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1 author:

Martin Ehler
University of Vienna

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The Multiresolution Structure of Pairs of Dual Wavelet Frames for a Pair of Sobolev Spaces

M. Ehler

Abstract

We study the multiresolution structure of wavelet frames. It is known that the internal structure of almost any nontrivial overcomplete dyadic tight wavelet frame's underlying multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ is degenerated in $L_2(\mathbb{R})$. More precisely, the relation $W_0 \oplus V_0 = V_1$, that would hold for wavelet bases, collapses into $W_0 = V_1$, where W_0 is the closed linear span of the wavelets' integer shifts.

In the present paper, we extend the latter result in three ways: First and most significantly, we don't require a tight wavelet frame and verify that the result still holds for a pair of dual wavelet frames. Secondly, we allow for general scaling matrices. Thirdly, the pair of dual wavelet frames is not required to form a frame for $L_2(\mathbb{R}^d)$ but only for a pair of dual Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Thus, the dual refinable function does not have to be in $L_2(\mathbb{R}^d)$. Finally, we construct pairs of dual wavelet frames for a pair of dual Sobolev spaces from any pair of multivariate refinable functions.

Keywords: Pairs of dual wavelet frames, multiresolution analysis, mixed extension principle, shift-invariant space, Sobolev space..

MSC: Primary 42C40; Secondary 42C15, 42B35.

§1. Introduction

Fast wavelet algorithms and their underlying multiresolution analysis are nowadays widely used in applied mathematics, signal and image processing, and the treatment of operator equations. Wavelets are commonly constructed from a refinable function that generates a multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$, i.e., an increasing sequence of closed subspaces in $L_2(\mathbb{R}^d)$. The wavelets are chosen such that the closed linear span W_0 of their integer shifts complement V_0 in V_1. Orthonormal wavelet bases are chosen in such a way that $W_0 \oplus V_0 = V_1$ is a direct orthogonal sum, and, for biorthogonal wavelet bases, one still has a direct algebraic sum. While V_0 reflects a coarse approximation, W_0 adds the details to represent any element in V_1. Thus, coarse scales and details are well-separated which is widely utilized when using the wavelet transform in applications.

The classical wavelet basis is nowadays often replaced with the more flexible concept of wavelet frames that allows for redundancy, cf. [2, 4, 7, 12, 20, 22, 25]. Tight wavelet frames generalize orthonormal wavelet bases, and pairs of dual wavelet frames are the redundant counterpart of biorthogonal wavelet bases. Kim, Kim, and Lim, however, have verified in [21] that almost any tight wavelet frame leads to a degenerated multiresolution structure, i.e., one has $W_0 = V_1$, for any nontrivial overcomplete tight wavelet frame for $L_2(\mathbb{R})$ with dyadic scaling. In other words, the wavelet space is no longer a complementary space, but rather fills the complete scale V_1. Thus, coarse scale approximation and details are no longer well-separated. In the present paper, we aim on extending this (negative) result to more general wavelet frame systems.

Before we explain the generalization in detail, we must briefly recall a rather new concept of wavelet frame systems: Han and Shen have introduced in [17] a new concept to simplify the construction of wavelet systems. They provided a construction recipe for pairs of dual wavelet frames for a pair of Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Constructing a pair of dual wavelet frames for a pair of Sobolev spaces turns out to be easier than for $L_2(\mathbb{R}^d)$, because smoothness and vanishing moment requirements can be separated from each other and split into primal and dual wavelets, respectively. Therefore, this construction scheme provides more flexibility than classical constructions for $L_2(\mathbb{R}^d)$.

The main result of the present paper is the generalization of the result by Kim, Kim, and Lim about the degenerated multiresolution structure. In fact, we verify that almost any pair of dual wavelet frames with general scaling matrices for a pair of Sobolev spaces leads to a degenerated MRA. This means that we extend the results in [21] in three ways: first and most significantly, we do not require the frames to be tight. We verify that the result still holds for a pair of dual wavelet frames. Secondly, we allow for general scaling matrices. Thirdly, the pair of dual wavelet frames is not required to form a frame for $L_2(\mathbb{R}^d)$, but only for the pair of Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Thus, the dual refinable function does not have to be in $L_2(\mathbb{R}^d)$.

The extension from $L_2(\mathbb{R}^d)$ to a pair of Sobolev spaces is relatively straight-forward. The same is true for the treatment of general scaling matrices. However, the extension from tight wavelet frames to pairs of dual wavelet frames is more difficult because the proof in [21] cannot directly be applied. As our main contribution we have identified an additional but natural condition under which the proof in [21] becomes much simpler and hence can be extended to pairs of dual wavelet frames. More precisely, we require that the underlying refinable functions satisfy an additional stability condition. It should be mentioned that the overwhelming examples of pairs of dual wavelet frames in literature satisfy the stability requirement, and the generalization means a significant extension of the results in [21].

We finally present a construction of pairs of dual wavelet frames for a pair of Sobolev spaces. We verify in the multivariate setting with isotropic scaling matrices that any pair of refinable functions give rise to a pair of dual wavelet frames for a pair of Sobolev spaces. This extends results in [6, 12], because the dual refinable function $\widetilde{\varphi}$ is not required to be contained in $L_2(\mathbb{R}^d)$.

The outline is as follows: Section 2 is dedicated to introducing pairs of dual wavelet frames for a pair of Sobolev spaces. The internal structure of pairs of dual wavelet frames is considered in Section 3. We present Kim, Kim, and Lim's results in Section 3.1. Section 3.2 recalls some results about shift-invariant spaces. We present our main result about the degenerated multiresolution structure of pairs of dual wavelet frames for a pair of Sobolev spaces in 3.3. We construct pairs of dual wavelet frames for a pair of Sobolev spaces in Section 4. A general construction recipe is recalled in Section 4.1, and Section 4.2 contains the construction from any pair of refinable functions, in which the dual refinable function does not have to be contained in $L_2(\mathbb{R}^d)$.

§2. The Setting

2.1. Frames for Hilbert Spaces

Let $\mathcal{K}$ be a countable index set. A collection $\{f_\kappa : \kappa \in \mathcal{K}\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist two positive constants A, B such that

$$A\|f\|^2_{\mathcal{H}} \leq \|(\langle f, f_\kappa\rangle_{\mathcal{H}})_{\kappa\in\mathcal{K}}\|^2_{\ell_2} \leq B\|f\|^2_{\mathcal{H}}, \quad \text{for all } f \in \mathcal{H}. \tag{2.1}$$

One observes that the *frame operator* $S := FF^*$, where

$$F : \ell_2(\mathcal{K}) \to \mathcal{H}, \quad (c_\kappa)_{\kappa\in\mathcal{K}} \mapsto \sum_{\kappa\in\mathcal{K}} c_\kappa f_\kappa,$$

is positive and boundedly invertible. The system $\{S^{-1}f_\kappa : \kappa \in \mathcal{K}\}$, which is also a frame for $\mathcal{H}$, is called the *canonical dual frame*, and it provides the expansion

$$f = \sum_{\kappa\in\mathcal{K}} \left\langle f, S^{-1}f_\kappa\right\rangle_{\mathcal{H}} f_\kappa, \quad \text{for all } f \in \mathcal{H}, \tag{2.2}$$

cf. Chapter 5 in [1]. If the frame operator S is a constant times the identity, then the frame is called tight and (2.2) resembles the expansion of f into an orthonormal basis.

