Discrete Mathematics:
Lecture 1. propositions

proposition

- A declarative sentence that is either true or false, but not both.
- A sentence that declares a fact.

- Washington DC is the capital of the United States of America   T
- $1 + 1 = 2$ T
- $2 + 2 = 3$ F
- What time is it?   X
- $x + 1 = 2$ X

compound proposition

- a proposition that combines one or more propositions using logical operators
- connectives (logical operators)
  - $\neg p$ (not p): negation of p
  - $p \wedge q$ (p and q): conjunction of p and q
  - $p \vee q$ (p or q): disjunction of p and q (inclusive or)
  - $p \oplus q$ (p xor q): exclusive or of p and q
- truth table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

compound proposition

- what is the conjunction propositions p and q where
  p is "Rebecca's PC has more than 16GB free hard disk space" and
  q is " The processor in Rebecca's PC runs faster than 1GHz"

compound proposition

- what is the conjunction propositions p and q where
  p is "Rebecca's PC has more than 16GB free hard disk space" and
  q is " The processor in Rebecca's PC runs faster than 1GHz"

Rebecca's PC has more than 16GB free hard disk space and
its processor runs faster than 1GHz

compound proposition

- connectives (logical operators)
  - $p \longrightarrow q$: <u>conditional</u> statement (implication)

    p: hypothesis (or premise)

    q: conclusion (or consequence)

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- If I love you, I will marry you
- If you get 100 points on the final, then you will get the grade A
- $(1 = 0) \longrightarrow$ pigs can fly
- If Tuesday is a day of the week, then I am a penguin

compound proposition

- “If I wear a red shirt tomorrow, then my professor will give me A+ “
- In logic, the sentence is TRUE so long as either
  - “ I don’t wear a red shirt ” or
  - “ My professor give me A+”

compound proposition

- p: It is below freezing

  q: It is snowing

  a) It is below freezing and snowing

  b) It is below freezing but not snowing

  c) It is not below freezing and it is not snowing

  d) It is either snowing or below freezing

  e) If it is below freezing, it is also snowing

  f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing

compound proposition

- p: It is below freezing

  q: It is snowing

  a) It is below freezing and snowing $p \wedge q$

  b) It is below freezing but not snowing $p \wedge \neg q$

  c) It is not below freezing and it is not snowing $\neg p \wedge \neg q$

  d) It is either snowing or below freezing $p \vee q$

  e) If it is below freezing, it is also snowing $p \longrightarrow q$

  f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing $(p \vee q) \wedge (p \longrightarrow \neg q)$

compound proposition

$$\begin{tikzcd}
p \longrightarrow q \arrow[r] \arrow[d] \arrow[dr] & q \longrightarrow p \\
\neg p \longrightarrow \neg q & \neg q \longrightarrow \neg p
\end{tikzcd}$$

converse: $q \longrightarrow p$

contrapositive: $\neg q \longrightarrow \neg p$

inverse: $\neg p \longrightarrow \neg q$

p	q	$p \longrightarrow q$	$\neg p$	$\neg q$	$\neg q \longrightarrow$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

compound proposition

$$(p \wedge q) \longrightarrow (p \vee q)$$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \longrightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

compound proposition

- connectives (logical operators)
  - $p \longleftrightarrow q$: <u>biconditional</u> statement

    p if and only if q, p iff q

    $(p \longrightarrow q) \wedge (q \longrightarrow p)$

p	q	$p \longrightarrow q$	$q \longrightarrow p$	$p \longleftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p: You get 100 points on the final

q: You get the grade A

$p \longleftrightarrow q$: You get 100 points on the final if and only if you get the grade A

precedence of logical operators

operator	precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\longrightarrow$	4
$\longleftrightarrow$	5

$p \wedge q \vee r \equiv (p \wedge q) \vee r$

bits

- A bit is a binary digit: 0 or 1
- By convention, 0 represents "false" and 1 represents "true"
- Boolean algebra is like ordinary algebra except that
  - variables stand for bits,
  - + means "or",
  - multiplication means "and"

bit operator

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
1	1	1	1	0
1	0	1	0	1
0	1	1	0	1
0	0	0	0	0

01 1011 0111

11 0001 1101

---

11 1011 1111 bitwise OR

01 0001 0101 bitwise AND

10 1010 1010 bitwise XOR

an example: system specification

1) The diagnostic message is stored in the buffer or it is retransmitted
2) The diagnostic message is not stored in the buffer
3) If the diagnostic message is stored in the buffer, then it is retransmitted

check the consistency!

p: The diagnostic message is stored in the buffer
q: The diagnostic message is retransmitted

p	q	1) $p \vee q$	2) $\neg p$	3) $p \longrightarrow q$
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	T

an example: system specification

1) The diagnostic message is stored in the buffer or it is retransmitted
2) The diagnostic message is not stored in the buffer
3) If the diagnostic message is stored in the buffer, then it is retransmitted

when we have
4) The diagnostic message is not retransmitted
check the consistency!

