CS181u Applied Logic & Automated Reasoning

Lecture 03: DPLL, Model Counting

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Harvey Mudd College
Department of Computer Science

Satisfiability for Boolean Logic

Given a formula ϕ, is it possible to assign all variables the values T or F so that the formula evaluates to T?

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

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$$(x, y, z, w, v) = (T, F, T, F, T)$$

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A satisfying assignment is called a model for ϕ.

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$$(x, y, z, w, v) = (T, F, T, F, T)$$

A satisfying assignment is called a model for ϕ.

A harder problem: *how many models are there?*

Model Counting for Boolean Logic

x	y	z	w	v	ϕ
F	F	F	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
T	F	F	T	T	F
T	F	T	F	F	F
T	F	T	F	T	T
T	F	T	T	F	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	F	F	T	F
T	T	F	T	F	F
T	T	F	T	T	F
T	T	T	F	F	T
T	T	T	F	T	T
T	T	T	T	F	T
T	T	T	T	T	T

Easy! Just make the truth table and count!

Model Counting for Boolean Logic

x	y	z	w	v	ϕ
F	F	F	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
T	F	F	T	T	F
T	F	T	F	F	F
T	F	T	F	T	T
T	F	T	T	F	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	F	F	T	F
T	T	F	T	F	F
T	T	F	T	T	F
T	T	T	F	F	T
T	T	T	F	T	T
T	T	T	T	F	T
T	T	T	T	T	T

Easy! Just make the truth table and count!

Model Counting for Boolean Logic

x	y	z	w	v	ϕ
F	F	F	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
T	F	F	T	T	F
T	F	T	F	F	F
T	F	T	F	T	T
T	F	T	T	F	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	F	F	T	F
T	T	F	T	F	F
T	T	F	T	T	F
T	T	T	F	F	T
T	T	T	F	T	T
T	T	T	T	F	T
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Easy! Just make the truth table and count!

ϕ has 6 models.

Model Counting for Boolean Logic

x	y	z	w	v	ϕ
F	F	F	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
T	F	F	T	T	F
T	F	T	F	F	F
T	F	T	F	T	T
T	F	T	T	F	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	F	F	T	F
T	T	F	T	F	F
T	T	F	T	T	F
T	T	T	F	F	T
T	T	T	F	T	T
T	T	T	T	F	T
T	T	T	T	T	T

Easy! Just make the truth table and count!

ϕ has 6 models.

This approach is $\Theta(2^n)$.

Bummer!

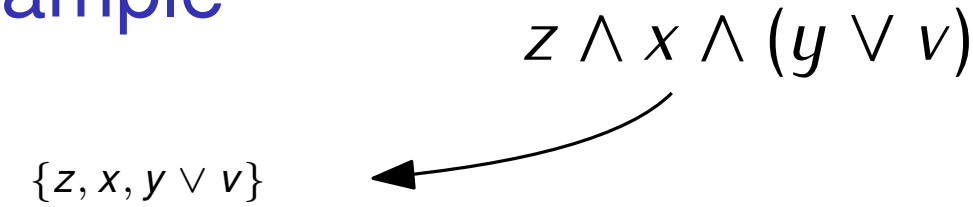
Model Counting for Boolean Logic

In 1962, Davis, Putnam, Logemann, Loveland published the DPLL algorithm for Boolean SAT.

DPLL is the backbone of modern industry-grade automated theorem proving (Amazon, Microsoft, NASA, ...)

DPLL is also a model counting algorithm!

DPLL Execution Example



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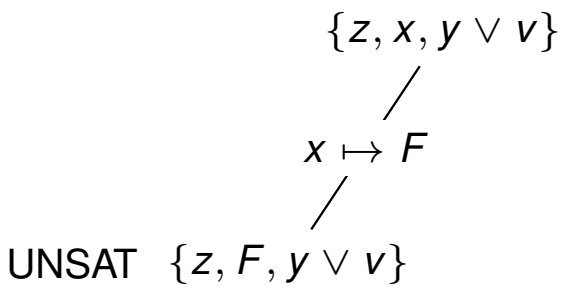

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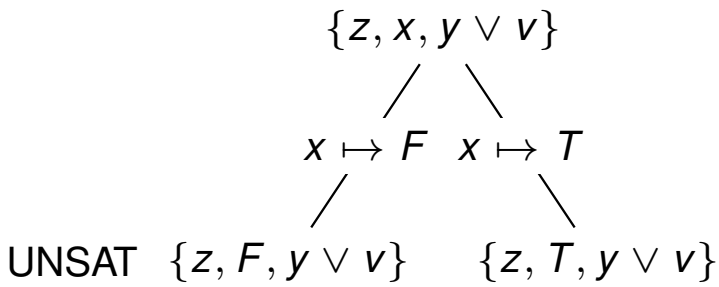

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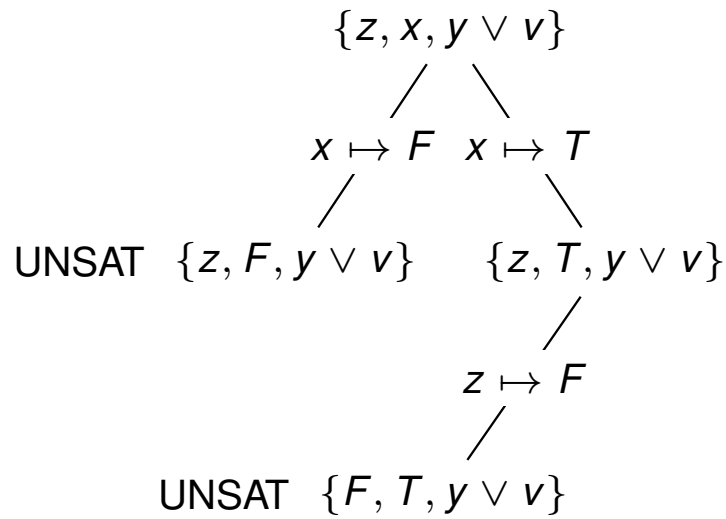

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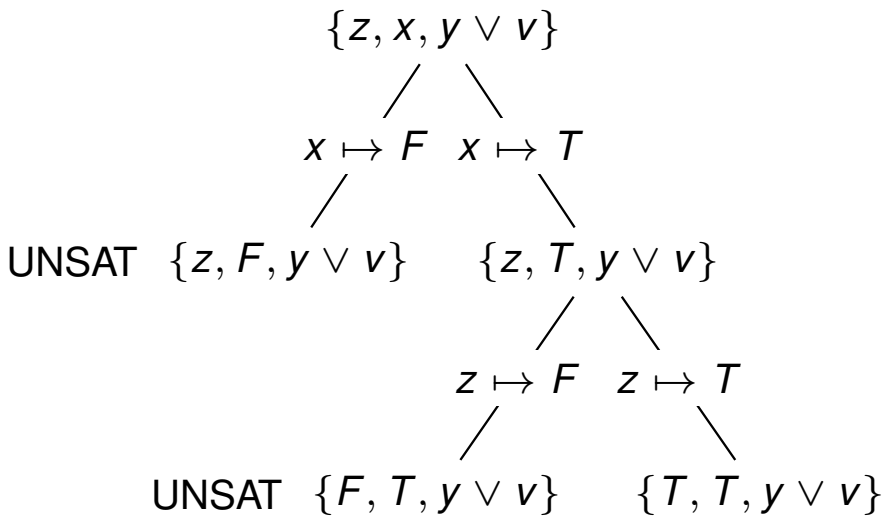

