Discrete Mathematics:
Lecture 2. Predicates

1.4 Predicates and Quantifiers

predicates

- predicate logic is an extension of propositional logic that permits concisely reasoning about whole classes of entities.
- propositional function $P(x) : x > 3$
  - proposition $P(4)$ is true
  - proposition $P(2)$ is false
- the result of applying a predicate P to an object x is the proposition $P(x)$

$R(x, y, z)$: $x + y = z$

what is the truth value of the proposition $R(1,2,3)$? True

quantifiers

quantification expresses the extent to which a predicate is true over a range of elements in a particular domain called the domain of discourse

- universal quantification means that a predicate is true for every element

  $\forall x\ P(x)$: for all x P(x), for every x P(x)

- existential quantification means that there is one or more elements for which the predicate is true

  $\exists x\ P(x)$: for some x P(x), for at least one x P(x)

$Q(x)$: $x<2$

$\forall x\ Q(x)$ is false since there is a counterexample $Q(3)$

$\exists x\ Q(x)$ is true when $x=1$

quantifiers

	when True?	when False?
$\forall x\ P(x)$	P(x) is true for every x	there is an x for which P(x) is false (Counterexample)
$\exists x\ P(x)$	there is an x for which P(x) is true.	P(x) is false for every x

quantifiers

$P(x): x < 2$

what is the truth value of the $\forall x\, P(x)$, where the domain consists of all real numbers?

$\forall x\, P(x)$ is false since $P(3)$ is not true

$P(x): x^2 > 10$

what is the truth value of the $\exists x\, P(x)$, where the universe of discourse consists of the positive integers not exceeding 4?

$\exists x\, P(x)$ is true since $P(1) \lor p(2) \lor p(3) \lor p(4)$ is true

quantifiers

- precedence of quantifiers: the quantifiers $\forall$ and $\exists$ have higher precedence than all logical operations

  $\forall x\, P(x) \vee Q(x) \equiv (\forall x\, P(x)) \vee Q(x)$

- logical equivalences involving quantifiers

  $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- negating quantified expressions

  $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

  $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$

what are the negations of the statements "There is an honest politician"?

$H(x)$: x is honest

$\neg \exists x\, H(x) \equiv \neg (\exists x\, H(x)) \equiv \forall x\, \neg H(x)$: Every politician is dishonest

quantifiers

every student in this class has studied calculus

- for every student x in this class, that student x has studied calculus

  $\forall x\ C(x)$

  C(x): x has studied calculus,

  the domain for x consists of the students in this class

- for every person x, if person x is a student in this class then x has studied calculus

  $\forall x\ (S(x) \longrightarrow C(x))$

  S(x): x is a student in this class

  C(x): x has studied calculus,

  the domain for x consists of all persons

$\neq\ \forall x\ (S(x) \land C(x))$

all people are students in this class and have studied calculus

quantifiers

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink coffee."

assume that the domain consists of creatures

P(x): x is a lion

Q(x): x is fierce

R(x): x drinks coffee

$\forall x\ (P(x) \longrightarrow Q(x))$

$\exists x\ (P(x) \land \neg R(x))$ $\neq$ $\exists x\ (P(x) \longrightarrow \neg R(x))$

this is true as long as there is at least one creature that is not a lion, even if every lion drink coffee

$\exists x\ (Q(x) \land \neg R(x))$

1.5 Nested Quantifiers

nested quantifiers

- $\forall x\ \exists y\ (x + y = 0)$

  for every real number x, there is a real number y such that $x + y = 0$

- Order of quantifiers

  $Q(x,y)$: $x + y = 0$

  domain for all variables consists of all real numbers

  - $\exists y\ \forall x\ Q(x,y)$: there is a real number such that for every real number x, $Q(x, y)$ ☞ false
  - $\forall x\ \exists y\ Q(x,y)$: for every real number x, there is a real number y such that $Q(x, y)$ ☞ true

nested quantifiers

statement	when true?	when false?
$\forall x\ \forall y\ P(x,y)$ $\forall y\ \forall x\ P(x,y)$	P(x,y) is true for every pair x, y	There is a pair x, y for which P(x, y) is false
$\forall x\ \exists y\ P(x,y)$	For every x, there is a y for which P(x, y) is true	There is an x such that P(x,y) is false for every y
$\exists x\ \forall y\ P(x,y)$	There is an x for which P(x, y) is true for every y	For every x, there is a y for which P(x, y) is false
$\exists x\ \exists y\ P(x,y)$ $\exists y\ \exists x\ P(x,y)$	There is a pair x, y for which P(x, y) is true	P(x, y) is false for every pair x, y

how to translate mathematical statements into logical expressions

The sum of two positive integers is always positive

$\forall x \, \forall y \, (x + y > 0)$,

where the domain for both variables consists of all positive integers

$\forall x \, \forall y \, ((x > 0) \wedge (y > 0) \longrightarrow (x + y > 0))$,

where the domain for both variables consists of all integers

how to translate English statements into logical expressions

If a person is female and is a parent, then this person is someone's mother

F(x): x is female

P(x): x is a parent

M(x, y): x is the mother of y

$\forall x\ ((F(x) \land P(x)) \longrightarrow \exists y\ M(x,y))$

$\forall x\ \exists y\ ((F(x) \land P(x)) \longrightarrow M(x,y))$

how to translate logical expressions to natural language

$\forall x\, (C(x) \vee \exists y (C(y) \wedge F(x, y)))$,
where $C(x)$ is "x has a computer," $F(x,y)$ is "x and y are friends", and the domain for x and y consists of all students in your school.

For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.

Every student in your school has a computer or has a friend who has a computer.

