Data Structure:

Disjoint Set

Skipped list

equivalence relations

- a relation on a set is equivalence relation if it is reflexive, symmetric, and transitive

show $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers

$a \equiv b \pmod{m}$ iff $b-a = km$

reflexive: $(a-a) = 0 \cdot m$

symmetric: if $a - b = k \cdot m$, $b - a = -k \cdot m$

transitive: $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$a - b = k \cdot m$, $b - c = l \cdot m$

$(a - b) + (b - c) = (a - c) = (k + l) \cdot m$

thus, $a \equiv c \pmod{m}$

equivalence classes

- the set of all elements that are related to an element a of A is the equivalence class of a
- the equivalence class of a with respect to R is denoted by $[a]_R$

  $[a]_R = \{s \mid (a, s) \in R\}$

what is the equivalence classes for congruence modulo 4?

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

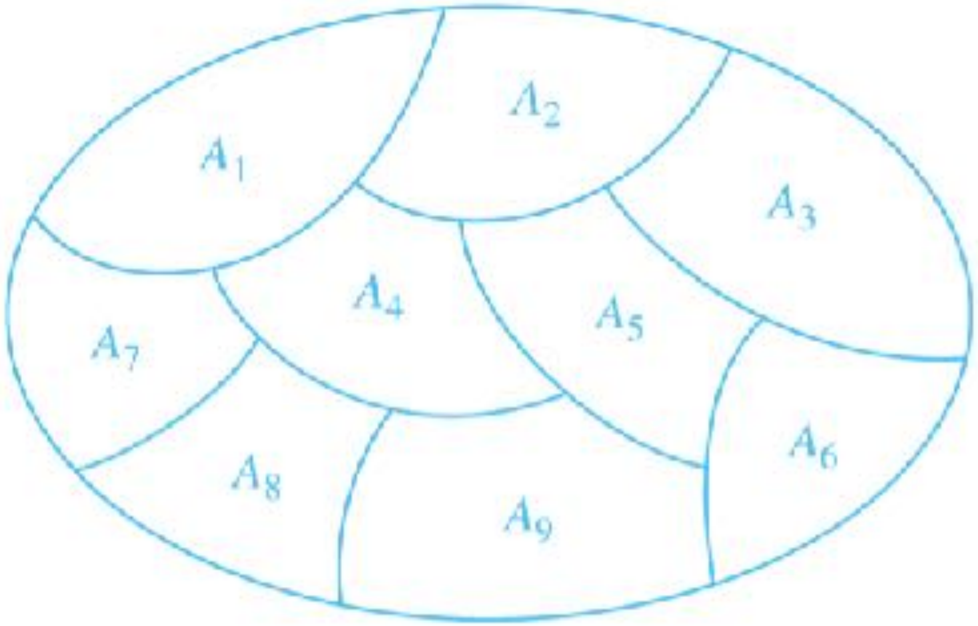
$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

equivalence classes and partitions

- R is an equivalence relation on a set A, a and b $\in$ A
  - aRb
  - [a] = [b]

- equivalence relation partitions a set, p and q $\in$ A
  - when $[p]_R \neq [q]_R$, $[p] \cap [q] = \varnothing$
  - $$\bigcup_{k \in A} [k]_R = A$$



Disjoint sets

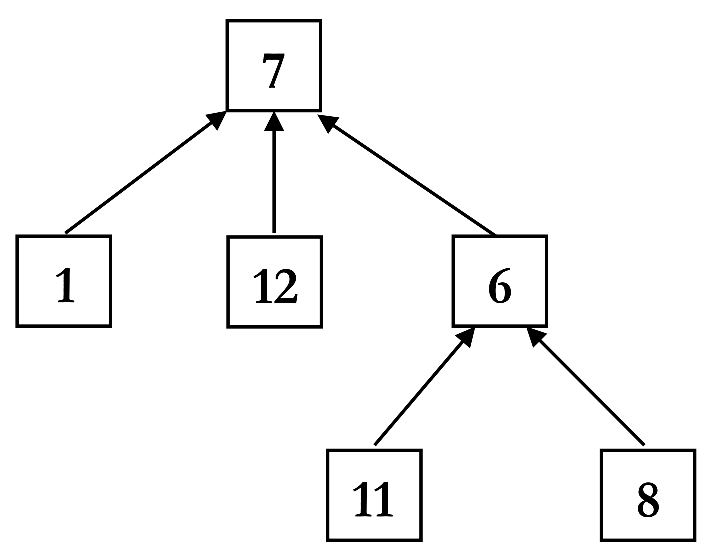
- We assume that the sets being represented are pairwise disjoint.
- If S_i and S_j are two sets and i != j, then there is no element that is in both S_i and S_j
- Basic operations needed for Disjoint Set
  - union
  - find

$S_1 = \{0, 6, 7, 8\}$, $S_2 = \{1, 4, 9\}$, $S_3 = \{2, 3, 5\}$

$S_1 \cup S_2 = \{0, 6, 7, 8, 1, 4, 9\}$

Union-Find in disjoint sets

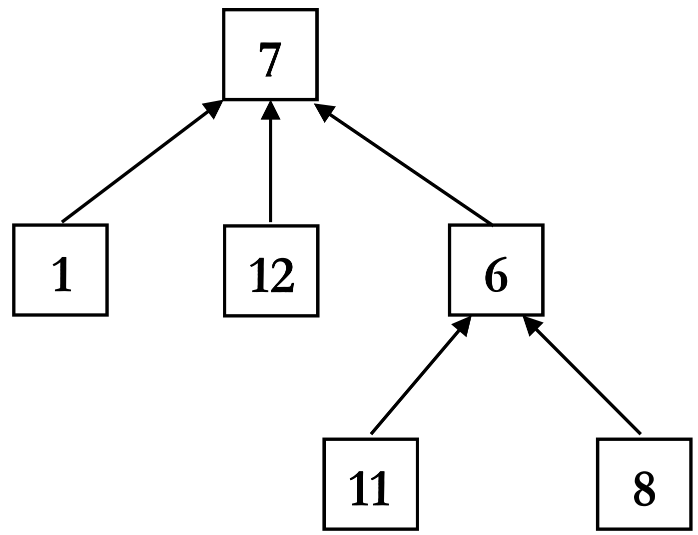
- maintain elements of S in a forest of inverted trees
  - pointers in the tree are directed towards the root.
  - the root of a tree has a NULL parent pointer
  - two elements are in the same set iff they are in the same tree.

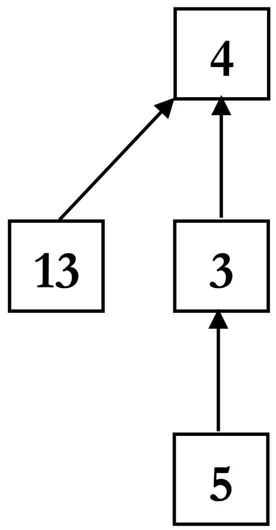


$S_1 = \{1, 6, 7, 8, 11, 12\}$

Union-Find in disjoint sets

- Find(S, i)
  - find the node containing i
  - follow the parent links up to the root.
  - return the root node as the "name" of the set.