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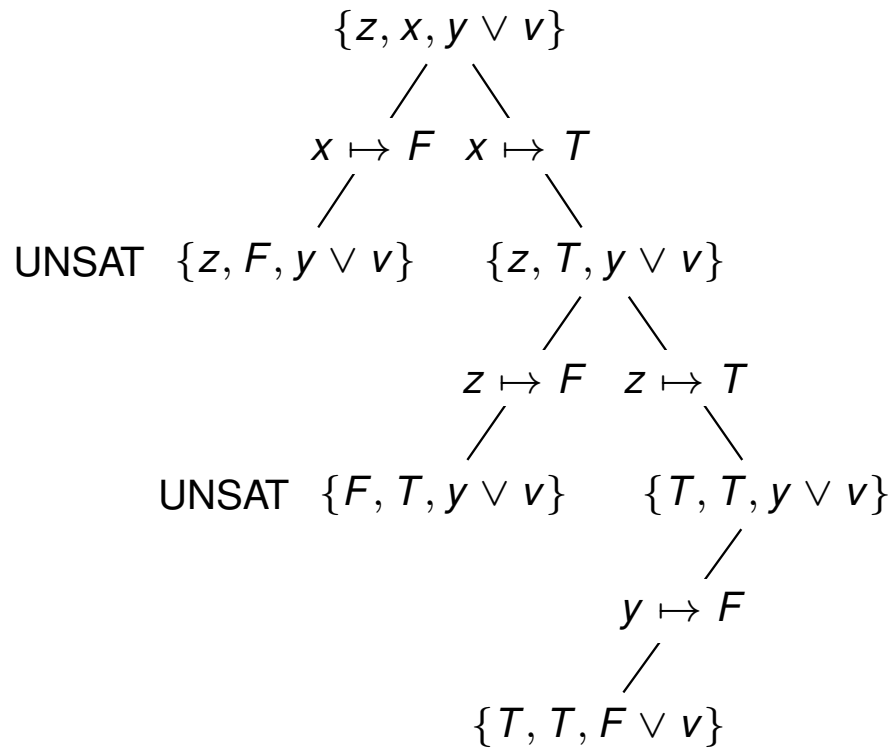

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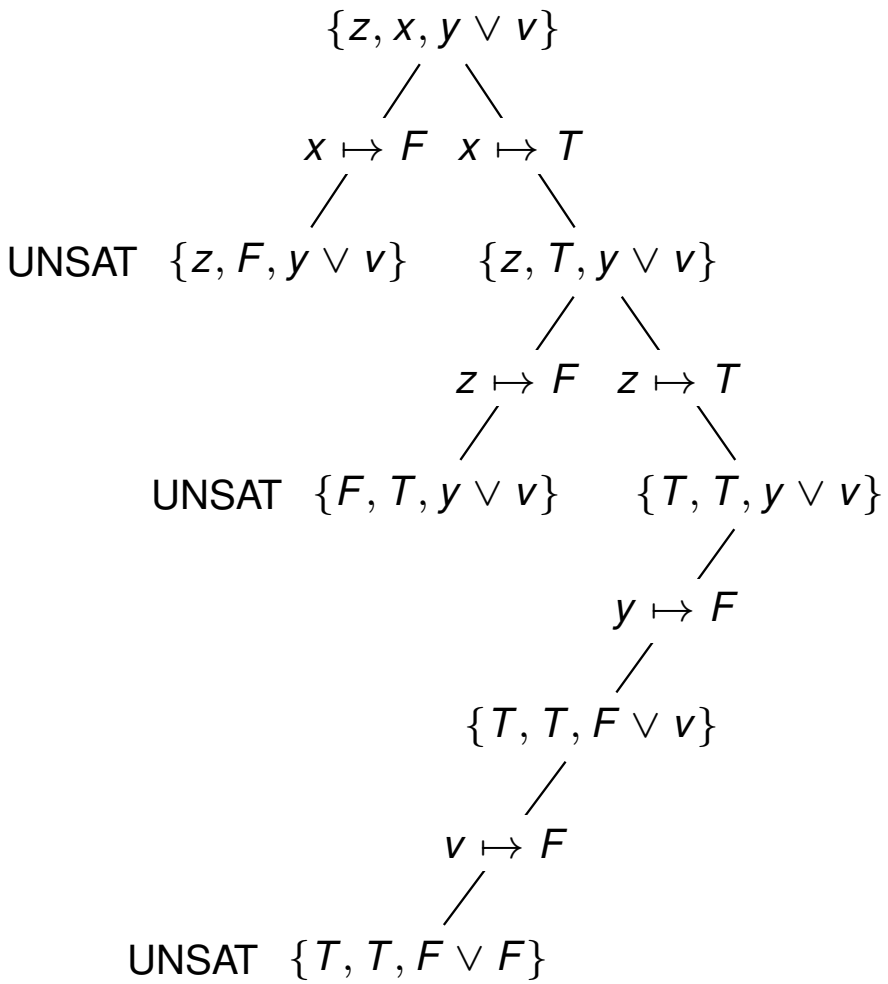

