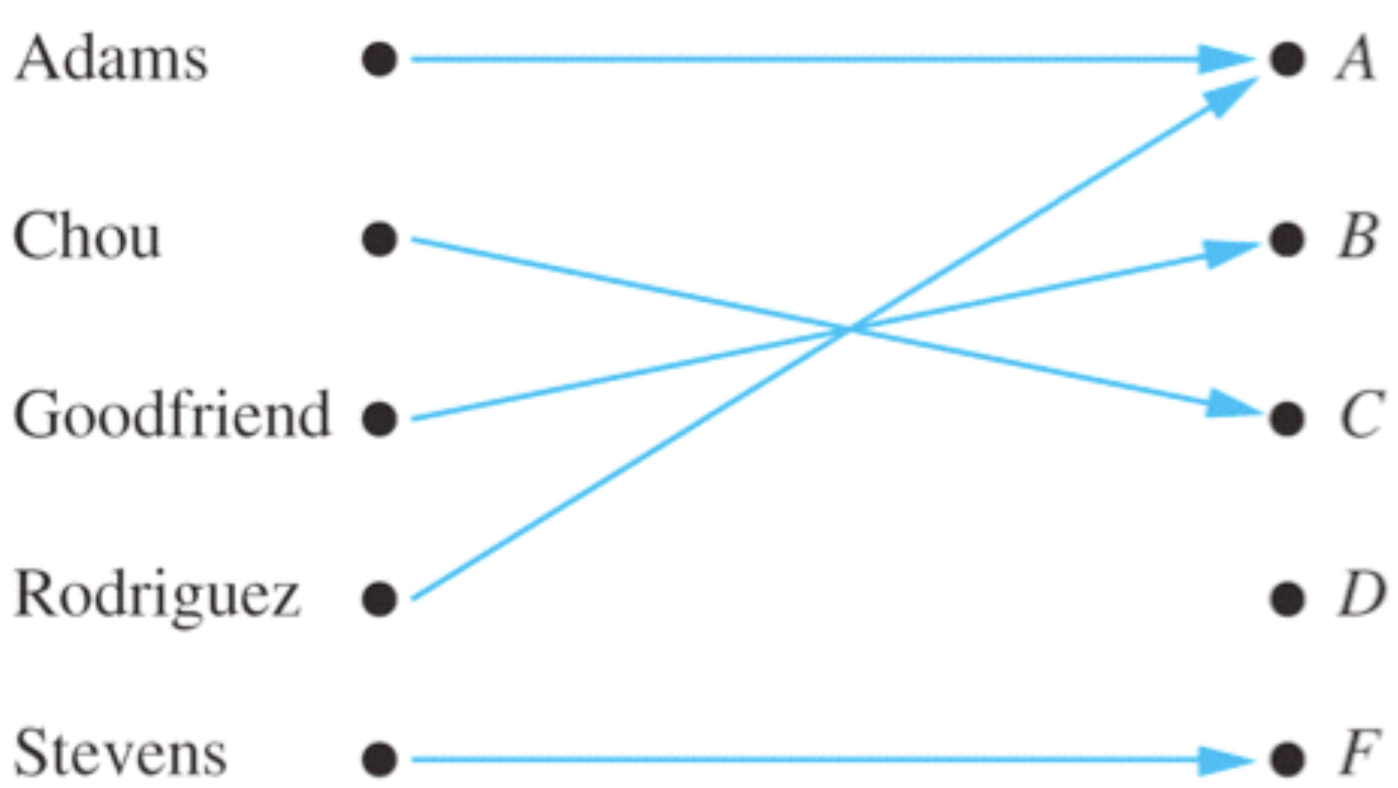
Discrete Mathematics:
Lecture 6. Functions

