Discrete Mathematics: Lecture 7. Sequence

sequences

- a discrete structure used to represent an ordered list

the sequence $\{a_n\}$, where $a_n = n$

$\{a_0, a_1, a_2, a_3, \ldots\} = \{0, 1, 2, 3, \ldots\}$

geometric progression

geometric progression is a sequence of the form

$a,\ ar,\ ar^2,\ ar^3, \ldots, ar^n, \ldots,$

where the initial term a and common ratio r are real numbers

a sequence $\{a_n\}$, where $a_n = 2 \cdot 5^n$

$\{a_n\} = \{2, 10, 50, 250, 1250, \ldots\}$

arithmetic progression

arithmetic progression is a sequence of the form

$a, \; a + d \;\; a + 2d, \; a + 3d, \ldots, a + nd, \ldots$

where the initial term a and common difference d are real numbers

a sequence $\{a_n\}$, where $a_n = -1 + 4n$

$\{a_n\} = \{-1, 3, 7, 11, \ldots\}$

recurrence relations

a recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence

a sequence $\{a_n\}$, where $a_n = a_{n-1} + 3$, $a_0 = 2$

$\{a_n\} = \{2, 5, 8, 11, \ldots\}$

Fibonacci sequence, $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$

$f_2 = 0 + 1 = 1$
$f_3 = 1 + 1 = 2$
$f_4 = 1 + 2 = 3$
$f_5 = 2 + 3 = 5$
$f_6 = 3 + 5 = 8$

recurrence relations

determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n=2,3,4,\ldots$

$$a_n = 2a_{n-1} - a_{n-2} = 2\,(3(n-1)) - 3(n-2) = 6n - 6 - 3n + 6 = 3n$$

recurrence relations

solve the recurrence relation and initial condition for $a_n = a_{n-1} + 3, a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3$
$a_3 = a_2 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
$a_4 = a_3 + 3 = 2 + 3 \cdot 3$
$\vdots$
$a_n = a_{n-1} + 3 = 2 + 3(n-1)$

forward substitution

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\vdots \\
&= a_{n-(n-1)} + 3\,(n-1) = 2 + 3(n-1) \\
&= a_1 + 3\,(n-1) = 2 + 3(n-1)
\end{aligned}
$$

backward substitution

recurrence relations

suppose that a person deposit \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

$p_n = p_{n-1} + 0.11P_{n-1} = 1.11P_{n-1}$, $p_0 = 10{,}000$

$p_1 = 1.11p_0$
$p_2 = 1.11P_1 = 1.11^2p_0$
$p_3 = 1.11p_2 = 1.11^3p_0$
$\vdots$
$p_n = 1.11^nP_0$

$p_{30} = 1.11^{30}p_0 = \$228{,}922.97$

summations

$$a_m + a_{m+1} + \ldots + a_n = \sum_{j=m}^{n} a_j$$

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

summations

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if r} \neq \text{1 and r} \neq 0 \\ (n+1)a & \text{if r = 1 and r} \neq 0 \end{cases}$$

when r ≠ 1

$$s_n = \sum_{j=0}^{n} ar^{j}$$

$$rs_n = r\sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} ar^{j+1} = \sum_{k=1}^{n+1} ar^{k} == \sum_{k=0}^{n} ar^{k} + (ar^{n+1} - a)$$

$$= s_n + (ar^{n+1} - a)$$

$$s_n = \frac{ar^{n+1} - a}{r-1}$$

double summations

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i$$

= 6 + 12 + 18 + 24 = 60

summations

$$\sum_{s\in\{0,2,4\}} s = 0 + 2 + 4 = 6$$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{149} k^2$$

$$= \frac{100 * 101 * 201}{6} - \frac{49 * 50 * 99}{6} = 29725$$

ref. summation formulae in table 2 on page 166

finite, countable, and uncountable sets

determine whether each of these sets is finite, countably infinite, or uncountable

- the negative integers
- the even integers
- the integers that are multiples of 7
- the integers less than 100
- the positive integers less than 1000000000
- the real numbers between 0 and 1/2

finite sets

A set S is finite with cardinality $n \in N$ if there is a bijection from the set $\{0, 1, \ldots n-1\}$ to S

- the integers that are multiples of 7 ?
- the integers less than 100 ?
- the negative integers ?
- the even integers ?

countable sets

- a set that is either finite or has the same cardinality as the set of positive integers is called countable

- the sets A and B have the same cardinality iff there is a one-to-one correspondence from A to B

countable sets

Show that the set of odd positive integers is a countable set

we can exhibit a one-to-one correspondence between the set of odd positive integers and the set of positive integers