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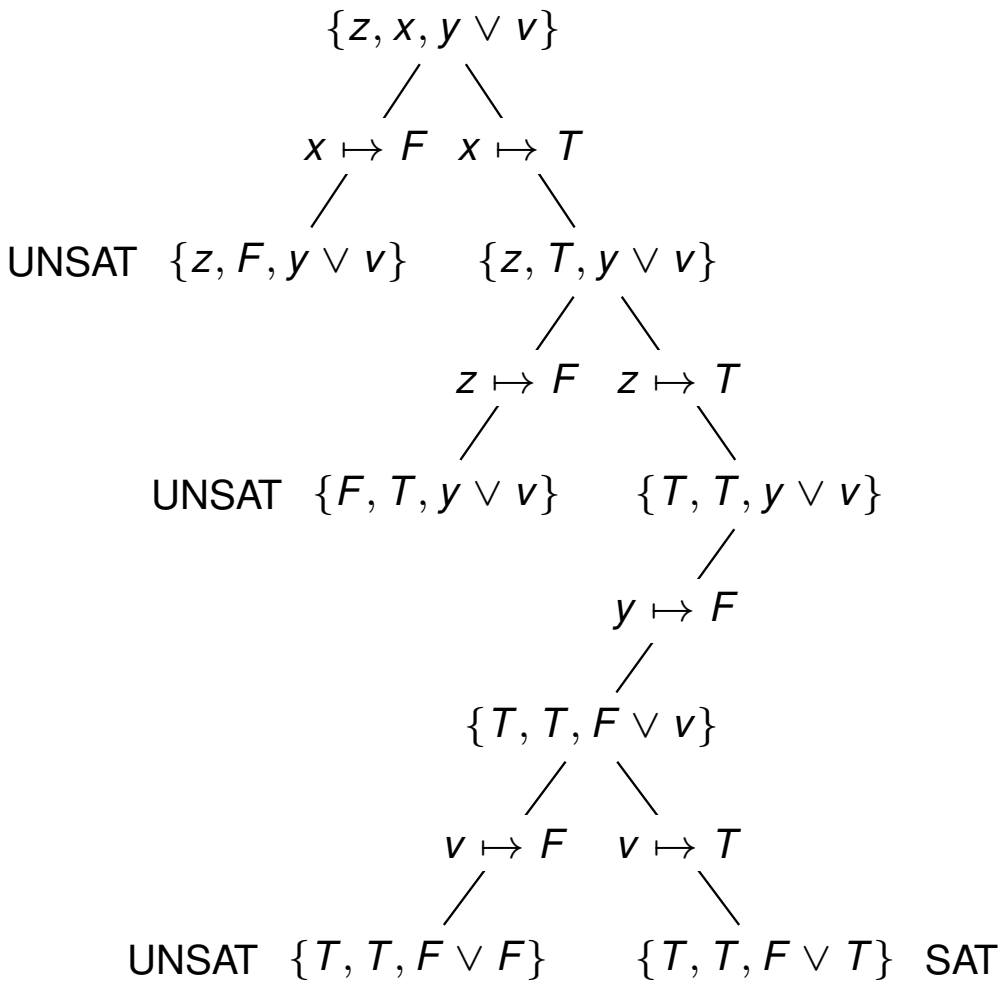

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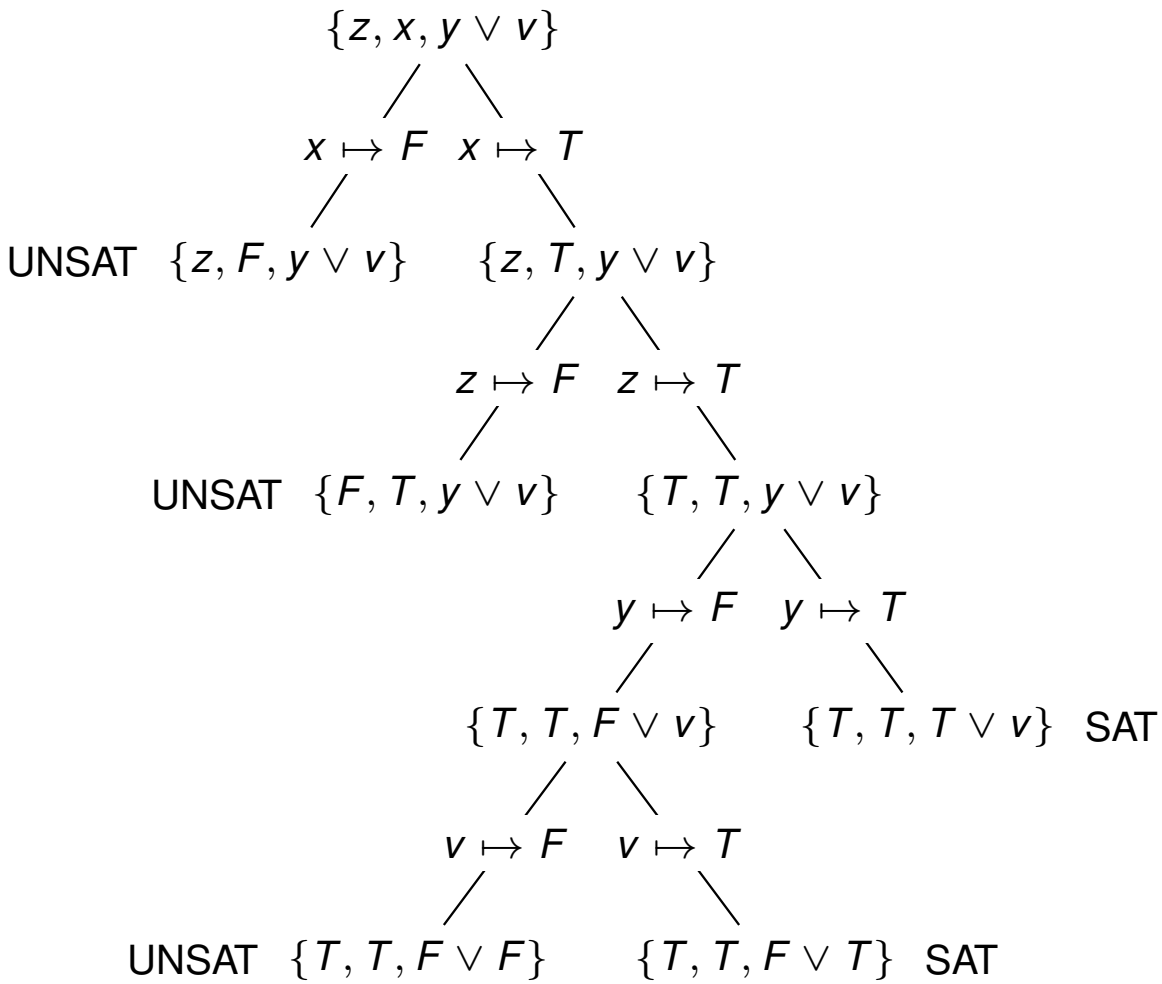