functions

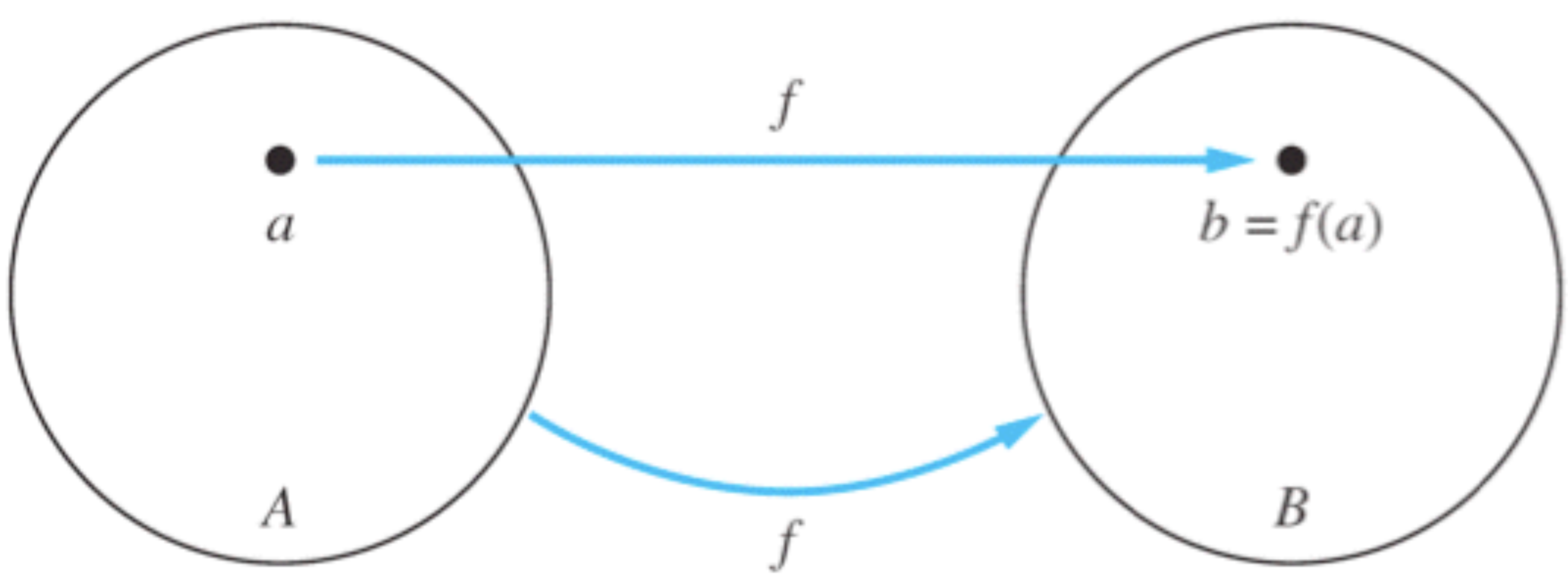


- a function f from A to B is an assignment of exactly one element of B to each element of A