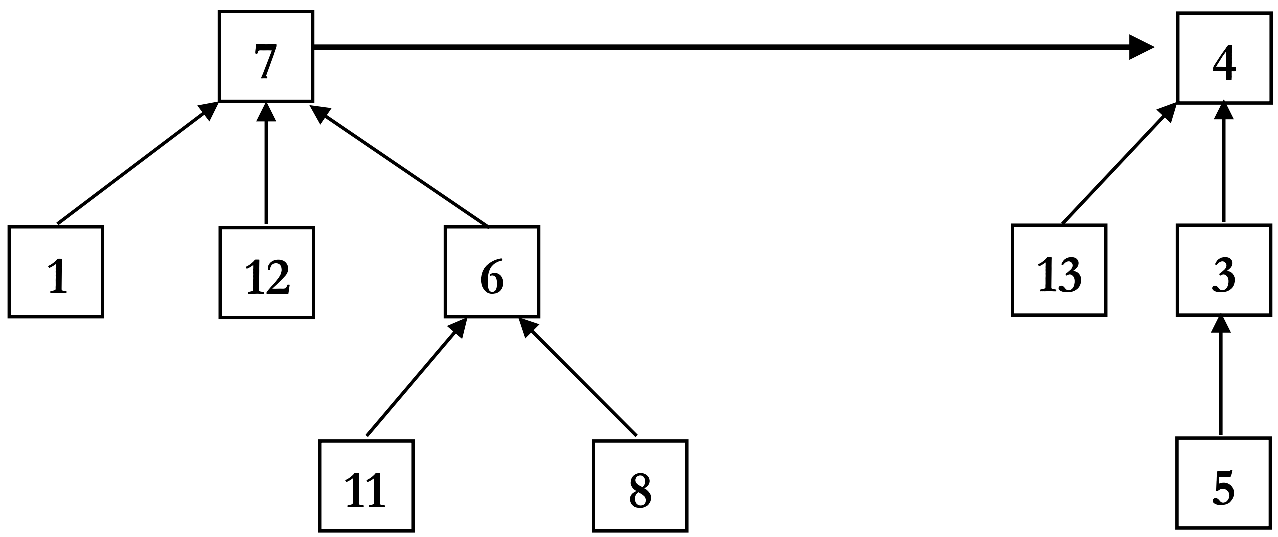


$S_1 = \{1, 6, 7, 8, 11, 12\}$

$S_2 = \{4, 3, 5, 13\}$

Union-Find in disjoint sets

- Union(i, j)



$S_1 = \{1, 6, 7, 8, 11, 12\}$

$S_2 = \{4, 3, 5, 13\}$

$S_1 \cup S_2 = \{1, 6, 7, 8, 11, 12, 4, 3, 5, 13\}$

Union-Find in disjoint sets

- Init(S): set all parent to 0

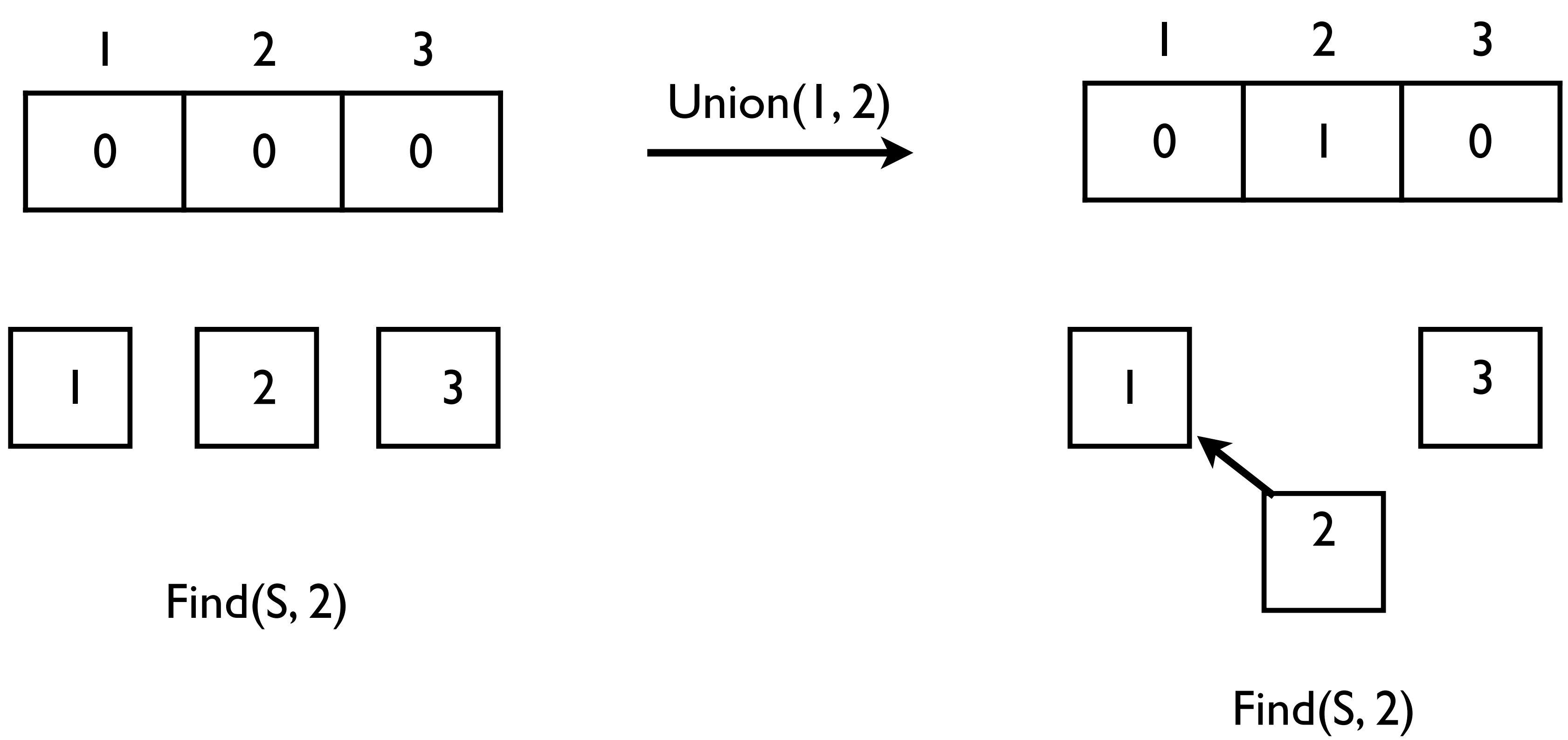
  ex) when $S = \{1, 2, 3\}$, [1] = {1}, [2] = {2}, [3] = {3}

- Find(S, i): follow the parent link

  ex) Union(1, 2): [1] = {1, 2}, [3] = {3}

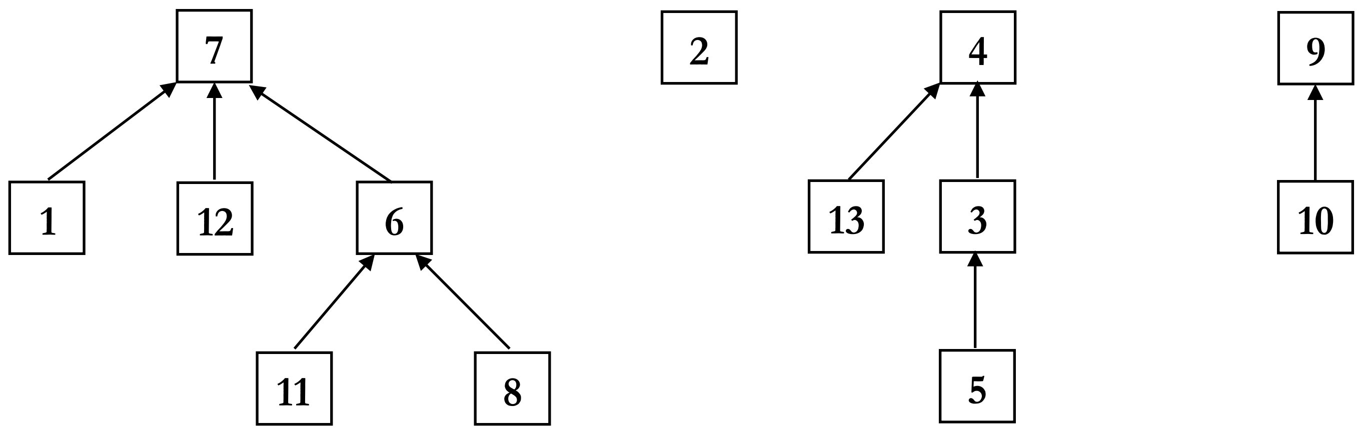
- Union(S, t) link the root of one tree into the root of the other tree

  ex) Find(S, 1) = 1, Find(S, 2) = 1, Find(S, 1) = Find(S, 2)



