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Internal structure of the multiresolution analyses defined by the unitary extension principle[☆]

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Abstract

We analyze the internal structure of the multiresolution analyses of $L^2(\mathbb{R}^d)$ defined by the unitary extension principle (UEP) of Ron and Shen. Suppose we have a wavelet tight frame defined by the UEP. Define V_0 to be the closed linear span of the shifts of the scaling function and W_0 that of the shifts of the wavelets. Finally, define V_1 to be the dyadic dilation of V_0. We characterize the conditions that $V_1 = W_0$, that $V_1 = V_0 \dot{+} W_0$ and $V_1 = V_0 \oplus W_0$. In particular, we show that if we construct a wavelet frame of $L^2(\mathbb{R})$ from the UEP by using two trigonometric filters, then $V_1 = V_0 \dot{+} W_0$; and show that $V_1 = W_0$ for the B-spline example of Ron and Shen. A more detailed analysis of the various 'wavelet spaces' defined by the B-spline example of Ron and Shen is also included.

Keywords: Shift-invariant space; Unitary extension principle; Multiresolution analysis

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1. Introduction and main results

The purpose of this article is to analyze the internal structure of the multiresolution analyses (MRAs) defined by the unitary extension principle (UEP) of Ron and Shen [7,10,21], which is a powerful generalization of the construction of orthonormal wavelets by Daubechies [8,9]. First, we introduce a dyadic version of the UEP. The following form of the Fourier transform is used throughout this article: $\hat{f}(x) := \int_{\mathbb{R}^d} f(t)\mathrm{e}^{-2\pi i x\cdot t}\mathrm{d}t$ if $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$; and $\wedge$ is extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem.

Let φ be a refinable function in $L^2(\mathbb{R}^d)$ such that

$$\hat{\varphi}(2x) = m_0(x)\hat{\varphi}(x) \quad \text{a.e. } x \in \mathbb{R}^d, \tag{1.1}$$

for some $m_0 \in L^2(\mathbb{T}^d)$, where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus which is conveniently identified with $[-1/2, 1/2]^d$ or $[0, 1]^d$. We further assume that:

$$\lim_{j\to-\infty} \hat{\varphi}(2^j x) = 1 \quad \text{a.e. } x \in \mathbb{R}^d; \tag{1.2}$$

$$\sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x + k)|^2 \in L^\infty(\mathbb{T}^d). \tag{1.3}$$

Moreover, define $\psi_1, \psi_2, \ldots, \psi_n$ via

$$\hat{\psi}_l(2x) := m_l(x)\hat{\varphi}(x), \quad 1 \le l \le n, \tag{1.4}$$

for some $m_1, m_2, \ldots, m_n \in L^\infty(\mathbb{T}^d)$. It is easy to see that $\psi_l \in L^2(\mathbb{R}^d)$ for each $l = 1, 2, \ldots, n$. The 1-periodic functions $m_0, \ldots, m_n$ are called the *filters* (or *masks*). Let

$$Q := \{q_1, q_2, \ldots, q_{2^d}\} := \{0, 1\}^d = \mathbb{Z}^d/2\mathbb{Z}^d, \tag{1.5}$$

and define

$$M(x) := \begin{pmatrix} m_0\left(x + \frac{q_1}{2}\right) & m_0\left(x + \frac{q_2}{2}\right) & \cdots & m_0\left(x + \frac{q_{2^d}}{2}\right) \\ m_1\left(x + \frac{q_1}{2}\right) & m_1\left(x + \frac{q_2}{2}\right) & \cdots & m_1\left(x + \frac{q_{2^d}}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_n\left(x + \frac{q_1}{2}\right) & m_n\left(x + \frac{q_2}{2}\right) & \cdots & m_n\left(x + \frac{q_{2^d}}{2}\right) \end{pmatrix}. \tag{1.6}$$

For $y \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, T_y denotes the unitary translation operator such that $T_y f(x) := f(x - y)$ and D denotes the unitary dyadic dilation operator such that $Df(x) := 2^{d/2} f(2x)$.

Proposition 1.1 (*UEP of Ron and Shen [21]*). *Suppose that the refinable function φ and the filters $m_0, m_1, \ldots, m_n$ satisfy (1.1)–(1.3). Define $\psi_1, \psi_2, \ldots, \psi_n$ by (1.4). If $M(x)^*M(x)$ is the identity matrix for a.e. $x \in \sigma(V_0) := \left\{x \in \mathbb{T}^d : \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x + k)|^2 \neq 0\right\}$, then $\{D^j T_k\psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \le l \le n\}$ is a tight frame (called wavelet frame) for $L^2(\mathbb{R}^d)$ with frame bound 1, where $M(x)$ is defined as in (1.6).*

The following is an example of UEP filters by Ron and Shen [23] (see also [5, Section 14.3]), which generate the B-spline wavelet frame of $L^2(\mathbb{R})$: For a fixed positive integer k, define trigonometric filters

$$m_l(x) := \binom{2k}{l}^{1/2} \sin^l(\pi x)\cos^{2k-l}(\pi x), \quad l = 0, 1, \dots, 2k. \tag{1.7}$$

Let $\varphi \in L^2(\mathbb{R})$, which is a cardinal B-spline of order $2k$, be the refinable function with the low-pass filter m_0 such that

$$\hat{\varphi}(x) := \left(\frac{\sin(\pi x)}{\pi x}\right)^{2k}. \tag{1.8}$$

It is known that $\varphi, m_0, m_1, \dots, m_{2k}$ satisfy all the requirements of the UEP. In this case the number n of the generators of the wavelet frame is $2k$, which is strictly greater than 1.

Suppose we have a scaling function $\varphi \in L^2(\mathbb{R}^d)$ and a collection of filters $m_0, m_1, \dots, m_n \in L^2(\mathbb{T}^d)$ satisfying Proposition 1.1. We define an MRA derived from the UEP.

$$\begin{cases} V_0 := \overline{\operatorname{span}}\{T_k\varphi : k \in \mathbb{Z}^d\}, \\ W_0 := \overline{\operatorname{span}}\{T_k\psi_l : k \in \mathbb{Z}^d, 1 \le l \le n\}, \\ V_1 := D(V_0). \end{cases} \tag{1.9}$$

In particular, $V_1 = \overline{\operatorname{span}}\{DT_k\varphi : k \in \mathbb{Z}^d\}$. Then, we have $V_0 \subset V_1$ and $W_0 \subset V_1$ by (1.1), (1.3) and (1.4). In this article we characterize the conditions that $V_1 = W_0$ and that $V_1 = V_0 \dot{+} W_0$, i.e., $V_1 = V_0 + W_0$ and $V_0 \cap W_0 = \{0\}$, and those that $V_1 = V_0 \oplus W_0$ (Section 3). In particular, we show that:

Theorem 1.2. *If we construct a wavelet tight frame of $L^2(\mathbb{R})$ by the UEP using two trigonometric filters m_0 and m_1 (a single wavelet case), then $V_1 = V_0 \dot{+} W_0$. On the other hand, suppose that we construct a wavelet tight frame of $L^2(\mathbb{R})$ by using $n+1$ trigonometric filters $m_0, \dots, m_n$ for some positive integer n greater than 1. Then either $V_1 = W_0$ or there exist a positive integer $i \in \{1, 2, \dots, n\}$ and scalars λ_j for $j \in \{1, 2, \dots, n\} \setminus \{i\}$ such that $m_j = \lambda_j m_i$. In particular, if we use the filters in (1.7), then $V_1 = W_0$.*

The following theorem gives more detailed information about the internal structure of the MRA derived from the B-spline example of Ron and Shen. Let $\psi_0 := \varphi$, where φ is defined as in (1.8), $m_l, l = 0, 1, \dots, n := 2k$, the filters as in (1.7), and $\psi_l, l = 1, \dots, n$, the 'wavelets' defined as in (1.4). Define

$$W_0^{(l)} := \overline{\operatorname{span}}\{T_k\psi_l : k \in \mathbb{Z}\}, \quad l = 0, 1, \dots, n.$$

For the sake of the consistency of the notation we let $V_0 := W_0^{(0)}$, $W_0 := \overline{\operatorname{span}}\{T_k\psi_l : k \in \mathbb{Z}, 1 \le l \le n\}$, and $V_1 = D(V_0)$, where D is now the 1-dimensional dilation operator. Recall that $W_0 = V_1$ by Theorem 1.2. We now have:

Theorem 1.3. *For $0 \le j < i \le n = 2k$, $W_0^{(j)} + W_0^{(i)}$ is closed if and only if $i - j$ is odd and $j \le k \le i$. In this case $W_0^{(j)} \dot{+} W_0^{(i)} = V_1$.*

The rest of this article is organized in the following manner: We first review those parts of shift-invariant space theory which are needed in our discussion in Section 2. Then we count the fiberwise dimensions of the three spaces, V_0, W_0 and V_1 by extending the ideas in [16,18,19] and give general results on the structure of the MRAs defined by the UEP in Section 3. More precisely, suppose we have an MRA of $L^2(\mathbb{R}^d)$ defined by the UEP. We first characterize the conditions that $V_1 = W_0$ (Theorem 3.4). We, then, characterize the conditions that $V_1 = V_0 \dot{+} W_0$ (Theorem 3.8) and that $V_1 = V_0 \oplus W_0$ (Theorem 3.9). A proof of Theorem 1.2 is given in Section 4 and another of Theorem 1.3 in Section 5.

2. A glimpse at shift-invariant space theory

In this section we review the necessary parts of the theory of shift-invariant subspaces of $L^2(\mathbb{R}^d)$ for the sake of completeness. All of the results we review, save Proposition 2.3 and Lemma 2.4, are contained in [1–3,11,12,20–22]. A closed subspace S is said to be a *shift-invariant subspace* of $L^2(\mathbb{R}^d)$ if it is invariant under each (multi-)integer shift, i.e., $T_k S \subset S$ for each $k \in \mathbb{Z}^d$. For $\Phi \subset L^2(\mathbb{R}^d)$, $\mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is obviously a shift-invariant subspace. It is said to be the shift-invariant subspace *generated by* Φ, and Φ is said to be a *generating set* of $\mathcal{S}(\Phi)$. Any shift-invariant subspace of $L^2(\mathbb{R}^d)$ has an at most countable generating set. If S is a shift-invariant subspace, define its *length* via $\operatorname{len} S := \min\{\#\Phi : S = \mathcal{S}(\Phi)\}$, where # denotes the cardinality. For $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $\hat{f}_{\|x}$ denotes the sequence $(\hat{f}(x+k))_{k\in\mathbb{Z}^d}$ which is in $\ell^2(\mathbb{Z}^d)$ a.e. For any subset A of $L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let $\hat{A}_{\|x} := \{\hat{f}_{\|x} : f \in A\}$. If S is a shift-invariant subspace, then $\hat{S}_{\|x}$ is called the *fiber* of S at $x \in \mathbb{R}^d$. We have the following formula for the length of S [1]:

$$\operatorname{len} S = \operatorname{ess-sup}\{\dim \hat{S}_{\|x} : x \in \mathbb{T}^d\}. \tag{2.1}$$

The following standard theorem is usually called the fundamental theorem of shift-invariant subspace of $L^2(\mathbb{R}^d)$.