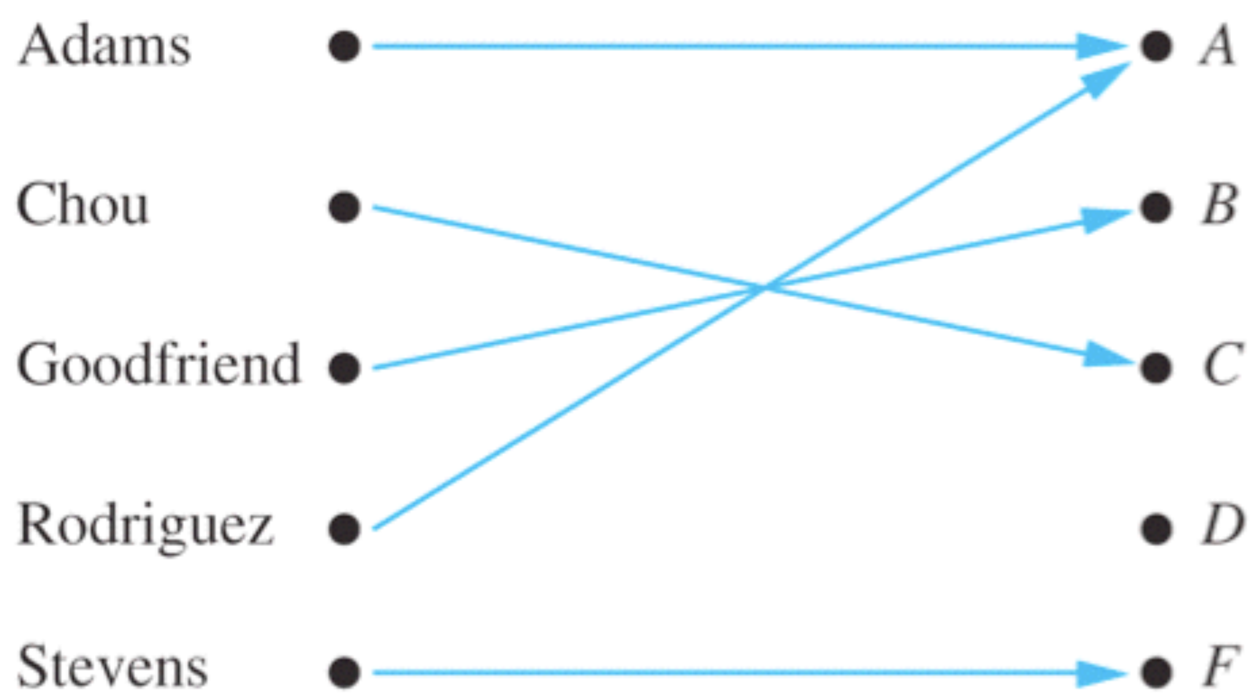
functions

- $f: A \to B$: A is the domain of f, B is the codomain of f
- $f(a) = b$: a is a preimage of b , b is the image of a
- range of f is the set of all images of elements of A

  range $R \subseteq B$ of f is $R = \{b \mid \exists a\ f(a) = b\}$



functions



- domain: {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- codomain: {A, B, C, D, F}
- range: {A, B, C, F}

functions

let f_1 and f_2 be functions from A to R

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $f_1 f_2 (x) = f_1(X) f_2 (x)$

f_1 and f_2: $R \rightarrow R$

$f_1(x) = x^2$

$f_2(x) = x - x^2$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1f_2)(x) = x^2(x - x^2) = x^3 - x^4$

different functions

- one-to-one (injunction)
- onto (surjection)
- one-to-one correspondence (bijection)

different functions

- one-to-one (injunction)
  - $f(a) \neq f(b)$ whenever $a \neq b$
  - $\forall a \ \forall b \ (a \neq b \rightarrow f(a) \neq f(b))$

If f is either strictly increasing or decreasing, then f is one-to-one

- f is strictly (or monotonically) increasing iff $x>y$ $f(x)>f(y)$ for all x,y
- f is strictly (or monotonically) decreasing iff $x>y$ $f(x)<f(y)$ for all x,y

$f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$

$f(a) = 4, f(b) = 5, f(c) = 1, f(d) = 3$

is f one-to-one?

$f(x) = x^2$

- $f: Z \rightarrow Z$: not one-to-one
- $f: Z+ \rightarrow Z$: one-to-one

different functions

- onto (surjection)
  - for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$
  - $\forall y \, \exists x \, (f(x) = y)$

$f: \{a, b, c, d\} \rightarrow \{1, 2, 3\}$

$f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 3$

is f onto?

$f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$

is f onto?

different functions

- one-to-one correspondence (bijection)
  - if it is one-to-one and onto

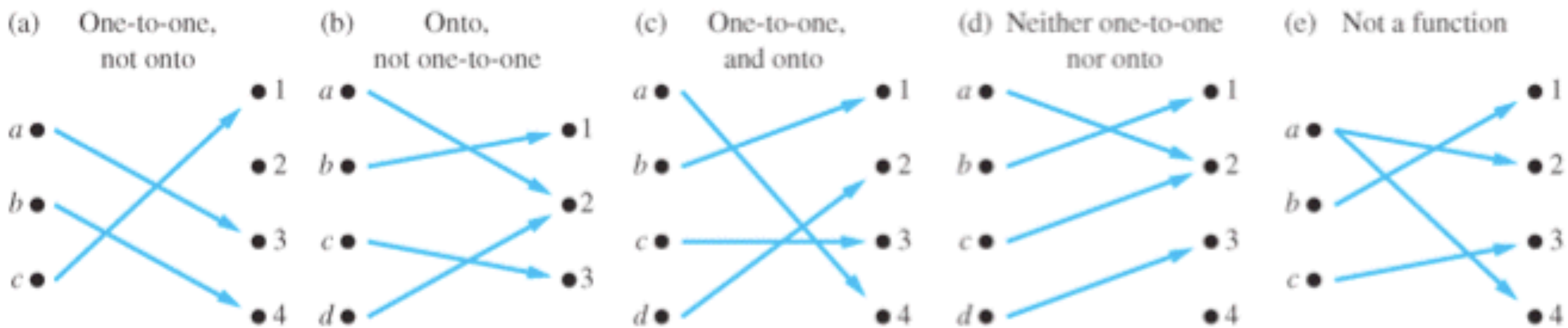
$f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$

$f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 4$

is f bijection?

