)\hat{c}(\xi)| \leqslant 2|\theta(\xi)|$. By the definition of $\nu_2(\hat{\tilde{a}})$ and [34, Theorem 4.1 and Proposition 4.5], since $\nu_2(2\theta) = \nu_2(\theta) - 1$ and $\nu_2(\theta) = 3/2$, we have

$$\nu_2(\hat{\tilde{a}}) \geqslant \nu_2(2\theta) = \nu_2(\theta) - 1 = 1/2 > 0.$$

Obviously we have $\hat{\tilde{a}}(0) = 1$. Define $\hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j}\xi), \xi \in \mathbb{R}^d$. By [29, Theorem 4.3] or [32, Theorem 4.1], we have $\nu_2(\tilde{\phi}) \geqslant \nu_2(\hat{\tilde{a}}) \geqslant 1/2 > -s$, since $s > 0$.

Denote $\{\gamma_1, \dots, \gamma_{2^d}\} = \{0, 1\}^d$ with $\gamma_1 = 0$. Define

$$\widehat{b^\ell}(\xi) = 2^{-s-d/2} e^{-i\gamma_\ell \cdot \xi}, \qquad \ell = 1, \dots, 2^d,\ \xi \in \mathbb{R}^d. \tag{2.24}$$

Note that the system of equations in (1.8) with $L = 2^d$ can be equivalently rewritten in the following matrix form:

$$\overline{E(\xi)} \begin{bmatrix} \widehat{\tilde{b}^1}(\xi) \\ \widehat{\tilde{b}^2}(\xi) \\ \vdots \\ \widehat{\tilde{b}^{2^d}}(\xi) \end{bmatrix} = \begin{bmatrix} 1 - \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) \\ -\overline{\hat{a}(\xi + \gamma_2\pi)}\hat{\tilde{a}}(\xi) \\ \vdots \\ -\overline{\hat{a}(\xi + \gamma_{2^d}\pi)}\hat{\tilde{a}}(\xi) \end{bmatrix} \tag{2.25}$$

$$\text{with } E(\xi) := \begin{bmatrix} \widehat{b^1}(\xi) & \cdots & \widehat{b^{2^d}}(\xi) \\ \widehat{b^1}(\xi + \gamma_2\pi) & \cdots & \widehat{b^{2^d}}(\xi + \gamma_2\pi) \\ \vdots & \vdots & \vdots \\ \widehat{b^1}(\xi + \gamma_{2^d}\pi) & \cdots & \widehat{b^{2^d}}(\xi + \gamma_{2^d}\pi) \end{bmatrix}. \tag{2.26}$$

It is straightforward to verify that $\overline{E(\xi)}E(\xi)^T = 2^{-2s}I_{2^d}$, where I_{2^d} denotes the $2^d \times 2^d$ identity matrix. So, define $\widehat{\tilde{b}^1}, \dots, \widehat{\tilde{b}^{2^d}}$ by

$$\begin{bmatrix} \widehat{\tilde{b}^1}(\xi) \\ \widehat{\tilde{b}^2}(\xi) \\ \vdots \\ \widehat{\tilde{b}^{2^d}}(\xi) \end{bmatrix} := \left[\overline{E(\xi)}\right]^{-1} \begin{bmatrix} 1 - \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) \\ -\overline{\hat{a}(\xi + \gamma_2\pi)}\hat{\tilde{a}}(\xi) \\ \vdots \\ -\overline{\hat{a}(\xi + \gamma_{2^d}\pi)}\hat{\tilde{a}}(\xi) \end{bmatrix}$$

$$= 2^{2s}E(\xi)^T \begin{bmatrix} 1 - \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) \\ -\overline{\hat{a}(\xi + \gamma_2\pi)}\hat{\tilde{a}}(\xi) \\ \vdots \\ -\overline{\hat{a}(\xi + \gamma_{2^d}\pi)}\hat{\tilde{a}}(\xi) \end{bmatrix}. \tag{2.27}$$

Clearly, all $\widehat{\tilde{b}^1}, \dots, \widehat{\tilde{b}^{2^d}}$ are 2π-periodic trigonometric polynomials. On the one hand, by (2.23), we see that

$$1 - \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) = 1 - \overline{\hat{a}(\xi)}\hat{\theta}(\xi)\hat{c}(\xi) = O\left(\|\xi\|^m\right), \quad \xi \to 0.$$

On the other hand, by $sr(\hat{a}) = m$, we see that $\hat{a}$ satisfies the sum rules of order m and therefore,

$$-\overline{\hat{a}(\xi + \gamma_\ell\pi)}\hat{\tilde{a}}(\xi) = O\left(\|\xi\|^m\right), \qquad \xi \to 0 \quad \text{for all } \ell = 2, \dots, 2^d.$$

Now by the definition of $\widehat{\tilde{b}^\ell}$ in (2.27), we conclude that there exists a positive constant C such that

$$\left|\widehat{\tilde{b}^\ell}(\xi)\right| \leqslant C \min\left(1, \|\xi\|^m\right), \quad \xi \in \mathbb{R}^d,\ \ell = 1, \dots, 2^d. \tag{2.28}$$

By the definition of $\widehat{b^\ell}$ in (2.24), it is evident that $|\widehat{b^\ell}(\xi)| \leqslant 2^{-s-d/2}$ for all $\xi \in \mathbb{R}$ and $\ell = 1, \dots, 2^d$. So, we have $\widehat{b^\ell} \in L_\infty(\mathbb{R}^d)$ for all $\ell = 1, \dots, 2^d$. Since $m = sr(\hat{a}) \geqslant \min(\nu_2(\phi), sr(\hat{a})) > s$ by $0 < s < \min(\nu_2(\phi), sr(\hat{a}))$, we see that the condition in (1.10) holds with $\alpha = 0$ and $\tilde{\alpha} = m$.