Proposition 2.1 (*[1,2,11,12]*). *If a shift-invariant subspace S is generated by Φ, i.e., $S = \mathcal{S}(\Phi)$, then, $\hat{S}_{\|x} = \overline{\operatorname{span}}\,\hat{\Phi}_{\|x}$ for a.e. $x \in \mathbb{T}^d$. Moreover, a square integrable function f is in S if and only if $\hat{f}_{\|x} \in \hat{S}_{\|x}$ a.e. $x \in \mathbb{T}^d$.*

For a shift-invariant subspace S, the *spectrum* $\sigma(S)$ of S is defined to be the set $\{x \in \mathbb{T}^d : \hat{S}_{\|x} \neq \{0\}\}$. We have the following obvious corollary.

Corollary 2.2. *Suppose that S_1, S_2 are finitely generated shift-invariant subspaces such that $S_1 \subset S_2$. Then $\hat{S}_{1\|x} \subset \hat{S}_{2\|x}$ a.e. $x \in \mathbb{T}^d$. In particular, $S_1 = S_2$ if and only if $\dim \hat{S}_{1\|x} = \dim \hat{S}_{2\|x}$ for a.e. $x \in \sigma(S_2)$.*

The following proposition, which is Theorem 2.3 of [14], characterizes the condition for the sum of two singly generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ to be closed. It was originally stated for $\mathbb{R}$, but the proof easily generalizes for $\mathbb{R}^d$. See [17] for a general characterization of the conditions for the sum of two, possibly infinitely generated, shift-invariant subspaces of $L^2(\mathbb{R}^d)$ to be closed.

Proposition 2.3 (*[14]*). *Let $U := \mathcal{S}(\{\varphi\})$, $V := \mathcal{S}(\{\psi\})$, and let*

$$E := \{x \in \sigma(U) \cap \sigma(V) : \hat{\varphi}_{\|x} \text{ and } \hat{\psi}_{\|x} \text{ are linearly independent}\}.$$

Then $U + V$ is closed if and only if either $|E| = 0$, or

$$\operatorname{ess-sup}_{x\in E} \left\{ \frac{\left|\left\langle \hat{\varphi}_{\|x}, \hat{\psi}_{\|x} \right\rangle\right|}{\|\hat{\varphi}_{\|x}\|_{\ell^2(\mathbb{Z})} \|\hat{\psi}_{\|x}\|_{\ell^2(\mathbb{Z})}} \right\} < 1. \tag{2.2}$$

We need the following lemma which extends [14, Lemma 2.2]. A proof can be found in [17, Lemma 3.4].

Lemma 2.4. *For two shift-invariant subspaces U and V of $L^2(\mathbb{R}^d)$, we have*

$$\sigma(U\cap V)=\left\{x\in\mathbb{T}^d:\hat{U}_{\|x}\cap\hat{V}_{\|x}\neq\{0\}\right\}\subset\sigma(U)\cap\sigma(V).$$

3. General results on the internal structure

Suppose that $\varphi, m_0,\ldots,m_n$ satisfy all the UEP requirements of Proposition 1.1. Suppose also that $\psi_1,\ldots,\psi_n$ are defined by (1.4) and that V_0, W_0 and V_1 are defined as in the paragraph preceding Theorem 1.2. Recall that $V_0=\mathcal{S}(\{\varphi\})$ and $W_0=\mathcal{S}(\{\psi_1,\psi_2,\ldots,\psi_n\})$. We now review the method of counting the fiberwise dimensions of finitely generated shift-invariant subspaces and improve some material contained in [16,18,19].

First note that each $k\in\mathbb{Z}^d$ can be written uniquely as $k=2k'+q$ for some $k'\in\mathbb{Z}^d$ and $q\in Q$, where Q is defined in (1.5). Note also that $DT_y=T_{y/2}D$ for each $y\in\mathbb{R}^d$. Therefore, $\{DT_k\varphi:k\in\mathbb{Z}^d\}=\{T_{k'}DT_q\varphi:k'\in\mathbb{Z}^d,q\in Q\}$. Hence $V_1=\mathcal{S}(\Pi)$, where $\Pi:=\{DT_q\varphi:q\in Q\}$. This shows that $\operatorname{len}V_1\le 2^d$. The following is Lemma 8 in [16] (see also [18, Lemma 3.2]), which can be checked easily by the readers (c.f. (3.5) and Lemma 3.2).

Proposition 3.1 (*[16]*). $\sigma(V_1)=2\sigma(V_0)(\mathrm{mod}1)$.

Recall that (1.1), (1.3) and (1.4) imply that V_0 and W_0 are contained in V_1. Therefore, V_0 and W_0 are shift-invariant subspaces of the shift-invariant space V_1. We now localize the shift-invariant subspaces V_0, W_0 and V_1. Since $(DT_q\varphi)^\wedge(x)=2^{-d/2}\mathrm{e}^{-2\pi iq\cdot(x/2)}\hat{\varphi}(x/2)$, we have $(DT_q\varphi)^\wedge_{\|x}=2^{-d/2}\mathrm{e}^{-\pi iq\cdot x}(\mathrm{e}^{-\pi iq\cdot k}\hat{\varphi}((x+k)/2))_{k\in\mathbb{Z}^d}$. Hence, for a.e. $x\in\mathbb{T}^d$,

$$\hat{V}_{1\|x}=\operatorname{span}\left\{\left((-1)^{q\cdot k}\hat{\varphi}\left(\frac{x+k}{2}\right)\right)_{k\in\mathbb{Z}^d}:q\in Q\right\},$$

$$\hat{V}_{0\|x}=\operatorname{span}\left\{(\hat{\varphi}(x+k))_{k\in\mathbb{Z}^d}\right\}=\operatorname{span}\left\{\left(m_0\left(\frac{x+k}{2}\right)\hat{\varphi}\left(\frac{x+k}{2}\right)\right)_{k\in\mathbb{Z}^d}\right\}. \tag{3.1}$$

Similarly,

$$\hat{W}_{0\|x}=\operatorname{span}\left\{\left(m_l\left(\frac{x+k}{2}\right)\hat{\varphi}\left(\frac{x+k}{2}\right)\right)_{k\in\mathbb{Z}^d}:1\le l\le n\right\}. \tag{3.2}$$

For $q\in Q$, define $P_q:\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ via

$$(P_qa)(k):=\begin{cases}a(k), & \text{if } k\in 2\mathbb{Z}^d+q,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\ell^2(\mathbb{Z}^d)=\oplus_{q\in Q}P_q(\ell^2(\mathbb{Z}^d))$. Define, for each $x\in\mathbb{T}^d$ and each $q\in Q$,

$$a_{x,q}:=P_q\left(\left(\hat{\varphi}\left(\frac{x+k}{2}\right)\right)_{k\in\mathbb{Z}^d}\right). \tag{3.3}$$

Note that $a_{x,q}$ is the 'up-sampled' version of $\hat{\varphi}_{\|(x+q)/2}$, i.e.,

$$\begin{cases}a_{x,q}(2k+q)=\hat{\varphi}_{\|(x+q)/2}(k), & \text{if } k\in\mathbb{Z}^d,\\ a_{x,q}(k)=0, & \text{if } k\notin 2\mathbb{Z}^d+q.\end{cases} \tag{3.4}$$

Therefore,

$$\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)} = \|\hat{\varphi}_{\|(x+q)/2}\|_{\ell^2(\mathbb{Z}^d)}. \tag{3.5}$$

With this notation we have $((-1)^{q\cdot k}\hat{\varphi}((x+k)/2))_{k\in\mathbb{Z}^d} = \sum_{p\in Q}(-1)^{q\cdot p}a_{x,p}$. On the other hand,

$$\hat{\varphi}_{\|x} = (\hat{\varphi}(x+k))_{k\in\mathbb{Z}^d} = \left(m_0\left(\frac{x+k}{2}\right)\hat{\varphi}\left(\frac{x+k}{2}\right)\right)_{k\in\mathbb{Z}^d} = \sum_{q\in Q} m_0\left(\frac{x+q}{2}\right)a_{x,q}. \tag{3.6}$$

Similarly, for each $l = 1, 2, \ldots, n$,

$$\hat{\psi}_{l\,\|x} = (\hat{\psi}_l(x+k))_{k\in\mathbb{Z}^d} = \sum_{q\in Q} m_l\left(\frac{x+q}{2}\right)a_{x,q}. \tag{3.7}$$

This shows that $\hat{V}_{1\,\|x} = \text{span}\{\sum_{p\in Q}(-1)^{q\cdot p}a_{x,p} : q \in Q\}$. Since the $2^d \times 2^d$ matrix $((-1)^{q\cdot p})_{q,p\in Q}$ is invertible, it is easy to see that $\hat{V}_{1\,\|x} = \text{span}\{a_{x,q} : q \in Q\}$ [16,18,19]. Combining this with (3.1) and (3.2) and the 1-periodicity of m_l, $0 \le l \le n$, we have:

Lemma 3.2. *For a.e. $x \in \mathbb{T}^d$*

$$\hat{V}_{0\,\|x} = \text{span}\left\{\sum_{q\in Q} m_0\left(\frac{x+q}{2}\right)a_{x,q}\right\}, \tag{3.8}$$

$$\hat{V}_{1\,\|x} = \text{span}\left\{a_{x,q} : q \in Q\right\}, \tag{3.9}$$

$$\hat{W}_{0\,\|x} = \text{span}\left\{\sum_{q\in Q} m_l\left(\frac{x+q}{2}\right)a_{x,q} : 1 \le l \le n\right\},$$

$$\begin{pmatrix} \hat{\varphi}_{\|x} \\ \hat{\psi}_{1\,\|x} \\ \vdots \\ \hat{\psi}_{n\,\|x} \end{pmatrix} = M\left(\frac{x}{2}\right)\begin{pmatrix} a_{x,q_1} \\ a_{x,q_2} \\ \vdots \\ a_{x,q_{2^d}} \end{pmatrix}. \tag{3.10}$$

In particular, for a.e. $x \in \mathbb{T}^d$,

$$\dim\hat{V}_{1\,\|x} = \#\{a_{x,q} : a_{x,q} \ne 0\}. \tag{3.11}$$

Proof. Everything except for (3.11) is already shown. Note that $\{a_{x,q} : q \in Q\} \subset \ell^2(\mathbb{Z}^d)$, which is defined in (3.3), is mutually orthogonal even though some $a_{x,q}$ may be 0. Now, (3.11) follows from (3.9). □