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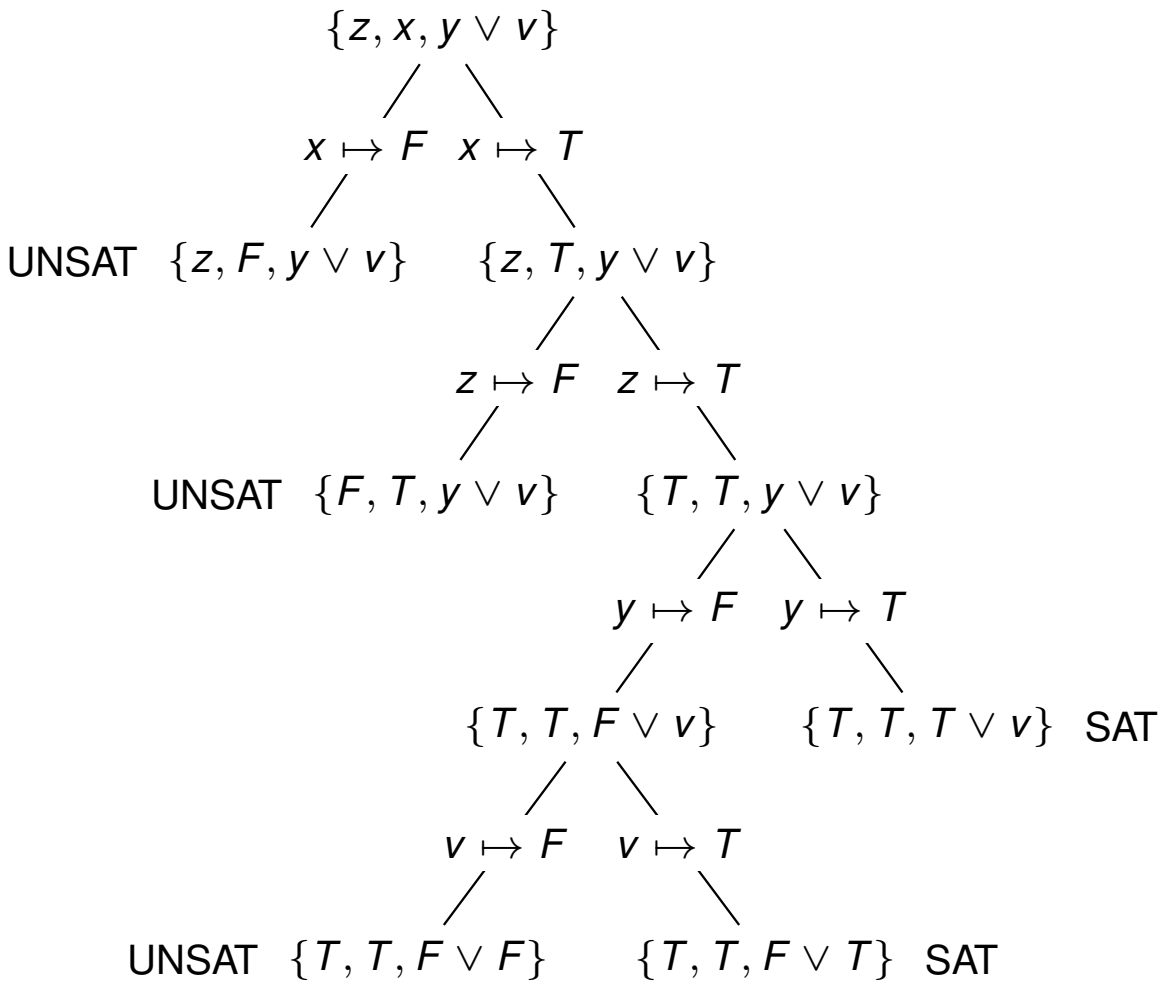

Conclusion: ϕ is satisfiable

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Function : DPLL(ϕ)
Input : CNF formula ϕ over n variables
Output : true or false, the satisfiability of F
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** false
  if all clauses of ϕ satisfied **then return** true
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto \mathit{true}]$) $\vee$ DPLL($\phi[x \mapsto \mathit{false}]$)
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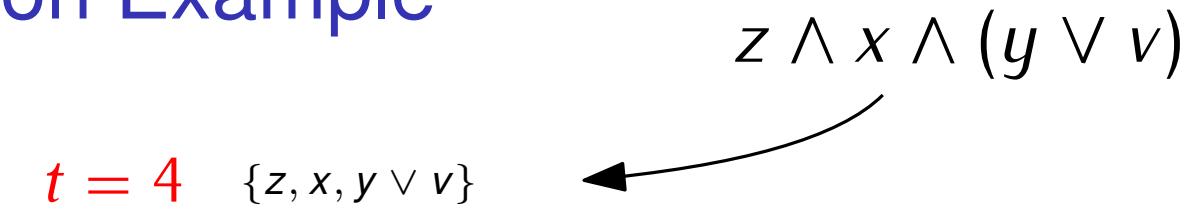

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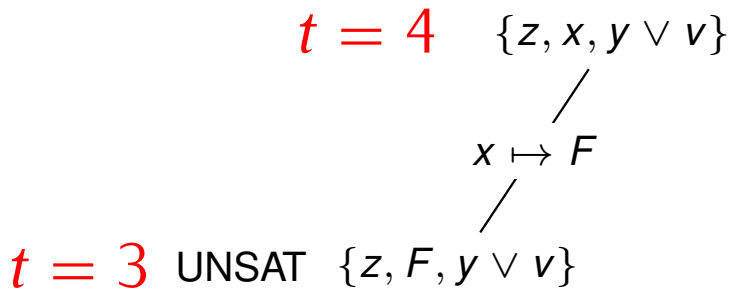

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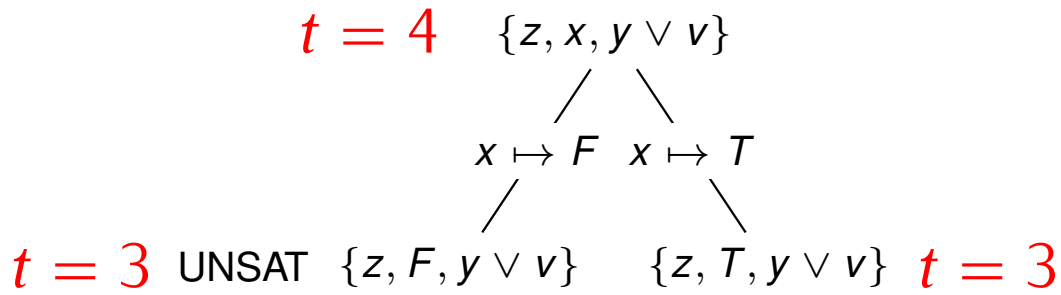

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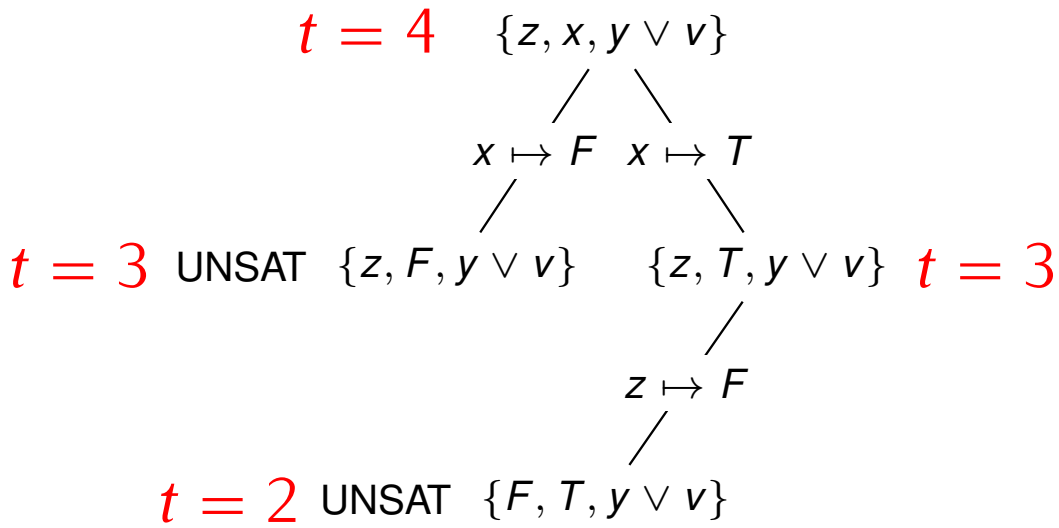

