

# Compactly Supported Wavelets in Sobolev Spaces of Integer Order

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We present a construction of regular compactly supported wavelets in any Sobolev space of integer order. It is based on the existence and suitable estimates of filters defined from polynomial equations. We give an implicit study of these filters and use the results obtained to construct scaling functions leading to multiresolution analysis and wavelets. Their regularity increases linearly with the length of their supports as in the  $L^2$  case. One technical problem is to prove that the intersection of the scaling spaces is reduced to 0. This is solved using sharp estimates of Littlewood–Paley type. © 1997 Academic Press, Inc.

## INTRODUCTION

The classical orthogonal wavelets in  $L^2$  characterize efficiently many functional spaces, for example the Sobolev ones, but are not orthogonal in these spaces. In [1], we give a general procedure to construct wavelets in any Sobolev space from a multiresolution analysis. In the same paper, we construct orthogonal wavelets in  $H^1(\mathbb{R})$  with compact support and arbitrary high regularity. This construction is performed from the point of view of filters and the key is to exhibit suitable families of filters.

In [1], an idea for a similar construction in Sobolev spaces of integer order already appears. Using this idea, we obtain here compactly supported wavelets in  $H^m(\mathbb{R})$  for every integer  $m$ . Their regularity increases with the degree of the filters as in the classical case.

Similar to [2], the construction of our filters is based on the resolution of a family of linear equations whose unknowns are polynomials. The existence and uniqueness of the solution is clear. However, we also need sharp estimations of these polynomials. In [1], we succeed because we have an explicit expression of the solution. Here, it does not seem possible to give a general form of the solutions for every  $m$  and scale index  $j$ . Hence we make an implicit study of the polynomial equations and prove two inequal-

ities involving their solutions. Using these results, we can proceed essentially along the same lines as in [1]. The only true difficulty is that for large  $m$  and low regularity, we must refine the estimations to prove that we fulfill all the conditions imposed on a multiresolution analysis.

To construct compactly supported wavelets in Sobolev spaces of negative integer orders, we can take sums of derivatives of wavelets of Sobolev spaces of higher order. In particular, it is sufficient to use Daubechies' wavelets of  $L^2(\mathbb{R})$ .

This paper is divided into three sections. The first section consists in a presentation of a general construction of wavelets in Sobolev spaces following the results of [1].

Section 2 is devoted to the construction of the families of filters used to define the wavelets. Some of their properties are given. More precisely, Proposition 2.2 is the main tool to prove the regularity and estimate the supports of the wavelets. Proposition 2.4 is used in Littlewood–Paley estimations to prove that the intersection of the spaces generated by the scaling functions is reduced to  $\{0\}$ .

The construction of wavelets is then presented in Section 3. We collect and apply the results obtained in the second section. Some parts of the proof follow the corresponding arguments in [1], and references are given to this paper.

## 1. GENERAL CONSTRUCTION

In order to make this paper self-contained, we recall here some of the results of [1].

**DEFINITION 1.1.** Let  $s$  be a real number. A multiresolution analysis of  $H^s(\mathbb{R})$  is a sequence  $V_j, j \in \mathbb{Z}$ , of closed linear subspaces of  $H^s(\mathbb{R})$  such that

- $V_j \subset V_{j+1}$  for every  $j$ ,
- $\bigcup_{j=-\infty}^{+\infty} V_j = H^s(\mathbb{R})$ ,
- $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$ ,
- for every  $j$ , there is a function  $\varphi^{(j)}$  such that the distributions  $2^{j/2} \varphi^{(j)}(2^j x - k)$ ,  $k \in \mathbb{Z}$ , form an orthonormal basis of  $V_j$ .

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Such functions  $\varphi^{(j)}$  are called *scaling functions* or *father wavelets*. The condition  $V_j \subset V_{j+1}$  for every  $j \in \mathbb{Z}$  is equivalent to the existence of  $2\pi$ -periodic functions  $m_0^{(j)} \in L_{loc}^2(\mathbb{R})$  such that the following scale relations hold:

$$\hat{\varphi}^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\hat{\varphi}^{(j+1)}(\xi).$$

These functions  $m_0^{(j)}$  are called *filters*. Note also that the filters satisfy

$$|m_0^{(j)}(\xi)|^2 + |m_0^{(j)}(\xi + \pi)|^2 = 1$$

since for every  $j$ , the distributions  $2^{j/2}\varphi^{(j)}(2^jx - k)$ ,  $k \in \mathbb{Z}$ , are orthonormal.

Defining  $\psi^{(j)}$ ,  $j \in \mathbb{Z}$ , by the formula

$$\hat{\psi}^{(j)}(2\xi) = -e^{-i\xi} \overline{m_0^{(j+1)}(\xi + \pi)} \hat{\varphi}^{(j+1)}(\xi)$$

we get that the functions

$$2^{j/2}\psi^{(j)}(2^jx - k), \quad j, k \in \mathbb{Z},$$

form an orthonormal basis for  $H^s(\mathbb{R})$ . This is what we call an *orthonormal basis of wavelets* for  $H^s(\mathbb{R})$ .