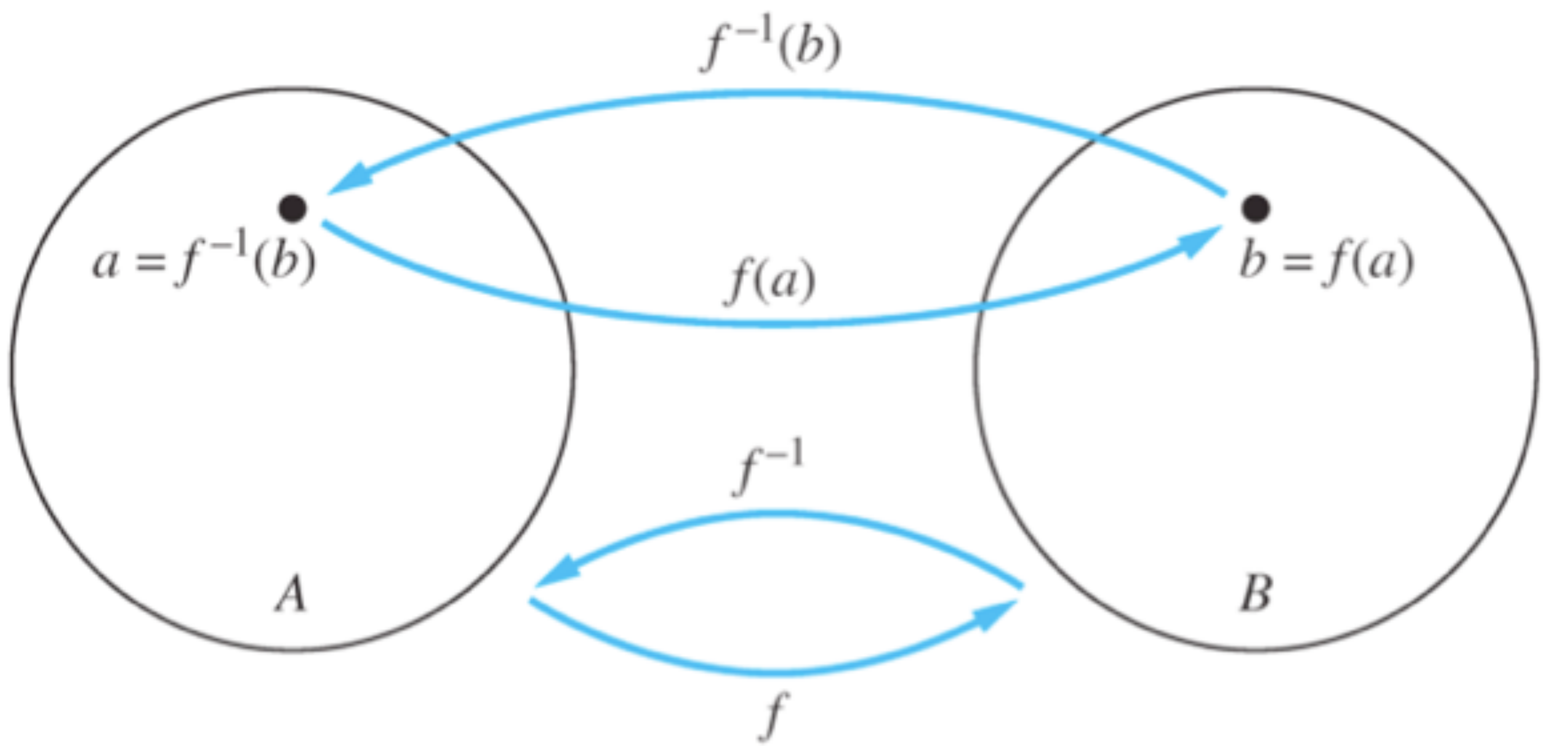
different functions

bipartite graph representation



different functions

- inverse function
  - $f^{-1}(b) = a$ when $f(a) = b$, f is one-to-one correspondence