Define $\psi^1, \dots, \psi^{2^d}, \tilde{\psi}^1, \dots, \tilde{\psi}^{2^d}$ as in (1.7) with $L = 2^d$. Now we see that all the conditions in Theorem 1.1 are satisfied with $L = 2^d$. So, $(X^s(\phi; \psi^1, \dots, \psi^{2^d}), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{2^d}))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. In particular, $X^s(\phi; \psi^1, \dots, \psi^{2^d})$ is a wavelet frame in $H^s(\mathbb{R}^d)$. By the choice of $\widehat{b^\ell}$, we see that $\psi^\ell = 2^{d/2-s}\phi(2 \cdot - \gamma_\ell)$. Consequently, we have

$$\psi^{\ell,s}_{j,k} = 2^{j(d/2-s)}\psi^\ell(2^j \cdot - k) = 2^{(j+1)(d/2-s)}\phi(2^{j+1} \cdot - 2k - \gamma_\ell) = \phi^s_{j+1,2k+\gamma_\ell},$$
$$j \in \mathbb{N}_0, k \in \mathbb{Z}^d.$$

Now we see that $X^s(\phi; \psi^1, \dots, \psi^{2^d})$ is exactly the same wavelet system as that in (1.13). □

3 Pairs of Dual Riesz Wavelet Bases in Sobolev Spaces

Based on the results in Sect. 2 on dual wavelet frames in Sobolev spaces, we study pairs of dual Riesz wavelet bases in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ in this section. Basically, we investigate when a pair of dual wavelet frames obtained in Sect. 2 will become a pair of dual Riesz wavelet bases in Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

3.1 Mixed Extension Principle for Dual Riesz Wavelet Bases

Let $\phi, \psi^1, \ldots, \psi^L \in H^s(\mathbb{R}^d)$. We say that $X^s(\phi; \psi^1, \ldots, \psi^L)$ is *a Riesz basis* in the Sobolev space $H^s(\mathbb{R}^d)$ if

(1) the set of all the linear combinations of elements in $X^s(\phi; \psi^1, \ldots, \psi^L)$ is dense in $H^s(\mathbb{R}^d)$,
(2) $X^s(\phi; \psi^1, \ldots, \psi^L)$ is a *Riesz sequence* in $H^s(\mathbb{R}^d)$: there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1\Bigg[\sum_{k\in\mathbb{Z}^d}|c_k|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|d_{j,k}^{\ell}\big|^2\Bigg] &\leqslant \Bigg\|\sum_{k\in\mathbb{Z}^d}c_k\phi_{0,k}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}d_{j,k}^{\ell}\psi_{j,k}^{\ell,s}\Bigg\|^2_{H^s(\mathbb{R}^d)} \\ &\leqslant C_2\Bigg[\sum_{k\in\mathbb{Z}^d}|c_k|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\big|d_{j,k}^{\ell}\big|^2\Bigg] \end{aligned} \tag{3.1}$$

holds for all finitely supported sequences $\{c_k\}_{k\in\mathbb{Z}^d}$ and $\{d_{j,k}^{\ell}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d,\ell=1,\ldots,L}$.

Let $\phi, \psi^1, \ldots, \psi^L$ belong to $H^s(\mathbb{R}^d)$ and let $\tilde{\phi}, \tilde{\psi}^1, \ldots, \tilde{\psi}^L$ belong to $H^{-s}(\mathbb{R}^d)$. We say that $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is *a pair of dual Riesz wavelet bases* in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if

(1) $X^s(\phi; \psi^1, \ldots, \psi^L)$ is a Riesz basis of the Sobolev space $H^s(\mathbb{R}^d)$,
(2) $X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ is a Riesz basis of the Sobolev space $H^{-s}(\mathbb{R}^d)$,
(3) $X^s(\phi; \psi^1, \ldots, \psi^L)$ and $X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L)$ are biorthogonal: for all $k, k' \in \mathbb{Z}^d$, $j, j' \in \mathbb{N}_0$ and $\ell, \ell' = 1, \ldots, L$,

$$\begin{aligned} &\langle\phi_{0,k}, \tilde{\phi}_{0,k'}\rangle = \delta_{k-k'}, \qquad \big\langle\psi_{j,k}^{\ell,s}, \tilde{\psi}_{j',k'}^{\ell',-s}\big\rangle = \delta_{j-j'}\delta_{k-k'}\delta_{\ell-\ell'}, \\ &\big\langle\phi_{0,k}, \tilde{\psi}_{j',k'}^{\ell',-s}\big\rangle = 0, \qquad \big\langle\psi_{j,k}^{\ell,s}, \tilde{\phi}_{0,k'}\big\rangle = 0, \end{aligned} \tag{3.2}$$

where δ denotes the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$.