So, a way to obtain wavelets is to construct a family of scaling functions  $\varphi^{(j)}$  leading to a multiresolution analysis. The following result gives sufficient conditions on these functions  $\varphi^{(j)}$  so that they define a multiresolution analysis.

**PROPOSITION 1.2.** *Let  $\varphi^{(j)}$ ,  $j \in \mathbb{Z}$ , be a sequence of elements of  $H^s(\mathbb{R})$  such that, for every  $j$ , the distributions*

$$\varphi_{jk}(x) = 2^{j/2}\varphi^{(j)}(2^jx - k), \quad k \in \mathbb{Z},$$

*are orthonormal in  $H^s(\mathbb{R})$ . Denote by  $V_j$  the closed linear hull of the  $\varphi_{jk}$ ,  $k \in \mathbb{Z}$ .*

(a) *If*

$$\lim_{j \rightarrow +\infty} |\hat{\varphi}^{(j)}(2^{-j}\xi)| = (1 + \xi^2)^{-s/2}$$

*then the union of the  $V_j$ 's is dense in  $H^s(\mathbb{R})$ .*

(b) *If there are  $A, \alpha > 0$  such that*

$$\int_{\mathbb{R}} (1 + |\xi|)^\alpha |\hat{\varphi}^{(j)}(\xi)|^2 d\xi \leq A$$

*for every  $j \leq 0$  then  $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$ .*

Let us now recall the construction of scaling functions starting from a family of filters.

**PROPOSITION 1.3.** *We fix  $m \in \mathbb{Z}$ . Let  $m_0^{(j)} \in L_{loc}^2(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , be a sequence of  $2\pi$ -periodic functions such that*

$$|m_0^{(j)}(\xi)|^2 + |m_0^{(j)}(\xi + \pi)|^2 = 1 \text{ a.e.,}$$

• *there are  $N > 0$  and  $c_j \in ]0, 1]$ ,  $j \in \mathbb{N}_0$ , such that  $\prod_{j=1}^{+\infty} c_j > 0$  and*

$$|m_0^{(j)}(\xi)| \geq c_j \cos^N\left(\frac{\xi}{2}\right), \quad \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], j \in \mathbb{N}_0.$$

*Fix  $j \in \mathbb{Z}$ . If the infinite product*

$$\hat{\varphi}^{(j)}(\xi) = \frac{1}{(1 + i2^j\xi)^m} \prod_{p=1}^{+\infty} m_0^{(j+p)}(2^{-p}\xi)$$

*converges almost everywhere then the  $\varphi_{jk}(x) = 2^{j/2}\varphi^{(j)}(2^jx - k)$ ,  $k \in \mathbb{Z}$ , are orthonormal in  $H^m(\mathbb{R})$ .*

Now the problem is to find a suitable family of filters.

## 2. CONSTRUCTION OF FILTERS

In what follows,  $m, N$  denote two positive integers with  $N > 0$ . The case  $m = 0, N = 1$  leads to the Haar system. We do not consider it.

We use the filters

$$m_{N,m}^{(j)}(\xi) = \left( \frac{e^{-i\xi/2} \cos((\xi + i2^{-j})/2)}{\sqrt{\text{ch}(2^{-j})}} \right)^m \times \left( \frac{1 + e^{-i\xi}}{2} \right)^N \mathcal{L}_{N,m}^{(j)}(\xi),$$

where the  $\mathcal{L}_{N,m}^{(j)}$  are trigonometric polynomials.

The first factor is used to cancel the singularity introduced in  $\hat{\psi}^{(j)}$  by the factor  $(1 + i2^j\xi)^{-m}$ . As in [2], the second factor is used to get higher regularity. For each  $j$ , the last factor is chosen such that the filter satisfies the orthogonality condition

$$|m_{N,m}^{(j)}(\xi)|^2 + |m_{N,m}^{(j)}(\xi + \pi)|^2 = 1.$$

If

$$P_{N,m}^{(j)}\left(\sin^2\left(\frac{\xi}{2}\right)\right) = |\mathcal{L}_{N,m}^{(j)}(\xi)|^2$$

this leads to the polynomial equation

$$(x_j - y)^m (1 - y)^N P_{N,m}^{(j)}(y) + (x_j - 1 + y)^m y^N P_{N,m}^{(j)}(1 - y) = (2x_j - 1)^m, \quad (1)$$

where  $x_j = \text{ch}^2(2^{-j-1})$ . Since  $x_j > 1$ , there are unique polynomials  $P_{N,m}^{(j)}, Q_{N,m}^{(j)}$  of degree at most  $N + m - 1$  such that

$$(x_j - y)^m (1 - y)^N P_{N,m}^{(j)}(y) + (x_j - 1 + y)^m y^N Q_{N,m}^{(j)}(y) = (2x_j - 1)^m. \quad (2)$$

The special form of the equation gives

$$Q_{N,m}^{(j)}(y) = P_{N,m}^{(j)}(1 - y).$$

Note that for  $m = 0$ , Eq. (1) is the one used by Daubechies in  $L^2(\mathbb{R})$ ; we have

$$P_{N,0}^{(j)}(y) = P_N(y) = \sum_{k=0}^{N-1} C_{N-1+k}^k y^k.$$

### 2.1. Some inequalities on the polynomials $P_{N,m}^{(j)}$

To obtain regularity of our wavelets, we need some estimations on the polynomials  $P_{N,m}^{(j)}$ . We first prove a general result on polynomials satisfying equations similar to (1).