Recall that $M(x)^*M(x)$ is assumed to be the identity matrix for a.e. $x \in \sigma(V_0)$. This forces, in particular, $n \ge 2^d - 1$. Let us define, for $0 \le i \le 2^d$,

$$\Delta_i := \{x \in \mathbb{T}^d : \dim\hat{V}_{1\,\|x} = i\}. \tag{3.12}$$

Then $\sigma(V_1) = \biguplus_{i=1}^{2^d}\Delta_i$, where $\biguplus$ denotes the disjoint union. Also, for each $1 \le l \le 2^d - 1$ and each $k_1, k_2, \ldots, k_l$ such that $1 \le k_1 < k_2 < \cdots < k_l \le 2^d$, define

$$\Delta_{2^d-l}^{k_1,\ldots,k_l} := \{x \in \Delta_{2^d-l} : a_{x,q_{k_1}} = \cdots = a_{x,q_{k_l}} = 0\}.$$

Then we have, by (3.11) and the orthogonality of $\{a_{x,q} : q \in Q\}$,

$$\Delta_{2^d-l} = \biguplus_{1\le k_1<k_2<\cdots<k_l\le 2^d} \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}.$$

Let M_1 be the matrix-valued mapping defined by

$$M_1(x) = \begin{pmatrix} m_1\left(x+\frac{q_1}{2}\right) & m_1\left(x+\frac{q_2}{2}\right) & \cdots & m_1\left(x+\frac{q_{2^d}}{2}\right) \\ m_2\left(x+\frac{q_1}{2}\right) & m_2\left(x+\frac{q_2}{2}\right) & \cdots & m_2\left(x+\frac{q_{2^d}}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_n\left(x+\frac{q_1}{2}\right) & m_n\left(x+\frac{q_2}{2}\right) & \cdots & m_n\left(x+\frac{q_{2^d}}{2}\right) \end{pmatrix}. \tag{3.13}$$

Note that, for a.e. $x \in \sigma(V_0)$,

$$I_{2^d} = M(x)^*M(x) = M_0(x) + M_1(x)^*M_1(x), \tag{3.14}$$

where

$$\begin{aligned} M_0(x) &:= \begin{pmatrix} \overline{m_0}\left(x+\frac{q_1}{2}\right) \\ \vdots \\ \overline{m_0}\left(x+\frac{q_{2^d}}{2}\right) \end{pmatrix} \begin{pmatrix} m_0\left(x+\frac{q_1}{2}\right) & \cdots & m_0\left(x+\frac{q_{2^d}}{2}\right) \end{pmatrix} \\ &= \left(\overline{m_0}\left(x+\frac{q_i}{2}\right) m_0\left(x+\frac{q_j}{2}\right)\right)_{1\le i,j\le 2^d}. \end{aligned}$$

Therefore

$$M_1(x)^*M_1(x) = I_{2^d} - M_0(x). \tag{3.15}$$

For a matrix A, let $A^{k_1,\ldots,k_l}$ denote the matrix obtained from A by deleting the $k_1,\ldots,k_l$-th columns of A, and let $A_{k_1,\ldots,k_l}$ denote the matrix obtained from A by deleting the $k_1,\ldots,k_l$-th rows and columns of A, simultaneously. Then, for a.e. $x \in \sigma(V_0)$,

$$(M_1(x)^{k_1,\ldots,k_l})^* M_1(x)^{k_1,\ldots,k_l} = I_{2^d-l} - M_0(x)_{k_1,\ldots,k_l}. \tag{3.16}$$

Lemma 3.3. *If $x \in \mathbb{T}^d \cap (\sigma(V_0) + \frac{1}{2}\mathbb{Z}^d)$, then the following hold for each $1 \le l \le 2^d - 1$ and for each choice of $k_1, k_2, \ldots, k_l$ such that $1 \le k_1 < k_2 < \cdots < k_l \le 2^d$.*

(1) rank $M(x) = 2^d$;
(2) rank $M_1(x) = 2^d$ *if and only if* $\sum_{j=1}^{2^d} \left|m_0\left(x+\frac{q_j}{2}\right)\right|^2 \neq 1$;
(3) rank $M_1(x) = 2^d - 1$ *if and only if* $\sum_{j=1}^{2^d} \left|m_0\left(x+\frac{q_j}{2}\right)\right|^2 = 1$;
(4) rank $M_1(x)^{k_1,\ldots,k_l} = 2^d - l$ *if and only if* $\sum_{1\le j\le 2^d, j\notin\{k_1,\ldots,k_l\}} \left|m_0\left(x+\frac{q_j}{2}\right)\right|^2 \neq 1$;
(5) rank $M_1(x)^{k_1,\ldots,k_l} = 2^d - l - 1$ *if and only if* $\sum_{1\le j\le 2^d, j\notin\{k_1,\ldots,k_l\}} \left|m_0\left(x+\frac{q_j}{2}\right)\right|^2 = 1$.

Proof. If $x \in \mathbb{T}^d \cap (\sigma(V_0) + \frac{1}{2}\mathbb{Z}^d)$, then there exist $x' \in \sigma(V_0)$ and $k \in \mathbb{Z}^d$ such that $x = x' + k/2$. We see that $M(x) = M(x' + k/2)$ is a suitable column permutation of the rectangular matrix $M(x')$ by (1.6) and by the 1-periodicity of the filters. Since $x' \in \sigma(V_0)$, $M(x')^*M(x') = I_{2^d}$.

Hence, $M(x)^*M(x) = I_{2^d}$ by the nature of the column permutation of a rectangular matrix. Therefore, (3.14)–(3.16) hold for such x.

(1): The column vectors of $M(x)$ are mutually orthonormal for a.e. $x \in \sigma(V_0)$. Hence $\operatorname{rank} M(x) = 2^d$.

(2), (3): It is easy to see that $\sum_{i=1}^{2^d} |m_0(x + \frac{q_i}{2})|^2$ is the eigenvalue of the matrix $M_0(x)$ with geometric multiplicity 1 and that 0 is another eigenvalue of $M_0(x)$ with geometric multiplicity $2^d - 1$. The spectral mapping theorem and (3.15) imply that the eigenvalues of $M_1(x)^*M_1(x)$ are $1 - \sum_{i=1}^{2^d} |m_0(x + \frac{q_i}{2})|^2$ with multiplicity 1 and 1 with multiplicity $2^d - 1$. Hence $\operatorname{rank} M_1(x) = 2^d$ if and only if $\sum_{i=1}^{2^d} |m_0(x + \frac{q_i}{2})|^2 \neq 1$, and $\operatorname{rank} M_1(x) = 2^d - 1$ if and only if $\sum_{i=1}^{2^d} |m_0(x + \frac{q_i}{2})|^2 = 1$.

(4), (5): The proof is similar to the above case if we use (3.16) instead of (3.15). □

Unlike the case of orthonormal wavelets, we may have $V_1 = W_0$ in most cases of the wavelet frames from the UEP if $n > 1$. See Corollary 3.5. We first find the conditions for $V_1 = W_0$. The corresponding conditions for $V_1 = V_0 \dot{+} W_0$ are given in Theorem 3.8, and those for $V_1 = V_0 \oplus W_0$ in Theorem 3.9.

Theorem 3.4. $V_1 = W_0$ *if and only if* $\sum_{q \in Q, a_{x,q} \neq 0} |m_0(\frac{x}{2} + \frac{q}{2})|^2 \neq 1$ *for a.e.* $x \in \mathbb{T}^d$.

Proof. Since $W_0 \subset V_1$, by Corollary 2.2, $V_1 = W_0$ if and only if $\dim \hat{V}_{1\|x} = \dim \hat{W}_{0\|x}$ for a.e. $x \in \sigma(V_1)$. (3.10) implies that

$$\begin{pmatrix} \hat{\psi}_{1\|x} \\ \vdots \\ \hat{\psi}_{n\|x} \end{pmatrix} = M_1\left(\frac{x}{2}\right) \begin{pmatrix} a_{x,q_1} \\ a_{x,q_2} \\ \vdots \\ a_{x,q_{2^d}} \end{pmatrix}. \tag{3.17}$$

Lemma 3.2, (1.9) and Proposition 2.1 imply that $\hat{V}_{1\|x} = \operatorname{span}\{a_{x,q} : q \in Q\}$, and $\hat{W}_{0\|x} = \operatorname{span}\{\hat{\psi}_{j\|x} : 1 \le j \le n\}$. Recall that $\{a_{x,q} : q \in Q\}$ is a set of mutually orthogonal vectors even though some elements may possibly be 0, and that

$$\sigma(V_1) = \Delta_{2^d} \biguplus \biguplus_{l=1}^{2^d-1} \biguplus_{1 \le k_1 < k_2 < \cdots < k_l \le 2^d} \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}. \tag{3.18}$$

If $x \in \mathbb{T}^d \setminus \sigma(V_1)$, then $\hat{V}_{1\|x} = \{0\}$ for a.e. such x. Therefore, $\hat{W}_{0\|x} = \{0\}$ for a.e. such x by Corollary 2.2. On the other hand, if $x \in \Delta_{2^d}$, then, by definition, $\dim \hat{V}_{1\|x} = 2^d$. Hence, $\{a_{x,q} : q \in Q\}$ forms an orthogonal basis for $\hat{V}_{1\|x}$ a.e. by Lemma 3.2. Therefore, for almost every such x, $\dim \hat{W}_{0\|x} = 2^d$ if and only if $\operatorname{rank} M(x/2) = 2^d$ by (3.17). Similarly, if $x \in \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}$, then $\dim \hat{V}_{1\|x} = 2^d - l$ and, $\{a_{x,q} : q \in Q \setminus \{q_{k_1}, q_{k_2}, \ldots, q_{k_l}\}\}$ forms an orthogonal basis for $\hat{V}_{1\|x}$. Hence, for almost every such x, $\dim \hat{W}_{0\|x} = 2^d - l$ if and only if $\operatorname{rank} M_1(x/2)^{k_1,k_2,\ldots,k_l} = 2^d - l$ by Eq. (3.17). On the other hand, if $x \in \sigma(V_1)$, then there exist $x' \in \sigma(V_0)$, $k \in \mathbb{Z}^d$ such that $x = 2x' + k$ by Proposition 3.1. Therefore, $x/2 = x' + k/2 \in \mathbb{T}^d$ (recall that we identified $\mathbb{T}^d$ with $[0, 1]^d$ or with $[-1/2, 1/2]^d$). Now, the claimed equivalence follows from Lemma 3.3. □

Corollary 3.5. *If* $\sum_{q\in Q}|m_0(\frac{x}{2}+\frac{q}{2})|^2 \neq 1$ *for a.e.* $x \in \sigma(V_1)$, *then* $V_1 = W_0$.