2.2. Wavelet Frames for Sobolev Spaces

Let M be a *scaling matrix*, i.e., an integer matrix, whose eigenvalues are larger than one in modulus. Throughout the present paper, we suppose that M is *isotropic*, i.e., it can be diagonalized and all of its eigenvalues have the same modulus, and we denote $m := |\det(M)|$. For a tempered distribution f, let

$$f_{j,k}(x) := m^{\frac{j}{2}} f(M^j x - k), \quad \text{for } j \in \mathbb{N}_0, k \in \mathbb{Z}^d.$$

The standard approach for the construction of wavelets is based on a *refinable function* φ, i.e., a solution of the *refinement equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(Mx - k), \tag{2.3}$$

where $(a_k)_{k\in\mathbb{Z}^d}$ is a finitely supported sequence. For the further discussion, let us mention that, for a function $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ used in this paper is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$ for $\xi \in \mathbb{R}^d$. In order to choose φ, one starts with a finitely supported sequence $(a_k)_{k\in\mathbb{Z}^d}$ satisfying $\sum_{k\in\mathbb{Z}^d} a_k = 1$. The *symbol* of the sequence $(a_k)_{k\in\mathbb{Z}^d}$ is the Laurent polynomial

$$a(z) := \sum_{k\in\mathbb{Z}^d} a_k z^k, \qquad z \in (\mathbb{C}\backslash\{0\})^d,$$

and

$$\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} a(e^{-2\pi i (M^T)^{-j}\xi}), \quad \xi \in \mathbb{R}^d \tag{2.4}$$

converges uniformly on compact sets, such that φ constitutes a compactly supported distributional solution of the refinement equation (2.3), normalized by $\widehat{\varphi}(0) = 1$, cf. [5].

Let Γ denote a complete set of representatives of $\left(M^{-\top}\mathbb{Z}^d\right)/\mathbb{Z}^d$ with $0 \in \Gamma$ and let $z_\gamma := (z_1 e^{-2\pi i\gamma_1}, \ldots, z_d e^{-2\pi i\gamma_d})$ for $\gamma = (\gamma_1, \ldots, \gamma_d) \in \Gamma$. We also make use of $\mathbf{1} := (1, \ldots, 1)$, and we say that a satisfies the *sum rules of order* s if a has a zero of order s at $\mathbf{1}_\gamma$ for all $\gamma \in \Gamma\backslash\{0\}$. Let $\mathrm{sr}(a)$ denote the maximal sum rule order of a.

For a real number s, the Sobolev space $H^s(\mathbb{R}^d)$ consists of all tempered distributions f such that

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + \|\xi\|^2)^s d\xi < \infty,$$

cf. [17]. Note that $\varphi \in L_2(\mathbb{R}^d)$ already implies

$$\nu_2(\varphi) = \sup\{s \in \mathbb{R} : \varphi \in H^s(\mathbb{R}^d)\} > 0,$$

i.e., $\varphi \in H^s(\mathbb{R}^d)$ for some $s > 0$, cf. [13]. This suggests that we should replace $L_2(\mathbb{R}^d)$ with the Sobolev space. By introducing the normalization in $H^s(\mathbb{R}^d)$, i.e.,

$$f^s_{j,k}(x) := m^{j(\frac{1}{2}-\frac{s}{d})} f(M^j x - k), \quad \text{for } j \in \mathbb{N}_0, k \in \mathbb{Z}^d,$$

this is also supported by the following fact: for $s = 0$, the collection

$$\{\varphi^s_{j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}^d\} \tag{2.5}$$

is not a frame for $L_2(\mathbb{R}^d) = H^0(\mathbb{R}^d)$, but it is a frame for $H^s(\mathbb{R}^d)$, for all $0 < s < \min(\nu_2(\varphi), \mathrm{sr}(a))$, cf. [17]. However, its canonical dual frame might not consist of shifts and dilations. In order to overcome this limitation, we recall that $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ form a pair of dual spaces, where

$$\langle f, g\rangle := \int_{\mathbb{R}^d} \widehat{f}(x)\overline{\widehat{g}(x)}dx, \qquad f \in H^s(\mathbb{R}^d),\ g \in H^{-s}(\mathbb{R}^d),$$

denotes the duality mapping. Han and Shen observed in [17] that (2.5) is a frame for $H^s(\mathbb{R}^d)$ iff there are positive constants A and B such that

$$A\|g\|^2_{H^{-s}} \le \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |\langle g, \varphi^s_{j,k}\rangle|^2 \le B\|g\|^2_{H^{-s}}, \quad \text{for all } g \in H^{-s}(\mathbb{R}^d). \tag{2.6}$$

Moreover, by defining

$$X^s(\varphi;\psi^{(1)},\ldots,\psi^{(n)}) := \{\varphi_{0,k} : k \in \mathbb{Z}^d\} \cup \{\psi^{(\mu),s}_{j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}^d, \mu = 1,\ldots,n\},$$

the system in (2.5) can be rewritten as $X^s(\varphi;\varphi_{1,\rho_1},\ldots,\varphi_{1,\rho_m})$, where $\widetilde{\Gamma} := \{\rho_1,\ldots,\rho_m\}$ is a complete set of representatives of $\mathbb{Z}^d/M\mathbb{Z}^d$. The characterization (2.6) leads us to the following alternative concept as introduced in [17]: A frame $X^s(\varphi;\psi^{(1)},\ldots,\psi^{(n)})$ for $H^s(\mathbb{R}^d)$ is called a tight wavelet frame for the pair $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ iff

$$f = \sum_{k\in\mathbb{Z}^d}\langle f, \varphi_{0,k}\rangle\varphi_{0,k} + \sum_{\mu=1}^{n}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f, \psi^{(\mu),-s}_{j,k}\rangle\psi^{(\mu),s}_{j,k}$$

holds for all $f \in H^s(\mathbb{R}^d)$. Two frames $X^s(\varphi;\psi^{(1)},\ldots,\psi^{(n)})$ and $X^{-s}(\widetilde{\varphi};\widetilde{\psi}^{(1)},\ldots,\widetilde{\psi}^{(n)})$ for $H^s(\mathbb{R}^d)$

and $H^{-s}(\mathbb{R}^d)$, respectively, are called a *pair of dual wavelet frames for the pair* $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ iff

$$f = \sum_{\mu=1}^{n} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{0,k}\rangle \varphi_{0,k} + \sum_{j=0}^{\infty}\sum_{\mu=1}^{n}\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}^{(\mu),-s}_{j,k}\rangle \psi^{(\mu),s}_{j,k}, \qquad \text{for all } f\in H^s(\mathbb{R}^d),$$

$$g = \sum_{\mu=1}^{n} \sum_{k\in\mathbb{Z}^d} \langle g, \varphi_{0,k}\rangle \widetilde{\varphi}_{0,k} + \sum_{j=0}^{\infty}\sum_{\mu=1}^{n}\sum_{k\in\mathbb{Z}^d} \langle g, \psi^{(\mu),s}_{j,k}\rangle \widetilde{\psi}^{(\mu),-s}_{j,k}, \qquad \text{for all } g\in H^{-s}(\mathbb{R}^d),$$

with the series converging unconditionally in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively, see [17].