Union-Find in disjoint sets

S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}
the current partition: {1, 6, 7, 8, 11, 12}, {2}, {3, 4, 5, 13}, {9, 10}



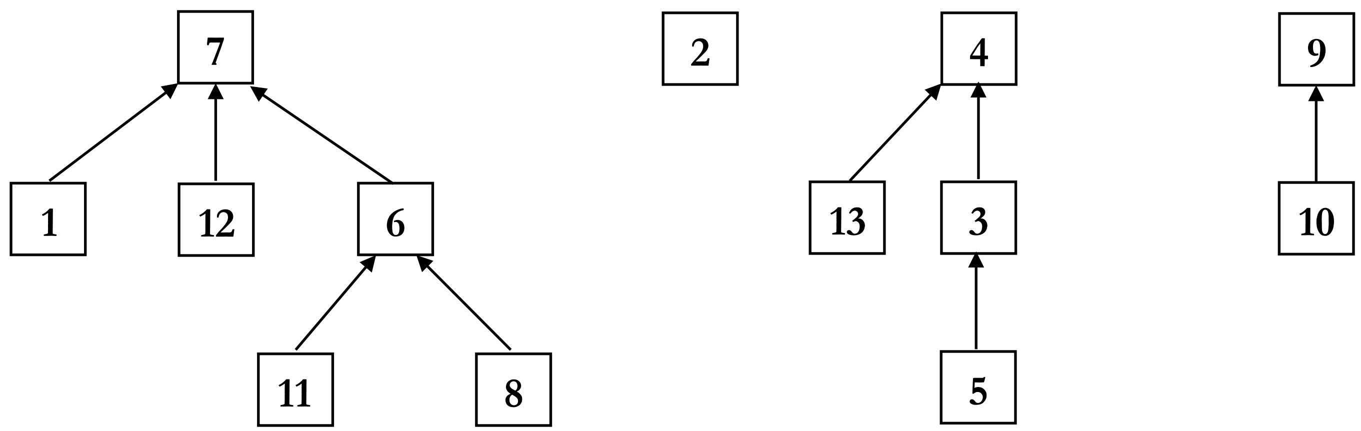
7	0	4	0	3	7	0	6	0	9	6	7	4
1	2	3	4	5	6	7	8	9	10	11	12	13

- there is no order how the tree should be structured
- the element is an index, not a *key*
- for each element, the array S[1..n] stores the index of the parent in the tree
- index of 0 means a null pointer

Union-Find in disjoint sets

S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}

the current partition: {1, 6, 7, 8, 11, 12}, {2}, {3, 4, 5, 13}, {9, 10}



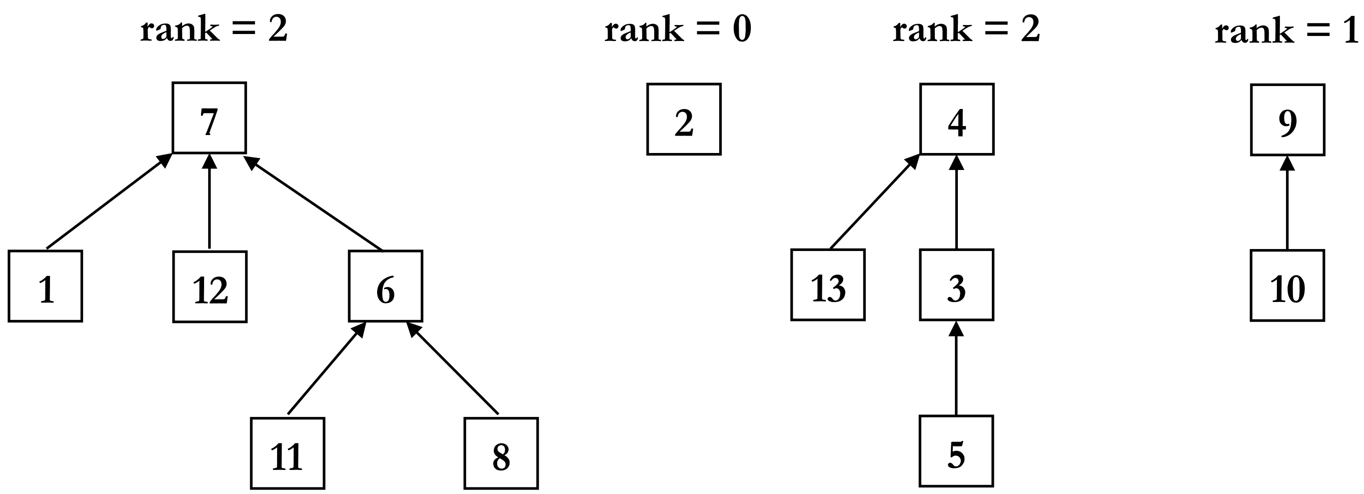
7	0	4	0	3	7	0	6	0	9	6	7	4
1	2	3	4	5	6	7	8	9	10	11	12	13

- What is the time complexity for union?
- Union({2}, {9, 10})

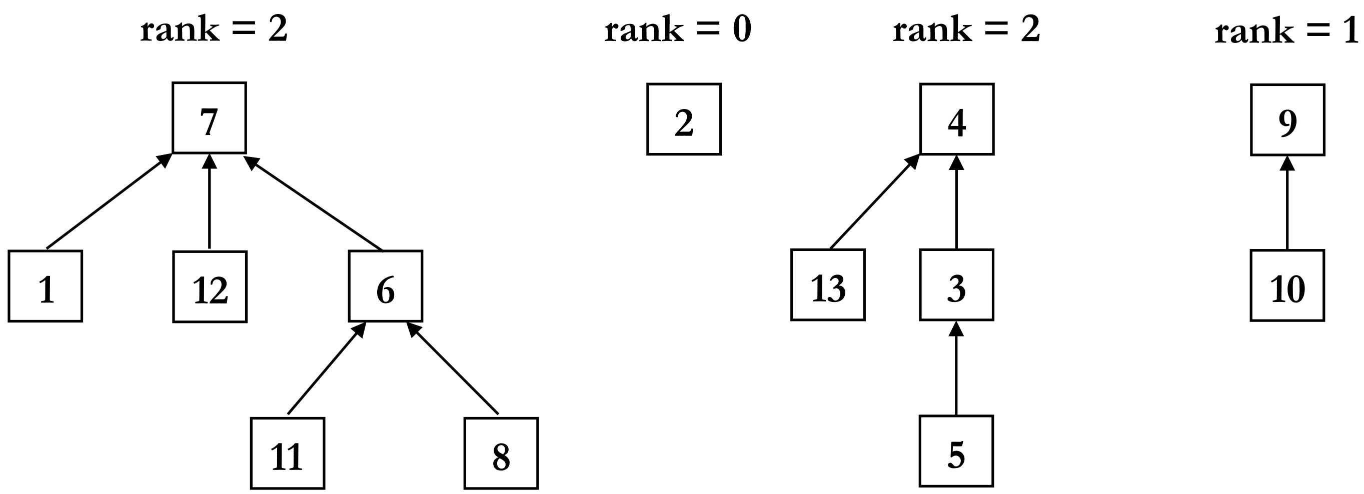
  If we link {9, 10} into {2}, the resulting height is 2

  If we link {2} into {9, 10}, the resulting height is 1
- How can we improve the simple union?

Union-Find in disjoint sets ADT



Union-Find in disjoint sets ADT



Array representation

7	0	4	-2	3	7	-2	6	-1	9	6	7	4
1	2	3	4	5	6	7	8	9	10	11	12	13

7	0	4	0	3	7	0	6	0	9	6	7	4
1	2	3	4	5	6	7	8	9	10	11	12	13

- to perform smart Union, each tree includes an extra information called *rank*, which is the height of the tree.
- link the tree with smaller rank to the tree with larger rank.
- where do we store the rank?
  - We only need to maintain the rank for the root nodes
  - One clever way is to store the negative of rank at the root node
    - if S[i] is strictly positive it is a parent pointer.
    - Otherwise, i is a root and -S[i] is the rank of the tree.