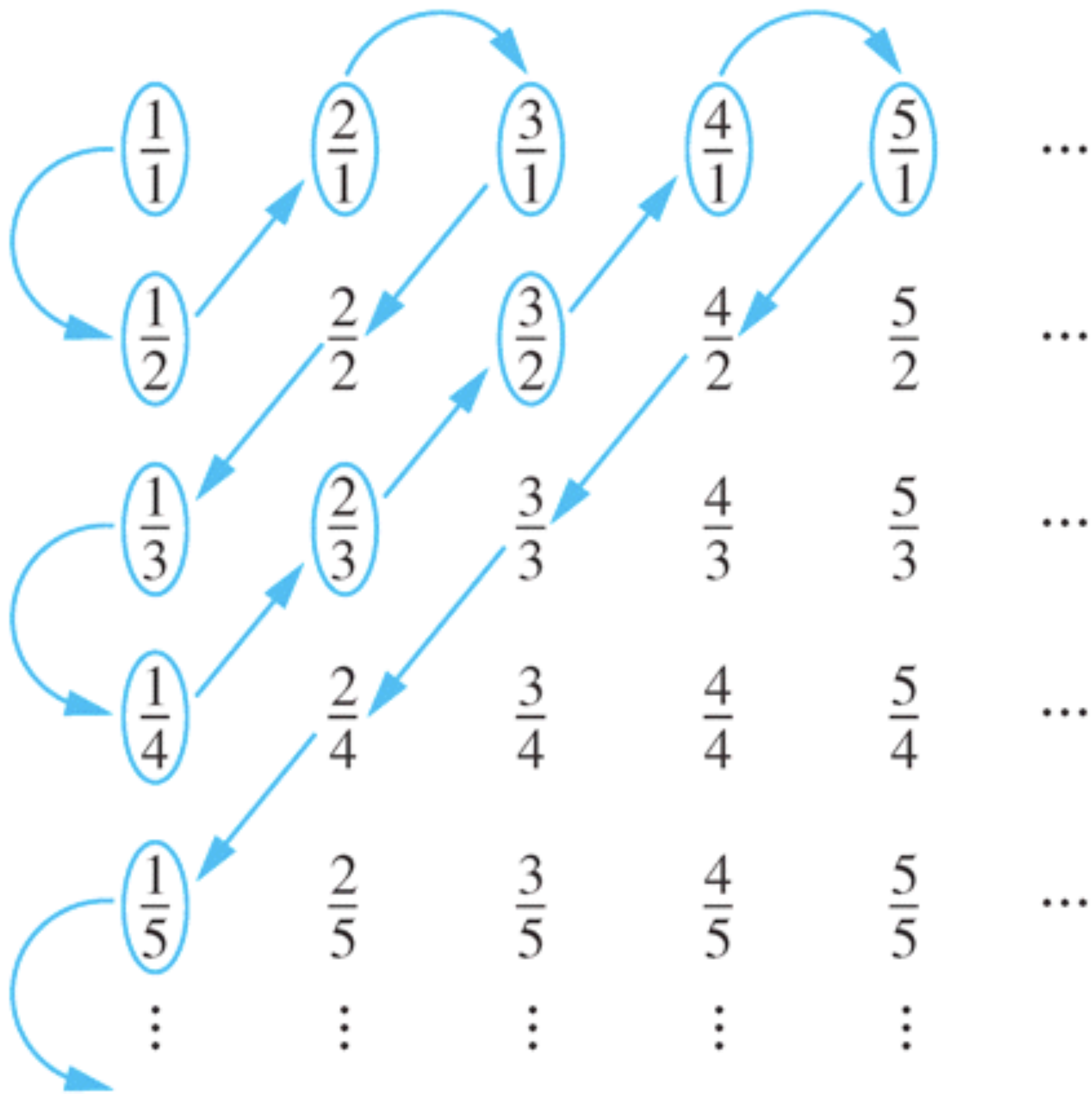
$f(n) = 2n - 1$
$f: Z+ \rightarrow \{\text{odd positive integers}\}$

1) one-to-one
if $(f(n) = f(m)) \rightarrow (n = m)$
suppose $f(n) = f(m)$ $2n-1 = 2m-1$ $n=m$

2) onto
if t is an odd positive integer, $t = 2k - 1$, where k is a natural number
since $2k - 1 = f(k)$, f is onto

countable sets

Show that the set of positive rational number is countable



uncountable sets

show that the set of real numbers is uncountable

$r_1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots = 0.23794102$
$r_2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots = 0.44590138$
$r_3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots = 0.09118764$
$r_4 = 0.d_{41}d_{42}d_{43}d_{44} \ldots = 0.80553900$
$\vdots$

$r = 0.d_1d_2d_3d_4 \ldots$

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

$r = 0.d_1d_2d_3d_4 \ldots = 0.4544$

matrices

a matrix is a rectangular array of numbers

3 x 2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$A = [a_{ij}]$, $B = [b_{ij}]$, $A + B = [a_{ij} + b_{ij}]$

A: m x k matrix, B: k x n matrix, $AB = [c_{ij}]$ (m x n matrix)

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$

matrices

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

matrices

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

matrices

$$AB \neq BA$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1*2 + 1*1 & 1*1 + 1*1 \\ 2*2 + 1*1 & 2*1 + 1*1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2*1 + 1*2 & 2*1 + 1*1 \\ 1*1 + 1*2 & 1*1 + 1*1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

identity matrices

identity matrix of order n is $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ if $i \neq j$

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$AI_n = I_mA = A$ A: m x n matrix

$A^0 = I_n$ A: n x n matrix

$A^r = AAA\dots A$ (r times)

transpose

$A = [a_{ij}]$

transpose of A, $A^t = [b_{ij}]$, $b_{ij} = a_{ji}$

transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

symmetric matrix

a square matrix A is called symmetric if $A = A^t$

$A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

zero-one matrix

$A = [a_{ij}]$ and $B = [b_{ij}]$ m x n zero-one matrices

join of A and B ($A \vee B$) is the zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$

meet of A and B ($A \wedge B$) is the zero-one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

boolean product

$A = [a_{ij}]$ (m x k zero-one matrix) and $B = [b_{ij}]$ (k x n zero-one matrix)

$A \odot B$ (Boolean product of A and B) = [cij] where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

boolean product

$A^{[r]} = A \odot A \odot A \odot \ldots \odot A$ (r times)

$A^{[0]} = I_n$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$