To construct wavelets, we choose two compactly supported refinable functions φ and $\widetilde{\varphi}$ and seek to define wavelets through

$$\psi^{(\mu)}(x) := m \sum_{k\in\mathbb{Z}^d} a^{(\mu)}_k \varphi(Mx-k) \quad \text{and} \quad \widetilde{\psi}^{(\mu)}(x) := m \sum_{k\in\mathbb{Z}^d} b^{(\mu)}_k \widetilde{\varphi}(Mx-k), \tag{2.7}$$

where $\big(a^{(\mu)}_k\big)_{k\in\mathbb{Z}^d}$ and $\big(b^{(\mu)}_k\big)_{k\in\mathbb{Z}^d}$, $\mu=1,\dots,n$, are suitable finitely supported sequences. Throughout the present paper, we apply the notation

$$a(z^M) := \sum_{k\in\mathbb{Z}^d} a_k z^{Mk}, \quad a(\tfrac{1}{z}) := \sum_{k\in\mathbb{Z}^d} a_k z^{-k},$$

and we say that a symbol a has s vanishing moments if a has a zero of order s at $z=\mathbf{1}$. Han and Shen have proven the following theorem in [17] for dyadic scaling. In fact, by following the lines in [17] one observes that it still holds with respect to an isotropic scaling matrix, see also [11] for the transition from dyadic to isotropic characterizations of function spaces:

Theorem 2.1. *Let M be an isotropic scaling matrix, and let $a^{(0)},\dots,a^{(n)}$ and $b^{(0)},\dots b^{(n)}$ be symbols with $a^{(0)}(\mathbf{1}) = b^{(0)}(\mathbf{1}) = 1$. Define φ and $\widetilde{\varphi}$ by (2.4), and the wavelet functions $\psi^{(1)},\dots,\psi^{(n)}$ and $\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)}$ by (2.7). Suppose that*

(1) *the following identity holds*

$$\sum_{\mu=0}^{n} a^{(\mu)}(z) b^{(\mu)}(\tfrac{1}{z_\gamma}) = \delta_{0,\gamma}, \quad \textit{for all } \gamma\in\Gamma, \tag{2.8}$$

(2) *for a real number $s\in\mathbb{R}$ satisfying $\nu_2(\varphi) > s$ and $\nu_2(\widetilde{\varphi}) > -s$, the symbols $a^{(1)},\dots,a^{(n)}$ and $b^{(1)},\dots,b^{(n)}$ have more than $-s$ and s vanishing moments, respectively.*

Then $X^s(\varphi;\psi^{(1)},\dots,\psi^{(n)})$ and $X^{-s}(\widetilde{\varphi};\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)})$ are a pair of dual wavelet frames for $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

§3. The Internal Structure of Wavelet Frames

3.1. The Internal Structure of Tight Wavelet Frames for $L_2(\mathbb{R}^d)$

This section is dedicated to recalling the results of Kim, Kim, and Lim in [21] about the structure of a multiresolution analysis of tight wavelet frames. Given a dyadic wavelet tight frame $X^0(\varphi; \psi^{(1)}, \dots, \psi^{(n)})$ for $L_2(\mathbb{R}^d)$, i.e., a tight wavelet frame for the pair of Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ with $s = 0$, then we define

$$\begin{aligned} V_0 &= \overline{\operatorname{span}}\{\varphi(x-k) : k \in \mathbb{Z}^d\}, \\ W_0 &= \overline{\operatorname{span}}\{\psi^{(\mu)}(x-k) : k \in \mathbb{Z}^d,\ \mu = 1, \dots, n\}, \\ V_1 &= \overline{\operatorname{span}}\{\varphi(2x-k) : k \in \mathbb{Z}^d\}, \end{aligned}$$

where the closure is taken in $L_2(\mathbb{R}^d)$. From the perspective of a wavelet basis one would expect that W_0 is an algebraic complement of V_0 in V_1, cf. [5]. In [21], however, it was shown that almost any dyadic tight wavelet frame for $L_2(\mathbb{R})$ leads to $W_0 = V_1$, i.e., the wavelet space completely fills V_1:

Theorem 3.1. *Let $X^s(\varphi; \psi^{(1)}, \dots, \psi^{(n)})$ be a tight wavelet frame for $L_2(\mathbb{R})$, that is constructed by Theorem 2.1 for $d = 1$, $s = 0$ with $M = 2$, and $b^{(0)} = a^{(0)}, \dots, b^{(n)} = a^{(n)}$. Suppose that there does not exist any number $\mu \in \{1, \dots, n\}$ such that any symbol $a^{(1)}, \dots, a^{(n)}$ other than $a^{(\mu)}$ is a multiple of $a^{(\mu)}$, then the equation $W_0 = V_1$ must hold.*

By analyzing the conditions in Theorem 2.1, one observes that the assumptions in Theorem 3.1 imply

$$a(z)a(1/z) + a(z_{1/2})a(1/z_{1/2}) \not\equiv 1. \tag{3.1}$$

To obtain an orthonormal wavelet basis, the underlying refinable function must have orthonormal integer shifts, which implies

$$a(z)a(1/z) + a(z_{1/2})a(1/z_{1/2}) \equiv 1, \tag{3.2}$$

see [5]. One major reason for constructing wavelet frames instead of orthonormal wavelet bases is that one wants to use refinable functions that do not satisfy (3.2) and hence (3.1) holds. Thus, condition (3.1) is natural in the context of tight wavelet frames.

3.2. Shift-invariant Subspaces of $H^s(\mathbb{R}^d)$

To extend Theorem 3.1 to wavelet frames for Sobolev spaces, we need to recall few results about shift-invariant subspaces of $L_2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$.

Let $\mathcal{T}$ be a topological function space that is invariant under integer shifts. For a subset $\Phi \subset \mathcal{T}$, we denote by $S_\Phi(\mathcal{T})$ the smallest closed integer shift-invariant subset of $\mathcal{T}$ that contains Φ. In the following we collect few results from [18] and [8], see also [19] for more emphasis on approximation order and [23, 24] for affine systems in $L_2(\mathbb{R}^d)$.

For $\Phi \subset L_2(\mathbb{R}^d)$, we obtain

$$S_\Phi(L_2(\mathbb{R}^d)) = \overline{\operatorname{span}}\{\varphi(\cdot - k) : k \in \mathbb{Z}^d,\ \varphi \in \Phi\}. \tag{3.3}$$

The shift-invariant space can be characterized on the Fourier side as

$$S_\Phi(L_2(\mathbb{R}^d)) = \{f \in L_2(\mathbb{R}^d) : \hat{f} = \tau^* \hat{\Phi}, \tau \text{ measurable and } \mathbb{Z}^d\text{-periodic}\}.$$

For $\Phi \subset H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, we obtain

$$S_\Phi(H^s(\mathbb{R}^d)) = \{f \in H^s(\mathbb{R}^d) : \hat{f} = \tau^* \hat{\Phi}, \tau \text{ measurable and } \mathbb{Z}^d\text{-periodic}\},$$

and, hence,

$$S_\Phi(H^s(\mathbb{R}^d)) = H^s(\mathbb{R}^d) \cap S_\Phi(L_2(\mathbb{R}^d)), \quad \text{for } s \geq 0.$$

Let us define the fiber of a space $\mathcal{T}$ at $\xi \in \mathbb{R}^d$ by

$$\mathcal{T}_{||\xi} := \{(\hat{f}(\xi + k))_k : f \in \mathcal{T}\}.$$

We have

$$\widehat{S}_\Phi(L_2(\mathbb{R}^d))_{||\xi} = \overline{\operatorname{span}}\{(\hat{\phi}(\xi + k))_k : \phi \in \Phi\} = \overline{\operatorname{span}}(\widehat{\Phi}_{||\xi}), \text{ for a.e. } \xi \in \mathbb{T}^d, \tag{3.4}$$

and $S_\Phi(L_2(\mathbb{R}^d))$ can be characterized in terms of its fibers:

$$S_\Phi(L_2(\mathbb{R}^d)) = \{f \in L_2(\mathbb{R}^d) : \hat{f}_{||\xi} = \widehat{S}_\Phi(L_2(\mathbb{R}^d))_{||\xi}, \text{ a.e. } \xi \in \mathbb{T}^d\}. \tag{3.5}$$

Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. The mapping

$$J_{-s} : L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d), \quad f \mapsto \mathcal{F}^{-1}\big((1 + \|\cdot\|^2)^{-s} \hat{f}\big) \tag{3.6}$$

is an isometry and yields

$$S_\Phi(H^s(\mathbb{R}^d)) = J_{-s} S_{J_s \Phi}(L_2(\mathbb{R}^d)). \tag{3.7}$$

In the following, we restrict us to finite sets Φ. In this case $\operatorname{span}(\widehat{\Phi}_{||\xi})$ is finite-dimensional and therefore closed. Let us state the first proposition:

Proposition 3.2. *Let Φ be finite and $s \geq 0$, then*

$$\begin{aligned} S_\Phi(H^s(\mathbb{R}^d)) &= \{f \in H^s(\mathbb{R}^d) : \hat{f}_{\|\xi} \in \operatorname{span}(\hat{\Phi}_{\|\xi}),\ \textit{a.e. } \xi \in \mathbb{T}^d\} \\ &= \{f \in H^s(\mathbb{R}^d) : \hat{f}_{\|\xi} \in \widehat{S}_\Phi(L_2(\mathbb{R}^d))_{\|\xi},\ \textit{a.e. } \xi \in \mathbb{T}^d\}. \end{aligned}$$

Proof. Making use of (3.7) and (3.5) with (3.4), we obtain:

$$\begin{aligned} S_\Phi(H^s(\mathbb{R}^d)) &= J_{-s} S_{J_s\Phi}(L_2(\mathbb{R}^d)) \\ &= J_{-s}\{f \in L_2(\mathbb{R}^d) : \hat{f}_{||\xi} \in \operatorname{span}(\widehat{J_s\Phi}_{\|\xi})\} \\ &= \{g \in H^s(\mathbb{R}^d) : \widehat{J_s g}_{||\xi} \in \operatorname{span}(\widehat{J_s\Phi}_{||\xi})\} \\ &= \{g \in H^s(\mathbb{R}^d) : ((1+\|\xi+k\|^2)^s \hat{g}(\xi+k))_k \in (1+\|\xi+k\|^2)^s \operatorname{span}(\hat{\Phi}(\xi+k))\} \\ &= \{g \in H^s(\mathbb{R}^d) : \hat{g}_{\|\xi} \in \operatorname{span}(\hat{\Phi}_{\|\xi})\} \\ &= \{g \in H^s(\mathbb{R}^d) : \hat{g}_{\|\xi} \in \widehat{S}_\Phi(L_2)_{\|\xi}\}. \end{aligned}$$

The latter equalities hold because, for $f \in L_2(\mathbb{R}^d)$, we have $\hat{f}_{||\xi} \in \ell_2(\mathbb{Z}^d)$, and hence, $f \in H^s(\mathbb{R}^d)$, $s \geq 0$ also implies $\hat{f}_{||\xi} \in \ell_2(\mathbb{Z}^d)$. ■

We also need the following statement about the fibers of Sobolev spaces:

Lemma 3.3. *For finite $\Phi \subset H^s(\mathbb{R}^d)$, $s \geq 0$, we have*

$$\hat{S}_\Phi(H^s(\mathbb{R}^d))_{\|\xi} = \operatorname{span}\hat{\Phi}_{\|\xi},\ \textit{for a.e. } \xi \in \mathbb{T}^d.$$

In other words, $\hat{S}_\Phi(H^s(\mathbb{R}^d))_{\|\xi} = \hat{S}_\Phi(L_2(\mathbb{R}^d))_{\|\xi}$, for a.e. $\xi \in \mathbb{T}^d$.

Proof. According to Proposition 3.2, we have $\hat{S}_\Phi(H^s(\mathbb{R}^d))_{\|\xi} \subset \operatorname{span}\hat{\Phi}_{\|\xi}$, for a.e. $\xi \in \mathbb{T}^d$. Choosing arbitrary coefficients $c_1, \ldots, c_r$ yields

$$\sum_{i=1}^r c_i \hat{\varphi}_{i\|\xi} = (\widehat{\sum_{i=1}^r c_i\varphi_i})_{\|\xi}.$$

Since $\sum_{i=1}^r c_i\varphi_i$ is contained in $H^s(\mathbb{R}^d)$, the reverse set inclusion holds as well. ■

The above results in the present section imply that (3.3) also holds for the Sobolev space $H^s(\mathbb{R}^d)$, i.e.,

$$S_\Phi(H^s(\mathbb{R}^d)) = \overline{\mathrm{span}}\{\varphi(\cdot - k) : k \in \mathbb{Z}^d,\ \varphi \in \Phi\}. \tag{3.8}$$

For a shift-invariant subspace S, the set $\sigma(S) = \{\xi \in \mathbb{T}^d : \hat{S}_{||\xi} \neq \{0\}\}$ is called the spectrum of S. Let us recall a standard result for shift-invariant spaces, see, for instance, [21] and references therein:

Theorem 3.4. *Suppose that S_1 and S_2 are finitely generated shift-invariant subspaces such that $S_1 \subset S_2$. Then $\hat{S}_{1||\xi} \subset \hat{S}_{2||\xi}$, for a.e. $\xi \in \mathbb{T}^d$. In particular, $S_1 = S_2$ iff $\dim(\hat{S}_{1||\xi}) = \dim(\hat{S}_{2||\xi})$, for a.e. $\xi \in \sigma(S_2)$.*

3.3. The Internal Structure of Pairs of Dual Wavelet Frames for a Pair of Dual Sobolev Spaces

Our main result is the generalization of Theorem 3.1 to wavelet bi-frames for pairs of Sobolev spaces with isotropic scaling matrices M. We consider the spaces

$$V_0 = \overline{\mathrm{span}}\{\varphi(x-k) : k \in \mathbb{Z}^d\}, \tag{3.9}$$

$$W_0 = \overline{\mathrm{span}}\{\psi^{(\mu)}(x-k) : k \in \mathbb{Z}^d,\ \mu = 1,\ldots,n\}, \tag{3.10}$$

$$V_1 = \overline{\mathrm{span}}\{\varphi(Mx-k) : k \in \mathbb{Z}^d\}, \tag{3.11}$$

where the closure is taken in $H^s(\mathbb{R}^d)$. For preparation, let us recall that φ is called stable in $H^s(\mathbb{R}^d)$ if its integer shifts form a Riesz-basis for their closed linear span in $H^s(\mathbb{R}^d)$. Biorthogonal wavelet bases for $L_2(\mathbb{R}^d)$ are constructed from two refinable functions φ and $\widetilde{\varphi}$ that have biorthogonal integer shifts and hence are stable in $L_2(\mathbb{R}^d)$. The geometrical conditions then imply that W_0 is an algebraic complement of V_0 in V_1. Let a and b be the symbols of φ and $\widetilde{\varphi}$, respectively, then the biorthogonality of their integer shifts imply

$$\sum_{\gamma\in\Gamma} a(z_\gamma)b(1/z_\gamma) \equiv 1. \tag{3.12}$$

Condition (3.12) means a significant limitation on the choice of refinable functions, and one constructs pairs of dual wavelet frames to obtain more flexibility. Thus the condition

$$\sum_{\gamma\in\Gamma} a(z_\gamma)b(1/z_\gamma) \not\equiv 1, \tag{3.13}$$

is natural in the context of pairs of dual wavelet frames. We now extend Theorem 3.1:

Theorem 3.5. *Let $X^s(\varphi;\psi^{(1)},\dots,\psi^{(n)})$ and $X^{-s}(\widetilde{\varphi};\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)})$, $s\geq 0$, be a pair of dual wavelet frames for $(H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d))$ that was constructed by Theorem 2.1. If φ is stable in $L_2(\mathbb{R}^d)$ and* (3.13) *holds, then we must have $W_0=V_1$.*

The complete proof of Theorem 3.5 shall be given on Page 13. Let us emphasize that the proof of Theorem 3.1 does not carry over to pairs of dual wavelet frames directly. The key ingredient making it work for frames that are not tight is the stability assumption. Note that it is only needed for the primal refinable function.