Proof. By Proposition 3.1, for a.e. $x \in \sigma(V_1)$, there exist $x' \in \sigma(V_0)$ and $k \in \mathbb{Z}^d$ such that $x = 2x' + k$, i.e., $x/2 = x' + k/2 \in \mathbb{T}^d$. Now, by the argument at the beginning of the proof of Lemma 3.3, $M(x/2)^*M(x/2) = I_{2^d}$ a.e. $x \in \sigma(V_1)$. Hence the rows of $M(x/2)$ form a tight frame for $\mathbb{C}^{2^d}$ with frame bound 1 by Corollary 1.3.6 of [5]. In particular, the $\mathbb{C}^{2^d}$-norm of any row of $M(x/2)$ is less than or equal to 1. Therefore, $\sum_{q\in Q}|m_0(\frac{x}{2}+\frac{q}{2})|^2 \le 1$. Hence, if $\sum_{q\in Q}|m_0(\frac{x}{2}+\frac{q}{2})|^2 \neq 1$, then $\sum_{q\in Q}|m_0(\frac{x}{2}+\frac{q}{2})|^2 < 1$ and hence $\sum_{q\in Q, a_{x,q}\neq 0}|m_0(\frac{x}{2}+\frac{q}{2})|^2 < 1$. The corollary now follows from Theorem 3.4. □

We remark that in most of the interesting examples of UEP wavelet frames we have $V_1 = W_0$ if it has more than one generator ($n > 1$). See [13] for one more such example.

Lemma 3.6. *Let* U, V *and* W *be shift-invariant subspaces of* $L^2(\mathbb{R}^d)$ *such that* $V \subset U$ *and* $W \subset U$. *Then* $U = V \dot{+} W$ *if and only if:*

(1) $V + W$ *is closed;*
(2) $\hat{U}_{\|x} = \hat{V}_{\|x} \dot{+} \hat{W}_{\|x}$, *for a.e.* $x \in \mathbb{T}^d$.

Proof. ($\Rightarrow$) (1) is trivial and (2) follows from a slight adaptation of Lemma 3.7 in [15], where (2) is proved for the case of finitely generated V and W. Recall that any shift-invariant subspace has a countable generating set.

($\Leftarrow$) Obviously, $V + W \subset U$. If $V + W$ is closed, then $V + W$ is a shift-invariant subspace since V and W are invariant under each shift. Now Condition (2) implies that $\hat{U}_{\|x} \subset (V+W)^{\wedge}_{\|x}$ a.e. Therefore, $U \subset V + W$ by Proposition 2.1. This shows that $U = V + W$. Condition (2), again, and Lemma 2.4 imply that $V \cap W = \{0\}$. Hence $U = V \dot{+} W$. □

Lemma 3.7. *The following equalities hold:*

$$V_0 = \overline{\operatorname{span}}\{D^j T_k \psi_l : j < 0, k \in \mathbb{Z}^d, 1 \le l \le n\} = \mathcal{S}(\{D^j \psi_l : j < 0, 1 \le l \le n\}).$$

In particular, $V_1 = \overline{V_0 + W_0}$.

Proof. Let $X := \overline{\operatorname{span}}\{D^j T_k \psi_l : j < 0, k \in \mathbb{Z}^d, 1 \le l \le n\}$. Since $\{D^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \le l \le n\}$ is a tight frame for $L^2(\mathbb{R}^d)$, X is a shift-invariant subspace by Proposition 3.1 in [4] (see also [14, Section 5] for the 1-dimensional case). Recall that V_0 and W_0 are subspaces of $V_1 = D(V_0)$. Therefore, $D^j(V_0)$ and $D^j(W_0)$ are subspaces of V_0 for each $j < 0$. In particular, $D^j T_k \psi_l \in V_0$ for each $j < 0, k \in \mathbb{Z}^d$ and $l = 1, 2, \ldots, n$. This shows that X is a shift-invariant subspace of V_0. Since $D^j \psi_l \in X$ for each $j < 0$ and $l = 1, 2, \ldots, n$, $\mathcal{S}(\{D^j \psi_l : j < 0, 1 \le l \le n\}) \subset X$. On the other hand, $D^j T_k \psi_l = T_{2^{-j}k} D^j \psi_l$. This shows that $X \subset \mathcal{S}(\{D^j \psi_l : j < 0, 1 \le l \le n\})$. Therefore, we have $X = \mathcal{S}(\{D^j \psi_l : j < 0, 1 \le l \le n\})$ and X is a subspace of V_0. To show that $X = V_0$ it is enough to show that $\sigma(V_0) \subset \sigma(X)$ since $V_0 = \mathcal{S}(\{\varphi\})$ is singly generated. Now, suppose that $x \in \sigma(V_0)$. Recall that we assumed $M(x)^*M(x) = I_{2^d}$ a.e. $x \in \sigma(V_0)$. Then, we have

$$\begin{aligned} 0 \neq \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k)|^2 &= \sum_{k\in\mathbb{Z}^d}\sum_{l=0}^{n} |m_l(x)\hat{\varphi}(x+k)|^2 \\ &= \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(2(x+k))|^2 + \sum_{l=1}^{n}\sum_{k\in\mathbb{Z}^d} |\hat{\psi}_l(2(x+k))|^2, \end{aligned}$$

where we have used the fact that $\sum_{l=0}^{n}|m_l(x)|^2 = 1$, which is the norm squared of one of the columns of $M(x)$ in Eqs. (1.6), (1.1) and (1.4). Suppose that $\sum_{l=1}^{n}\sum_{k\in\mathbb{Z}^d}|\hat{\psi}_l(2(x+k))|^2 \neq 0$. Then, obviously, $x \in \sigma(X)$, since $\widehat{D^{-1}\psi_l}(x) = D\hat{\psi}_l$. Suppose, on the other hand, that $\sum_{l=1}^{n}\sum_{k\in\mathbb{Z}^d}|\hat{\psi}_l(2(x+k))|^2 = 0$. Then, we have $0 \neq \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(x+k)|^2 = \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(2(x+k))|^2$. This implies that $2x \pmod 1 \in \sigma(V_0)$. Therefore, $\sum_{l=0}^{n}|m_l(2x)|^2 = 1$. Hence we have

$$\begin{aligned}0 \neq \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(x+k)|^2 &= \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(2(x+k))|^2 = \sum_{k\in\mathbb{Z}^d}\sum_{l=0}^{n}|m_l(2x)\hat{\varphi}(2(x+k))|^2\\ &= \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(2^2(x+k))|^2 + \sum_{l=1}^{n}\sum_{k\in\mathbb{Z}^d}|\hat{\psi}_l(2^2(x+k))|^2.\end{aligned}$$

By repeating the previous argument, we have either $x \in \sigma(X)$ or $2^2x \pmod 1 \in \sigma(V_0)$. Either this process stops after a finite number of times to give the desired result that $x \in \sigma(X)$, or we have

$$0 \neq \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(x+k)|^2 = \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(2^j(x+k))|^2 \tag{3.19}$$

for all $j \in \mathbb{N}$. Now, let $E := \{x \in \sigma(V_0) : (3.19) \text{ holds for all } j \in \mathbb{N}\}$. If we integrate the middle term of (3.19) over E, then we have $\int_{E+\mathbb{Z}^d}|\hat{\varphi}(x)|^2\,dx$. On the other hand, if we integrate the right-most term of (3.19) over E, we have

$$\int_E \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(2^jx+2^jk)|^2 dx = 2^{-jd}\sum_{k\in\mathbb{Z}^d}\int_{2^jE+2^jk}|\hat{\varphi}(t)|^2 dt \leq 2^{-jd}\|\hat{\varphi}\|^2,$$

for all $j \in \mathbb{N}$. This shows that the Lebesgue measure of $E+\mathbb{Z}^d$ is 0 and so is the Lebesgue measure of E. Hence we have shown that $\sigma(V_0) \subset \sigma(X)$ a.e. This completes the first part of the lemma. For the second part of the lemma we argue as follows. $V_1 = D(V_0) = D(X)$. Hence

$$\begin{aligned}V_1 &= \overline{\operatorname{span}}\{D^jT_k\psi_l : j \leq 0, k \in \mathbb{Z}^d, 1 \leq l \leq n\}\\ &= \overline{\operatorname{span}\{T_k\psi_l : k \in \mathbb{Z}^d, 1 \leq l \leq n\} + \operatorname{span}\{D^jT_k\psi_l : j < 0, k \in \mathbb{Z}^d, 1 \leq l \leq n\}}\\ &= \overline{W_0 + X} = \overline{W_0 + V_0}. \quad \square\end{aligned}$$

We now characterize the case where $V_1 = V_0 \dotplus W_0$ under the assumption that $V_0 + W_0$ is closed. See [17] for the condition for the closedness of the sum of two shift-invariant spaces.

Theorem 3.8. *Suppose $V_0 + W_0$ is closed. Then $V_1 = V_0 \dotplus W_0$ if and only if, for a.e. $x \in \mathbb{T}^d$, $\sum_{1\leq j\leq 2^d, a_{x,q_j}\neq 0}|m_0(\frac{x+q_j}{2})|^2 = 1$ or 0.*

Proof. From the assumption that $V_0 + W_0$ is closed we have $V_1 = V_0 + W_0$ by Lemma 3.7. Now, for a.e. $x \in \mathbb{T}^d$, $\hat{V}_{0\|x} + \hat{W}_{0\|x} \subset \hat{V}_{1\|x}$ since V_0 and W_0 are shift-invariant subspaces of V_1; and also $\hat{V}_{1\|x} = (V_0 + W_0)^{\wedge}_{\|x} \subset \hat{V}_{0\|x} + \hat{W}_{0\|x}$. Hence $\hat{V}_{1\|x} = \hat{V}_{0\|x} + \hat{W}_{0\|x}$ a.e. Therefore, $V_1 = V_0 \dotplus W_0$ if and only if $\hat{V}_{0\|x} \cap \hat{W}_{0\|x} = \{0\}$ a.e. by Lemma 3.6. Since we already have $\hat{V}_{1\|x} = \hat{V}_{0\|x} + \hat{W}_{0\|x}$ a.e., and since $\hat{V}_{1\|x}$ is finite dimensional a.e., $V_1 = V_0 \dotplus W_0$ if and only if $\dim\hat{V}_{1\|x} = \dim\hat{V}_{0\|x} + \dim\hat{W}_{0\|x}$ a.e. We first show that $\dim\hat{V}_{1\|x} = \dim\hat{V}_{0\|x} + \dim\hat{W}_{0\|x}$ a.e. if and only if the following two conditions are satisfied:

(1) $\sum_{j=1}^{2^d}|m_0(\frac{x+q_j}{2})|^2 = 1$ or 0 for a.e. $x \in \Delta_{2^d}$;

(2) For each $l = 1, 2, \ldots, 2^d - 1$ and for each choice of $k_1, k_2, \ldots, k_l$ such that $1 \le k_1 < k_2, \ldots, < k_l \le 2^d$,

$$\sum_{1\le j\le 2^d,\, j\notin\{k_1,\ldots,k_l\}} \left| m_0\left(\frac{x+q_j}{2}\right)\right|^2 = 1 \text{ or } 0 \quad \text{for a.e. } x \in \Delta_{2^d-l}^{k_1,\ldots,k_l}.$$

If $x \in \mathbb{T}^d \setminus \sigma(V_1)$, then obviously $\hat{V}_{1\,\|x} = \hat{V}_{0\,\|x} = \hat{W}_{0\,\|x} = \{0\}$. Hence there is nothing to prove.

