Discrete Mathematics: Lecture 5. Set theory

sets

- a set is an unordered collection of zero or more distinct objects.
- this object is called an element or a member of the set
- $a \in A$: a is an element of the set A, $A = \{a\}$
- $a \notin B$: a is not an element of the set B
- $A = B$: two sets are equal iff $\forall x\ (x \in A \longleftrightarrow x \in B)$
- empty set, null set $\varnothing = \{\ \}$, $\neg\exists x\ x \in \varnothing$

sets

- a set V is denoted by listing all of its elements in curly braces
  - $V = \{1, 3, 5, 7, 9\}$
- set builder notation
  - $\{x \mid P(x)\}$ is the set of all x such that P(x)
  - $V = \{x \mid x$ is an odd positive integer less than $10\}$
  - $V = \{x \in Z^+ \mid x$ is odd and $x < 10\}$
- sets are unordered
  - $\{1, 3, 5\} = \{3, 5, 1\}$
- all elements are distinct.
  - $\{3, 5, 1\} = \{1, 3, 3, 5, 5, 5\}$

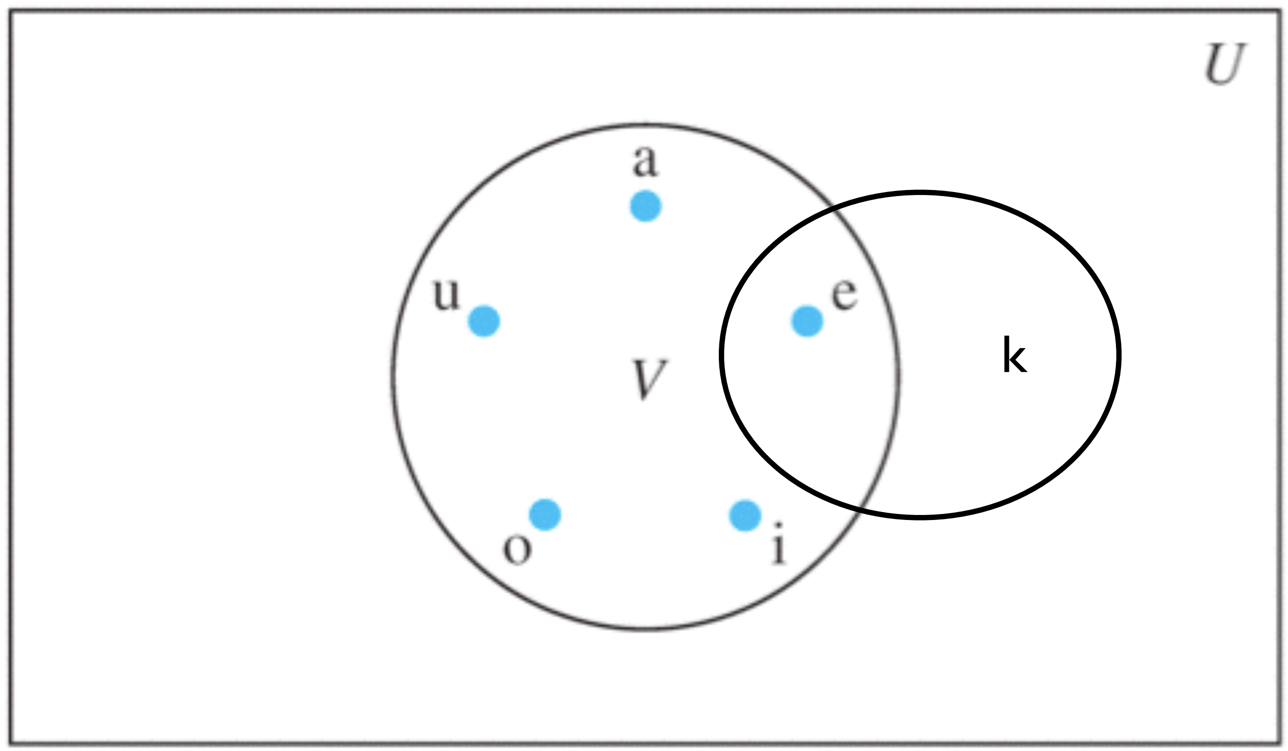
sets

- set equality
  - two sets are equal if and only if they contain exactly the same elements
  - {1, 2, 3, 4}