Remark 3.6. It should be mentioned that the overwhelming number of wavelet examples in literature - including frames - are based on stable refinable functions. Therefore, the assumption about φ's stability in Theorem 3.5 does not seem to be too restrictive.

Let us shed some light on the impact of the stability assumption on the proof of Theorem 3.5: Kim, Kim, and Lim define in [21] the following numbers $A_{\xi,\gamma}$ (in fact, they only consider $M=2$, and we generalize the approach to general scaling matrices):

$$A_{\xi,\gamma}(M^\top(k+\gamma)) \quad := \quad \hat{\varphi}_{\|M^{-\top}\xi+\gamma}(k),\ \text{for } k\in\mathbb{Z}^d \tag{3.14}$$

$$A_{\xi,\gamma}(k) \quad := \quad 0 \text{ for } k\notin M^\top(\mathbb{Z}^d+\gamma). \tag{3.15}$$

They must explicitly take care of those cases in which there is $\xi\in\mathbb{R}^d$ such that $A_{\xi,\gamma}=0$. In these cases, the proof in [21] cannot be extended to pairs of dual wavelet frames. Our main contribution is the identification of the stability condition to prevent occurrence of such cases:

Proposition 3.7. *If φ is compactly supported and stable in $H^s(\mathbb{R}^d)$, $s\geq 0$ an even integer, then $A_{\xi,\gamma}\neq 0$, for all $\xi\in\mathbb{T}^d$ and $\gamma\in\Gamma$.*

Proof. We first verify that $\hat{\varphi}$ does not have any $\mathbb{Z}^d$-periodical zeros on $\mathbb{T}^d$: If $f\in H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, then $(\hat{f}(\xi+k))_{k\in\mathbb{Z}^d}$ is in a weighted ℓ_2-space with weights $\omega_k^\xi=(1+\|\xi+k\|^2)^s$. For two functions $f,g:\mathbb{R}^d\to\mathbb{C}$, we define

$$[f,g]_s(\xi) := \sum_{k\in\mathbb{Z}^d} f(\xi+k)\overline{g(\xi+k)}(1+\|\xi+k\|^2)^s,\quad \xi\in\mathbb{R}^d,\ s\in\mathbb{R}.$$

Due to [17], a function $f\in H^s(\mathbb{R}^d)$ is stable in $H^s(\mathbb{R}^d)$ iff there are constants $0<C_1\leq C_2<\infty$ such that

$$C_1\leq [\hat{f},\hat{f}]_s(\xi)\leq C_2,\quad \text{for a.e. } \xi\in\mathbb{T}^d. \tag{3.16}$$

If $f \in L_2(\mathbb{R}^d)$, then the Fourier coefficients of $[\hat{f}, \hat{f}]_0$ are $\langle f(\cdot - k), f \rangle$, see [9]. In case that f is additionally compactly supported, then $[\hat{f}, \hat{f}]_0$ is a trigonometric polynomial and hence (3.16) holds for $s = 0$ and for all $\xi \in \mathbb{T}^d$.

The function g defined by $\hat{g}(\xi) = \hat{\varphi}(\xi)(1 + \|\xi\|^2)^{s/2}$ is contained in $L_2(\mathbb{R}^d)$ and $[\hat{g}, \hat{g}]_0 = [\hat{\varphi}, \hat{\varphi}]_s$. Since φ is compactly supported and $s \geq 0$ is an even integer, $\hat{g}$ has an analytic extension. According to the Paley-Wiener Theorem g is compactly supported and hence $[\hat{\varphi}, \hat{\varphi}]_s$ is a trigonometric polynomial. This means that the upper estimate always holds. Since we have a trigonometric polynomial, the a.e. is void and the estimates must hold everywhere on $\mathbb{T}^d$. The lower estimate is then equivalent to the condition that $\hat{\varphi}$ does not have any $\mathbb{Z}^d$-periodical zeros on $\mathbb{R}^d$.

We have verified that φ's stability is equivalent to $\hat{\varphi}$ not having any $\mathbb{Z}^d$-periodical zeros on $\mathbb{T}^d$. Suppose now that $A_{\xi,\gamma} = 0$. In particular, we then have $A_{\xi,\gamma}(M^\top(\xi + \gamma)) = 0$. According to (3.14), this yields $\hat{\varphi}(M^{-\top}\xi + \gamma + k) = 0$, for all $k \in \mathbb{Z}^d$. Hence, $M^{-\top}\xi + \gamma$ is a $\mathbb{Z}^d$-periodical zero of $\hat{\varphi}$, which contradicts the stability of φ. ∎

The proof of Proposition 3.7 essentially requires that (3.16) holds for all $\xi \in \mathbb{R}^d$. This is satisfied if φ is stable in $H^s(\mathbb{R}^d)$ and $[\hat{\varphi}, \hat{\varphi}]_s$ is continuous. We have verified that $[\hat{\varphi}, \hat{\varphi}]_s$ is a trigonometric polynomial if s is an even integer. In dimension one, Han has shown that $[\hat{\varphi}, \hat{\varphi}]_s$ is continuous for general $s \geq 0$, cf. [15]. An extension to higher dimensions does not seem to be trivial. In fact, an extension is not necessary to prove Theorem 3.5 since we shall use Proposition 3.7 only for $s = 0$.

Let us finally mention that the stability of φ in $H^s(\mathbb{R}^d)$ can be expressed in terms of its symbol a. To make this precise we need some preparation: for two measurable sets $P, Q \subset \mathbb{R}^d$, we denote $P \simeq Q$ if they are equal up to a set of measure zero. A measurable set $Q \subset \mathbb{R}^d$ is called a tiling for $(\mathbb{R}^d, \mathbb{Z}^d)$ if

(1) $\mathbb{R}^d \simeq \bigcup_{k \in \mathbb{Z}^d} (Q + k)$,

(2) $(Q + k) \cap Q \simeq \emptyset$, for all $k \in \mathbb{Z}^d \setminus \{0\}$.