Union-Find in disjoint sets ADT

S = {0, 1, 2, 3}
Perform the following operation in order
union(0,1), union(1,2), union(2,3), union(3, 4)

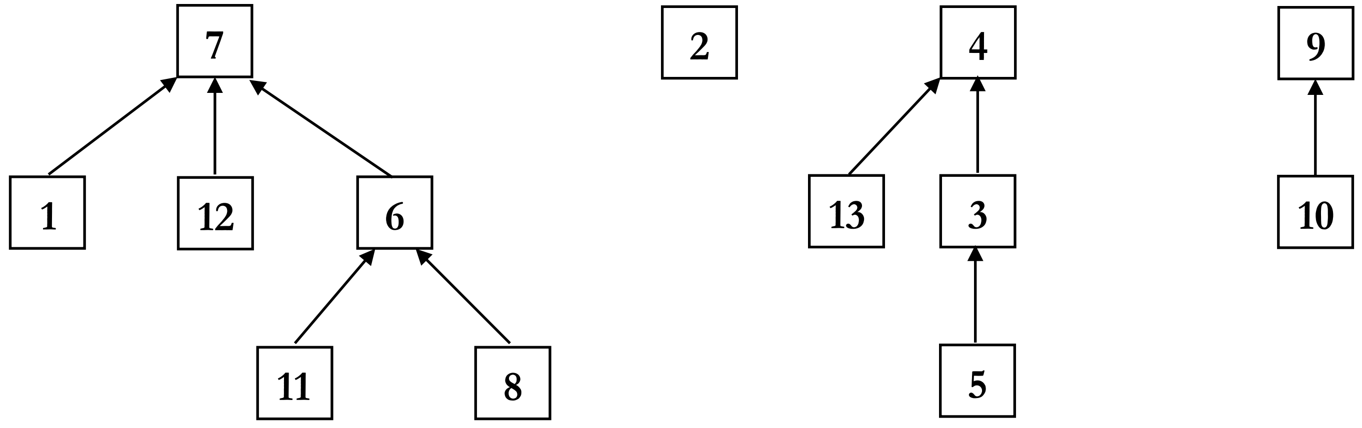
Union-Find in disjoint sets ADT

```
Disj_Sets S[n];
void Init(Disj_Sets *S)
{
        for (i = 1; i <= n; i++)  S[i] = 0;
}
```

```
Set_Type Find1(Elt_Type x, Disj_Sets *S)
{
        while (S[x] > 0)
           x = S[x];
        return x;
}
```

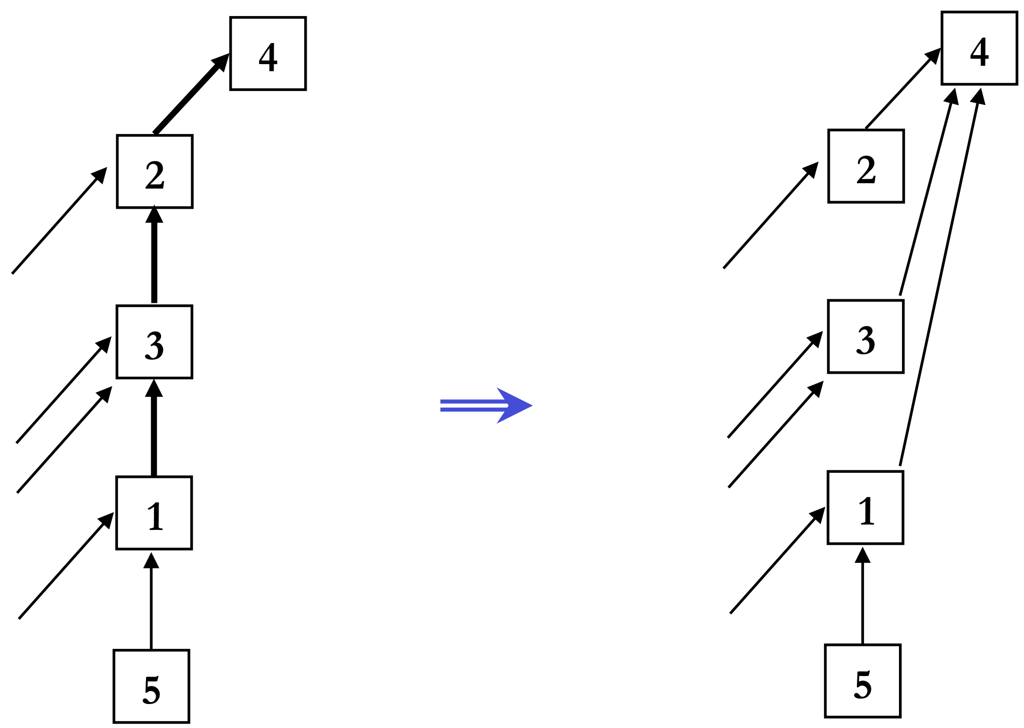
Union-Find in disjoint sets ADT

```
void Union(Disj_Sets *S, Set_Type r1, Set_Type r2)
{
        if (S[r2] < S[r1])
          S[r1] = r2;                /* if |S[r2]| > |S[r1]|,   add r1 to r2 */
        else
        {
               if (S[r2] == S[r1])
                    S[r1]--;
               S[r2] = r1;                /* add r2 to r1 */
         }
}
```



path compression

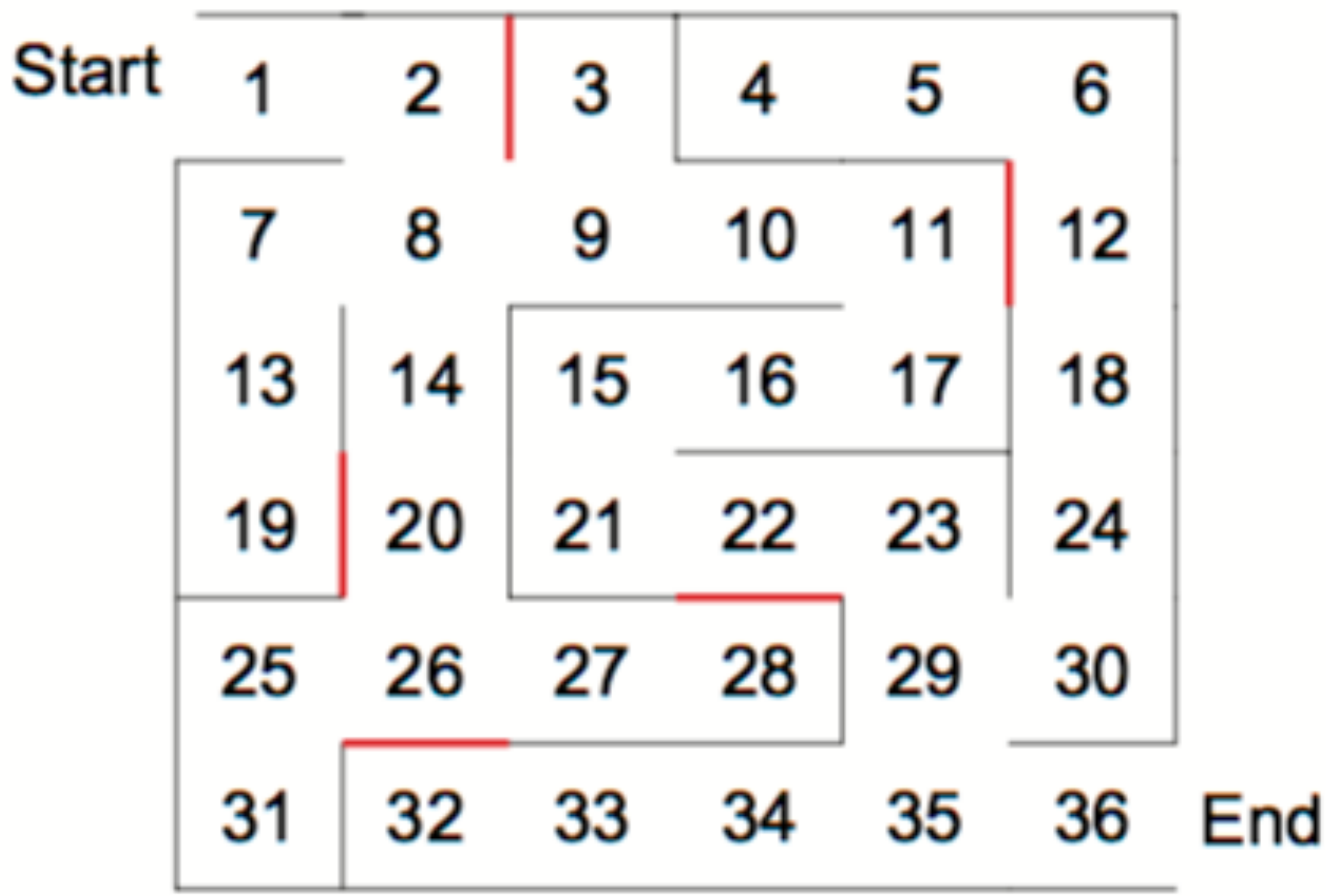
- a simple heuristic to improve running time significantly (ALMOST gets rid of the log n factor in the running time $O(n \log n)$)
- If we compress the paths on each Find(), subsequent Find() will go much faster.
- "Compress the path" means that when we find the root we set all parent pointers of the node on our find path to the root.