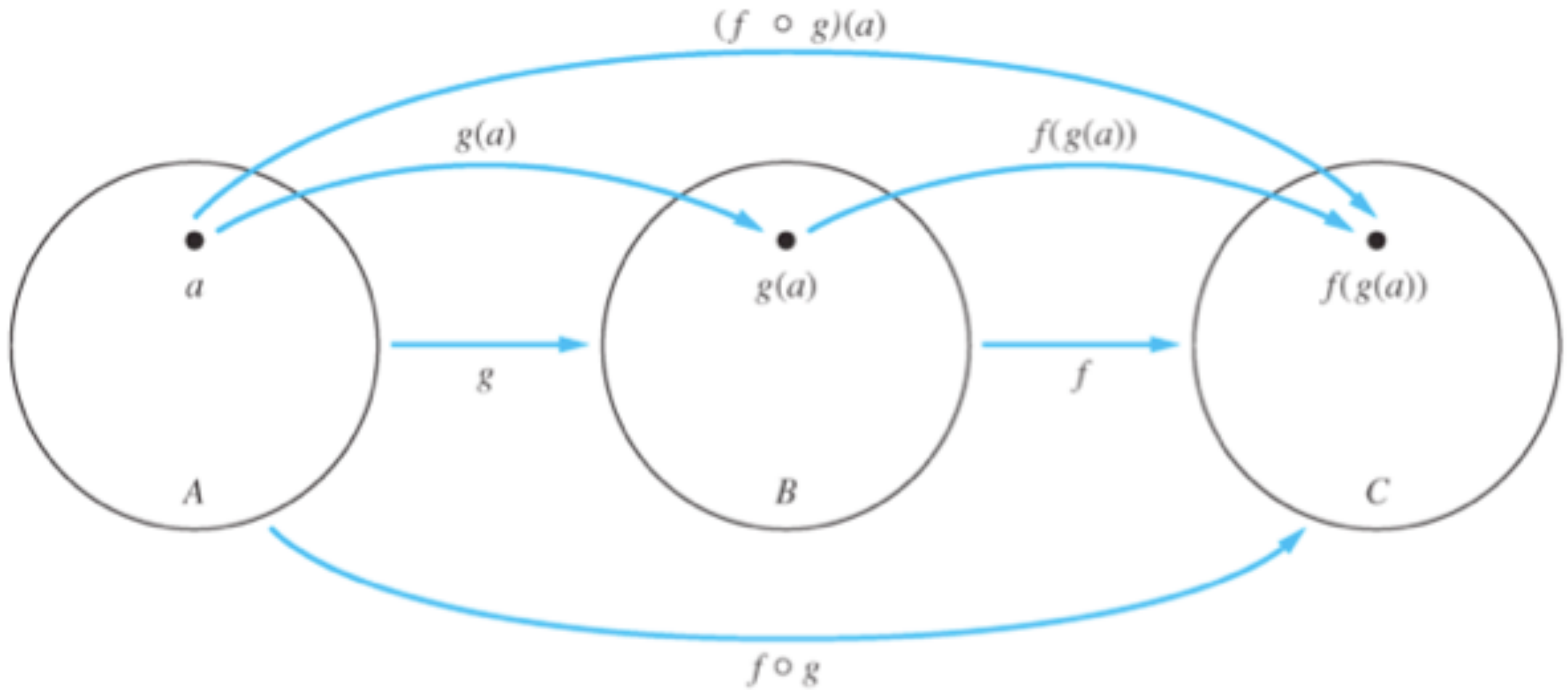
f: {a, b, c, d} → {1, 2, 3, 4}

f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 4

is f bijection?

composition of the functions

$(f \circ g)(a) = f(g(a))$



f, g: Z → Z

$f(x) = 2x + 3$, $g(x) = 3x + 2$

what is the composition of f and g? composition of g and f?