    = {x | x is an integer where x>0 and x<5 }

    = {x | x is a positive integer whose square is >0 and <25}
- infinite set
  - **N** = {0, 1, 2, …}   the natural numbers.
  - **Z** = {…, -2, -1, 0, 1, 2, …} the integers
  - **R** = the "real" numbers, such as 374.1828471929498181917…

Venn Diagrams

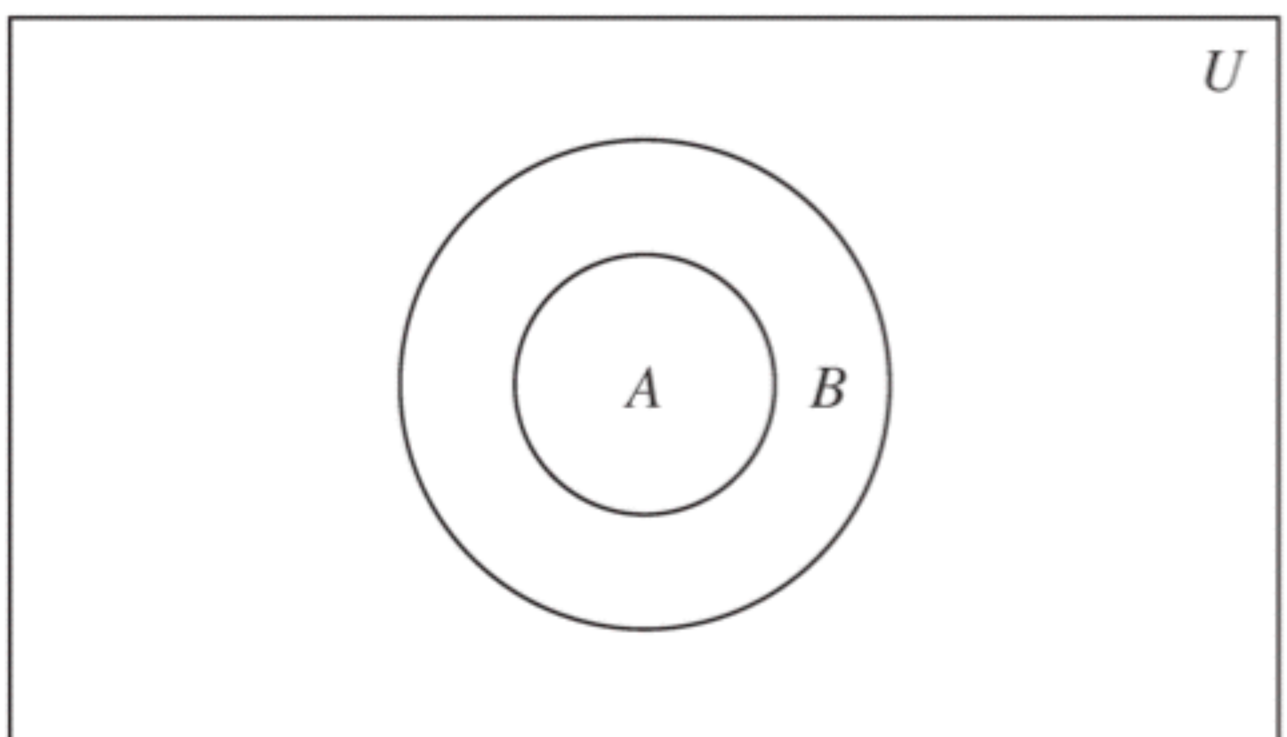


subsets

- the set A is a subset of B iff every element of A is also an element of B

  $A \subseteq B$: $\forall x\ (x \in A \rightarrow x \in B)$
- for every set S, $\varnothing \subseteq S$ and $S \subseteq S$
- $A \subset B$: A is a proper subset of B