Let  $N$  be a strictly positive integer and  $r_1, \dots, r_N \in \mathbb{R}$  such that  $1 \leq r_1 \leq \dots \leq r_N$ . Consider

$$R_N(x) = \prod_{j=1}^N (r_j - x).$$

Since  $R_N(x)$  and  $R_N(1-x)$  do not vanish simultaneously, there is a unique polynomial  $Q_{N-1}$  of degree at most  $N-1$  such that

$$R_N(x)Q_{N-1}(x) + R_N(1-x)Q_{N-1}(1-x) = 1. \quad (3)$$

In the same way, for every  $t \geq 1$ , we denote by  $Q_{N,t}$  the polynomial of degree at most  $N$  satisfying

$$(t-x)R_N(x)Q_{N,t}(x) + (t-1+x)R_N(1-x)Q_{N,t}(1-x) = 2t-1. \quad (4)$$

It turns out that  $Q_{N,t}$  is increasing with respect to  $t$  if  $1-r_1 \leq x \leq 1/2$  and decreasing if  $x \geq 1/2$ .

LEMMA 2.1. For  $x \in [1-r_1, +\infty[$  and  $s > t \geq 1$ , the solutions of (3) and (4) satisfy

$$\frac{Q_{N,t}(x) - 2Q_{N-1}(x)}{2x-1} > 0$$

$$\frac{Q_{N,t}(x) - Q_{N,s}(x)}{2x-1} > 0.$$

*Proof.* By definition we get

$$\begin{aligned} & (t-x)R_N(x)(Q_{N,t}(x) - 2Q_{N-1}(x)) \\ & + (t-1+x)R_N(1-x)(Q_{N,t}(1-x) - 2Q_{N-1}(1-x)) \\ & = 2t-1 - 2((t-x)R_N(x)Q_{N-1}(x) \\ & + (t-1+x)R_N(1-x)Q_{N-1}(1-x)) \\ & = (2x-1)(1-2R_N(1-x)Q_{N-1}(1-x)). \end{aligned}$$

Hence

$$\begin{aligned} & (t-x)R_N(x) \frac{Q_{N,t}(x) - 2Q_{N-1}(x)}{2x-1} \\ & = 1 - R_N(1-x) \left( 2Q_{N-1}(1-x) + (t-1+x) \right. \\ & \quad \left. \times \frac{Q_{N,t}(1-x) - 2Q_{N-1}(1-x)}{2x-1} \right). \end{aligned}$$

This is a polynomial of degree at most  $2N$  since

$$Q_{N,t} \left( \frac{1}{2} \right) = 2Q_{N-1} \left( \frac{1}{2} \right) = R_N \left( \frac{1}{2} \right)^{-1}.$$

It is strictly positive for  $1-r_1$ . Assume it vanishes for some  $r \in ]1-r_1, +\infty[$ . Then it is equal to 0 for  $t, r, r_1, \dots, r_N$  and equal to 1 for  $1-r_1, \dots, 1-r_N$ . Hence its derivative has  $2N$  roots. This is a contradiction and proves the first inequality.

We proceed in the same way for the second inequality. Here we get

$$\begin{aligned} & (t-x)R_N(x)(Q_{N,t}(x) - Q_{N,s}(x)) \\ & + (t-1+x)R_N(1-x)(Q_{N,t}(1-x) - Q_{N,s}(1-x)) \\ & = 2t-1 - \frac{t-x}{s-x}(2s-1 - (s-1+x)R_N(1-x) \\ & \quad \times Q_{N,s}(1-x)) - (t-1+x)R_N(1-x)Q_{N,s}(1-x) \\ & = \frac{s-t}{s-x}(2x-1)(1-R_N(1-x)Q_{N,s}(1-x)). \end{aligned}$$

Hence

$$\begin{aligned} & (s-x)(t-x)R_N(x) \frac{Q_{N,t}(x) - Q_{N,s}(x)}{2x-1} \\ & = s-t - R_N(1-x) \left( (s-t)Q_{N,s}(1-x) \right. \\ & \quad \left. + (t-1+x)(s-x) \frac{Q_{N,t}(1-x) - Q_{N,s}(1-x)}{2x-1} \right). \end{aligned}$$