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

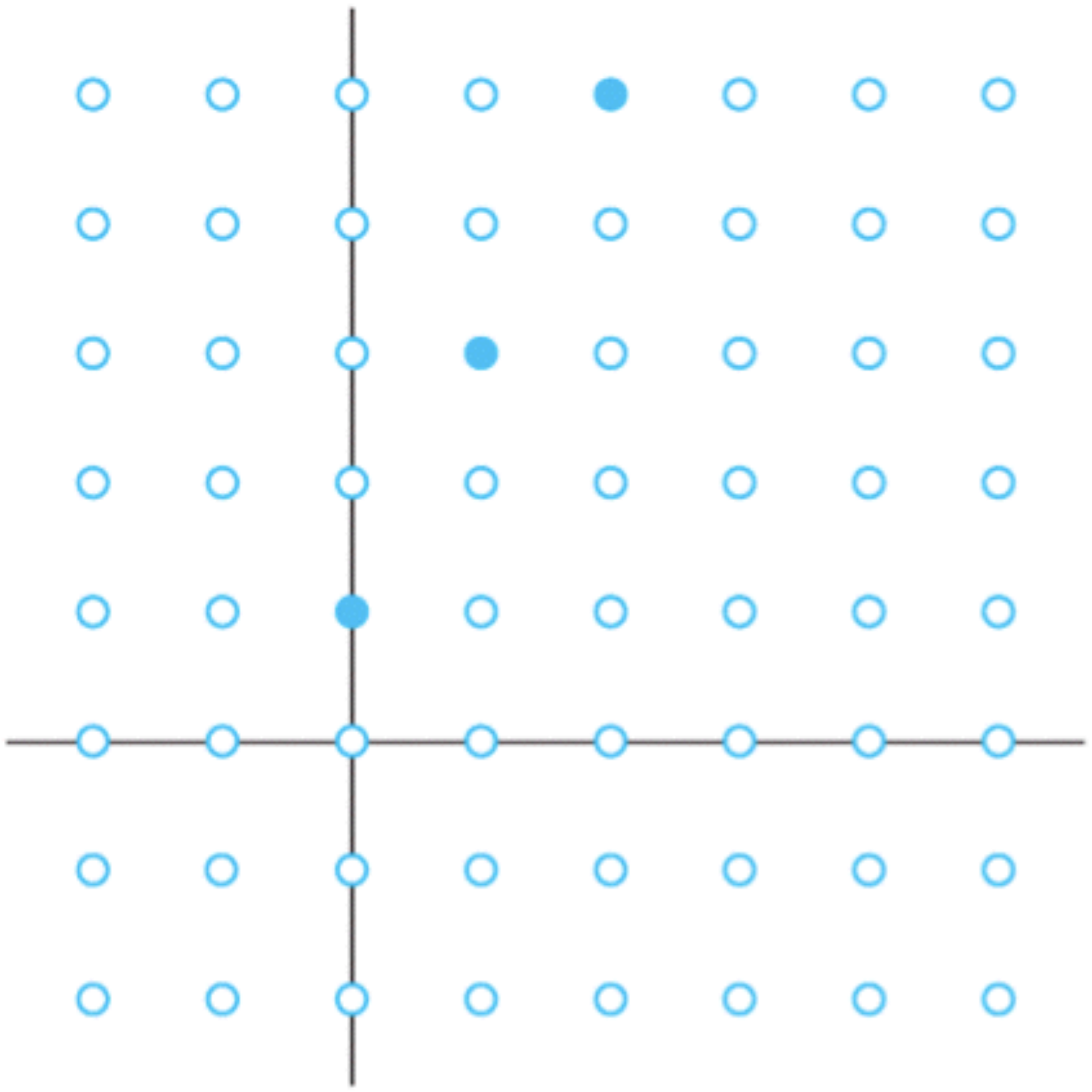
graphs of functions

f: A → B,

the graph of the function f is the set of ordered pairs

$\{(a, b) \mid a \in A \text{ and } f(a) = b\}$

f: Z → Z, f(n) = 2n + 1



floor and ceiling functions