p: The diagnostic message is stored in the buffer
q: The diagnostic message is retransmitted

p	q	1) $p \vee q$	2) $\neg p$	3) $p \longrightarrow q$	4) $\neg q$
T	T	T	F	T	F
T	F	T	F	F	T
F	T	T	T	T	F
F	F	F	T	T	T

an example: logic puzzles

There is an island that has two kinds of people: knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if
A says "B is a knight",
B says " The two of us are opposite types"

p: A is a knight
q: B is a knight

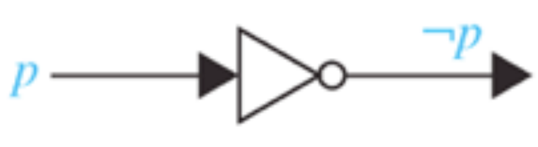
an example: logic puzzles

There is an island that has two kinds of inhabitants: knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if
A says "The two of us are knights"
B says "A is a knave"

p: A is a knight
q: B is a knight

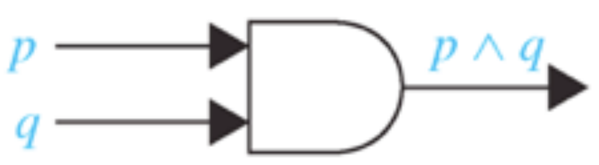
an example: logic circuits

basic logic gates



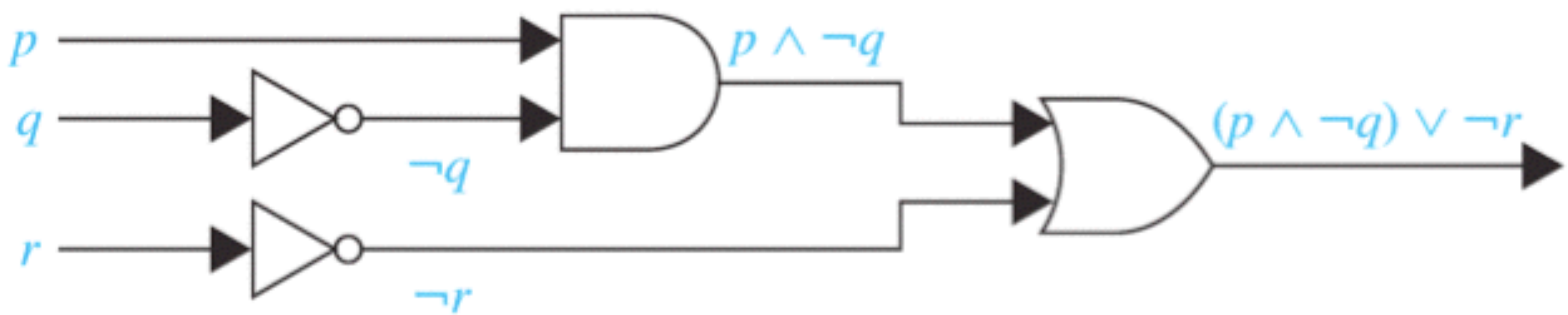
Inverter

OR gate



AND gate

an example: logic circuits



propositional equivalences

- A tautology is a compound proposition that is always true.
- A contradiction is a compound proposition that is always false.
- A contingency is a compound proposition that is neither a tautology nor a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

		↑ tautology	↑ contradiction

logical equivalences

- two syntactically different compound propositions may be semantically identical, which is equivalent
- when compound propositions have the same truth values in all possible cases, they are logically equivalent
- $p \equiv q$: p and q are logically equivalent
- a truth table can be used to determine whether two compound propositions are equivalent

logical equivalences

$p \longrightarrow q \equiv \neg p \vee q$

p	q	$p \longrightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

ref. table 7 and 8 on page 26

De Morgan laws

- $\neg (p \lor q) \equiv \neg p \land \neg q$
- $\neg (p \land q) \equiv \neg p \lor \neg q$

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

logical equivalences

identity laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
domination laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
double negation law	$\neg (\neg p) \equiv p$
commutative laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
negation laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$

logical equivalences

$\neg (p \lor (\neg p \land q))$	$\equiv \neg p \land \neg (\neg p \land q)$	De Morgan law
	$\equiv \neg p \land (p \lor \neg q)$	De Morgan law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	distributive law
	$\equiv F \lor (\neg p \land \neg q)$	negation law
	$\equiv \neg p \land \neg q$	identity law

propositional satisfiability

- A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true.
- A compound proposition is unsatisfiable when the compound proposition is false for all assignments.

$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is satisfiable?

p	q	r	$p \lor \neg q$	$q \lor \neg r$	$r \lor \neg p$	$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	T	F
F	T	F	F	T	T	F
F	F	T	T	F	T	F
F	F	F	T	T	T	T