This polynomial of degree at most  $2N+1$  is strictly positive at  $1-r_1$ . Assume it vanishes for  $r \in ]1-r_1, +\infty[$ . Then it is equal to 0 for  $s, t, r, r_1, \dots, r_N$  and equal to  $s-t$  for  $1-r_1, \dots, 1-r_N$ . Hence its derivative has  $2N+1$  roots. This is a contradiction. ■

PROPOSITION 2.2. The solutions  $P_{N,m}^{(j)}$  of Eq. (1) satisfy

$$2^m P_N(y) \leq P_{N,m}^{(j)}(y) \leq P_{m+N}(y) \quad \text{for } y \in [1/2, 1],$$

$$P_{m+N}(y) \leq P_{N,m}^{(j)}(y) \leq 2^m P_N(y) \quad \text{for } y \in [0, 1/2].$$

*Proof.* For every strictly positive integer  $m$ , we apply Lemma 2.1 with

$$R_{N+m-1}(x) = (t-x)^{m-1}(1-x)^N.$$

For every  $t \geq 1$ , we get polynomials  $Q_{N,m-1}(x)$  and  $Q_{N,m-1,t}(x)$ , respectively, of degree at most  $m+N-2$  and  $m+N-1$  such that

$$\frac{Q_{N,m-1,t}(x) - 2Q_{N,m-1}(x)}{2x-1} > 0, \quad (5)$$

$$\frac{Q_{N,m-1,t}(x) - Q_{N,m-1,s}(x)}{2x-1} > 0 \quad (6)$$

for  $x \in [0, +\infty[$  and  $s > t \geq 1$ . By definition, we have

$$Q_{N,m-1,x_j} = (2x_j - 1)Q_{N,m} \quad (7)$$

$$Q_{N,m-1,1} = Q_{N+1,m-1}. \quad (8)$$

Let us consider  $x \in [1/2, 1]$ ; the case  $x \in [0, 1/2]$  is similar. From (5) and (7), we get

$$\begin{aligned} Q_{N,m-1,x_j}(x) &\geq 2Q_{N,m-1}(x) = 2(2x_j - 1)^{-1}Q_{N,m-2,x_j}(x) \\ &\geq 2^2(2x_j - 1)^{-1}Q_{N,m-2}(x) \\ &\geq \dots \geq 2^m(2x_j - 1)^{-m+1}Q_{N,0}(x), \end{aligned}$$

and from (6), (7), (8), we get

$$\begin{aligned} Q_{N,m-1,x_j}(x) &\leq Q_{N,m-1,1}(x) = Q_{N+1,m-1}(x) \\ &= (2x_j - 1)^{-1}Q_{N+1,m-2,x_j}(x) \\ &\leq (2x_j - 1)^{-1}Q_{N+2,m-2}(x) \\ &\leq \dots \leq (2x_j - 1)^{-m+1}Q_{N+m,0}(x). \end{aligned}$$

The conclusion follows since we have by definition

$$P_{N,m}^{(j)} = (2x_j - 1)^m Q_{N,m} = (2x_j - 1)^{m-1} Q_{N,m-1,x_j}. \quad \blacksquare$$

## 2.2. Estimations on associated operators

Let

$$\begin{aligned} (T_N f)(\xi) &= P_N \left( \sin^2 \left( \frac{\xi}{4} \right) \right) f \left( \frac{\xi}{2} \right) \\ &\quad + P_N \left( \cos^2 \left( \frac{\xi}{4} \right) \right) f \left( \frac{\xi}{2} + \pi \right) \end{aligned}$$

and  $T_{N,m}^{(j)}$  the similar operators associated to the polynomials  $P_{N,m}^{(j)}$ .

To prove that the scaling functions we construct below define linear spaces  $V_{N,m}^{(j)}$  with an intersection reduced to  $\{0\}$ , we use results on regularity for large  $N$  and Littlewood–Paley techniques for the others. To achieve this, we need sharp estimations of the norm of the operators  $T_{N,m}^{(j+1)} \dots T_{N,m}^{(j+n)}$ . The object of this subsection is to establish them.

An elementary computation shows that the spaces

$$V_M = \left\{ \sum_{k=0}^M f_k \cos(k\xi) : f_k \in \mathbb{R} \right\}$$

are invariant under  $T_{N,m}^{(j)}$  for all  $j \in \mathbb{Z}$  and  $M \geq N + m - 2$ . We fix an arbitrary norm on each of these finite dimensional spaces.