If $x \in \sigma(V_1)$, then there exist $x' \in \sigma(V_0)$ and $k \in \mathbb{Z}$ such that $x = 2x' + k$ by Proposition 3.1. Hence $x/2 = x' + k/2$. Recall (3.18). Suppose that $x \in \Delta_{2^d}$. In this case $\dim \hat{V}_{1\,\|x} = 2^d$. Hence $\{a_{x,q} : q \in Q\}$ is an orthogonal basis for $\hat{V}_{1\,\|x}$ by Lemma 3.2. Recall that V_0 is singly generated. Therefore, $\dim \hat{V}_{0\,\|x} = 0$ or 1. This shows that:

$$\begin{aligned}
&\dim \hat{V}_{0\,\|x} = 0, \qquad \dim \hat{W}_{0\,\|x} = 2^d \\
&\Leftrightarrow \sum_{j=1}^{2^d} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 0, \qquad \operatorname{rank} M_1\left(\frac{x}{2}\right) = 2^d \quad \text{(by (3.8), (3.10) and (3.13))} \\
&\Leftrightarrow \sum_{j=1}^{2^d} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 0 \quad \text{(by Lemma 3.3)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\dim \hat{V}_{0\,\|x} = 1, \qquad \dim \hat{W}_{0\,\|x} = 2^d - 1 \\
&\Leftrightarrow \sum_{j=1}^{2^d} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 \neq 0, \qquad \operatorname{rank} M_1\left(\frac{x}{2}\right) = 2^d - 1 \quad \text{(by (3.8), (3.10) and (3.13))} \\
&\Leftrightarrow \sum_{j=1}^{2^d} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 1 \quad \text{(by Lemma 3.3)}.
\end{aligned}$$

If $x \in \Delta_{2^d-l}^{k_1,\ldots,k_l}$, then $\dim \hat{V}_{1\,\|x} = 2^d - l$ and $a_{x,q_{k_1}} = \cdots = a_{x,q_{k_l}} = 0$. Therefore, $\{a_{x,q} : q \notin \{q_{k_1}, q_{k_2}, \ldots, q_{k_l}\}\}$ is an orthogonal basis for $\hat{V}_{1\,\|x}$. Now, a similar argument shows that:

$$\begin{aligned}
&\dim \hat{V}_{0\,\|x} = 0, \qquad \dim \hat{W}_{0\,\|x} = 2^d - l \\
&\Leftrightarrow \sum_{1\le j\le 2^d,\, j\notin\{k_1,\ldots,k_l\}} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 0, \qquad \operatorname{rank} M_1\left(\frac{x}{2}\right)^{k_1,\ldots,k_l} = 2^d - l \\
&\Leftrightarrow \sum_{1\le j\le 2^d,\, j\notin\{k_1,\ldots,k_l\}} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 0.
\end{aligned}$$

Likewise, we have:

$$\begin{aligned}
&\dim \hat{V}_{0\,\|x} = 1, \qquad \dim \hat{W}_{0\,\|x} = 2^d - l - 1 \\
&\Leftrightarrow \sum_{1\le j\le 2^d,\, j\notin\{k_1,\ldots,k_l\}} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 \neq 0, \qquad \operatorname{rank} M_1\left(\frac{x}{2}\right)^{k_1,\ldots,k_l} = 2^d - l - 1
\end{aligned}$$

$$\Leftrightarrow \sum_{1\le j\le 2^d, j\notin\{k_1,\ldots,k_l\}} \left| m_0\left(\frac{x}{2}+\frac{q_j}{2}\right)\right|^2 = 1.$$

This shows that $V_1 = V_0 \dot{+} W_0$ if and only if Conditions (1) and (2) are satisfied. Obviously, Conditions (1) and (2) are equivalent to $\sum_{1\le j\le 2^d, a_{x,q_j}\ne 0} |m_0(\frac{x+q_j}{2})|^2 = 1$ or 0 a.e. □

We now characterize conditions for $V_1 = V_0 \oplus W_0$, which are simpler than those for $V_1 = V_0 \dot{+} W_0$.

Theorem 3.9. *$V_1 = V_0 \oplus W_0$ if and only if*

$$\sum_{j=1}^{2^d} m_0\left(\frac{x+q_j}{2}\right)\overline{m_i}\left(\frac{x+q_j}{2}\right)\|a_{x,q_j}\|^2_{\ell^2(\mathbb{Z}^d)} = 0 \tag{3.20}$$

for a.e. x and for each $i = 1, 2, \ldots, n$.

Proof. Recall that $V_1 = \overline{V_0 + W_0}$ by Lemma 3.7. If $V_0 \perp W_0$, then $V_0 + W_0$ is closed. Therefore, $V_1 = V_0 \oplus W_0$. On the other hand, If $V_1 = V_0 \oplus W_0$, then obviously $V_0 \perp W_0$. It is easy to see that $V_0 \perp W_0$ if and only if $\hat{\varphi}_{\|x} \perp \hat{\psi}_{i\,\|x}$ for each $i = 1, 2, \ldots, n$ and for a.e. $x \in \mathbb{T}^d$ [1]. The orthogonality of $\{a_{x,q_j} : 1 \le j \le 2^d\}$ and Eqs. (3.6) and (3.7) imply that $\hat{\varphi}_{\|x} \perp \hat{\psi}_{i\,\|x}$ if and only if (3.20) holds. □

4. Proof of Theorem 1.2

In this section we apply the results in Section 3 to give a proof of Theorem 1.2. Note that we only deal with the univariate case in this section.

We first prove the first part of Theorem 1.2. Let m_0 and m_1 be trigonometric filters such that

$$M(x) := \begin{pmatrix} m_0(x) & m_0\left(x+\frac{1}{2}\right) \\ m_1(x) & m_1\left(x+\frac{1}{2}\right) \end{pmatrix}$$

is a unitary matrix for each $x \in \mathbb{T}$. Note that $m_0(0) = 1$ by (1.1) and (1.2). If we define φ, as usual, via $\hat{\varphi}(x) := \prod_{j=1}^{\infty} m_0(2^{-j}x)$ for $x \in \mathbb{R}$, then φ is a compactly supported refinable function in $L^2(\mathbb{R})$ [9, Section 6.2]. In particular, $\hat{\varphi}$ is an entire function of exponential type by a theorem of Paley and Wiener. Thus, φ satisfies the following conditions:

$$\hat{\varphi}(2x) = m_0(x)\hat{\varphi}(x), \quad x \in \mathbb{R};$$
$$\hat{\varphi}(0) = 1;$$
$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+k)|^2 \quad \text{is a trigonometric polynomial.}$$

In particular, $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+k)|^2$ has at most a finite number of zeros since it is not identically zero. If we define $\psi \in L^2(\mathbb{R})$ via $\hat{\psi}(2x) := m_1(x)\hat{\varphi}(x)$, then $\{D^j T_k \psi : j, k \in \mathbb{Z}\}$ is a wavelet frame for $L^2(\mathbb{R})$ by the UEP (Proposition 1.1). Moreover, ψ is also compactly supported, and $\sum_{k\in\mathbb{Z}} |\hat{\psi}(x+k)|^2$ is also a trigonometric polynomial [9, Section 6.2]. Define, as usual, $V_0 := \mathcal{S}(\{\varphi\})$, $W_0 := \mathcal{S}(\{\psi\})$, $V_j := D^j(V_0)$, and $W_j := D^j(W_0)$ for $j \in \mathbb{Z}$. We then have the following direct sum decomposition of V_1. Surprisingly enough, this simple fact has a rather long proof throughout this subsection.

Proposition 4.1. $V_1 = V_0 \dot{+} W_0$. *In particular,* $V_0 + W_0$ *is closed.*

For the convenience of computation we adopt the following notations. Let $F(x) := \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+k)|^2$, $P(x) := \sum_{k\in\mathbb{Z}} |\hat{\psi}(x+k)|^2$, $\mu_0(x) := |m_0(x)|^2$ and $\mu_1(x) := |m_1(x)|^2$. Note that F, P, μ_0 and μ_1 are all non-negative trigonometric polynomials. Finally, if $f(x) = 0$, we let $\operatorname{ord}(f, x)$ denote the order of the zero at x of f.

The following lemma holds since the matrix $M(x)$ is unitary for each $x \in \mathbb{T}$.

Lemma 4.2. *For each* $x \in \mathbb{T}$ *the following hold:*

(1) $\mu_0(x) + \mu_0\left(x + \frac{1}{2}\right) = 1$;
(2) $m_0(x)\overline{m_1}(x) + m_0\left(x + \frac{1}{2}\right)\overline{m_1}\left(x + \frac{1}{2}\right) = 0$;
(3) $\mu_1(x) + \mu_1\left(x + \frac{1}{2}\right) = 1$;
(4) $\mu_0(x) + \mu_1(x) = 1$.

Note that $\sigma(V_0) = \mathbb{T}$. Hence $\sigma(V_1) = \mathbb{T}$ by Proposition 3.1. If we assume that $V_0 + W_0$ is closed, then $V_1 = V_0 \dot{+} W_0$ by Theorem 3.8 since $|m_0(x)|^2 + |m_0(x + \frac{1}{2})|^2 = 1$ a.e. Hence, we only need to show that $V_0 + W_0$ is closed in order to complete the proof of Proposition 4.1. The proof is rather long and technical.

Lemma 4.3. *Suppose that* $F(x_0) = 0$ *for some* $x_0 \in \mathbb{T}$. *Then,*

(1) $F\left(x_0 + \frac{1}{2}\right) \neq 0$ *and* $\mu_0\left(x_0 + \frac{1}{2}\right) = 0$;
(2) *There exists a non-trivial cycle* $\{x_0, x_1, \ldots, x_n\} \subset \mathbb{T}$ *of zeros of* F *such that* $x_j = 2x_{j+1} \pmod 1$, $0 \le j \le n$, *where* $x_{n+1} := x_0$;
(3) $\operatorname{ord}(F, x_k) \le \operatorname{ord}\left(\mu_0\left(\cdot + \frac{1}{2}\right), x_{k+1}\right)$, $0 \le k \le n$;
(4) $\operatorname{ord}(F, x_k)$ *is the same for all* $k = 0, 1, \ldots, n$;
(5) $\operatorname{ord}(F, x_k) \le \operatorname{ord}\left(\mu_0\left(\cdot + \frac{1}{2}\right), x_k\right)$.