It is easy to check that $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if and only if (3.2) holds and $(X^s(\phi; \psi^1, \ldots, \psi^L), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^L))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

To make a pair of dual frame systems derived from the mixed extension principle to be a pair of Riesz bases in Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, we first require $L = 2^d - 1$, otherwise, they are often a pair of overcomplete systems. Secondly, it is clear that we need the stability on both refinable functions ϕ and $\tilde{\phi}$ (see [34, Theorem 3.2]). This can be stated in terms of conditions on masks. In fact, we just need to replace the conditions on $\mu_2(\phi)$ and $\mu_2(\tilde{\phi})$ in Theorem 2.4 by similar but stronger conditions on $\mu_2(\hat{a})$ and $\mu_2(\hat{\tilde{a}})$. Note that $\mu_2(\phi) \geqslant \mu_2(\hat{a})$ is always true. In particular, when both $\hat{a}$ and $\hat{\tilde{a}}$ are 2π-periodic trigonometric polynomials, we replace the conditions on $\nu_2(\phi)$ and $\nu_2(\tilde{\phi})$ in Theorem 1.1 by similar conditions on $\nu_2(\hat{a})$ and $\nu_2(\hat{\tilde{a}})$. It is known that when ϕ is stable, then $\nu_2(\phi) = \nu_2(\hat{a})$. However, when only the masks are given, it is a highly nontrivial task to know the stability of their refinable functions ([32]). Our result shows that conditions imposed on both masks here can guarantee the stability of the refinable functions which in turn guarantees the Riesz properties of both systems. All these give the following mixed extension principle for a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

Recall that a 2π-periodic function $\hat{a}$ in d-variables has *exponential decay* if its Fourier coefficients decay exponentially. More precisely, letting $\hat{a}(\xi) = \sum_{k\in\mathbb{Z}^d} a_k e^{-ik\cdot\xi}$ be its Fourier series, then there exist some positive numbers r and C such that $|a_k| \leqslant Ce^{-r\|k\|}$ for all $k \in \mathbb{Z}^d$.

Theorem 3.1 *Let $\hat{a}, \widehat{b^1}, \dots, \widehat{b^{2^d-1}}$ and $\hat{\tilde{a}}, \widehat{\tilde{b}^1}, \dots, \widehat{\tilde{b}^{2^d-1}}$ be 2π-periodic functions in d-variables with exponential decay such that $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. Define ϕ and $\tilde{\phi}$ as in* (1.6). *Assume that*

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi+\gamma\pi) + \sum_{\ell=1}^{2^d-1} \overline{\widehat{b^\ell}(\xi)}\widehat{\tilde{b}^\ell}(\xi+\gamma\pi) = \delta_\gamma, \quad \gamma \in \{0,1\}^d,\ \xi \in \mathbb{R}^d \tag{3.3}$$

and for some $s \in \mathbb{R}$, (1.26) *holds. Then $(X^s(\phi; \psi^1, \dots, \psi^{2^d-1}), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{2^d-1}))$ is a pair of dual Riesz wavelet bases in the pair of Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.*

Proof By (1.26) and the definition of $\mu_2(\hat{a})$, there exist 2π-periodic trigonometric polynomials $\mathring{\hat{a}}$ and $\mathring{\hat{\tilde{a}}}$ with $\mathring{\hat{a}}(0) = \mathring{\hat{\tilde{a}}}(0) = 1$ such that

$$\begin{aligned} &\nu_2(\mathring{\hat{a}}) > s, \qquad \nu_2(\mathring{\hat{\tilde{a}}}) > -s \quad \text{and} \\ &\bigl|\hat{a}(\xi)\bigr| \leqslant \bigl|\mathring{\hat{a}}(\xi)\bigr|, \qquad \bigl|\hat{\tilde{a}}(\xi)\bigr| \leqslant \bigl|\mathring{\hat{\tilde{a}}}(\xi)\bigr|, \quad \xi \in \mathbb{R}^d. \end{aligned} \tag{3.4}$$

Let $\mathring{\phi}$ and $\mathring{\tilde{\phi}}$ be the compactly supported refinable functions associated with $\mathring{\hat{a}}$ and $\mathring{\hat{\tilde{a}}}$, respectively. Then we have $\nu_2(\mathring{\phi}) \geqslant \nu_2(\mathring{\hat{a}}) > s$ and $\nu_2(\mathring{\tilde{\phi}}) \geqslant \nu_2(\mathring{\hat{\tilde{a}}}) > -s$ ([29, 32, 34]). Now by (3.4), we have $|\hat{\phi}(\xi)| \leqslant |\hat{\mathring{\phi}}(\xi)|$ and $|\hat{\tilde{\phi}}(\xi)| \leqslant |\hat{\mathring{\tilde{\phi}}}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Consequently, by Proposition 2.6, we have

$$\mu_2(\phi) \geqslant \mu_2(\mathring{\phi}) = \nu_2(\mathring{\phi}) > s \quad \text{and} \quad \mu_2(\tilde{\phi}) \geqslant \mu_2(\mathring{\tilde{\phi}}) = \nu_2(\mathring{\tilde{\phi}}) > -s.$$

Therefore, the condition in (1.26) of Theorem 3.1 implies Item (ii) of Theorem 2.4.

Denote $m = sr(\hat{\mathring{a}})$ and $\tilde{m} = sr(\hat{\tilde{\mathring{a}}})$. Now we show that (3.3) and (1.26) imply (1.10) with $\alpha = \tilde{m}$ and $\tilde{\alpha} = m$. Since $\hat{\mathring{a}}$ and $\hat{\tilde{\mathring{a}}}$ are 2π-periodic trigonometric polynomials with $\hat{\mathring{a}}(0) = \hat{\tilde{\mathring{a}}}(0) = 1$, by [29, Proposition 4.1 or Theorem 4.3], we conclude that $m = sr(\hat{\mathring{a}}) \geqslant \nu_2(\hat{\mathring{a}}) > s$ and $\tilde{m} = sr(\hat{\tilde{\mathring{a}}}) \geqslant \nu_2(\hat{\tilde{\mathring{a}}}) > -s$. Define $\alpha := \tilde{m}$ and $\tilde{\alpha} := m$. we conclude $\alpha > -s$ and $\tilde{\alpha} > s$.