We will need some information on the behavior of  $P_{N,m}^{(j)}$  if  $j \rightarrow +\infty$  and  $j \rightarrow -\infty$ . To obtain this, we consider again Eq. (2)

$$(t - y)^m (1 - y)^N P_{N,m,t}(y) + (t - 1 + y)^m y^N Q_{N,m,t}(y) = (2t - 1)^m$$

for  $t \leq 1$ . This equation can be written as a linear system of  $2N + 2m$  equations with  $2N + 2m$  unknowns. Because of the existence and uniqueness of the solution, the matrix of the system has an inverse. Moreover, the form of the equation shows that the elements of this inverse are rational fractions in the parameter  $t$ . It follows that

$$\lim_{t \rightarrow 1} P_{N,m,t}(y) = \lim_{j \rightarrow +\infty} P_{N,m}^{(j)}(y) = P_{N+m}(y).$$

For every  $y$ , hence uniformly on compact sets. If we note that for  $t \in ]0, 1]$ , the polynomials  $P_{N,m,1/t}$ ,  $Q_{N,m,1/t}$  are the unique solution of

$$\begin{aligned} (1 - ty)^m (1 - y)^N P_{N,m,1/t}(y) \\ + (1 - t + ty)^m y^N Q_{N,m,1/t}(y) = (2 - t)^m, \end{aligned}$$

we also obtain

$$\lim_{t \rightarrow 0} P_{N,m,1/t}(y) = \lim_{j \rightarrow -\infty} P_{N,m}^{(j)}(y) = 2^m P_N(y)$$

for every  $y$ , hence uniformly on compact sets.

These results prove that if we write

$$P_{N,m}^{(j)}(y) = \sum_{k=0}^{N+m-1} c_{N,m,k}(x_j) y^k$$

the  $c_{N,m,k}(x_j)$  are rational fractions in  $x_j$  for which the degree of the numerator is strictly less than that of the denominator for  $k \geq N$  and the same for  $k < N$ .

After these preparations, we prove the announced estimations.

LEMMA 2.3. *For every  $N, m$  there is  $C > 0$  such that for operator norm on  $V_{N+m-2}$ , we have*

$$\|T_{N,m}^{(j)} - 2^m T_N\| \leq C 2^{2j} \quad \text{if } j \leq 0,$$

$$\|T_{N,m}^{(j)} - T_{N+m}\| \leq C 2^{-2j} \quad \text{if } j \geq 0.$$

*Proof.* We write as above

$$P_N(y) = \sum_{k=0}^{N+m-1} c_{N,k} y^k.$$

Using the special form of the coefficients  $c_{N,m,k}(x_j)$ , we obtain

$$\begin{aligned} |c_{N,m,k}(x_j) - c_{N+m,k}| &\leq C(x_j - 1) \leq C 2^{-2j} \quad \text{if } j \geq 0, \\ |c_{N,m,k}(x_j)| &\leq \frac{C}{x_j} \leq C 2^{2j} \quad \text{if } j \leq 0 \text{ and } k \geq N, \\ |c_{N,m,k}(x_j) - 2^m c_{N,k}| &\leq \frac{C}{x_j} \leq C 2^{2j} \quad \text{if } j \leq 0 \text{ and } k < N. \end{aligned}$$

It follows that for every  $f \in V_{N+m-2}$

$$|(T_{N,m}^{(j)}f)(\xi) - (T_{N+m}f)(\xi)| \leq C2^{2j}\|f\| \quad \text{for } j \leq 0$$

and

$$|(T_{N,m}^{(j)}f)(\xi) - (2^m T_N f)(\xi)| \leq C2^{-2j}\|f\| \quad \text{for } j \geq 0.$$

The conclusion follows because of the equivalence between any norm on finite dimensional spaces. ■

**PROPOSITION 2.4.** *For every  $N, m$ , there is  $R > 0$  such that, for  $n \in \mathbb{N}_0$  and  $j \leq 0$*

$$\|T_{N,m}^{(j+1)} \dots T_{N,m}^{(j+n)}\| \leq R2^{n(m+2N-1)} \quad \text{if } j+n \leq 0,$$

$$\|T_{N,m}^{(j+1)} \dots T_{N,m}^{(j+n)}\| \leq R2^{jm+2nN+2mn-n} \quad \text{if } j+n > 0.$$

*Proof.* The operator  $T_{N,m}^{(j)}$  is approximated by  $2^m T_N$  if  $j$  is negative and by  $T_{N+m}$  if  $j$  is positive. More precisely, for every  $M \in \mathbb{N}_0$  we have

$$\begin{aligned} |(T_M f)(\xi)| &\leq \left( P_M \left( \sin^2 \left( \frac{\xi}{4} \right) \right) + P_M \left( \cos^2 \left( \frac{\xi}{4} \right) \right) \right) \|f\| \\ &\leq 22^{2(M-1)} \|f\| = 2^{2M-1} \|f\| \end{aligned}$$

since  $P_M \leq 2^{2(M-1)}$  on  $[0, 1]$ . By Proposition (2.3), there is  $J \in \mathbb{N}_0$  such that

$$\|T_{N,m}^{(j)} - 2^m T_N\| \leq 2^j \quad \text{if } j \leq -J$$

$$\|T_{N,m}^{(j)} - T_{N+m}\| \leq 2^{-j} \quad \text{if } j \geq J.$$

We use now these estimations and Lemma 10 of [1]. For  $j+n \leq 0$ , we get

$$\|T_{N,m}^{(j+1)} \dots T_{N,m}^{(j+n)}\| \leq R2^{(m+2N-1)n}$$

and for  $j+n > 0$ , we get

$$\|T_{N,m}^{(j+1)} \dots T_{N,m}^{(j+n)}\| \leq R2^{(2m+2N-1)(n+j)} 2^{(m+2N-1)(-j)}.$$