Proof. By a standard argument we note that

$$F(x) = \mu_0\left(\frac{x}{2}\right) F\left(\frac{x}{2}\right) + \mu_0\left(\frac{x}{2} + \frac{1}{2}\right) F\left(\frac{x}{2} + \frac{1}{2}\right). \tag{4.1}$$

Then (1) and (2) follow if we apply Steps 2, 3, 4 and 5 in the proof of Theorem 6.3.5 in [9] to (4.1). Suppose that $F(x_k) = 0$. Since $x_k = 2x_{k+1} \pmod 1$ and F is 1-periodic, we have

$$0 = F(x_k) = F(2x_{k+1}) = \mu_0(x_{k+1})F(x_{k+1}) + \mu_0\left(x_{k+1} + \frac{1}{2}\right) F\left(x_{k+1} + \frac{1}{2}\right). \tag{4.2}$$

Moreover, (1) implies that $F\left(x_{k+1} + \frac{1}{2}\right) \neq 0$ and $\mu_0\left(x_{k+1} + \frac{1}{2}\right) = 0$ since $F(x_{k+1}) = 0$. Thus $\mu_0(x_{k+1}) = 1 \neq 0$ by Lemma 4.2 (1). From (4.2) we have, for $k = 0, 1, \ldots, n$,

$$\operatorname{ord}(F, x_k) = \min\left\{\operatorname{ord}(F, x_{k+1}), \operatorname{ord}\left(\mu_0\left(\cdot + \frac{1}{2}\right), x_{k+1}\right)\right\}. \tag{4.3}$$

Therefore (3) holds. (4.3) also implies that

$$\operatorname{ord}(F, x_0) \le \operatorname{ord}(F, x_1) \le \cdots \le \operatorname{ord}(F, x_n) \le \operatorname{ord}(F, x_0),$$

which proves (4). Finally, (3) and (4) imply (5). $\square$

Lemma 4.4. *$\{T_k\psi : k \in \mathbb{Z}\}$ is a Riesz basis for W_0. In particular, $P(x) \neq 0$ for each $x \in \mathbb{T}$.*

Proof. By a standard result (see [9, Equation (5.3.2)] for example) it suffices to show that there exist positive constants A and B such that $A \leq P(x) \leq B$ a.e. $x \in \mathbb{T}$ in order to prove that $\{T_k\psi : k \in \mathbb{Z}\}$ is a Riesz basis for W_0. Since P is a non-negative trigonometric polynomial, we only need to show that P has no zero. Suppose, on the contrary, that $P(x) = 0$ for some $x \in \mathbb{T}$. Again, by a standard argument, we have

$$0 = P(x) = \mu_1\left(\frac{x}{2}\right) F\left(\frac{x}{2}\right) + \mu_1\left(\frac{x}{2} + \frac{1}{2}\right) F\left(\frac{x}{2} + \frac{1}{2}\right).$$

$\mu_1\left(\frac{x}{2}\right) F\left(\frac{x}{2}\right) = 0$ and $\mu_1\left(\frac{x}{2} + \frac{1}{2}\right) F\left(\frac{x}{2} + \frac{1}{2}\right) = 0$ since μ_1 and F are non-negative trigonometric polynomials,. Suppose that $F\left(\frac{x}{2}\right) = 0$. Then, by Lemma 4.3(1), $F\left(\frac{x}{2} + \frac{1}{2}\right) \neq 0$ and $\mu_0\left(\frac{x}{2} + \frac{1}{2}\right) = 0$. Lemma 4.2(4) implies that $\mu_1\left(\frac{x}{2} + \frac{1}{2}\right) = 1$. Therefore, $\mu_1\left(\frac{x}{2} + \frac{1}{2}\right) F\left(\frac{x}{2} + \frac{1}{2}\right) \neq 0$, which is a contradiction. Suppose, on the other hand, that $\mu_1\left(\frac{x}{2}\right) = 0$. Then $\mu_1\left(\frac{x}{2} + \frac{1}{2}\right) = 1 \neq 0$ by Lemma 4.2 (3). This forces that $F\left(\frac{x}{2} + \frac{1}{2}\right) = 0$. Thus, $\mu_0\left(\frac{x}{2}\right) = 0$ by Lemma 4.3(1). Lemma 4.2(4) now implies that $\mu_1\left(\frac{x}{2}\right) = 1 \neq 0$, which is a contradiction. These contradictions show that P has no zeros, which completes the proof. $\square$

We now give a proof of the first part of Theorem 1.2. As noted before it remains to show that $V_0 + W_0$ is closed. We use Proposition 2.3. The Cauchy–Schwarz inequality implies that the 1-periodic trigonometric rational function

$$f(x) := \frac{\left|\left\langle \hat{\varphi}_{\|x}, \hat{\psi}_{\|x} \right\rangle\right|^2}{\|\hat{\varphi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} \|\hat{\psi}_{\|x}\|^2_{\ell^2(\mathbb{Z})}}$$

has a removable singularity at the point where the denominator vanishes. Hence it defines a continuous function on the compact set $\mathbb{T}$. If we show that at each point $x \in \mathbb{T}$ the value of the periodic function $f(x)$ is strictly less than 1, then the proof will be complete. Recall that $Q = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ since we are working in $L^2(\mathbb{R})$. For $x \in \mathbb{T}$, define $a_{x,0}$ and $a_{x,1}$ as in (3.4), i.e.,

$$\begin{cases} a_{x,0}(2k) = \hat{\varphi}\left(\dfrac{x+2k}{2}\right); \\ a_{x,0}(2k+1) = 0; \\ a_{x,1}(2k+1) = \hat{\varphi}\left(\dfrac{x+2k+1}{2}\right); \\ a_{x,1}(2k) = 0. \end{cases} \tag{4.4}$$

Recall that

$$\hat{\varphi}_{\|x} = m_0\left(\frac{x}{2}\right) a_{x,0} + m_0\left(\frac{x}{2} + \frac{1}{2}\right) a_{x,1},$$

$$\hat{\psi}_{\|x} = m_1\left(\frac{x}{2}\right) a_{x,0} + m_1\left(\frac{x}{2} + \frac{1}{2}\right) a_{x,1},$$

$\|a_{x,0}\|^2_{\ell^2(\mathbb{Z})} = F\left(\frac{x}{2}\right)$, and also that $\|a_{x,1}\|^2_{\ell^2(\mathbb{Z})} = F\left(\frac{x}{2}+\frac{1}{2}\right)$. The orthogonality of $\{a_{x,0}, a_{x,1}\}$ and Lemma 4.2(1) imply that

$$\begin{aligned}\|\hat{\varphi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} &= \mu_0\left(\frac{x}{2}\right)F\left(\frac{x}{2}\right)+\mu_0\left(\frac{x+1}{2}\right)F\left(\frac{x+1}{2}\right)\\ &= \left(1-\mu_0\left(\frac{x+1}{2}\right)\right)F\left(\frac{x}{2}\right)+\mu_0\left(\frac{x+1}{2}\right)F\left(\frac{x+1}{2}\right)\\ &= \mu_0\left(\frac{x+1}{2}\right)\left(F\left(\frac{x+1}{2}\right)-F\left(\frac{x}{2}\right)\right)+F\left(\frac{x}{2}\right). \qquad (4.5)\end{aligned}$$

By symmetry,

$$\|\hat{\varphi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} = \mu_0\left(\frac{x}{2}\right)\left(F\left(\frac{x}{2}\right)-F\left(\frac{x+1}{2}\right)\right)+F\left(\frac{x+1}{2}\right). \qquad (4.6)$$

Similarly,

$$\begin{aligned}\|\hat{\psi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} &= \mu_1\left(\frac{x+1}{2}\right)\left(F\left(\frac{x+1}{2}\right)-F\left(\frac{x}{2}\right)\right)+F\left(\frac{x}{2}\right) \qquad (4.7)\\ &= \mu_1\left(\frac{x}{2}\right)\left(F\left(\frac{x}{2}\right)-F\left(\frac{x+1}{2}\right)\right)+F\left(\frac{x+1}{2}\right). \qquad (4.8)\end{aligned}$$

Likewise, the orthogonality of $\{a_{x,0}, a_{x,1}\}$ and Lemma 4.2(2) imply that

$$\begin{aligned}\left|\left\langle\hat{\varphi}_{\|x},\hat{\psi}_{\|x}\right\rangle\right|^2 &= \left|m_0\left(\frac{x}{2}\right)\overline{m_1}\left(\frac{x}{2}\right)F\left(\frac{x}{2}\right)+m_0\left(\frac{x+1}{2}\right)\overline{m_1}\left(\frac{x+1}{2}\right)F\left(\frac{x+1}{2}\right)\right|^2\\ &= \left|-m_0\left(\frac{x+1}{2}\right)\overline{m_1}\left(\frac{x+1}{2}\right)F\left(\frac{x}{2}\right)+m_0\left(\frac{x+1}{2}\right)\overline{m_1}\left(\frac{x+1}{2}\right)F\left(\frac{x+1}{2}\right)\right|^2\\ &= \mu_0\left(\frac{x+1}{2}\right)\mu_1\left(\frac{x+1}{2}\right)\left|F\left(\frac{x+1}{2}\right)-F\left(\frac{x}{2}\right)\right|^2. \qquad (4.9)\end{aligned}$$

By symmetry,

$$\left|\left\langle\hat{\varphi}_{\|x},\hat{\psi}_{\|x}\right\rangle\right|^2 = \mu_0\left(\frac{x}{2}\right)\mu_1\left(\frac{x}{2}\right)\left|F\left(\frac{x+1}{2}\right)-F\left(\frac{x}{2}\right)\right|^2. \qquad (4.10)$$

Now let x_0 be any element of $\mathbb{T}$. Since f has at most a finite number of removable singularities, $f(x_0) = \lim_{x\to x_0} f(x)$.

We divide the problem into two cases. Let us define, temporarily, $a(x) := F(\frac{x+1}{2}) - F(\frac{x}{2})$. Suppose that $a(x_0) \geq 0$. In this case, we again consider two subcases. First, suppose that $F(\frac{x_0}{2}) \neq 0$. Then, since F, μ_0, μ_1 and a have at most a finite number of zeros, by using (4.5), (4.7) and (4.9), we have

$$\begin{aligned}f(x_0) &= \lim_{x\to x_0} f(x)\\ &= \lim_{x\to x_0}\frac{\mu_0\left(\frac{x+1}{2}\right)\mu_1\left(\frac{x+1}{2}\right)a(x)^2}{\mu_0\left(\frac{x+1}{2}\right)\mu_1\left(\frac{x+1}{2}\right)a(x)^2+\mu_0\left(\frac{x+1}{2}\right)F\left(\frac{x}{2}\right)a(x)+\mu_1\left(\frac{x+1}{2}\right)F\left(\frac{x}{2}\right)a(x)+F(\frac{x}{2})^2}\end{aligned}$$

$$= \frac{\mu_0\left(\frac{x_0+1}{2}\right)\mu_1\left(\frac{x_0+1}{2}\right)a(x_0)^2}{\mu_0\left(\frac{x_0+1}{2}\right)\mu_1\left(\frac{x_0+1}{2}\right)a(x_0)^2+\mu_0\left(\frac{x_0+1}{2}\right)F\left(\frac{x_0}{2}\right)a(x_0)+\mu_1\left(\frac{x_0+1}{2}\right)F\left(\frac{x_0}{2}\right)a(x_0)+F(\frac{x_0}{2})^2}$$
$$< 1.$$