  $\forall x\ (x \in A \rightarrow x \in B) \land \exists x\ (x \in B \land x \notin A)$
- $A = \{\varnothing\ \{a\}, \{b\}, \{a,b\}\} = \{x \mid x \text{ is a subset of the set } \{a,b\}\}$
- $a \neq \{a\} \neq \{\{a\}\}$



size of a set

- when a set S has n distinct elements,
  - S is a finite set
  - n is the cardinality of S
  - $|S| = n$

$A = \{x \mid x$ is odd positive integers, $x < 10\}$, $|A|=5$

$|\varnothing| = 0$
$|\{\{a,b,c\}, \{d,e,f\}\}| = 2$

power sets

- $P(S)$: the power set of S is the set of all subsets of the set S
- $P(S) = \{x \mid x \subseteq S\}$
- $|P(S)| = 2^{|s|}$

$p(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{2,0\}, \{0,1,2\}\}$

$p(\emptyset) = \{\emptyset\}$
$p(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
$p(\{1\}) = \{\emptyset, \{1\}\}$

Cartesian products

- ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,... and a_n as its nth element
- ordered 2-tuples are called ordered pairs

  $(a, b) = (c, d)$ *iff* $a=c$ and $b=d$

- $A \times B$: Cartesian product of A and B

  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

- $A^2 = A \times A$
- Cartesian product is not commutative.

  $\neg\forall A, B, \quad A \times B = B \times A$

Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

set with quantifiers, truth sets, ...

- $\forall x \in S \ (P(x))$: $\forall x\ (x \in S \rightarrow P(x))$
  $\exists x \in S \ (P(x))$: $\exists x\ (x \in S \wedge P(x))$

- given a predicate P, and a domain D, the truth set of P is the set of elements x in D for which p(x) is true

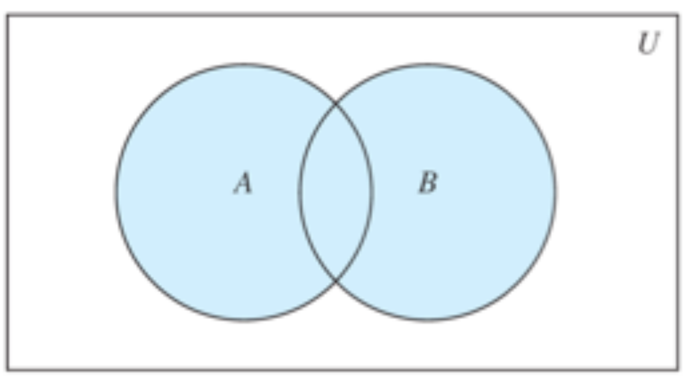
what are the truth sets of the predicates P(x), Q(x), and R(x)?

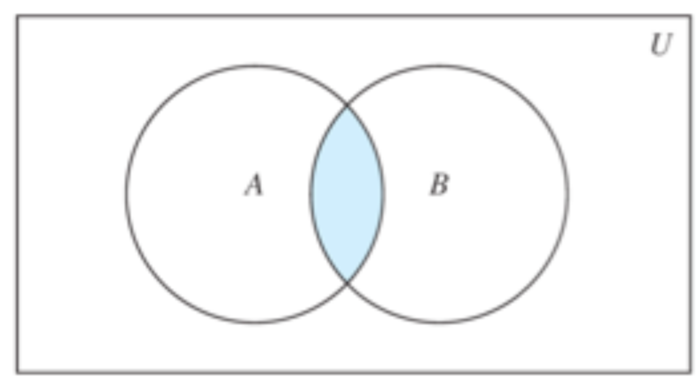
the domain is the set of integers
P(x): $|x| = 1$
Q(x): $x^2 = 2$
R(x): $|x| = x$