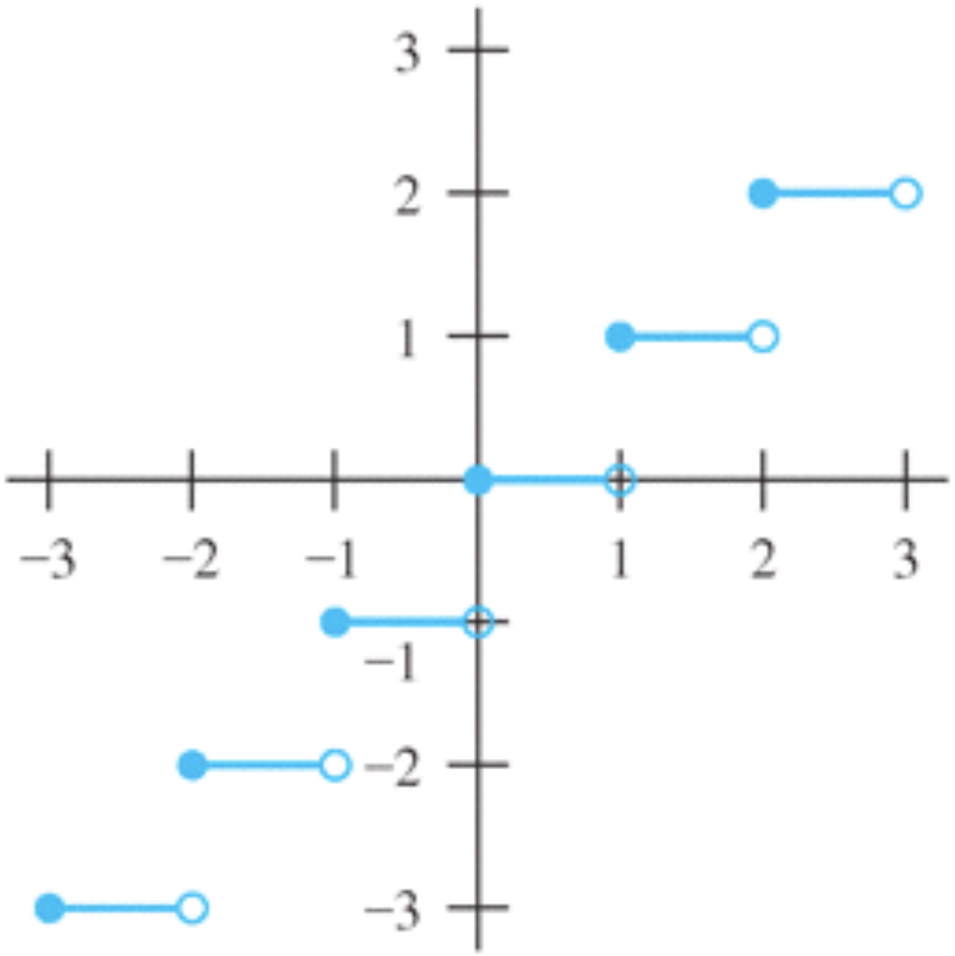
- floor function $\lfloor x \rfloor$ assigns to the real number x the largest integer that is less than or equal to x
- ceiling function $\lceil x \rceil$ assigns to the real number x the smallest integer that is greater than or equal to x