The last inequality follows since all terms in the numerator and the denominator are non-negative and $F(\frac{x_0}{2})$ is assumed to be positive. Suppose, on the other hand, that $F\left(\frac{x_0}{2}\right) = 0$. Then, $F\left(\frac{x_0+1}{2}\right) \neq 0$, $\mu_0\left(\frac{x_0+1}{2}\right) = 0$ by Lemma 4.3(1), $\mu_0\left(\frac{x_0}{2}\right) = 1$ by Lemma 4.2(1), and $\mu_1\left(\frac{x_0+1}{2}\right) = 1$ by Lemma 4.2(4). Moreover, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \lim_{x\to x_0} \frac{\mu_0\left(\frac{x+1}{2}\right)}{F\left(\frac{x}{2}\right)}$$

by Lemma 4.3(5). In particular, $a(x_0) > 0$. Since F, μ_0, μ_1 and a have at most a finite number of zeros, by using (4.5), (4.7) and (4.9), we have

$$\begin{aligned}
f(x_0) &= \lim_{x\to x_0} f(x)\\
&= \lim_{x\to x_0} \frac{\mu_0\left(\frac{x+1}{2}\right)\mu_1\left(\frac{x+1}{2}\right)a(x)^2}{\mu_0\left(\frac{x+1}{2}\right)\mu_1\left(\frac{x+1}{2}\right)a(x)^2+\mu_0\left(\frac{x+1}{2}\right)F\left(\frac{x}{2}\right)a(x)+\mu_1\left(\frac{x+1}{2}\right)F\left(\frac{x}{2}\right)a(x)+F(\frac{x}{2})^2}\\
&= \lim_{x\to x_0} \frac{\frac{\mu_0\left(\frac{x+1}{2}\right)}{F(\frac{x}{2})}\mu_1\left(\frac{x+1}{2}\right)a(x)^2}{\frac{\mu_0\left(\frac{x+1}{2}\right)}{F(\frac{x}{2})}\mu_1\left(\frac{x+1}{2}\right)a(x)^2+\mu_0\left(\frac{x+1}{2}\right)a(x)+\mu_1\left(\frac{x+1}{2}\right)a(x)+F(\frac{x}{2})}\\
&= \frac{\alpha\mu_1\left(\frac{x_0+1}{2}\right)a(x_0)^2}{\alpha\mu_1\left(\frac{x_0+1}{2}\right)a(x_0)^2+\mu_0\left(\frac{x_0+1}{2}\right)a(x_0)+\mu_1\left(\frac{x_0+1}{2}\right)a(x_0)+F(\frac{x_0}{2})}\\
&= \frac{\alpha a(x_0)^2}{\alpha a(x_0)^2+a(x_0)} < 1.
\end{aligned}$$

The last inequality follows since all terms involved are non-negative and $a(x_0) > 0$.

The case that $F(\frac{x_0+1}{2}) - F(\frac{x_0}{2}) \le 0$ can be handled similarly by using (4.6), (4.8) and (4.10). This completes the proof of the first part of Theorem 1.2.

We now prove the latter part of Theorem 1.2. Suppose we construct a wavelet tight frame for $L^2(\mathbb{R})$ by using $n+1$ trigonometric filters with $n >$. Recall that $Q = \{0, 1\}$ since we are now working in $L^2(\mathbb{R})$. Note that $|m_0(x)|^2 + |m_0\left(x+\frac{1}{2}\right)|^2 - 1$ is a trigonometric polynomial. Hence either it is identically zero or it has only finitely many zeros. Therefore, either $|m_0(x)|^2 + |m_0\left(x+\frac{1}{2}\right)|^2 = 1$ for each $x \in \mathbb{T}$ or $|m_0(x)|^2 + |m_0\left(x+\frac{1}{2}\right)|^2 \neq 1$ a.e. Suppose that $|m_0(x)|^2 + |m_0\left(x+\frac{1}{2}\right)|^2 = 1$ for each $x \in \mathbb{T}$. Then Lemma 3.3(3) implies that $\operatorname{rank} M_1(x) = 1$ for a.e. $x \in \sigma(V_0)$. Since $M_1(x)$ is an $n \times 2$ matrix and $|\sigma(V_0)| > 0$, the linear dependence condition in Theorem 1.2 holds on a set of positive measure. Since m_i's are trigonometric polynomials, the linear dependence condition holds on $\mathbb{T}$. Suppose, on the other hand, that $|m_0(x)|^2 + |m_0\left(x+\frac{1}{2}\right)|^2 \neq 1$ a.e. Then $V_1 = W_0$ by Corollary 3.5. In particular, if $m_0(x) = \cos^{2k}\pi x$, then $\cos^{4k}(\frac{\pi x}{2}) + \cos^{4k}(\frac{\pi x}{2}+\frac{\pi}{2}) < 1$ a.e. $x \in \mathbb{T}$. Therefore, $V_1 = W_0$ by Corollary 3.5.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Recall that $\psi_0 := \varphi$ is defined in (1.8) and $\hat{\psi}_l(2x) := m_l(x)\hat{\varphi}(x)$ for $l = 1, 2, \ldots, n := 2k$, where the filters m_l's are defined in (1.7). Since φ is compactly supported and the filters are trigonometric polynomials, ψ_l is also compactly supported for each l. Moreover, $\hat{\varphi}_{\|x}$ is defined for each $x \in \mathbb{T}$ and $\hat{\varphi}_{\|x} \neq 0$ for each $x \in \mathbb{T}$ [6, Theorem 4.5]. Define $F(x) := \|\hat{\varphi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+k)|^2$. A well-known result shows that F is a trigonometric polynomial, and so is $\|\hat{\psi}_{l\,\|x}\|^2_{\ell^2(\mathbb{Z})}$ for each $l = 1, 2, \ldots, 2k$. Hence F is strictly positive and continuous, and $\|\hat{\psi}_{l\,\|x}\|^2_{\ell^2(\mathbb{Z})}$ is non-negative and continuous on the compact set $\mathbb{T}$. Recall that we define $W_0^{(l)} := \mathcal{S}(\{\psi_l\})$, $V_0 := W_0^{(0)}$, $W_0 := \mathcal{S}(\{\psi_l : l = 1, \ldots, 2k\})$, and $V_1 = D(V_0)$, where D is now the 1-dimensional dyadic dilation operator. Hence $\sigma(V_0) = \mathbb{T}$, and Theorem 1.2 implies that $W_0 = V_1$. Recall also that $Q = \{0, 1\}$ since we are working in $L^2(\mathbb{R})$ and that $a_{x,0}$ and $a_{x,1}$ are defined in (4.4).

In particular $a_{x,0} \perp a_{x,1}$ and $(\hat{\varphi}(\frac{x+k}{2}))_{k\in\mathbb{Z}} = a_{x,0} \oplus a_{x,1}$. Since $\hat{\psi}_l(x) = m_l(x/2)\hat{\varphi}(x/2)$, $\hat{\psi}_{l\,\|x} = m_l(x/2)a_{x,0} \oplus m_l(x/2+1/2)a_{x,1}$ for each $l = 0, 1, \ldots, 2k$. (3.9) implies that $\hat{V}_{1\,\|x} = \operatorname{span}\{a_{x,0}, a_{x,1}\}$, and (3.5) implies that $\dim \hat{V}_{1\,\|x} = 2$ for each $x \in \mathbb{T}$. For $0 \le j < i \le 2k$ define

$$M_{ij}(x) := \begin{pmatrix} m_i(x) & m_i\left(x+\frac{1}{2}\right) \\ m_j(x) & m_j\left(x+\frac{1}{2}\right) \end{pmatrix}$$
$$= \begin{pmatrix} c_i^{1/2}\sin^i\left(\frac{\pi x}{2}\right)\cos^{2k-i}\left(\frac{\pi x}{2}\right) & c_i^{1/2}\cos^i\left(\frac{\pi x}{2}\right)(-1)^{2k-i}\sin^{2k-i}\left(\frac{\pi x}{2}\right) \\ c_j^{1/2}\sin^j\left(\frac{\pi x}{2}\right)\cos^{2k-j}\left(\frac{\pi x}{2}\right) & c_j^{1/2}\cos^j\left(\frac{\pi x}{2}\right)(-1)^{2k-j}\sin^{2k-j}\left(\frac{\pi x}{2}\right) \end{pmatrix},$$

where $c_i := \binom{2k}{i}$ and $c_j := \binom{2k}{j}$. Then we have

$$\begin{pmatrix} \hat{\psi}_{i\,\|x} \\ \hat{\psi}_{j\,\|x} \end{pmatrix} = M_{ij}\left(\frac{x}{2}\right)\begin{pmatrix} a_{x,0} \\ a_{x,1} \end{pmatrix}. \tag{5.1}$$

Now,

$$\det M_{ij}\left(\frac{x}{2}\right) = c_i^{1/2}c_j^{1/2}\left(\sin^{2k-i+j}\left(\frac{\pi x}{2}\right)\cos^{2k-i+j}\left(\frac{\pi x}{2}\right)(-1)^{2k-i}\right)$$
$$\times\left((-1)^{i-j}\sin^{2(i-j)}\left(\frac{\pi x}{2}\right) - \cos^{2(i-j)}\left(\frac{\pi x}{2}\right)\right).$$

Let us identify $\mathbb{T}$ with $[-1/2, 1/2]$. Then, $\det M_{ij}(x/2) = 0$ if and only if

$$\begin{cases} x = \pm 1/2 \text{ or } 0, & \text{if } i-j \text{ is even and } i-j \neq 2k, \\ x = \pm 1/2, & \text{if } i-j = 2k, \\ x = 0, & \text{if } i-j \text{ is odd.} \end{cases} \tag{5.2}$$

In particular, $\det M_{ij}(x/2) \neq 0$ a.e., and hence $\operatorname{rank} M(x/2) = 2$ a.e. This shows that $\dim \operatorname{span}\{\hat{\psi}_{i\,\|x}, \hat{\psi}_{j\,\|x}\} = 2$ a.e. Recall that $\operatorname{span}\{\hat{\psi}_{i\,\|x}, \hat{\psi}_{j\,\|x}\} \subset \hat{V}_{1\,\|x}$ and that $\dim \hat{V}_{1\,\|x} = 2$ for each $x \in \mathbb{T}$. Since $\mathcal{S}(\{\psi_i, \psi_j\}) = \overline{W_0^{(i)} + W_0^{(j)}}$, $\overline{W_0^{(i)} + W_0^{(j)}} = V_1$ by Corollary 2.2. The determinant condition also implies that $\widehat{W_0^{(i)}}_{\|x} \cap \widehat{W_0^{(j)}}_{\|x} = \{0\}$ a.e. Hence $W_0^{(i)} \cap W_0^{(j)} = \{0\}$ by Lemma 2.4. This shows that, for $0 \le j < i \le 2k$, $W_0^{(i)} \dotplus W_0^{(j)} = V_1$ if and only if $W_0^{(i)} + W_0^{(j)}$ is closed. Therefore, we only need to show that $W_0^{(i)} + W_0^{(j)}$ is closed if and only if $i - j$ is odd and $j \le k \le i$ in order to complete the proof of Theorem 1.3.