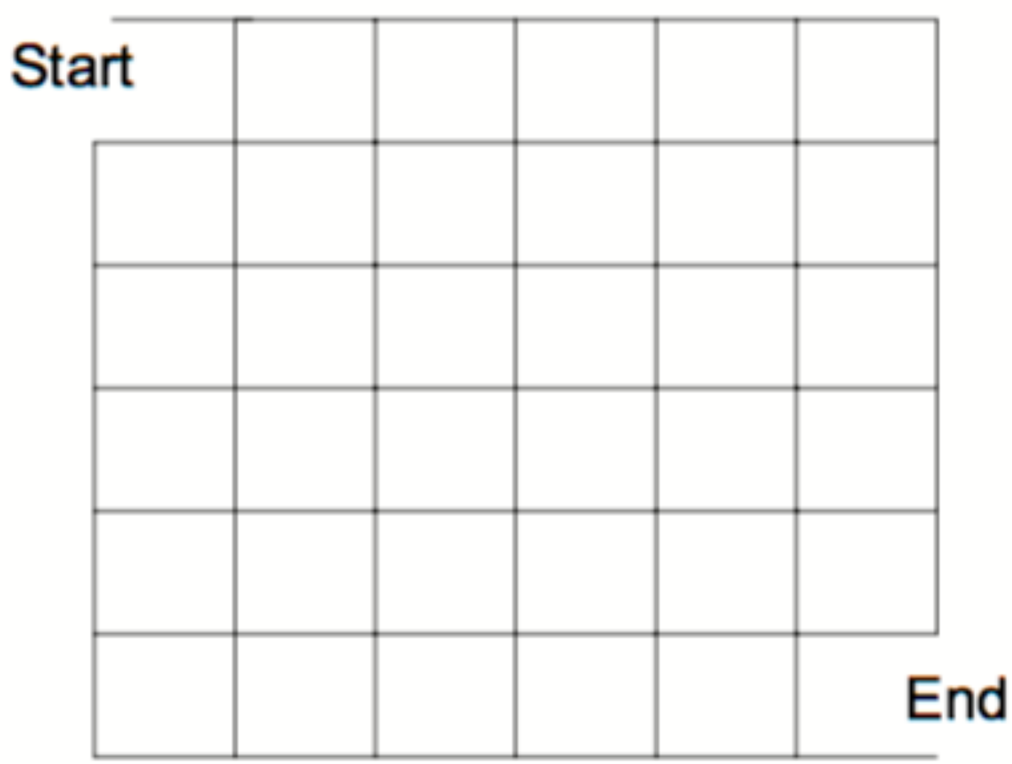

```
Set_Type Find2(Elt_Type x, Disj_Sets S)
{
        if (S[x] <= 0)
                return x;
        else
                return (S[x] = Find2(S[x], S));
}
```

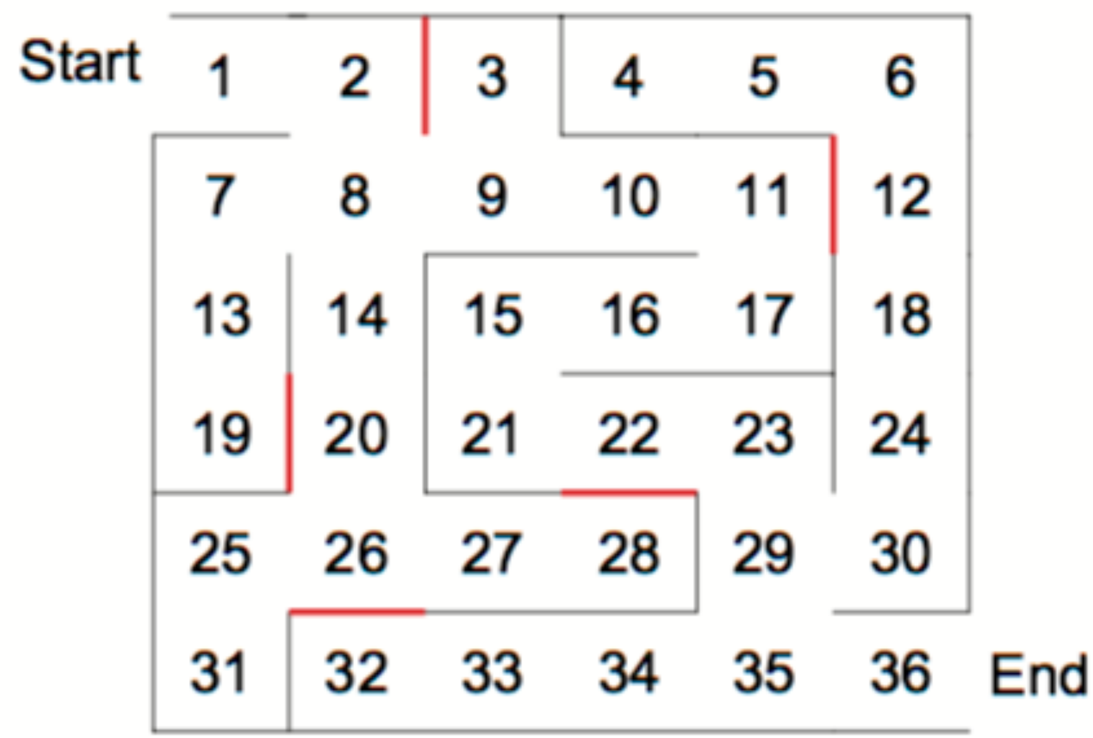
- running time of Find2() is still proportional to the height of the tree
- each time you spend lots of time in Find2(), you make the tree flatter, thus making subsequent Find2() faster.

disjoint sets ADT



disjoint sets ADT





Union-Find in disjoint sets ADT

1	2	3	4	5	6	7	8	…	14	15	16	17	18	19	20	21	…..
						-5	7		20						-3		

{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,30,32
33,34,35,36}

Union(8, 14)