■

### 3. MULTIREOLUTION ANALYSIS AND WAVELETS

To define the trigonometric polynomials  $\mathcal{L}_{N,m}^{(j)}$ , we use a result of Riesz (see [2, p. 972]). For every  $N, m$ , and  $j$ , there is a trigonometric polynomial of degree at most  $N+m-1$

$$\mathcal{L}_{N,m}^{(j)}(\xi) = \sum_{k=0}^{N+m-1} b_{k,N,m}^{(j)} e^{-ik\xi}$$

with real coefficients, such that

$$|\mathcal{L}_{N,m}^{(j)}(\xi)|^2 = P_N^{(j)} \left( \sin^2 \left( \frac{\xi}{2} \right) \right).$$

This polynomial is uniquely defined if we require that the roots of  $\sum_{k=0}^{N+m-1} b_{k,N,m}^{(j)} z^k$  satisfy  $|z| \geq 1$  and

$$\mathcal{L}_{N,m}^{(j)}(0) = \sqrt{P_N^{(j)}(0)} = (2 - x_j^{-1})^{m/2}.$$

From now on, we use these trigonometric polynomials to define  $\varphi_{N,m}^{(j)}$  and  $\psi_{N,m}^{(j)}$ . Their explicit expression is

$$\begin{aligned} \hat{\varphi}_{N,m}^{(j)}(\xi) &= \frac{1}{(1 + i2^j \xi)^m} \prod_{p=1}^{+\infty} m_{N,m}^{(j+p)} (2^{-p} \xi) \\ &= (\hat{\varphi}_0^{(j)}(\xi))^m e^{-iN\xi/2} \left( \frac{\sin(\xi/2)}{\xi/2} \right)^N \prod_{p=1}^{+\infty} \mathcal{L}_{N,m}^{(j+p)} (2^{-p} \xi) \end{aligned}$$

with

$$\hat{\varphi}_0^{(j)}(\xi) = \frac{2^{1+j/2}}{1 + 2^{2j} \xi^2} \frac{e^{-i\xi/2}}{i\sqrt{\text{sh}(2^{-j})}} \sin \left( \frac{\xi + i2^{-j}}{2} \right).$$

The wavelets are given by

$$\hat{\psi}_{N,m}^{(j)}(2\xi) = -e^{-i\xi} \overline{m_{N,m}^{(j+1)}(\xi + \pi)} \hat{\varphi}_{N,m}^{(j+1)}(\xi).$$

**THEOREM 3.1.** *Let  $N, m$  be two strictly positive integers.*

1. *For every  $j$ , the functions  $2^{j/2} \varphi_{N,m}^{(j)}(2^j x - k)$ ,  $k \in \mathbb{Z}$ , are orthonormal in  $H^m(\mathbb{R})$  and the spaces*

$$V_{N,m}^{(j)} = \overline{\langle \varphi_{N,m}^{(j)}(2^j x - k) : k \in \mathbb{Z} \rangle}$$

*form a multiresolution analysis of  $H^m(\mathbb{R})$ .*

2. *We have  $\text{supp}(\varphi_{N,m}^{(j)}) \subset [0, +\infty]$  and there are polynomials  $p_{N,m,j}$  of degree at most  $m-1$  with real coefficients such that*

$$\varphi_{N,m}^{(j)}(x) = p_{N,m,j}(x) e^{-x2^{-j}} \quad \text{if } x \in [2(N+m-1) + 1, +\infty[.$$

3. *The functions  $2^{j/2} \psi_{N,m}^{(j)}(2^j x - k)$ ,  $j, k \in \mathbb{Z}$ , form an orthonormal basis of  $H^m(\mathbb{R})$ .*

4. *We have  $\text{supp}(\psi_{N,m}^{(j)}) \subset [-(N+m-1), N+m]$ .*

5. *We have*

$$\varphi_{N,m}^{(j)}, \psi_{N,m}^{(j)} \in C^\alpha(\mathbb{R})$$

$$\text{if } \alpha < (N+m-1) \left( 1 - \frac{\ln 3}{2 \ln 2} \right) \approx 0.2075(N+m-1).$$

*Proof.* The second and fourth points of the theorem are obtained directly using the explicit form of the functions and Paley–Wiener’s theorem. To apply this theorem, we need to know that the infinite product in the definition of  $\hat{\varphi}_{N,m}^{(j)}$  is an entire function with an exponential growth. This is achieved with the estimation

$$|\mathcal{L}_{N,m}^{(j)}(\xi) - 1| \leq C_{N,m} |\xi| e^{N|\mathcal{J}\xi|} + C_m \text{sh}^2(2^{-j-1}) \quad (9)$$

for all  $\xi \in \mathbb{C}$ . This is obtained as in [1] using Proposition 2.2.