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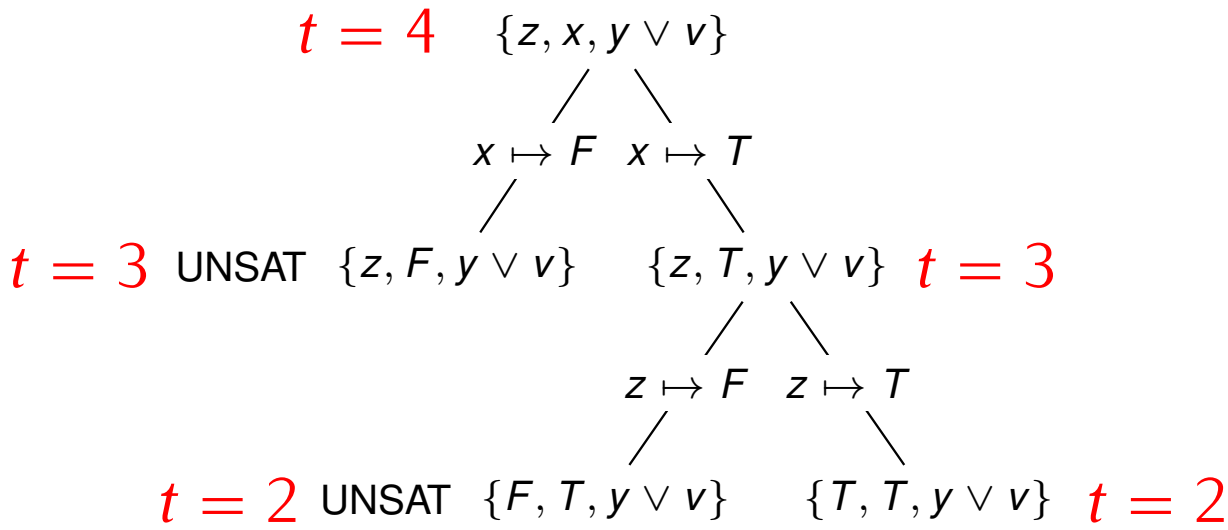

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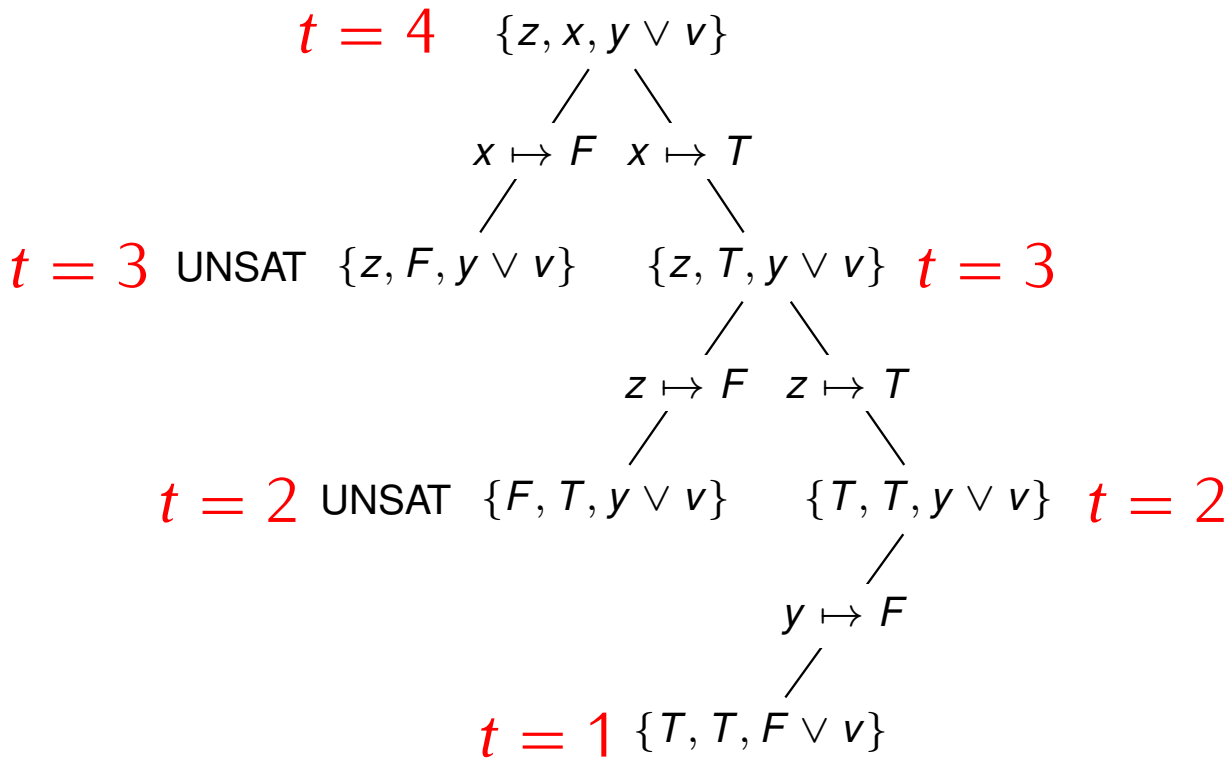