Condition (3.3) can be rewritten in a matrix form. Let $\widehat{b^0} = \hat{a}$ and $\widehat{\tilde{b}^0} = \hat{\tilde{a}}$ and define matrices $U(\xi) := (\widehat{b^\ell}(\xi + \gamma\pi))_{\ell=0,\dots,2^d-1,\gamma\in\{0,1\}^d}$ and $\widetilde{U}(\xi) := (\widehat{\tilde{b}^\ell}(\xi + \gamma\pi))_{\ell=0,\dots,2^d-1,\gamma\in\{0,1\}^d}$. Then condition (3.3) is equivalent to $U(\xi)\overline{\widetilde{U}(\xi)}^T = I_{2^d}$. Since the cardinality of the set $\{0,1\}^d$ is 2^d, both $U(\xi)$ and $\widetilde{U}(\xi)$ are square matrices. Hence, $U(\xi)\overline{\widetilde{U}(\xi)}^T = I_{2^d}$ implies $\overline{\widetilde{U}(\xi)}^T U(\xi) = I_{2^d}$. This immediately leads to that (3.3) implies

$$\widehat{b^\ell}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \widehat{b^\ell}(\xi + \gamma\pi)\overline{\hat{\tilde{a}}(\xi + \gamma\pi)} = 0, \quad \ell = 1, \dots, 2^d - 1,$$

and

$$\sum_{\gamma\in\{0,1\}^d} \overline{\hat{a}(\xi + \gamma\pi)}\hat{\tilde{a}}(\xi + \gamma\pi) = 1. \tag{3.5}$$

By (3.4), we have $sr(\hat{\tilde{a}}) \geqslant sr(\hat{\tilde{\mathring{a}}}) = \tilde{m} = \alpha$. By the definition of sum rules, we have $\hat{\tilde{a}}(\xi + \gamma\pi) = O(\|\xi\|^\alpha)$ as $\xi \to 0$ for all $\gamma \in \{0,1\}^d\setminus\{0\}$. Now it follows from the above identity that

$$\widehat{b^\ell}(\xi)\overline{\hat{\tilde{a}}(\xi)} = -\sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \widehat{b^\ell}(\xi + \gamma\pi)\overline{\hat{\tilde{a}}(\xi + \gamma\pi)} = O\big(\|\xi\|^\alpha\big), \quad \xi \to 0.$$

Since $\hat{\tilde{a}}(0) = 1$ and $\hat{\tilde{a}}(\xi)$ is continuous, we conclude from the above relation that $\widehat{b^\ell}(\xi) = O(\|\xi\|^\alpha)$ as $\xi \to 0$. Similarly, we have $\widehat{\tilde{b}^\ell}(\xi) = O(\|\xi\|^{\tilde{\alpha}})$ as $\xi \to 0$. So (1.10) is satisfied.

Obviously, (3.3) is just (1.8) with $L = 2^d - 1$. Therefore, all the conditions in Theorem 2.4 are satisfied with $L = 2^d - 1$. So, by Theorem 2.4, $(X^s(\phi; \psi^1, \dots, \psi^{2^d-1}), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{2^d-1}))$ is a pair of dual wavelet frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

In order to show that $(X^s(\phi; \psi^1, \dots, \psi^{2^d-1}), X^{-s}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^{2^d-1}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, it now suffices to prove the biorthogonality condition in (3.2).

Let η be a compactly supported tensor-product interpolatory refinable function such that $\eta(k) = \delta_k$ for all $k \in \mathbb{Z}^d$, $(1 + \|\cdot\|)^{2d}\hat{\eta}(\cdot) \in L_\infty(\mathbb{R}^d)$, and

$$\hat{\eta}(2\xi) = \hat{c}(\xi)\hat{\eta}(\xi) \quad \text{with } \hat{c}(\xi) = 1 + O\big(\|\xi\|^{m+\tilde{m}}\big), \ \xi \to 0, \tag{3.6}$$

where $\hat{c}$ is a 2π-periodic trigonometric polynomial in d-variables. Such a function η always exists (see Sect. 1.2). Define a sequence of functions $\{\widehat{f_n}\}_{n=1}^{\infty}$ by $f_0 := \eta$ and

$$\widehat{f_n}(\xi) := \overline{\hat{a}(\xi/2)}\hat{\tilde{a}}(\xi/2)\widehat{f_{n-1}}(\xi/2) = \hat{\eta}(2^{-n}\xi)\prod_{j=1}^{n}\overline{\hat{a}(2^{-j}\xi)}\prod_{j=1}^{n}\hat{\tilde{a}}(2^{-j}\xi),$$
$$\xi \in \mathbb{R}^d,\ n \in \mathbb{N} \cup \{0\}.$$

Denote $\hat{\Phi}(\xi) := \overline{\hat{\phi}(\xi)}\hat{\tilde{\phi}}(\xi)$. Since $\phi \in H^s(\mathbb{R}^d)$ and $\tilde{\phi} \in H^{-s}(\mathbb{R}^d)$, by the Cauchy–Schwarz inequality, it is evident that $\hat{\Phi} \in L_1(\mathbb{R}^d)$. Using a similar idea given in [29] for trigonometric polynomials, we shall prove that

$$\lim_{n\to\infty} \|\widehat{f_n} - \hat{\Phi}\|_{L_1(\mathbb{R}^d)} = 0. \tag{3.7}$$