Using (5.1) we have

$$\begin{aligned} \left|\left\langle \hat{\psi}_{i\,\|x}, \hat{\psi}_{j\,\|x}\right\rangle_{\ell^2(\mathbb{Z})}\right|^2 &= c_i c_j \left| \sin^{i+j}\left(\frac{\pi x}{2}\right)\cos^{4k-i-j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right)\right. \\ &\quad \left. + (-1)^{4k-i-j}\cos^{i+j}\left(\frac{\pi x}{2}\right)\sin^{4k-i-j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right)\right|^2, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \|\hat{\psi}_{i\,\|x}\|^2_{\ell^2(\mathbb{Z})} &= c_i \sin^{2i}\left(\frac{\pi x}{2}\right)\cos^{4k-2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) \\ &\quad + c_i \cos^{2i}\left(\frac{\pi x}{2}\right)\sin^{4k-2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right), \end{aligned} \tag{5.4}$$

$$\begin{aligned} \|\hat{\psi}_{j\,\|x}\|^2_{\ell^2(\mathbb{Z})} &= c_j \sin^{2j}\left(\frac{\pi x}{2}\right)\cos^{4k-2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) \\ &\quad + c_j \cos^{2j}\left(\frac{\pi x}{2}\right)\sin^{4k-2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right). \end{aligned} \tag{5.5}$$

For $0 \le j < i \le 2k$, define

$$E_{ij} := \left\{x \in \sigma\left(W_0^{(i)}\right) \cap \sigma\left(W_0^{(i)}\right) : \hat{\psi}_{i\,\|x} \text{ and } \hat{\psi}_{j\,\|x} \text{ are linearly independent}\right\},$$

and

$$g_{ij}(x) := \frac{\left|\left\langle \hat{\psi}_{i\,\|x}, \hat{\psi}_{j\,\|x}\right\rangle_{\ell^2(\mathbb{Z})}\right|^2}{\|\hat{\psi}_{i\,\|x}\|^2_{\ell^2(\mathbb{Z})}\|\hat{\psi}_{j\,\|x}\|^2_{\ell^2(\mathbb{Z})}}.$$

Recall that $F(x) > 0$ for each $x \in \mathbb{T}$. Therefore, (5.4) or (5.5) implies that $\sigma(W_0^{(l)}) = \mathbb{T}$ if $l = 0$ or $l = 2k$ and $\sigma(W_0^{(i)}) = \mathbb{T} \setminus \{0\}$ if $0 < l < 2k$. Hence we have $E_{ij} = \mathbb{T} \setminus \{x \in \mathbb{T} : \det M_{ij}(x/2) = 0\}$. Note that, by (5.1), neither $\hat{\psi}_{i\,\|x}$ nor $\hat{\psi}_{j\,\|x}$ is 0 if $x \in E_{ij}$ since $a_{x,0} \ne 0$, $a_{x,1} \ne 0$ and $a_{x,0} \perp a_{x,1}$ for each $x \in \mathbb{T}$. We note that the numerator and both terms in the denominator of g_{ij} are trigonometric polynomials. The Cauchy–Schwarz inequality implies that g_{ij} has a removable singularity at the points where the denominator vanishes. In particular, g_{ij} defines a continuous function on the compact set $\mathbb{T}$.

Now, suppose that $0 \le j < i \le 2k$ and $i - j$ is even. (5.2) implies that $\hat{\psi}_{i\,\|x}$ and $\hat{\psi}_{j\,\|x}$ are linearly dependent at $x = \pm 1/2$. Moreover, (5.4) and (5.5) imply that neither $\hat{\psi}_{i\,\|x}$ nor $\hat{\psi}_{j\,\|x}$ is 0 at $x = \pm 1/2$ since $F(x) > 0$ for each $x \in \mathbb{T}$. Hence $g_{ij}(\pm 1/2) = 1$ by the equality condition of the Cauchy–Schwarz inequality. Since g_{ij} is continuous, $\lim_{x \to \pm 1/2} g_{ij}(x) = 1$. Hence $\text{ess-sup}_{x \in E_{ij}} g(x) = 1$. Proposition 2.3 implies that $W_0^{(i)} + W_0^{(j)}$ is not closed.

Suppose, on the other hand, that $0 \le j < i \le 2k$ and $i - j$ is odd. (5.2) implies that $E_{ij} = \mathbb{T} \setminus \{0\}$ and $g_{ij}(x) < 1$ for $x \in E_{ij}$ by the equality condition of the Cauchy–Schwarz inequality. Since g_{ij} is continuous on the compact set $\mathbb{T}$, $W_0^{(i)} + W_0^{(j)}$ is closed if and only if $\lim_{x \to 0} g_{ij}(x) < 1$ by Proposition 2.3. We divide this into two subcases: $i + j < 2k$ and $i + j > 2k$. ($i + j$ cannot be $2k$ since $i - j$ is assumed to be odd.)

We first consider the case: $j < i$, $i - j$ is odd, and $i + j < 2k$. In particular, $i + j < 4k - i - j$. Hence the right-hand side of (5.3) becomes

$$\alpha := c_i c_j \sin^{2(i+j)}\left(\frac{\pi x}{2}\right) \left| \cos^{4k-i-j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) - \cos^{i+j}\left(\frac{\pi x}{2}\right) \sin^{4k-2(i+j)}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right|^2.$$

If, moreover, $2i < 4k - 2i$ and $2j < 4k - 2j$, then the right-hand side of (5.4) becomes

$$\beta := c_i \sin^{2i}\left(\frac{\pi x}{2}\right) \left(\cos^{4k-2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) + \cos^{2i}\left(\frac{\pi x}{2}\right) \sin^{4k-4i}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right),$$

and that of (5.5) becomes

$$\gamma := c_j \sin^{2j}\left(\frac{\pi x}{2}\right) \left(\cos^{4k-2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) + \cos^{2j}\left(\frac{\pi x}{2}\right) \sin^{4k-4j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right).$$

Hence

$$\lim_{x\to 0} g_{ij}(x) = \lim_{x\to 0} \frac{\alpha}{\beta\gamma} = \frac{F(0)^2}{F(0)^2} = 1,$$

since $2k-2j > 0$, $2k-2i > 0$ and $F(0) > 0$. Hence $W_0^{(i)} + W_0^{(j)}$ is not closed by Proposition 2.3.

If, $2i \geq 4k - 2i$ and $2j < 4k - 2j$, then the right-hand side of (5.4) becomes

$$\delta := c_i \sin^{4k-2i}\left(\frac{\pi x}{2}\right) \left(\sin^{4i-4k}\left(\frac{\pi x}{2}\right) \cos^{4k-2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) + \cos^{2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right).$$

Then, $\alpha/(\delta\gamma) = \sin^{4i-4k}\left(\frac{\pi x}{2}\right) \times (*)$ if $2i > 4k - 2i$ and $2j < 4k - 2j$, where $(*)$ denotes a quantity that is irrelevant in our calculation. In this case $\lim_{x\to 0} \alpha/(\delta\gamma) = 0$. On the other hand, if $2i = 4k - 2i$ and $2j < 4k - 2j$, then $\alpha/(\delta\gamma)$ becomes

$$\frac{\left|\cos^{4k-i-j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) - \sin^{4k-2(i+j)}\left(\frac{\pi x}{2}\right) \times (*)\right|^2}{\cos^{2i}\left(\frac{x}{2}\right)\left(F\left(\frac{x}{2}\right) + F\left(\frac{x+1}{2}\right)\right)\left(\cos^{4k-2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) + \sin^{4k-4j}\left(\frac{\pi x}{2}\right) \times (*)\right)}.$$

As x approaches 0 the above quantity becomes $F(0)/(F(0) + F(1/2)) < 1$ since $F(0) > 0$ and $F(1/2) > 0$. Hence

$$\lim_{x\to 0} g_{ij}(x) = \lim_{x\to 0} \frac{\alpha}{\delta\gamma} = \begin{cases} 0, & \text{if } 2i > 4k - 2i, \quad 2j < 4k - 2j, \\ \dfrac{F(0)}{F(0) + F(1/2)} < 1, & \text{if } 2i = 4k - 2i, 2j < 4k - 2j. \end{cases}$$

If $2i \geq 4k - 2i$ and $2j \geq 4k - 2j$, then $i + j \geq 2k$, which is impossible since we assumed that $i + j < 2k$. We have just shown that, under the assumptions that $j < i$, $i - j$ is odd, and $i + j < 2k$, $W^{(j)} + W^{(i)}$ is closed if and only if $j < k \leq i$.

We now consider the case that $j < i$, $i - j$ is odd, and $i + j > 2k$. In particular, $i + j > 4k - i - j$. Then, the right-hand side of (5.3) becomes

$$\zeta := c_i c_j \sin^{2(4k-i-j)}\left(\frac{\pi x}{2}\right) \left| \sin^{2i+2j-4k}\left(\frac{\pi x}{2}\right) \cos^{4k-i-j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) - \cos^{i+j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right|^2.$$

If $2i > 4k - 2i$ and $2j > 4k - 2j$, then the right-hand side of (5.4) becomes δ and that of (5.5) becomes

$$\eta := c_j \sin^{4k-2j}\left(\frac{\pi x}{2}\right) \left(\sin^{4j-4k}\left(\frac{\pi x}{2}\right) \cos^{4k-2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x}{2}\right) + \cos^{2j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right).$$

Hence

$$\lim_{x \to 0} g_{ij}(x) = \lim_{x \to 0} \frac{\zeta}{\delta \eta} = \frac{F(1/2)^2}{F(1/2)^2} = 1$$

since $F(1/2) > 0$.

On the other hand, if $2i > 4k - 2i$ and $2j < 4k - 2j$, then $g_{ij}(x) = \zeta/(\delta\gamma) = \sin^{4k-4j} \times (*)$. Hence $\lim_{x \to 0} g_{ij}(x) = 0$.

If $2i > 4k - 2i$ and $2j = 4k - 2j$, then $g_{ij}(x) = \zeta/(\delta\gamma)$ and it becomes

$$\frac{\left| \sin^{2i-2k}\left(\frac{\pi x}{2}\right) \times (*) - \cos^{i+j}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right|^2}{\left(\sin^{4i-4k}\left(\frac{\pi x}{2}\right) \times (*) + \cos^{2i}\left(\frac{\pi x}{2}\right) F\left(\frac{x+1}{2}\right) \right) \cos^{2j}\left(\frac{\pi x}{2}\right) \left(F\left(\frac{x}{2}\right) + F\left(\frac{x+1}{2}\right) \right)}.$$

Hence

$$\lim_{x \to 0} g_{ij}(x) = \frac{F(1/2)}{F(0) + F(1/2)} < 1$$

since $F(0) > 0$ and $F(1/2) > 0$. Finally, if $2i \leq 4k - 2i$ and $2j \leq 4k - 2j$ then $i + j \leq 2k$, which is impossible since we assumed that $i + j > 2k$. We have just shown that, under the assumptions that $j < i$, $i - j$ is odd, and $i + j > 2k$, $W^{(j)} + W^{(i)}$ is closed if and only if $j \leq k < i$. If we combine this fact with that we proved earlier, we have: If $j < i$, then $W^{(j)} + W^{(i)}$ is closed, and hence $V_1 = W^{(j)} \dot{+} W^{(i)}$, if and only if $i - j$ is odd and $j \leq k \leq i$. $\square$

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