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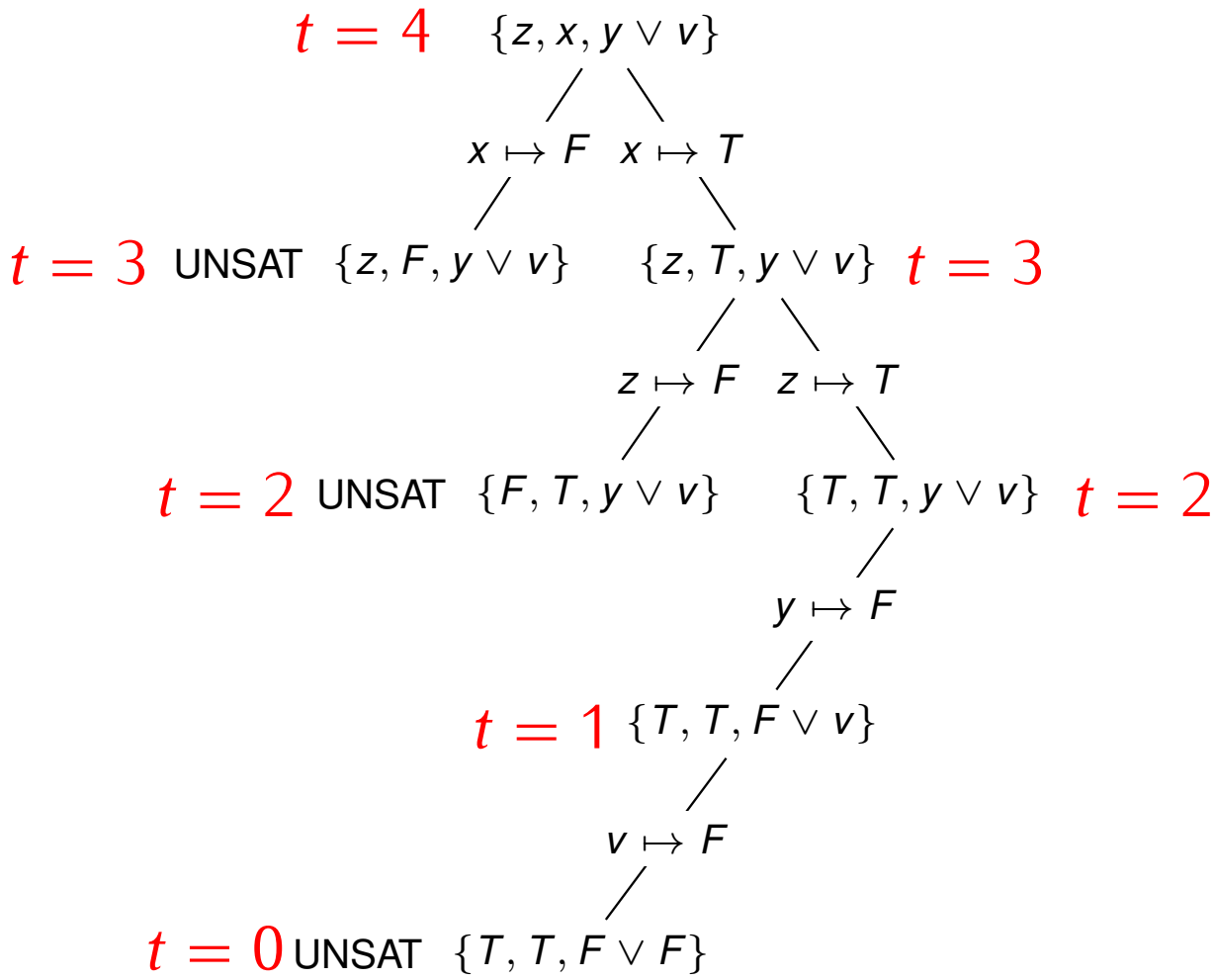

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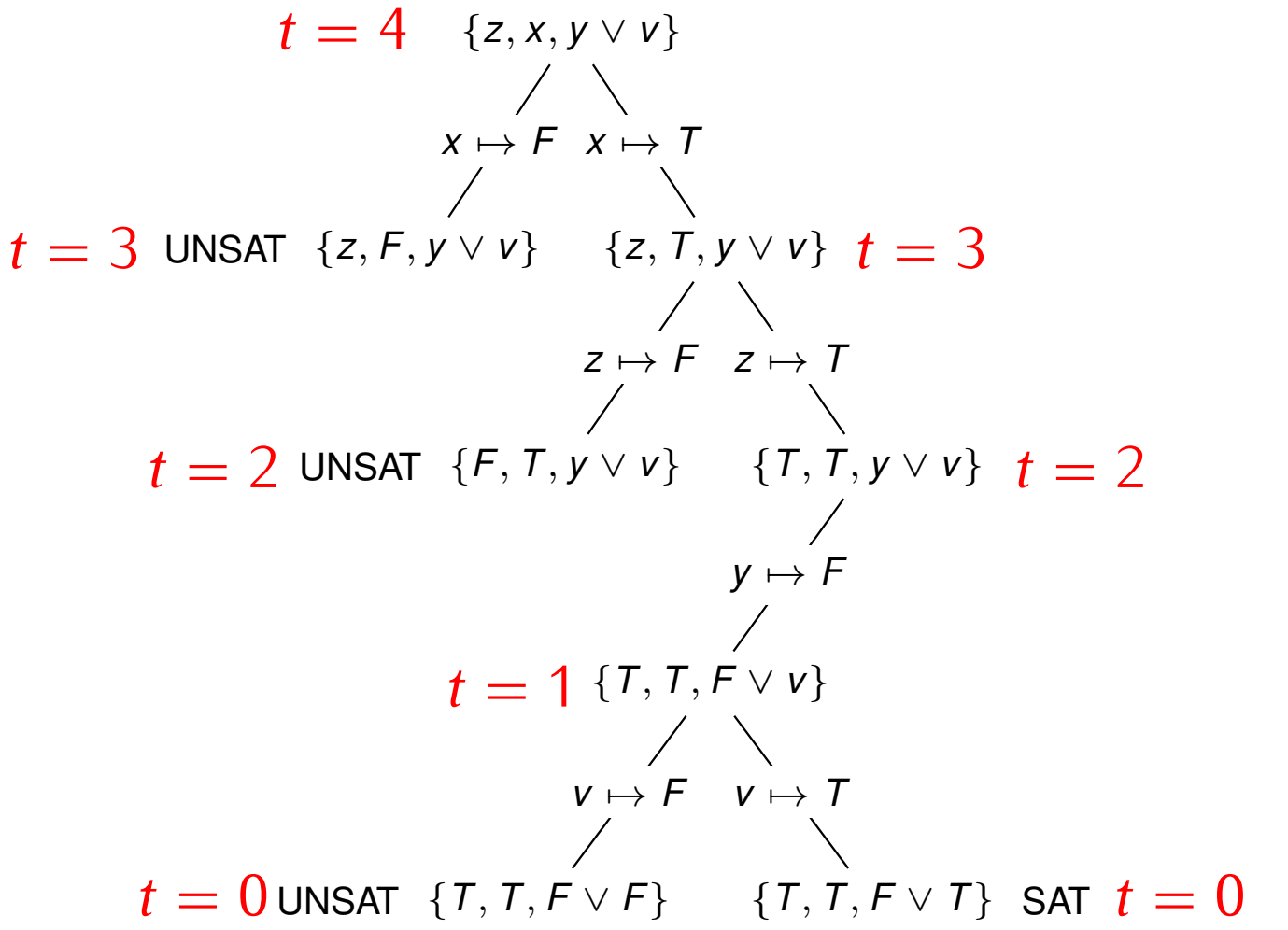
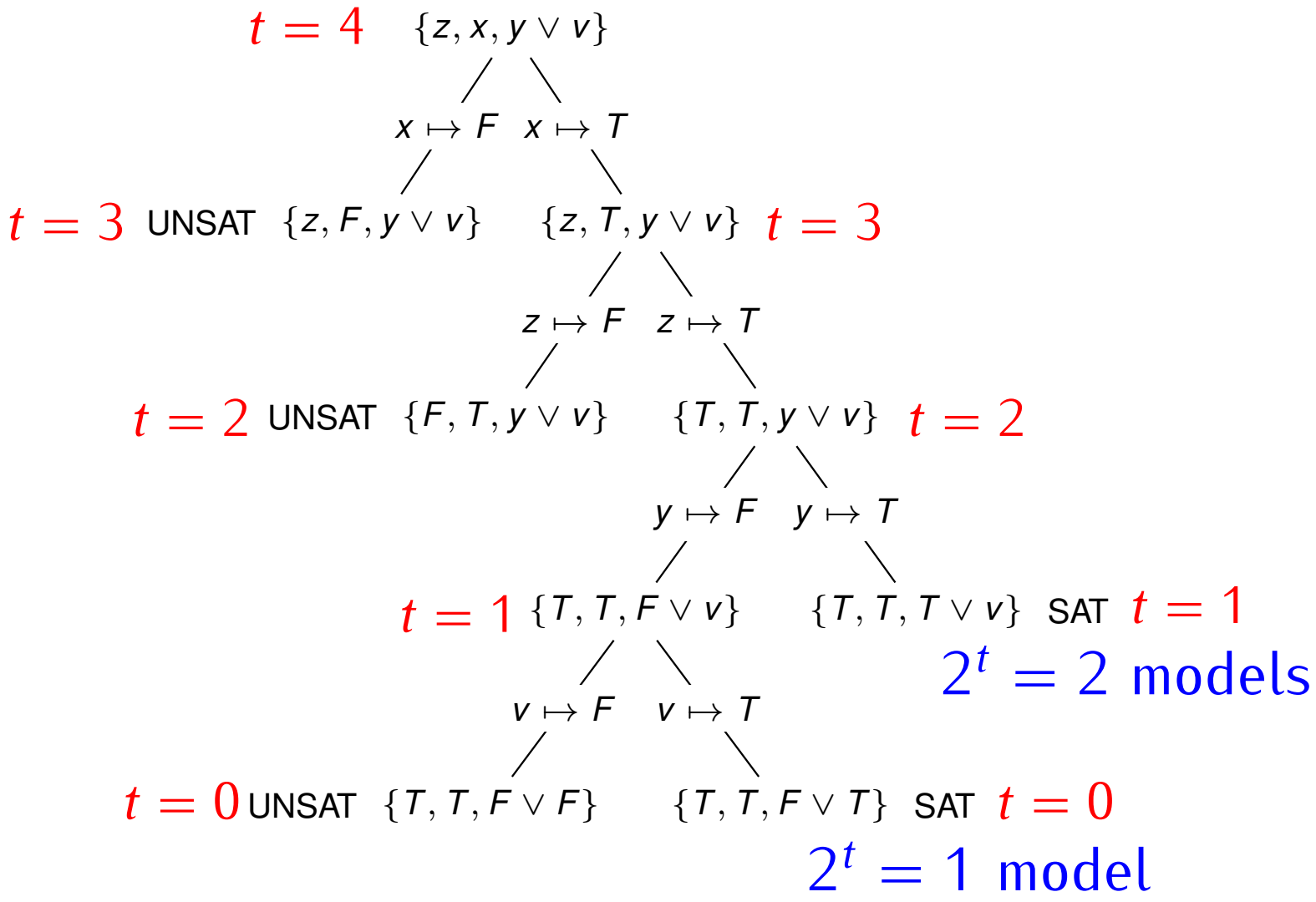