Find(8) = 7, Find(14) = 20

{1,2,7,8,9,13,19, 14, 20, 26, 27}

skip lists

skip lists

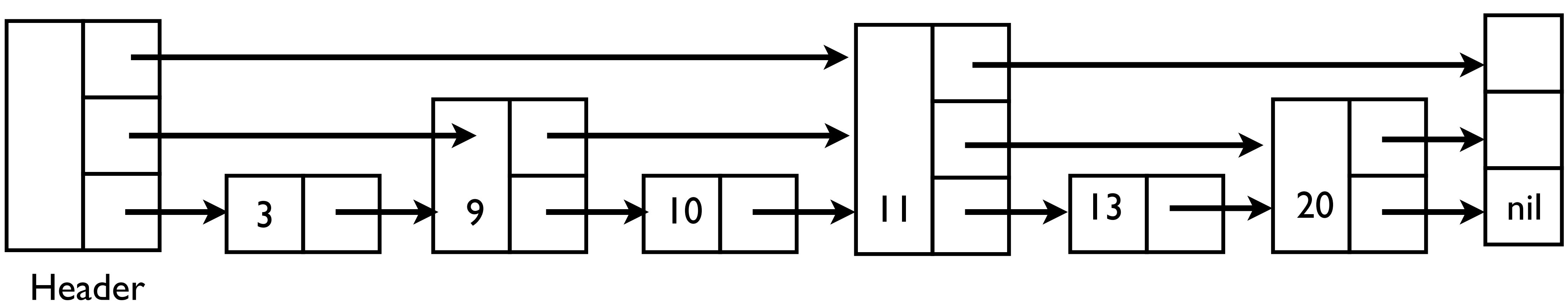
- linked lists do
    - insertion and deletion in O(1), but find in O(n)
    - not store sorted lists
- how can we make linked lists better?

- store in sorted linked lists?
- a randomized data structure: it uses the random number generator

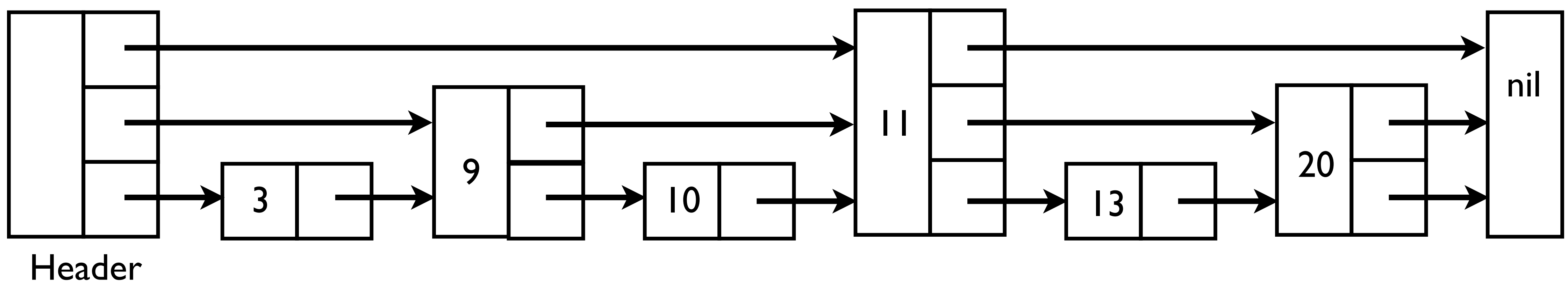
- skip lists
    - use hierarchy of sorted linked lists
    - skip over lots of items to find an element
    - expected search time is O(log n) with high probability

perfect skip lists

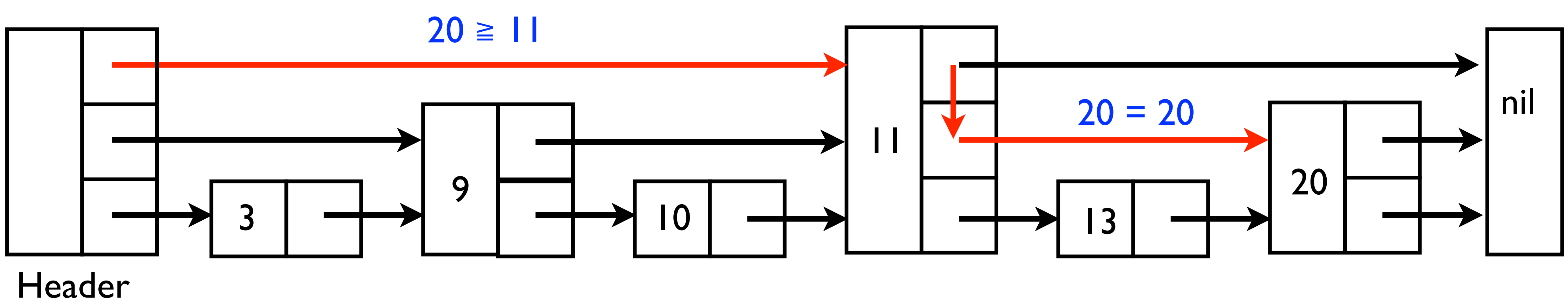
- nodes are of variable size, including 1 and O(log n) pointers
- search(k)
  - if k = key, done
  - if k < next key, go down a level
  - if k ≧ next key, go right
- In the worst case,
  - we have to go through all log *n* levels
  - at each level, we visit at most 2 nodes: $O(\log n)$



perfect skip lists: search



search(20)



How about search(14)?

randomized skip lists

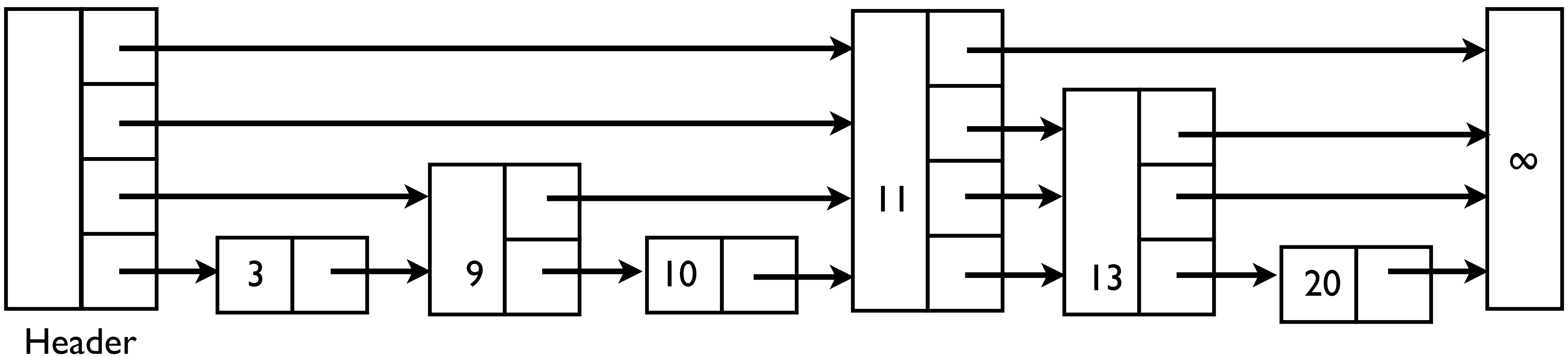
- perfect skip lists need to rearrange the entire list after insertion and deletion
- to insert or delete x,
  - search for x in the skip list
  - find the position $p_0, p_1, \ldots, p_i$ of the items that has the largest key less than x in each level $0, 1, \ldots, i$
- The maximum level (the size of header node) should be log *n* when *n* is the maximum number of nodes allowed