The orthogonality follows from Propositions 1.3 and 2.2. Indeed, we have

$$\begin{aligned} |m_{N,m}^{(j)}(\xi)| &= |m_0^{(j)}(\xi)|^m \cos^N\left(\frac{\xi}{2}\right) \sqrt{P_{N,m}^{(j)}(\sin^2(\xi/2))} \\ &\geq \left(\frac{\text{ch}(2^{-j}) + \cos(\xi)}{2 \text{ch}(2^{-j})}\right)^{m/2} \cos^N\left(\frac{\xi}{2}\right) \sqrt{P_{N+m}(0)} \\ &\geq \left(\frac{1 + \cos(\xi)}{2 \text{ch}(2^{-j})}\right)^{m/2} \cos^N\left(\frac{\xi}{2}\right) = \frac{\cos^{N+m}(\xi/2)}{\text{ch}^{m/2}(2^{-j})}. \end{aligned}$$

To obtain the density of the union of the spaces  $V_{N,m}^{(j)}$ , we use Proposition 1.2. We have to verify that

$$\lim_{j \rightarrow +\infty} |\hat{\varphi}_{N,m}^{(j)}(2^{-j}\xi)| = (1 + \xi^2)^{-m/2}.$$

Using (9), we see that for fixed  $\xi \in \mathbb{R}$ , the infinite product

$$\prod_{p=1}^{+\infty} |\mathcal{L}_{N,m}^{(j+p)}(2^{-j-p}\xi)|$$

is uniformly convergent on  $\mathbb{N}$ . Moreover, we have

$$\lim_{j \rightarrow +\infty} \prod_{p=1}^{+\infty} |\mathcal{L}_{N,m}^{(j+p)}(2^{-j-p}\xi)| = \prod_{p=1}^{+\infty} \lim_{j \rightarrow +\infty} |\mathcal{L}_{N,m}^{(j+p)}(2^{-j-p}\xi)| = 1. \quad \text{And for } j+n \leq 0, \text{ we get}$$

This fact and

$$\begin{aligned} &\lim_{j \rightarrow +\infty} \frac{2^{1+j/2}}{\sqrt{\text{sh}(2^{-j})}} \left| \sin\left(\frac{\xi + i2^{-j}}{2}\right) \right| \\ &= \lim_{j \rightarrow +\infty} \left( \frac{\text{ch}(2^{-j}) - \cos(2^{-j}\xi)}{2^{-j-1} \text{sh}(2^{-j})} \right)^{1/2} \\ &= (1 + \xi^2)^{1/2} \end{aligned}$$

lead to the conclusion.

We also have the following result of regularity. For every  $\epsilon > 0$  there is  $C = C_{\epsilon,N,m} > 0$  such that

$$|\hat{\varphi}_{N,m}^{(j)}(\xi)| \leq C(1 + 2^{2j}\xi^2)^{-m/2}(1 + |\xi|)^{-N+(N+m-1)(\ln(3)/2 \ln(2)) + \epsilon}$$

for every  $j \in \mathbb{Z}$ . The proof is similar to the case  $m = 1$  in [1]. It uses here the fact that

$$\begin{aligned} P_{N,m}^{(j)}(y) &\leq 3^{N+m-1} \quad \text{if } 0 \leq y \leq \frac{3}{4} \\ P_{N,m}^{(j+1)}(y)P_{N,m}^{(j)}(4y(1-y)) &\leq 3^{2(N+m-1)} \quad \text{if } \frac{3}{4} \leq y \leq 1, \end{aligned}$$

which is easily obtained from Proposition 2.2. This proves the fifth point.

To prove that  $\cap_j V_{N,m}^{(j)} = \{0\}$  we apply Proposition 1.2; i.e., we have to find  $\alpha, A > 0$  such that, for all  $j \leq 0$ ,

$$\int_{\mathbb{R}} (1 + |\xi|)^\alpha |\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 d\xi \leq A.$$

For large  $N$ , i.e., for  $-N+(N+m-1)(\ln(3)/2 \ln(2)) < -1/2$ , it is direct using the previous regularity estimation.

For the other values, we use Littlewood–Paley estimations. First, we remark that using Lemma 3.2 we obtain

$$|\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 \leq C2^{-n(2N+m)}(1 + 2^{2j+2n})^{-m} \prod_{l=1}^n |\mathcal{L}_{N,m}^{(j+l)}(2^{-l}\xi)|^2$$

for  $n \in \mathbb{N}_0, j \leq 0$  and  $\xi \in \mathbb{R}$  such that  $2^{n-1}\pi \leq |\xi| \leq 2^n\pi$ .