$2^t = 1$ model

DPLL Execution Example



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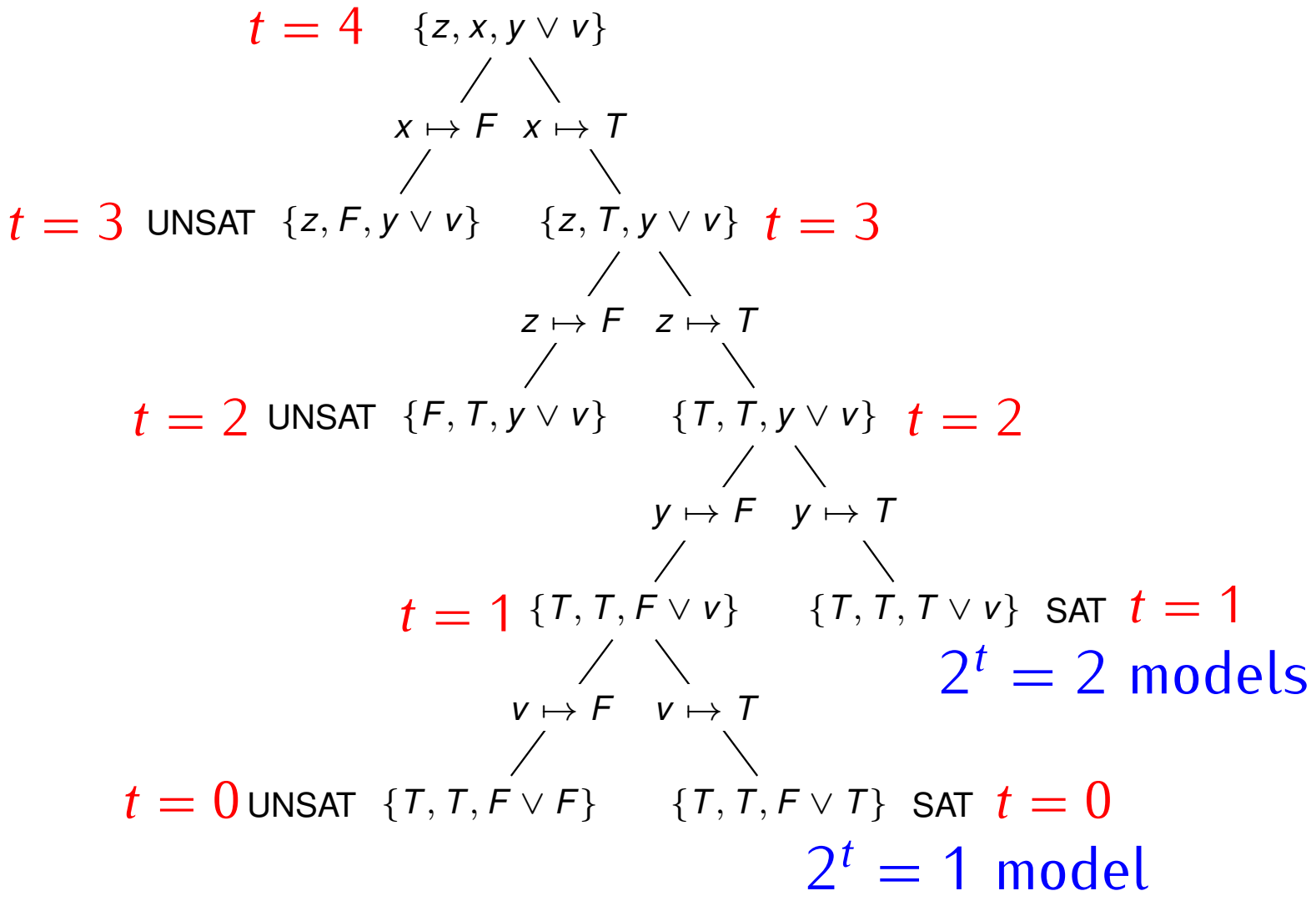

Conclusion: ϕ has 3 models

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

DPLL can be converted into a procedure for #CNF-SAT.