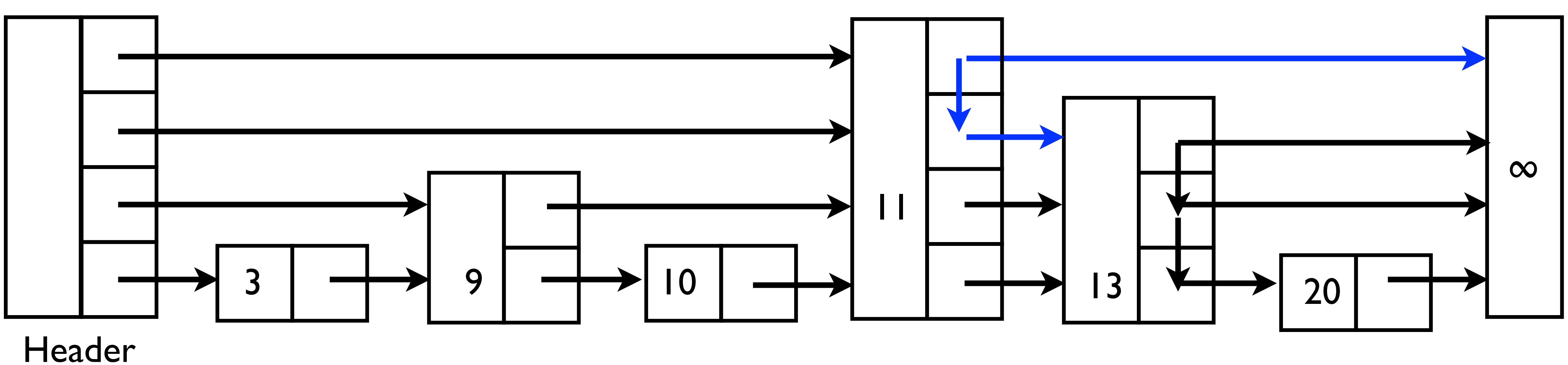
```
struct skip_node {
        element_type     element;
        int     level;
        struct skip_node      **forward;
} *s;

s = (skip_node*)malloc( sizeof(struct skip_node) );
s->forward = (skip_node**)malloc( sizeof(skip_node *)*(level+1) );
```

randomized skip lists: search

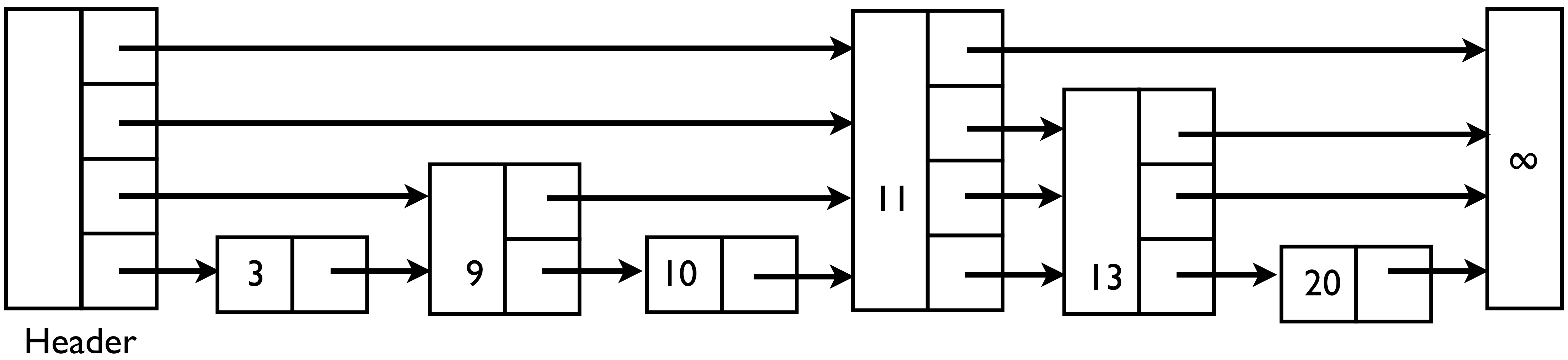
search(13)

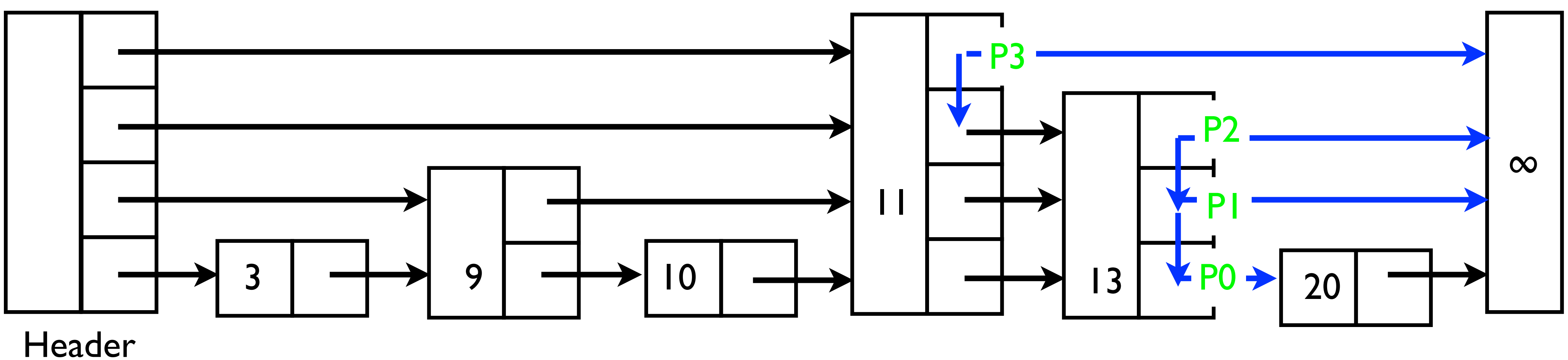




randomized skip lists: insert

insert(14)

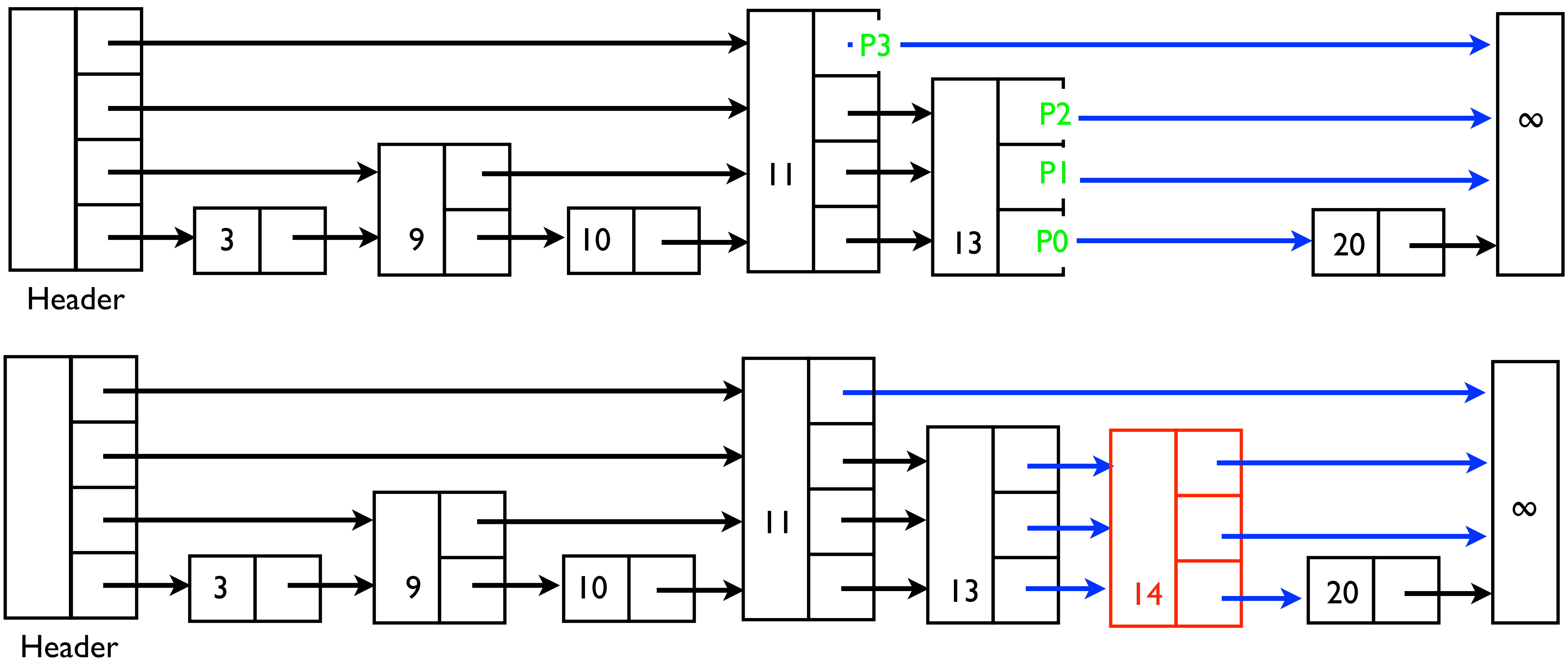




found the place for insertion!