Denote $\widehat{g_n} := \widehat{f_{n+1}} - \widehat{f_n}$ and by (3.6), we define

$$\begin{aligned}\hat{g}(\xi) &:= \overline{\hat{a}(\xi/2)}\hat{\tilde{a}}(\xi/2)\hat{\eta}(\xi/2) - \hat{\eta}(\xi)\\ &= \big[\overline{\hat{a}(\xi/2)}\hat{\tilde{a}}(\xi/2) - \hat{c}(\xi/2)\big]\hat{\eta}(\xi/2) = \hat{A}(\xi/2)\hat{\eta}(\xi/2)\end{aligned} \tag{3.8}$$

with

$$\hat{A}(\xi) := \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) - \hat{c}(\xi). \tag{3.9}$$

By (3.4) and the definition of sum rules, we have $sr(\hat{a}) \geqslant sr(\hat{\mathring{a}}) = m$ and $sr(\hat{\tilde{a}}) \geqslant sr(\hat{\mathring{\tilde{a}}}) = \tilde{m}$. Now it follows from (3.5) that

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) - 1 = -\sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \overline{\hat{a}(\xi+\gamma\pi)}\hat{\tilde{a}}(\xi+\gamma\pi) = O\big(\|\xi\|^{m+\tilde{m}}\big), \quad \xi \to 0.$$

Now by (3.6), we have

$$\hat{A}(\xi) = \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) - \hat{c}(\xi) = \big[\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) - 1\big] + \big[1 - \hat{c}(\xi)\big] = O\big(\|\xi\|^{m+\tilde{m}}\big), \quad \xi \to 0,$$

that is, $\partial^\beta \hat{A}(\gamma\pi) = 0$ for all $\gamma \in \{0,1\}^d$ and $|\beta| < m + \tilde{m}$. Generalizing a result as in [29, Page 60], we can deduce that there exist $\widehat{A_\beta}$, $|\beta| = m + \tilde{m}$ of 2π-periodic functions with exponential decay such that

$$\hat{A}(\xi) = \sum_{|\beta|=m+\tilde{m}} \widehat{\nabla^\beta\delta}(2\xi)\widehat{A_\beta}(\xi), \tag{3.10}$$

where $\widehat{\nabla^\beta\delta}(\xi_1,\ldots,\xi_d) := (1-e^{-i\xi_1})^{\beta_1}\cdots(1-e^{-i\xi_d})^{\beta_d}$ for $\beta = (\beta_1,\ldots,\beta_d)$. Consequently, by the definition of $\hat{g}$ in (3.8), we have

$$\hat{g}(\xi) = \sum_{|\beta|=m+\tilde{m}} \widehat{\nabla^\beta\delta}(\xi)\hat{\eta}_\beta(\xi) \quad \text{with } \widehat{\eta_\beta}(\xi) := \widehat{A_\beta}(\xi/2)\hat{\eta}(\xi/2). \tag{3.11}$$

By the definition of f_n, we deduce that

$$\widehat{g_n}(\xi) = \widehat{f_{n+1}}(\xi) - \widehat{f_n}(\xi) = \hat{g}\big(2^{-n}\xi\big)\prod_{j=1}^{n}\overline{\hat{a}(2^{-j}\xi)}\prod_{j=1}^{n}\hat{\tilde{a}}\big(2^{-j}\xi\big). \tag{3.12}$$

Since $(1+\|\cdot\|)^{2d}\hat{\eta}(\cdot) \in L_\infty(\mathbb{R}^d)$ and $\widehat{A_\beta}$ has exponential decay, it is straightforward to see that $\sum_{k\in\mathbb{Z}^d}|\widehat{\eta_\beta}(\cdot+2\pi k)| \in L_\infty(\mathbb{R}^d)$. Now by (3.12) and (3.4), we deduce that

$$\begin{aligned}
\|\widehat{g_n}\|_{L_1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\big|\hat{g}\big(2^{-n}\xi\big)\big|\prod_{j=1}^{n}\big|\hat{a}\big(2^{-j}\xi\big)\big|\prod_{j=1}^{n}\big|\hat{\tilde{a}}\big(2^{-j}\xi\big)\big|\,d\xi \\
&= 2^{dn}\int_{\mathbb{R}^d}\big|\hat{g}(\xi)\big|\prod_{j=0}^{n-1}\big|\hat{a}\big(2^{j}\xi\big)\big|\prod_{j=0}^{n-1}\big|\hat{\tilde{a}}\big(2^{j}\xi\big)\big|\,d\xi \\
&\leqslant 2^{dn}\sum_{|\beta|=m+\tilde{m}}\int_{\mathbb{R}^d}\big|\widehat{\nabla^\beta\delta}(\xi)\big|\big|\widehat{\eta_\beta}(\xi)\big|\prod_{j=0}^{n-1}\big|\hat{a}\big(2^{j}\xi\big)\big|\prod_{j=0}^{n-1}\big|\hat{\tilde{a}}\big(2^{j}\xi\big)\big|\,d\xi \\
&= 2^{dn}\sum_{|\beta|=m+\tilde{m}}\int_{[-\pi,\pi]^d}\big|\widehat{\nabla^\beta\delta}(\xi)\big|\prod_{j=0}^{n-1}\big|\hat{a}\big(2^{j}\xi\big)\big|\prod_{j=0}^{n-1}\big|\hat{\tilde{a}}\big(2^{j}\xi\big)\big| \\
&\quad\times\sum_{k\in\mathbb{Z}^d}\big|\widehat{\eta_\beta}(\xi+2\pi k)\big|d\xi \\
&\leqslant C_1 2^{dn}\sum_{|\beta|=m}\sum_{|\gamma|=\tilde{m}}\int_{[-\pi,\pi]^d}\left(\big|\widehat{\nabla^\beta\delta}(\xi)\big|\prod_{j=0}^{n-1}\big|\hat{a}\big(2^{j}\xi\big)\big|\right) \\
&\quad\times\left(\big|\widehat{\nabla^\gamma\delta}(\xi)\big|\prod_{j=0}^{n-1}\big|\hat{\tilde{a}}\big(2^{j}\xi\big)\big|\right)d\xi \\
&\leqslant C_1 2^{dn}\sum_{|\beta|=m}\sum_{|\gamma|=\tilde{m}}\left(\int_{[-\pi,\pi]^d}\big|\widehat{\nabla^\beta\delta}(\xi)\big|^2\prod_{j=0}^{n-1}\big|\hat{a}\big(2^{j}\xi\big)\big|^2\,d\xi\right)^{1/2} \\
&\quad\times\left(\int_{[-\pi,\pi]^d}\big|\widehat{\nabla^\gamma\delta}(\xi)\big|^2\prod_{j=0}^{n-1}\big|\hat{\tilde{a}}\big(2^{j}\xi\big)\big|^2\,d\xi\right)^{1/2} \\
&\leqslant C_1 2^{dn}\sum_{|\beta|=m}\sum_{|\gamma|=\tilde{m}}\left(\int_{[-\pi,\pi]^d}\big|\widehat{\nabla^\beta\delta}(\xi)\big|^2\prod_{j=0}^{n-1}\big|\hat{\mathring{a}}\big(2^{j}\xi\big)\big|^2\,d\xi\right)^{1/2} \\
&\quad\times\left(\int_{[-\pi,\pi]^d}\big|\widehat{\nabla^\gamma\delta}(\xi)\big|^2\prod_{j=0}^{n-1}\big|\hat{\mathring{\tilde{a}}}\big(2^{j}\xi\big)\big|^2\,d\xi\right)^{1/2}
\end{aligned}$$