We say that a satisfies the Cohen-criterion if there is a compact set $K \subset \mathbb{R}^d$ containing an open set around the origin, that is a tiling for $(\mathbb{R}^d, \mathbb{Z}^d)$, and

$$a(\xi) \neq 0, \quad \text{for all } \xi \in \bigcup_{j=1}^{\infty} M^{-\top} K.$$

It turns out that the Cohen-criterion is equivalent to φ's stability in $H^s(\mathbb{R}^d)$:

Proposition 3.8. *Let a be a Laurent polynomial satisfying $a(\mathbf{1}) = 1$ and let φ be defined through (2.4). Then there is $s \in \mathbb{R}$ such that $\varphi \in H^s(\mathbb{R}^d)$. If $s \geq 0$ is an even integer, then the following points are equivalent:*

(i) *φ is stable in $H^s(\mathbb{R}^d)$,*

(ii) *a satisfies the Cohen-criterion,*

(iii) *$\widehat{\varphi}$ does not have any $\mathbb{Z}^d$-periodical zeros on $\mathbb{R}^d$.*

Proof. Due to the definition (2.4), the Fourier transform of φ is polynomially bounded. Hence, there is $s \in \mathbb{R}$ such that $\varphi \in H^s(\mathbb{R}^d)$. We have already verified the equivalence between (i) and (iii) in the proof of Proposition 3.7. It is known that (ii) is equivalent to (iii), see [26]. ■

By applying Proposition 3.7, we can essentially follow the ideas in [21] to prove Theorem 3.5:

Proof of Theorem 3.5. The previous section has shown that the results about shift-invariant spaces and fibers in $L_2(\mathbb{R}^d)$ carry over to $H^s(\mathbb{R}^d)$, $s \geq 0$. Although the results in [21] address only dyadic scaling, we can essentially follow the lines. To do so, we need to change notation and consider a symbol a rather as a trigonometric polynomial $a(\xi) = \sum_{k\in\mathbb{Z}^d} a_k e^{-2\pi i\langle k,\xi\rangle}$ than as a Laurent polynomial $a(z) = \sum_{k\in\mathbb{Z}^d} a_k z^k$. According to (3.8), the spaces (3.9), (3.10), and (3.11) are finitely generated shift-invariant spaces. For $\gamma \in \Gamma = \mathbb{Z}^d/M^{-\top}\mathbb{Z}^d$, and $\rho \in \widetilde{\Gamma} = M\mathbb{Z}^d/\mathbb{Z}^d$, we then obtain, for a.e. $\xi \in \mathbb{T}^d$,

$$
\begin{aligned}
\hat{V}_{0||\xi} &= \operatorname{span}\{\big(a^{(0)}(M^{-\top}(\xi+k))\hat{\varphi}(M^{-\top}(\xi+k))\big)_{k\in\mathbb{Z}^d}\}, && (3.17)\\
\hat{W}_{0||\xi} &= \operatorname{span}\{\big(a^{(\mu)}(M^{-\top}(\xi+k))\hat{\varphi}(M^{-\top}(\xi+k))\big)_{k\in\mathbb{Z}^d} : \mu = 1,\dots,n\}, && (3.18)\\
\hat{V}_{1||\xi} &= \operatorname{span}\{\big(e^{-2\pi i\langle\rho,M^{-\top}k\rangle}\hat{\varphi}(M^{-\top}(\xi+k))\big)_{k\in\mathbb{Z}^d} : \rho\in\widetilde{\Gamma}\}. && (3.19)
\end{aligned}
$$

Since $W_0 \subset V_1$, we only have to derive the reverse inclusion. According to Theorem 3.4, it is sufficient to verify $\dim(\hat{W}_{0||\xi}) = \dim(\hat{V}_{1||\xi})$, for a.e. $\xi \in \sigma(V_1)$. Since φ is stable, $\hat{\varphi}$ does not have any $\mathbb{Z}^d$-periodical zeros on $\mathbb{T}^d$ which yields $\sigma(V_1) = \mathbb{T}^d$. By applying the refinement equation and the definition of $A_{\xi,\gamma}$, we can verify

$$
\begin{aligned}
\hat{\varphi}_{||\xi} &= \sum_{\gamma\in\Gamma_M} a^{(0)}(M^{-\top}\xi+\gamma)A_{\xi,\gamma}, && (3.20)\\
\hat{\psi}^{(\mu)}_{||\xi} &= \sum_{\gamma\in\Gamma_M} a^{(\mu)}(M^{-\top}\xi+\gamma)A_{\xi,\gamma}. && (3.21)
\end{aligned}
$$

Since we have

$$
e^{-2\pi i\langle M^{-1}\rho,k\rangle}\hat{\varphi}(M^{-\top}(\xi+k)) = \sum_{\gamma\in\Gamma} e^{-2\pi i\langle\rho,\gamma\rangle}A_{\xi,\gamma},
$$

and the matrix $(e^{-2\pi i\langle \rho,\gamma\rangle})_{\rho,\gamma}$ is invertible, the identity (3.19) implies

$$\hat{V}_{1\|\xi} = \operatorname{span}\{A_{\xi,\gamma} : \gamma \in \Gamma\}.$$

Due to Proposition 3.7, $\{A_{\xi,\gamma} : \gamma \in \Gamma\}$ is a collection of nonzero vectors, and their definition in (3.14) and (3.15) implies that they in fact form an orthogonal system. Hence, we have

$$\dim(\hat{V}_{1\|\xi}) = \operatorname{card}(\Gamma) = m.$$

The assumption $a^{(0)}(0) = 1$ implies that the symbol $a^{(0)}$ is not identically zero, and one observes that, for a.e. $\xi \in \mathbb{T}^d$,

$$\alpha(\xi) := \sum_{\gamma\in\Gamma} a^{(0)}(\xi+\gamma)\overline{b^{(0)}(\xi+\gamma)}$$

is an eigenvalue of the matrix

$$\mathcal{C}_0(\xi) = (a^{(0)}(\xi+\gamma_i)\overline{b^{(0)}(\xi+\gamma_j)})_{i,j}$$

with eigenvector

$$\Lambda(\xi) := (a^{(0)}(\xi), \dots, a^{(0)}(\xi+\gamma_{m-1}))^\top. \tag{3.22}$$

Although we only need the latter statement for a.e. $\xi \in \mathbb{T}^d$, it should be mentioned that the stability of φ implies that the vector in (3.22) is never zero which yields that the latter statement holds for all $\xi \in \mathbb{T}^d$. Let

$$\mathcal{A}(\xi) := \begin{pmatrix} a^{(0)}(\xi) & a^{(0)}(\xi+\gamma_1) & \dots & a^{(0)}(\xi+\gamma_{m-1}) \\ a^{(1)}(\xi) & a^{(1)}(\xi+\gamma_2) & \dots & a^{(0)}(\xi+\gamma_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ a^{(n)}(\xi) & a^{(n)}(\xi+\gamma_1) & \dots & a^{(n)}(\xi+\gamma_{m-1}) \end{pmatrix}$$

and define $\mathcal{B}(\xi)$ in the same way. We denote by $\mathcal{A}_1(\xi)$ and $\mathcal{B}_1(\xi)$ the matrices that occur if we delete the first row of $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$, respectively. Since (2.8) is equivalent to $\mathcal{A}(\xi)\overline{\mathcal{B}^\top(\xi)} = \mathcal{I}_m$, the matrix equality

$$\mathcal{A}_1(\xi)\overline{\mathcal{B}_1^\top(\xi)} = \mathcal{I}_m - \mathcal{C}_0(\xi)$$

holds. Suppose now that $\operatorname{rank}(\mathcal{A}_1(\xi)\overline{\mathcal{B}_1^\top(\xi)}) < m$, then there is $0 \neq v(\xi) \in \mathbb{R}^m$ such that

$$0 = \mathcal{A}_1(\xi)\overline{\mathcal{B}_1^\top(\xi)}v(\xi) = v(\xi) - \mathcal{C}_0(\xi)v(\xi). \tag{3.23}$$