randomized skip lists: insert

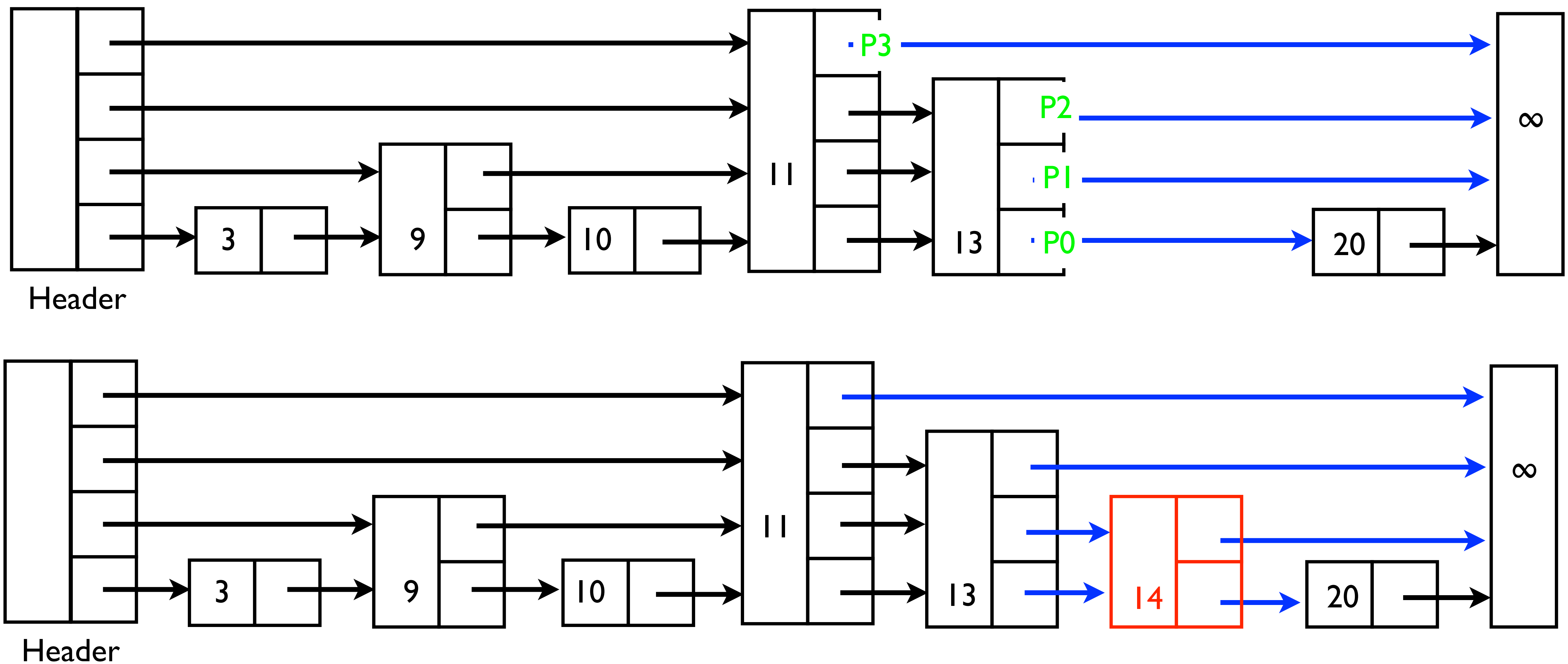
insert(14) at level 2




```
search(x);                          # find x
level = 0;                          # insert node in level 0
while (FLIP()  ==  “heads”)
        level ++;                   # move the level of the new node up
```

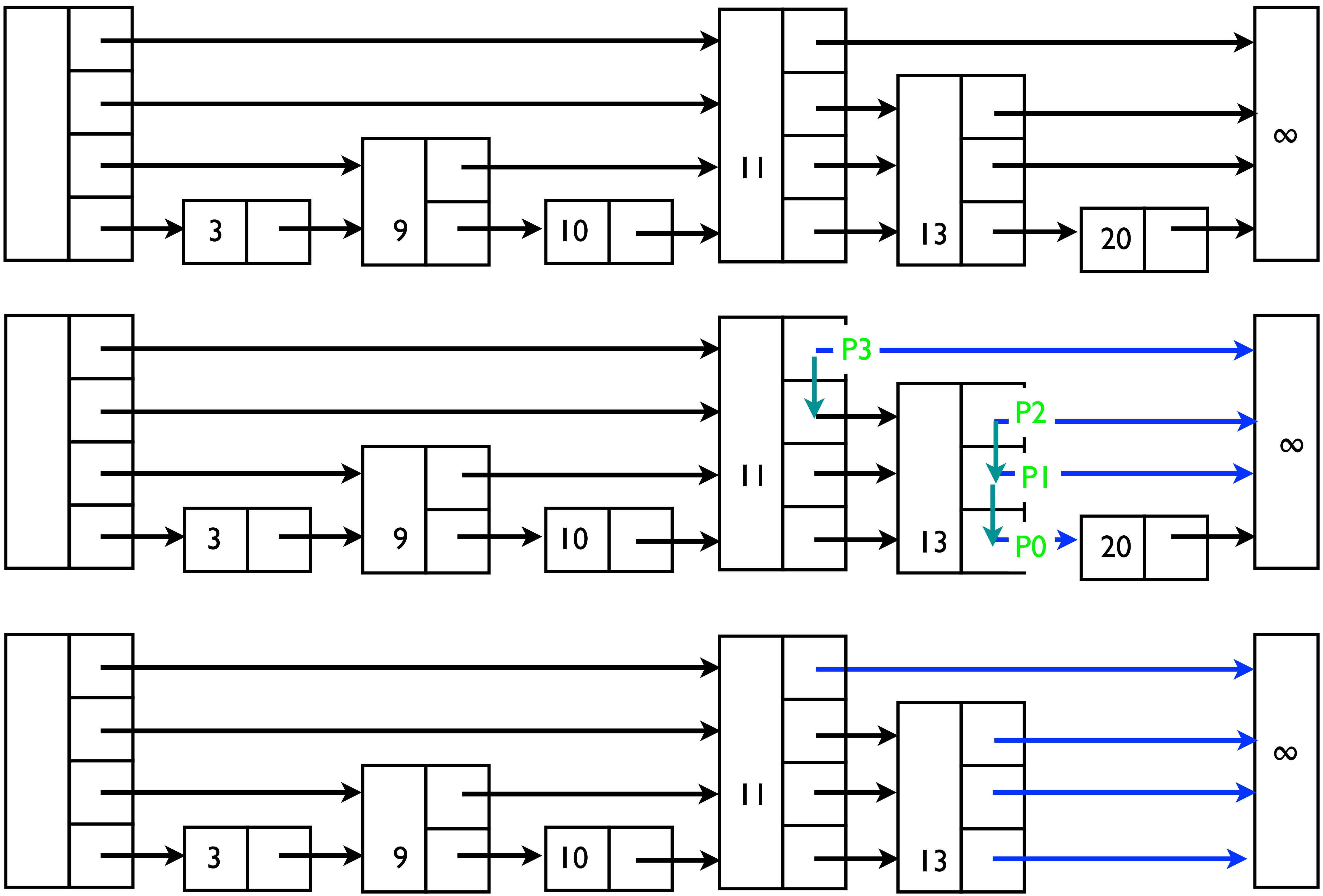
randomized skip lists: insert

insert(14) at level 1



randomized skip lists: delete

delete(20)



randomized skip lists: delete

delete(9)