Then, we use Proposition 2.4 and the equality

$$\int_{-\pi}^{\pi} (T_{N,m}^{(j+1)} \cdots T_{N,m}^{(j+n)} 1)(\xi) d\xi = \int_{-2^n\pi}^{2^n\pi} \left| \prod_{p=1}^n \mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi) \right|^2 d\xi.$$

For  $j+n > 0$ , we get

$$\begin{aligned} &\int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} (1 + |\xi|)^\alpha |\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 d\xi \\ &\leq C2^{n\alpha} 2^{-2nN-nm} (1 + 2^{2j+2m})^{-m} \|T_{N,m}^{(j+1)} \cdots T_{N,m}^{(j+n)}\| \\ &\leq CR2^{n\alpha} 2^{-2nN-nm} (2^{j+n})^{-m} 2^{jm+2nN+2mn-n} \\ &\leq CR2^{n\alpha} 2^{-n} \\ &\leq C' 2^{n(\alpha-1)}. \end{aligned}$$

And for  $j+n \leq 0$ , we get

$$\begin{aligned} &\int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} (1 + |\xi|)^\alpha |\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 d\xi \\ &\leq C2^{n\alpha} 2^{-2nN-nm} \|T_{N,m}^{(j+1)} \cdots T_{N,m}^{(j+n)}\| \\ &\leq RC2^{n\alpha} 2^{-2nN-nm} 2^{n(m+2N-1)} \\ &\leq RC2^{n\alpha} 2^{-n}. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\mathbb{R}} (1 + |\xi|)^\alpha |\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 d\xi \\ &\leq C + \sum_{n=1}^{+\infty} \int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} (1 + |\xi|)^\alpha |\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 d\xi, \end{aligned}$$

the proof is complete. ■

LEMMA 3.2. For  $n \in \mathbb{N}_0, j \leq 0, \xi \in \mathbb{R}, 2^{n-1}\pi \leq |\xi| \leq 2^n\pi$ , we have

$$|\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 \leq C2^{-n(2N+m)}(1 + 2^{2j+2n})^{-m} \prod_{l=1}^n |\mathcal{L}_{N,m}^{(j+l)}(2^{-l}\xi)|^2,$$

where  $C$  depends only on  $N, m$ .

*Proof.* We recall that

$$\hat{\varphi}_{N,m}^{(j)}(\xi) = (\hat{\varphi}_0^{(j)}(\xi))^m e^{-iN\xi/2} \left( \frac{\sin(\xi/2)}{\xi/2} \right)^N \prod_{p=1}^{+\infty} \mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi).$$

We have

$$|\hat{\varphi}_0^{(j)}(\xi)|^2 = \frac{2^{1+j}(\operatorname{ch}(2^{-j}) - \cos(\xi))}{\operatorname{sh}(2^{-j})(1 + 2^{2j}\xi^2)^2} \leq C2^j(1 + 2^{2j}\xi^2)^{-2} \\ \leq C'2^j(1 + 2^{2j+2n})^{-2}.$$

As in [1], from the estimations

$$|\mathcal{L}_{N,m}^{(j)}(\xi)| \leq e^{C|\xi|} |\mathcal{L}_{N,m}^{(j)}(0)|$$

and

$$\prod_{p=n+1}^{+\infty} |\mathcal{L}_{N,m}^{(j+p)}(0)| \\ = 2^{-m(j+n+1)/2} \coth^{m/2}(2^{-j-n-1}) \leq (1 + 2^{-(j+n+1)})^{m/2},$$

it follows that

$$|\hat{\varphi}_{N,m}^{(j)}(\xi)|^2 \leq C2^{jm-2nN}(1 + 2^{2j+2n})^{-2m} \\ \times \prod_{p=1}^n |\mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi)|^2 e^{C2^{-n}|\xi|} \\ \times \prod_{p=n+1}^{+\infty} |\mathcal{L}_{N,m}^{(j+p)}(0)|^2 \\ \leq C2^{jm-2nN}(1 + 2^{2j+2n})^{-2m} \\ \times \prod_{p=1}^n |\mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi)|^2 (1 + 2^{-(j+n+1)})^m \\ \leq C'2^{-2nN-nm} \left( \frac{2^{j+n} + 1}{1 + 2^{2j+2n}} \right)^m (1 + 2^{2j+2n})^{-m} \\ \times \prod_{p=1}^n |\mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi)|^2 \\ \leq C''2^{-2nN-nm}(1 + 2^{2j+2n})^{-m} \\ \times \prod_{p=1}^n |\mathcal{L}_{N,m}^{(j+p)}(2^{-p}\xi)|^2.$$

This proves the lemma. ■

*Remark.* Wavelets in Sobolev spaces with negative integer order are easy to construct. Indeed, for every  $m \in \mathbb{Z}$  and  $k \in \mathbb{N}$ , the isometries

$$H^m(\mathbb{R}) \rightarrow H^{m-k}(\mathbb{R}) : f \mapsto (2\pi)^{-1} \mathcal{F}^+((1 + i\xi)^k \hat{f}(\xi))$$

preserve the compacity of the support.

For  $m = 0$ , there exist filters independent of  $j$ : the Daubechies ones. Using the previous isometries we can use them to construct wavelets in  $H^{-k}(\mathbb{R})$ . It follows that the dependence on  $j$  in  $\varphi^{(j)}$  comes only from the Sobolev weight.

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