Thus, $v(\xi)$ is an eigenvector of $\mathcal{C}_0(\xi)$, and therefore contained in the range of $\mathcal{C}_0(\xi)$. Since

$$\mathcal{C}_0(\xi) = \begin{pmatrix} a^{(0)}(\xi) \\ \vdots \\ a^{(0)}(\xi+\gamma_{m-1}) \end{pmatrix} (\overline{b^{(0)}(\xi)}, \dots, \overline{b^{(0)}(\xi+\gamma_{m-1})}),$$

the range of $\mathcal{C}_0(\xi)$ is one-dimensional. The eigenvector $\Lambda(\xi)$ is also in its range and hence there must exist a number $0 \neq \lambda(\xi) \in \mathbb{R}$ such that $v(\xi) = \lambda(\xi)\Lambda(\xi)$. Equation (3.23) implies that

$$0 = \lambda(\xi)\Lambda(\xi) - \lambda(\xi)\mathcal{C}_0(\xi)\Lambda(\xi) = \lambda(\xi)\Lambda(\xi) - \lambda(\xi)\alpha(\xi)\Lambda(\xi).$$

This yields $\alpha(\xi) = 1$, for all $\xi \in \mathbb{T}^d$, which contradicts the assumption $\alpha(\xi) \neq 1$. Therefore, $\operatorname{rank}(\mathcal{A}_1(\xi)\overline{\mathcal{B}_1^\top(\xi)}) = m$ must hold, which implies $\operatorname{rank}(\mathcal{A}_1(\xi)) = m$.

Due to (3.18), we have

$$\hat{W}_{0||\xi} = \operatorname{span}\{\hat{\psi}^{(\mu)}_{||\xi} : \mu = 1, \dots, n\}.$$

By applying (3.21), the following identity holds

$$\begin{pmatrix} \hat{\psi}^{(1)}_{||\xi} \\ \vdots \\ \hat{\psi}^{(n)}_{||\xi} \end{pmatrix} = \mathcal{A}_1(M^{-\top}\xi) \begin{pmatrix} A_{\xi,\gamma_0} \\ \vdots \\ A_{\xi,\gamma_{m-1}} \end{pmatrix}.$$

Since $\mathcal{A}_1(\xi)$ has rank m, we obtain $\dim(\hat{W}_{0||\xi}) = m$. Therefore, $\dim(\hat{W}_{0||\xi}) = \dim(\hat{V}_{1||\xi})$, and hence $W_0 = V_1$ due to Theorem 3.4.

■

§4. The Construction of Pairs of Dual Wavelet Frames

Our second goal is to verify that any two multivariate refinable functions φ and $\widetilde{\varphi}$ give rise to a compactly supported wavelet bi-frame for a pair of Sobolev spaces $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$.

4.1. A General Construction Recipe

The following theorem from [17] is a generalization of Theorem 2.1:

Theorem 4.1. *Let $a^{(0)},\dots,a^{(n)}$ and $b^{(0)},\dots b^{(n)}$ be symbols with $a^{(0)}(\mathbf{1})=b^{(0)}(\mathbf{1})=1$. Define φ and $\widetilde{\varphi}$ by (2.4), and the wavelet functions $\psi^{(1)},\dots,\psi^{(n)}$ and $\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)}$ by (2.7). Suppose there is an additional symbol θ with $\theta(\mathbf{1})=1$ such that*

(1) *the following identity holds*

$$\theta(z^M)a^{(0)}(z)b^{(0)}(\tfrac{1}{z_\gamma})+\sum_{\mu=1}^{n_1}a^{(\mu)}(z)b^{(\mu)}(\tfrac{1}{z_\gamma})=\delta_{0,\gamma}\theta(z),\quad \text{for all } \gamma\in\Gamma, \tag{4.1}$$

(2) *for a real number $s\in\mathbb{R}$ satisfying $\nu_2(\varphi)>s$ and $\nu_2(\widetilde{\varphi})>-s$, the symbols $a^{(1)},\dots,a^{(n)}$ and $b^{(1)},\dots,b^{(n)}$ have more than $-s$ and s vanishing moments, respectively.*

Then $X^s(\varphi;\psi^{(1)},\dots,\psi^{(n)})$ and $X^{-s}(\widetilde{\phi};\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)})$ is a pair of dual wavelet frames for $(H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d))$, where $\hat{\widetilde{\phi}}(\xi)=\theta(e^{-2\pi i\xi})\hat{\widetilde{\varphi}}(\xi)$.

It was verified in [10] that (S1) and (S2) in the following theorem imply the conditions (4.1). We can therefore state the following construction recipe that shall be used in the proof of Theorem 4.3 in Section 4.2.

Theorem 4.2. *Given $0<s\in\mathbb{R}$, let $a^{(0)},\dots,a^{(n)}$ and $b^{(0)},\dots b^{(n)}$ be symbols with $a^{(0)}(\mathbf{1})=b^{(0)}(\mathbf{1})=1$. Define φ and $\widetilde{\varphi}$ by (2.4), and the wavelet functions $\psi^{(1)},\dots,\psi^{(n)}$ and $\widetilde{\psi}^{(1)},\dots,\widetilde{\psi}^{(n)}$ by (2.7). Suppose that $\min(\mathrm{sr}(a),\nu_2(\varphi))>s$ and $\nu_2(\widetilde{\varphi})>-s$ and define θ by*

$$\theta(z):=\sum_{\gamma\in\Gamma}a^{(0)}(z_\gamma)\,b^{(0)}(\tfrac{1}{z_\gamma}). \tag{4.2}$$

Assume that the following conditions hold ($n=n_1+n_2$):

(S1) *the following identity holds*

$$\sum_{\mu=0}^{n_1}a^{(\mu)}(z)b^{(\mu)}(\tfrac{1}{z_\gamma})=\delta_{0,\gamma}\theta(z),\quad \text{for all } \gamma\in\Gamma, \tag{4.3}$$

and all $b^{(1)},\dots,b^{(n_1)}$ have more than s vanishing moments,

(S2) *there are symbols* $\eta^{(\nu)}$, $\widetilde{\eta}^{(\nu)}$, $\nu = 1, \ldots, n_2$, *such that*

$$\eta(z) = \sum_{\nu=1}^{n_2} \eta^{(\nu)}(z)\widetilde{\eta}^{(\nu)}(\tfrac{1}{z}), \qquad \textit{where} \qquad \eta(z) := 1 - \theta(z), \tag{4.4}$$

that all $\widetilde{\eta}^{(1)}, \ldots, \widetilde{\eta}^{(n_2)}$ *have more than* s *vanishing moments and, for* $\nu = 1, \ldots, n_2$,

$$a^{(n_1+\nu)}(z) := \eta^{(\nu)}(z^M)a^{(0)}(z), \qquad b^{(n_1+\nu)}(z) := \widetilde{\eta}^{(\nu)}(z^M)b^{(0)}(z). \tag{4.5}$$

Then $X^s\big(\varphi; \psi^{(1)}, \ldots, \psi^{(n_1+n_2)}\big)$ *and* $X^{-s}\big(\widetilde{\phi}; \widetilde{\psi}^{(1)}, \ldots, \widetilde{\psi}^{(n_1+n_2)}\big)$ *are a pair of dual wavelet frames for* $(H^s(\mathbb{R}^d)$, $H^{-s}(\mathbb{R}^d))$, *where* $\hat{\widetilde{\phi}}(\xi) = \theta(e^{-2\pi i\xi})\hat{\widetilde{\varphi}}(\xi)$.

Note that since θ is a Laurent polynomial, the function $\widetilde{\phi}$ in Theorem 4.2 is a finite linear combination of integer shifts of $\widetilde{\varphi}$, i.e., $\widetilde{\phi}(x) = \sum_{k\in\mathbb{Z}^d} \theta_k \widetilde{\varphi}(x-k)$.