1.6 Rules of Inference

valid arguments

If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.

- an argument is a sequence of statements that end with a conclusion
- an argument is valid if the truth of all its premises implies that the conclusion is true

valid arguments

If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.

p: You have a current password

q: You can log onto the network

$$\begin{array}{l} p \longrightarrow q \\ \underline{p \qquad\qquad} \\ \therefore\ q \end{array}$$

premises

conclusion

valid arguments

If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.

p: You have a current password
q: You can log onto the network

$$\begin{array}{l} p \longrightarrow q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

← premises
← conclusion

$$((p \longrightarrow q) \wedge p) \longrightarrow q$$

valid arguments

$$((p \longrightarrow q) \wedge p) \longrightarrow q$$

p	q	$p \longrightarrow q$	$((p \longrightarrow q) \wedge p)$	$((p \longrightarrow q) \wedge p) \longrightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- when $p \longrightarrow q$ and p are true, q is true
- this argument form is valid because whenever all its premises are true, the conclusion must be true
- an argument form is valid when it is tautology

rule of inference

rule of inference

$$\begin{array}{l} p \longrightarrow q \\ \underline{p \qquad\qquad} \\ \therefore\ q \end{array}$$

tautology

$$((p \longrightarrow q) \land p) \longrightarrow q$$

p	q	$p \longrightarrow q$	$((p \longrightarrow q) \land p)$	$((p \longrightarrow q) \land p) \longrightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

rule of inference

rule of inference

$$\begin{array}{rl} & p \longrightarrow q \\ & \neg\, q \\ \hline \therefore & \neg\, p \end{array}$$

tautology

$$(\neg q \wedge (p \longrightarrow q)) \longrightarrow \neg\, p$$

p	q	$\neg p$	$\neg q$	$p \longrightarrow q$	$((p \longrightarrow q) \wedge \neg q)$	$((p \longrightarrow q) \wedge \neg q) \longrightarrow$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

rule of inference

rule of inference	tautology	name
$p \longrightarrow q$ $\underline{p \quad\quad}$ $\therefore q$	$((p \longrightarrow q) \wedge p) \longrightarrow q$	Modus ponens
$p \longrightarrow q$ $\underline{\neg q \quad\quad}$ $\therefore \neg p$	$((p \longrightarrow q) \wedge \neg q) \longrightarrow \neg p$	Modus tollens
$p \longrightarrow q$ $\underline{q \longrightarrow r}$ $\therefore p \longrightarrow r$	$((p \longrightarrow q) \wedge (q \longrightarrow r)) \longrightarrow (p \longrightarrow r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p \quad\quad}$ $\therefore q$	$((p \vee q) \wedge \neg p) \longrightarrow q$	Disjunctive syllogism

rule of inference

rule of inference	tautology	name
$\dfrac{p}{\therefore\ p \lor q}$	$p \longrightarrow (p \lor q)$	Addition
$\dfrac{p \land q}{\therefore\ p}$	$(p \land q) \longrightarrow p$	Simplification
$\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \land q}$	$(p \land q) \longrightarrow (p \land q)$	Conjunction
$\dfrac{\begin{array}{c} p \lor q \\ \neg p \lor r \end{array}}{\therefore\ q \lor r}$	$((p \lor q) \land (\neg p \lor r)) \longrightarrow (q \lor r)$	Resolution

rule of inference

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow

p: it is raining today

q: we will not have a barbecue today

r: we will have a barbecue tomorrow

$$\begin{array}{rl} & p \longrightarrow q \\ & q \longrightarrow r \\ \hline \therefore & p \longrightarrow r \end{array}$$

Hypothetical syllogism

rule of inference

premises:

It is not sunny this afternoon and it is colder than yesterday.

We will go swimming only if it is sunny.

If we do not go swimming, then we will take a canoe trip.

If we take a canoe trip, then we will be home by sunset.

conclusion:

We will be home by sunset

(1) $\neg p \wedge q$

(2) $r \longrightarrow p$

(3) $\neg r \longrightarrow s$

(4) $s \longrightarrow t$

p: It is sunny this afternoon

q: It is colder than yesterday

r: we will go swimming

s: We will take a canoe trip

t: We will be home by sunset

$\neg p \wedge q$	$\longrightarrow$	$\neg p$	simplification
$\neg p \wedge (r \longrightarrow p)$	$\longrightarrow$	$\neg r$	Modus tollens
$\neg r \wedge (\neg r \longrightarrow s)$	$\longrightarrow$	s	Modus ponens
$s \wedge (s \longrightarrow t)$	$\longrightarrow$	t	Modus ponens

rules of inference for quantified statements

rule of inference	name
$\dfrac{\forall x\ P(x)}{\therefore\ P(c)}$	universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\ P(x)}$	universal generalization
$\dfrac{\exists x\ P(x)}{\therefore\ P(c) \text{ for some element } c}$	existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x\ P(x)}$	existential generalization

rules of inference for quantified statements

premises:

Everyone in this discrete mathematics class has taken a course in computer science

Marla is a student in this class

conclusion

Marla has taken a course in computer science

D(x): x is in the discrete mathematics class

C(x): x has taken a course in computer science

1. $\forall x\, (D(x) \longrightarrow C(x))$
2. D(Marla)
3. D(Marla) $\longrightarrow$ C(Marla) universal instantiation from 1
4. C(Marla) Modus ponens from 2 and 3

$$\begin{array}{l} \forall x\, (P(x) \longrightarrow Q(x)) \\ \underline{P(a), \text{ where a is a particular element in the domain}} \\ \therefore\ Q(a) \end{array}$$

universal modus ponens