$$= (2\pi)^d C_1 2^{dn} \sum_{|\beta|=m} \sum_{|\gamma|=\tilde{m}} \left\| \nabla^\beta \mathring{a}^n \right\|_{\ell_2(\mathbb{Z}^d)} \left\| \nabla^\gamma \mathring{\tilde{a}}^n \right\|_{\ell_2(\mathbb{Z}^d)},$$

where $C_1 := \max_{|\beta|=m+\tilde{m}} \| \sum_{k\in\mathbb{Z}^d} |\widehat{\eta_\beta}(\cdot + 2\pi k)| \|_{L_\infty(\mathbb{R}^d)} < \infty$ and

$$\widehat{\nabla^\beta \mathring{a}^n}(\xi) := \widehat{\nabla^\beta \delta}(\xi) \prod_{j=0}^{n-1} \hat{\mathring{a}}\big(2^j \xi\big) \quad \text{and} \quad \widehat{\nabla^\gamma \mathring{\tilde{a}}^n}(\xi) := \widehat{\nabla^\gamma \delta}(\xi) \prod_{j=0}^{n-1} \hat{\mathring{\tilde{a}}}\big(2^j \xi\big).$$

By the definition of $\nu_2(\hat{\mathring{a}})$ and (3.4), for any ν and $\tilde{\nu}$ such that $\nu_2(\hat{\mathring{a}}) > \nu > s$ and $\nu_2(\hat{\mathring{\tilde{a}}}) > \tilde{\nu} > -s$, there exists a positive constant C_2 such that

$$\begin{aligned} \left\| \nabla^\beta \mathring{a}^n \right\|_{\ell_2(\mathbb{Z}^d)} &\leqslant C_2 2^{-(\nu+d/2)n} \quad \forall |\beta| = m,\ n \in \mathbb{N}, \\ \left\| \nabla^\gamma \mathring{\tilde{a}}^n \right\|_{\ell_2(\mathbb{Z}^d)} &\leqslant C_2 2^{-(\tilde{\nu}+d/2)n} \quad \forall |\gamma| = \tilde{m},\ n \in \mathbb{N}. \end{aligned} \tag{3.13}$$

Now we can conclude that

$$\|\widehat{g_n}\|_{L_1(\mathbb{R}^d)} \leqslant (2\pi)^d C_1 2^{dn} \sum_{|\beta|=m} \sum_{|\gamma|=\tilde{m}} \left\| \nabla^\beta \mathring{a}^n \right\|_{\ell_2(\mathbb{Z}^d)} \left\| \nabla^\gamma \mathring{\tilde{a}}^n \right\|_{\ell_2(\mathbb{Z}^d)} \leqslant C_3 2^{-(\nu+\tilde{\nu})n},$$
$$n \in \mathbb{N},$$

where $C_3 := (2\pi)^d C_1 C_2 \sum_{|\beta|=m} \sum_{|\gamma|=\tilde{m}} 1 < \infty$. Since $\nu + \tilde{\nu} > s - s = 0$, by the above inequality, it follows from $\widehat{g_n} = \widehat{f_{n+1}} - \widehat{f_n}$ and $\lim_{n\to\infty} \widehat{f_n}(\xi) = \hat{\Phi}(\xi)$ for all $\xi \in \mathbb{R}^d$ that (3.7) holds.