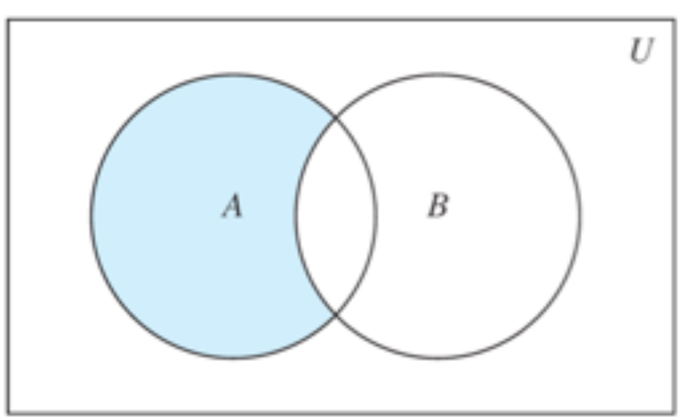
$P = \{x \in Z \mid |x| = 1\} = \{-1, 1\}$
$Q = \{x \in Z \mid x^2 = 2\} = \{\}$
$R = \{x \in Z \mid |x| = x\} = \{y \in N\}$

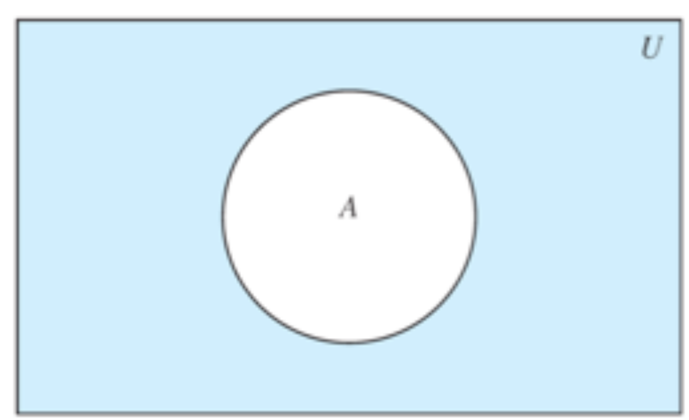
set operations

- union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- disjoint: $A \cap B = \varnothing$
- difference of A and B: $A - B = \{x \mid x \in A \wedge x \notin B\}$
- complement of A with respect to U: $\overline{A} = U - A = \{x \in U \mid x \notin A\}$
- inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$









set identity

$A \cap U = A$ $A \cup \varnothing = A$	identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	domination laws
$A \cup A = A$ $A \cap A = A$	idempotent laws
$\overline{\left(\overline{A}\right)}$	complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	commutative laws

set identity

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	complement laws

proving set identity

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

if $x \in \overline{A \cap B}$, then $x \in (\overline{A} \cup \overline{B})$

$x \notin (A \cap B)$ => $\neg\,((x \in A) \wedge (x \in B))$ => $\neg\,(x \in A) \vee \neg\,(x \in B)$

=> $(x \notin A) \vee (x \notin B)$ => $(x \in \overline{A}) \vee (x \in \overline{B})$ => $x \in (\overline{A} \cup \overline{B})$

2) $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$

if $x \in (\overline{A} \cup \overline{B})$ then $x \in \overline{A \cap B}$

$x \in (\overline{A} \cup \overline{B})$ => $((x \notin A) \vee (x \notin B))$ => $\neg\,(x \in A) \vee \neg\,(x \in B)$

=> $\neg\,((x \in A) \wedge (x \in B))$ => $\neg\,(x \in (A \cap B))$ => $x \in \overline{(A \cap B)}$

proving set identity

use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

generalized unions and intersections

- the union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- the intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

representation of sets with bit strings

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$.

- a bit string that represents the set of all odd integers in U

  1010101010

- a bit string that represents the set of integers not exceeding 5 in U

  1111100000

representation of sets with bit strings

the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$
use bit string to find the union and intersection of these sets

$\{1, 2, 3, 4, 5\}$: 1 1 1 1 1 0 0 0 0
$\{1, 3, 5, 7, 9\}$: 1 0 1 0 1 0 1 0 1

- union

$$\begin{array}{r} 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ \vee\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \\ =\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1 \end{array}$$

$\{1, 2, 3, 4, 5, 7, 9\}$

- intersection

$$\begin{array}{r} 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ \wedge\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \\ =\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0 \end{array}$$

$\{1, 3, 5\}$