4.2. Pairs of Dual Wavelet Frames From Arbitrary Pairs of Refinable Functions

The following Theorem extends results in [6, 12], because $\widetilde{\varphi}$ is not required to be contained in $L_2(\mathbb{R}^d)$:

Theorem 4.3. *Given a compactly supported refinable function* $\varphi \in L_2(\mathbb{R}^d)$ *with* $a(\mathbf{1}) = 1$ *and* $\mathrm{sr}(a) \geq 1$, *let* $\widetilde{\varphi}$ *be any compactly supported refinable distribution with* $b(\mathbf{1}) = 1$ *such that* $\nu_2(\widetilde{\varphi}) > -\nu_2(\varphi)$. *Then there are compactly supported wavelets* $\psi^{(1)}, \ldots, \psi^{(m+d)}$ *and* $\widetilde{\psi}^{(1)}, \ldots, \widetilde{\psi}^{(m+d)}$ *derived from* (2.7) *as well as* $s > 0$ *such that* $X^s(\varphi; \psi^{(1)}, \ldots, \psi^{(m+d)})$ *and* $X^{-s}(\widetilde{\phi}; \widetilde{\psi}^{(1)}, \ldots, \widetilde{\psi}^{(m+d)})$ *form a pair of dual wavelet frames for* $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, *where* $\widetilde{\phi}$ *is a finite linear combination of integer shifts of* $\widetilde{\varphi}$.

For preparation, we recall that

$$\sum_{\rho\in\widetilde{\Gamma}} \frac{1}{\mathbf{1}_\gamma^\rho} = \begin{cases} m, & \text{for } \gamma \in \mathbb{Z}^d, \\ 0, & \text{for } \gamma \in M^{-\top}\mathbb{Z}^d \setminus \mathbb{Z}^d, \end{cases} \tag{4.6}$$

which follows from a result about character sums, cf. [3].

Proof of Theorem 4.3. Due to [13], $\varphi \in L_2(\mathbb{R}^d)$ implies $\nu_2(\varphi) > 0$. Let

$$s := \frac{1}{2}\min(1, \nu_2(\varphi), \nu_2(\varphi) + \nu_2(\widetilde{\varphi})),$$

then according to $\nu_2(\widetilde{\varphi}) > -\nu_2(\varphi)$, we have $0 < s < 1$. For $\mu = 1, \dots, m$, let us define wavelet symbols by

$$a^{(\mu)}(z) := \frac{1}{m} z^{\rho_\mu} \tag{4.7}$$

$$b^{(\mu)}(z) := z^{\rho_\mu} \theta(\tfrac{1}{z}) - b(z) \sum_{\gamma \in \Gamma} z_\gamma^{\rho_\mu} a(\tfrac{1}{z_\gamma}), \tag{4.8}$$

where $\{\rho_1, \dots, \rho_m\} = \widetilde{\Gamma}$. According to $a(\mathbf{1}) = b(\mathbf{1}) = 1$ and the sum rules of a, all of the symbols $b^{(\mu)}$ have at least one vanishing moment. Next, we verify that (4.3) holds. Let $a^{(0)} = a$, $b^{(0)} = b$, and $\gamma \in \Gamma$, then

$$\sum_{\mu=0}^{m} a^{(\mu)}(z) b^{(\mu)}(\tfrac{1}{z_\gamma}) = a(z) b(\tfrac{1}{z_\gamma}) + \frac{1}{m} \sum_{\rho \in R} z^\rho \Big(\frac{1}{z_\gamma^\rho} \theta(z_\gamma) - b(\tfrac{1}{z_\gamma}) \sum_{\tilde{\gamma} \in \Gamma} \frac{1}{z_{\tilde{\gamma}+\gamma}^\rho} a(z_{\tilde{\gamma}+\gamma}) \Big).$$

Since $\sum_{\tilde{\gamma} \in \Gamma} \frac{1}{z_{\tilde{\gamma}+\gamma}^\rho} a(z_{\tilde{\gamma}+\gamma}) = \sum_{\tilde{\gamma} \in \Gamma} \frac{1}{z_{\tilde{\gamma}}^\rho} a(z_{\tilde{\gamma}})$, we apply (4.6) and obtain

$$\begin{aligned} \sum_{\mu=0}^{m} a^{(\mu)}(z) b^{(\mu)}(\tfrac{1}{z_\gamma}) &= a(z) b(\tfrac{1}{z_\gamma}) + \theta(z) \frac{1}{m} \sum_{\rho \in R} \frac{1}{\mathbf{1}_\gamma^\rho} - b(\tfrac{1}{z_\gamma}) \frac{1}{m} \sum_{\tilde{\gamma} \in \Gamma} a(z_{\tilde{\gamma}}) \sum_{\rho \in R} \frac{1}{\mathbf{1}_{\tilde{\gamma}}^\rho} \\ &= a(z) b(\tfrac{1}{z_\gamma}) + \theta(z) \delta_{0,\gamma} - b(\tfrac{1}{z_\gamma}) a(z) \\ &= \theta(z) \delta_{0,\gamma}. \end{aligned}$$

In order to find a convenient sum of products for (4.4), we observe that η has a zero at $z = \mathbf{1}$. A standard result in algebra implies that there are Laurent polynomials $q_1, \dots, q_d$ such that

$$\eta(z) = \sum_{i=1}^{d} (z_i - 1) q_i(z).$$

Choosing $\eta^{(i)}(z) := q_i(z)$ and $\widetilde{\eta}^{(i)}(z) := \frac{1}{z_i} - 1$, we can define the wavelet symbols $a^{(m+1)}, \dots, a^{(m+d)}$ and $b^{(m+1)}, \dots, b^{(m+d)}$ by (4.5). Therefore, the assumptions of Theorem 4.2 are satisfied, which concludes the proof of Theorem 4.3. ■

Remark 4.4. The proof of Theorem 4.3 generates the systems

$$X^s(\varphi; \varphi_{1,\rho_1}, \dots, \varphi_{1,\rho_m}, \psi^{(m+1)}, \dots, \psi^{(m+d)}) \quad \text{and} \quad X^{-s}(\widetilde{\phi}; \widetilde{\psi}^{(1)}, \dots, \widetilde{\psi}^{(m+d)})$$

that form a pair of dual wavelet frames for $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. Since

$$\widehat{\psi^{(m+\nu)}}(\xi) = \eta^{(\nu)}(e^{-2\pi i\xi})a(e^{-2\pi i M^{-\top}\xi})\widehat{\varphi}(M^{-\top}\xi),$$

the refinement equation yields $\widehat{\psi^{(m+\nu)}}(\xi) = \eta^{(\nu)}(e^{-2\pi i\xi})\widehat{\varphi}(\xi)$. Therefore, we obtain $\psi^{(m+\nu)}(x) = \sum_{k\in\mathbb{Z}^d} \eta_k^{(\nu)}\varphi(x-k)$, and, hence, the wavelets $\psi^{(m+\nu)}$, $\nu = 1, \dots, m$, are already contained in the finite linear span of $\{\varphi_{0,k} : k \in \mathbb{Z}^d\}$.

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M. Ehler,
University of Maryland,
Department of Mathematics,
Norbert Wiener Center,
College Park, MD 20742.
ehlermar@math.umd.edu

National Institutes of Health,
Eunice Kennedy Shriver National Institute of Child Health and Human Development,
Section on Medical Biophysics,
9 Memorial Drive, Bethesda, MD 20892.
ehlermar@mail.nih.gov