Since $(1 + \|\cdot\|)^{2d} \hat{\eta}(\cdot) \in L_\infty(\mathbb{R}^d)$, we see that $\sum_{k\in\mathbb{Z}^d} |\widehat{f_n}(\xi + 2\pi k)| \in L_1([-\pi, \pi]^d)$. Now we prove that

$$\sum_{k\in\mathbb{Z}^d} \widehat{f_n}(\xi + 2\pi k) = 1, \quad \text{a.e. } \xi \in \mathbb{R}^d \text{ and } n \in \mathbb{N} \cup \{0\}. \tag{3.14}$$

Since η is an interpolatory refinable function, we have $\sum_{k\in\mathbb{Z}^d} \hat{\eta}(\xi + 2\pi k) = 1$. By the definition $f_0 = \eta$, it is evident that (3.14) holds for $n = 0$. Suppose that (3.14) holds for $n - 1$. Then by $\widehat{f_n}(\xi) = \overline{\hat{a}(\xi/2)} \hat{\tilde{a}}(\xi/2) \widehat{f_{n-1}}(\xi/2)$ and (3.5), we deduce that

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} \widehat{f_n}(\xi + 2\pi k) &= \sum_{k\in\mathbb{Z}^d} \sum_{\gamma\in\{0,1\}^d} \overline{\hat{a}(\xi/2 + \gamma\pi)} \hat{\tilde{a}}(\xi/2 + \gamma\pi) \widehat{f_{n-1}}(\xi/2 + \gamma\pi + 2\pi k) \\ &= \sum_{\gamma\in\{0,1\}^d} \overline{\hat{a}(\xi/2 + \gamma\pi)} \hat{\tilde{a}}(\xi/2 + \gamma\pi) = 1. \end{aligned}$$

Therefore, by induction, (3.14) holds. Now it follows from (3.14) that $\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f_n}(\xi) e^{ik\cdot\xi}\, d\xi = \delta_k$ for all $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}^d$. Therefore, by (3.7), we conclude that

$$\langle \phi, \tilde{\phi}(\cdot - k) \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) \overline{\hat{\tilde{\phi}}(\xi)} e^{ik\cdot\xi}\, d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\Phi}(\xi) e^{ik\cdot\xi}\, d\xi = \delta_k,$$
$$k \in \mathbb{Z}^d.$$

Now by (3.3) and a standard argument in wavelet analysis, we see that (3.2) holds. This completes the proof. □

3.2 Proofs of Corollaries 1.4 and 1.7

Proof of Corollary 1.4 Let $\hat{b}(\xi) := \overline{\eta} e^{-i\xi}\overline{\hat{\tilde{a}}(\xi+\pi)}$ and $\hat{\tilde{b}}(\xi) := \eta^{-1} e^{-i\xi}\overline{\hat{a}(\xi+\pi)}$. By (1.22), it is easy to check that (3.3) is satisfied with $d = 1$. Note that for a 2π-periodic trigonometric polynomial $\hat{a}$ with $\hat{a}(0) = 1$, we always have $\mu_2(\hat{a}) = \nu_2(\hat{a})$. Now the sufficiency part of Corollary 1.4 follows directly from Theorem 3.1.

Now we prove the necessity of the conditions in (1.22) and (1.23). By a standard argument in wavelet analysis, it is easy to see that the condition in (1.22) is a direct consequence of the biorthogonality property in (3.2) of the two systems $X^s(\phi;\psi)$ and $X^{-s}(\tilde{\phi};\tilde{\psi})$.

Since $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$, we see that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz sequence in $H^s(\mathbb{R})$. By induction, it follows from the refinement equation $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ that

$$\hat{\phi}\big(2^n\xi\big) = \widehat{a^n}(\xi)\hat{\phi}(\xi) \quad \text{with } \widehat{a^n}(\xi) := \hat{a}\big(2^{n-1}\xi\big)\cdots\hat{a}(2\xi)\hat{a}(\xi),\ n \in \mathbb{N}_0.$$

Consequently, we deduce that

$$\big(1 - e^{-i\xi}\big)^m \hat{\phi}\big(2^n\xi\big) = \widehat{\nabla^m a^n}(\xi)\hat{\phi}(\xi) \quad \text{with } \widehat{\nabla^m a^n}(\xi) := \big(1 - e^{-i\xi}\big)^m \widehat{a^n}(\xi), \tag{3.15}$$

that is, in the time domain, we have

$$2^{-n}\big[\nabla^m_{2^{-n}}\phi\big]\big(2^{-n}\cdot\big) = \sum_{k\in\mathbb{Z}} \big[\nabla^m a^n\big]_k \phi(\cdot - k), \quad m, n \in \mathbb{N}_0, \tag{3.16}$$

where $\nabla_y f := f - f(\cdot - y)$ and $\nabla^m := (\nabla)^m$. Now we consider the two cases $s \geqslant 0$ and $s < 0$.

Suppose that $s \geqslant 0$. Since $\phi \in H^s(\mathbb{R})$ and $s \geqslant 0$, we have $\phi \in L_2(\mathbb{R})$. Since $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz sequence in $H^s(\mathbb{R})$ and $\hat{\phi}$ is a continuous function, we see that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is also a Riesz sequence in $L_2(\mathbb{R})$. Now by (3.16), there exists a positive constant C, depending only on ϕ, such that

$$\big\|\nabla^m a^n\big\|^2_{\ell_2(\mathbb{Z})} \leqslant C\big\|2^{-n}\big[\nabla^m_{2^{-n}}\phi\big]\big(2^{-n}\cdot\big)\big\|^2_{L_2(\mathbb{R})} = C2^{-n}\big\|\nabla^m_{2^{-n}}\phi\big\|^2_{L_2(\mathbb{R})}. \tag{3.17}$$

Since $\phi \in H^s(\mathbb{R})$ and $s \geqslant 0$, for any $m > s$, we see that $\lim_{n\to\infty} 2^{sn}\|\nabla^m_{2^{-n}}\phi\|_{L_2(\mathbb{R})} = 0$. Now it follows from (3.17) that

$$\lim_{n\to\infty} 2^{n+2sn}\big\|\nabla^m a^n\big\|^2_{\ell_2(\mathbb{Z})} = 0 \quad \forall m \in \mathbb{N} \text{ and } m > s. \tag{3.18}$$

Since $\hat{a}$ is a 2π-periodic trigonometric polynomial, by [29, Proposition 4.1] we see that (3.18) implies

$$\limsup_{n\to\infty}\big\|\nabla^m a^n\big\|^{1/n}_{\ell_2(\mathbb{Z})} < 2^{-1/2-s} \quad \text{and} \quad sr(\hat{a}) > s. \tag{3.19}$$

Now taking $m = sr(\hat{a})$, by the definition of $\nu_2(\hat{a})$ in (1.18), we conclude that $\nu_2(\hat{a}) > s$.