$\lfloor 0.5 \rfloor = 0$ $\quad$ $\lceil 0.5 \rceil = 1$

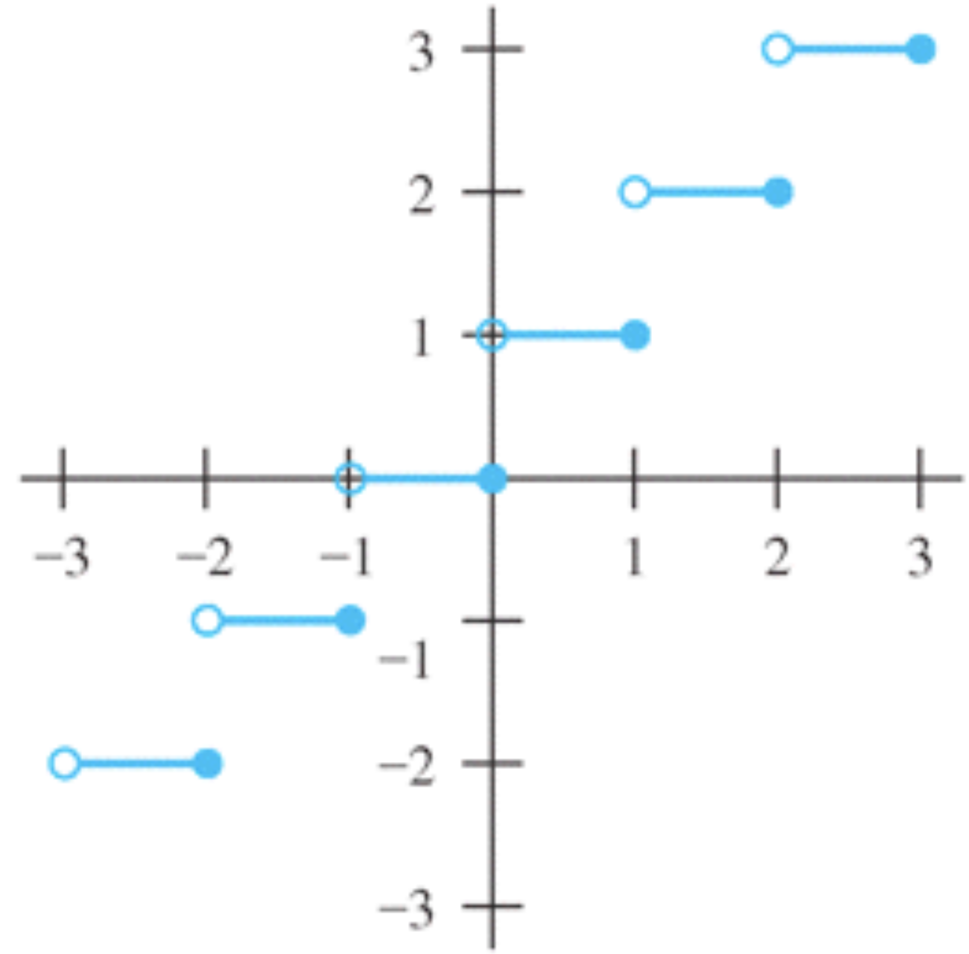
$\lfloor -0.5 \rfloor = -1$ $\quad$ $\lceil -0.5 \rceil = 0$

$\lfloor 7 \rfloor = 7$ $\quad$ $\lceil 7 \rceil = 7$

floor and ceiling functions



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

floor and ceiling functions

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8bits. How many bytes are required to encode 100 bits of data?

$\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes

floor and ceiling functions

properties of the floor and ceiling functions
(n is an integer, x is a real number)

(1a) $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$

(1b) $\lceil x \rceil = n$ iff $n - 1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ iff $x - 1 < n \leq x$

(1d) $\lceil x \rceil = n$ iff $x \leq n < x + 1$

(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = - \lceil x \rceil$

(3b) $\lceil -x \rceil = - \lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

floor and ceiling functions

prove $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

suppose $\lfloor x \rfloor = m$, m is a positive integer

$m \leq x < m + 1$ by property (1a)

$m + n \leq x + n < m + 1 + n$

$\lfloor x+n \rfloor = m + n = \lfloor x \rfloor + n$

floor and ceiling functions

ref. $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$

prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 0.5 \rfloor$

suppose $x = n + \epsilon$, when n is integer and $0 \leq \epsilon < 1$

1) $0 \leq \epsilon < 0.5$ $\quad (0 \leq 2\epsilon < 1,\ 0.5 \leq \epsilon + 0.5 < 1)$

$2x = 2n + 2\epsilon \qquad \lfloor 2x \rfloor = 2n$

$\lfloor x + 0.5 \rfloor = \lfloor n + \epsilon + 0.5 \rfloor = n$

$\lfloor x \rfloor + \lfloor x + 0.5 \rfloor = n + n = 2n$

2) $0.5 \leq \epsilon < 1$ $\quad (1 \leq 2\epsilon < 2,\ 1 \leq \epsilon + 0.5 < 1.5)$

$2x = 2n + 2\epsilon = (2n + 1) + (2\epsilon - 1) \qquad \lfloor 2x \rfloor = 2n + 1$

$\lfloor x + 0.5 \rfloor = \lfloor n + \epsilon + 0.5 \rfloor = \lfloor n + 1 + \epsilon - 0.5 \rfloor = n + 1$

$\lfloor x \rfloor + \lfloor x + 0.5 \rfloor = n + n + 1 = 2n + 1$

partial functions

$f: A \rightarrow B$

a total function f when A is the domain of definition

a partial function $f: A' \rightarrow B$, where A' is a subset of A, domain of definition

$f: Z \rightarrow R$ where $f(n) = \sqrt{n}$

f is a partial function from Z to R where domain of definition is the set of nonnegative integers