Suppose that $s < 0$. Let $m = sr(\hat{a})$. Then $m \geqslant 0 > s$. Since $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz sequence in $H^s(\mathbb{R})$, it follows from (3.16) that there exists a positive constant C, depending only on ϕ, such that

$$\|\nabla^m a^n\|_{\ell_2(\mathbb{Z})}^2 \leqslant C \|2^{-n}[\nabla_{2^{-n}}^m \phi](2^{-n}\cdot)\|_{H^s(\mathbb{R})}^2. \tag{3.20}$$

On the other hand, we have

$$\begin{aligned}\|2^{-n}[\nabla_{2^{-n}}^m \phi](2^{-n}\cdot)\|_{H^s(\mathbb{R})}^2 &= \frac{1}{2\pi}\int_{\mathbb{R}} \big|\widehat{\nabla_{2^{-n}}^m \phi}(2^n\xi)\big|^2 \big(1+|\xi|^2\big)^s \, d\xi \\ &= 2^{-n}\frac{1}{2\pi}\int_{\mathbb{R}} \big|\widehat{\nabla_{2^{-n}}^m \phi}(\xi)\big|^2 \big(1+|2^{-n}\xi|^2\big)^s \, d\xi \\ &= 2^{-n-2sn}\frac{1}{2\pi}\int_{\mathbb{R}} \big|\widehat{\nabla_{2^{-n}}^m \phi}(\xi)\big|^2 \big(2^{2n}+|\xi|^2\big)^s \, d\xi.\end{aligned}$$

Note that $\widehat{\nabla_{2^{-n}}^m \phi}(\xi) = (1 - e^{-i2^{-n}\xi})^m \hat{\phi}(\xi)$. Therefore, $|\widehat{\nabla_{2^{-n}}^m \phi}(\xi)| \leqslant 2^m |\hat{\phi}(\xi)|$. Hence, by (3.20) and the above inequality, we have the estimate

$$\|\nabla^m a^n\|_{\ell_2(\mathbb{Z})}^2 \leqslant 2^{2m} C 2^{-n-2sn}\frac{1}{2\pi}\int_{\mathbb{R}} \big|\hat{\phi}(\xi)\big|^2 \big(2^{2n}+|\xi|^2\big)^s \, d\xi. \tag{3.21}$$

Since $s < 0$, we observe that $|\hat{\phi}(\xi)|^2(2^{2n} + |\xi|^2)^s \leqslant |\hat{\phi}(\xi)|^2(1 + |\xi|^2)^s$ for all $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $\phi \in H^s(\mathbb{R})$, we have $|\hat{\phi}(\xi)|^2(1 + |\xi|^2)^s \in L_1(\mathbb{R})$. Now by the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \big|\hat{\phi}(\xi)\big|^2 \big(2^{2n}+|\xi|^2\big)^s \, d\xi = \int_{\mathbb{R}} \big|\hat{\phi}(\xi)\big|^2 \lim_{n\to\infty}\big(2^{2n}+|\xi|^2\big)^s \, d\xi = 0.$$

Hence, it follows from the above inequality and (3.21) that (3.18) still holds. Now by [29, Proposition 4.1], we still have (3.19). Since $m = sr(\hat{a})$, by the definition of $\nu_2(\hat{a})$ in (1.18), we conclude that $\nu_2(\hat{a}) > s$.

Noting that $\{\tilde{\phi}(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz sequence in $H^{-s}(\mathbb{R})$, using the same argument for the proof of $\nu_2(\hat{a}) > s$, we can similarly prove that $\nu_2(\hat{\tilde{a}}) > -s$. $\square$

Proof of Corollary 1.7 Let $\hat{\tilde{a}}(\xi) := \overline{\hat{b}(\xi+\pi)}/\overline{d(\xi)}$ and $\hat{\tilde{b}}(\xi) := -\overline{\hat{a}(\xi+\pi)}/\overline{d(\xi)}$. Since $d(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ and $\hat{a}$, $\hat{b}$ are 2π-periodic functions with exponential decay, we see that $\hat{\tilde{a}}$ and $\hat{\tilde{b}}$ are 2π-periodic functions with exponential decay. Now it is straightforward to check that (3.3) is satisfied. Note that we assumed $\hat{a}(\pi)\hat{b}(0) = 0$ and $\hat{a}(0) = 1$. By the definition of $\hat{\tilde{a}}$, we have

$$\hat{\tilde{a}}(0) = \frac{\overline{\hat{b}(\pi)}}{\overline{\hat{a}(0)\hat{b}(\pi) - \hat{a}(\pi)\hat{b}(0)}} = \frac{\overline{\hat{b}(\pi)}}{\overline{\hat{b}(\pi)}} = 1.$$

Hence, all the conditions in Theorem 3.1 are satisfied in a univariate setting. Now by Theorem 3.1, $(X^s(\phi;\psi), X^{-s}(\tilde{\phi};\tilde{\psi}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$. In particular, $X^s(\phi;\psi)$ is a Riesz wavelet basis in $H^s(\mathbb{R})$. $\square$

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