Function : DPLL(ϕ, t)
Input : CNF formula ϕ over n variables; $t \in \mathbb{Z}$
Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** *false*
  if all clauses of ϕ satisfied **then return** *true*
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto true]$, $t-1$) $\vee$ DPLL($\phi[x \mapsto true]$, $t-1$)
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  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto true], t-1$) $\vee$ DPLL($\phi[x \mapsto true], t-1$)
end

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Input : CNF formula ϕ over n variables; $t \in \mathbb{Z}$
Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** 0
  if all clauses of ϕ satisfied **then return** *true*
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto true], t-1$) $\vee$ DPLL($\phi[x \mapsto true], t-1$)
end

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DPLL can be converted into a procedure for #CNF-SAT.

Function : DPLL(ϕ, t)
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Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** 0
  if all clauses of ϕ satisfied **then return** *true*
  x $\leftarrow$ SelectBranchVariable(ϕ)
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end

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DPLL can be converted into a procedure for #CNF-SAT.

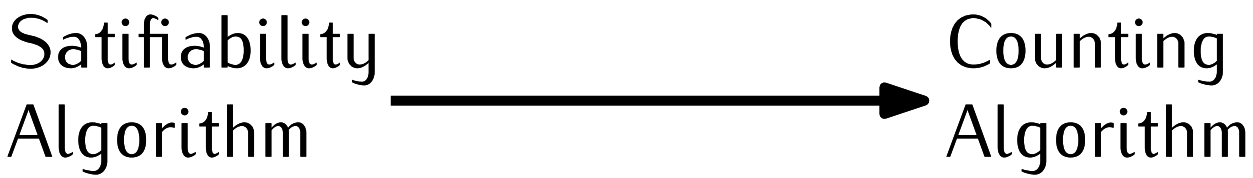
Function : DPLL(ϕ, t)
Input : CNF formula ϕ over n variables; $t \in \mathbb{Z}$
Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** 0
  if all clauses of ϕ satisfied **then return** 2^t
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto \mathit{true}], t-1) \vee$ DPLL($\phi[x \mapsto \mathit{true}], t-1$)
end

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

DPLL can be converted into a procedure for #CNF-SAT.

Function : DPLL(ϕ, t)
Input : CNF formula ϕ over n variables; $t \in \mathbb{Z}$
Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** 0
  if all clauses of ϕ satisfied **then return** 2^t
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto \mathit{true}], t-1$) + DPLL($\phi[x \mapsto \mathit{true}], t-1$)
end

The Big Idea



Model Counting for Arrays

$$\forall i : 0 \leq a[i] < k \wedge 0 \leq b[i] < k$$

$$\forall i : a[i] \neq b[i]$$

$$\text{length}(a) = n$$

$$\text{length}(b) = n$$

Model Counting for Arrays

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$$\text{length}(a) = n$$

$$\text{length}(b) = n$$

First, is this SAT?

$k = 4, n = 5, a = [1, 1, 1, 1, 1], b = [0, 2, 3, 2, 0]$

Model Counting for Arrays

$\forall i : 0 \leq a[i] < k \wedge 0 \leq b[i] < k$

$\forall i : a[i] \neq b[i]$

$\text{length}(a) = n$

$\text{length}(b) = n$

First, is this SAT?

$k = 4, n = 5, a = [1, 1, 1, 1, 1], b = [0, 2, 3, 2, 0]$

For a given k and n, how many models?

$$\#\phi(k, n) = (k^2 - k)^n$$

Model Counting for Arrays

$$\left.\begin{array}{l} \forall i : 0 \leq a[i] < k \\ \forall i, j : i < j \Rightarrow a[i] < a[j] \\ \text{length}(a) = n \\ \text{length}(b) = n \end{array}\right\} \; a \text{ is sorted}$$

Model Counting for Arrays

$$\forall i : 0 \leq a[i] < k$$
$$\forall i : i < pivot \Rightarrow a[i] < a[pivot]$$
$$\forall i : i > pivot \Rightarrow a[i] > a[pivot]$$
$$\text{length}(a) = n$$

quicksort partitioning

Model Counting for Arrays

$$\left.\begin{array}{l} \forall i : 0 \leq a[i] < k \\ \forall i : i < pivot \Rightarrow a[i] < a[pivot] \\ \forall i : i > pivot \Rightarrow a[i] > a[pivot] \\ \text{length}(a) = n \end{array}\right\} \text{quicksort partitioning}$$

We could do ad-hoc analysis for every constraint, but we want an algorithm!

Model Counting for Arrays

The SAT algorithm
for arrays

$\phi \in$ Array Theory

Model Counting for Arrays

The SAT algorithm
for arrays

$\phi \in$ Array Theory

$T_1 : A \rightarrow UIF$

$\phi' \in$ UIF Theory

Model Counting for Arrays

The SAT algorithm
for arrays

$\phi \in$ Array Theory

$\downarrow$ $T_1 : A \rightarrow UIF$

$\phi' \in$ UIF Theory

$\downarrow$ $T_2 : UIF \rightarrow EL$

$\phi'' \in$ Equality Logic

Model Counting for Arrays

The SAT algorithm
for arrays

$\phi \in$ Array Theory

$\downarrow$ $T_1 : A \rightarrow UIF$

$\phi' \in$ UIF Theory

$\downarrow$ $T_2 : UIF \rightarrow EL$

$\phi'' \in$ Equality Logic

There is a SAT
algorithm for this!

Model Counting for Arrays

The SAT algorithm for arrays

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$\downarrow$ $T_2 : UIF \rightarrow EL$

$\phi'' \in$ Equality Logic

There is a SAT algorithm for this!

(Proposed) counting algorithm for arrays

$\phi \in$ Array Theory

Model Counting for Arrays

The SAT algorithm for arrays

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(Proposed) counting algorithm for arrays

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Model Counting for Arrays

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(Proposed) counting algorithm for arrays

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$\phi' \in$ UIF Theory

$\downarrow$ $T_2' : UIF \rightarrow EL$

$\phi'' \in$ Equality Logic

$\downarrow$ $T_3' : EL \rightarrow LIA$

$\phi''' \in$ Linear Integer Arithmetic

Model Counting for Arrays

The SAT algorithm for arrays

$\phi \in$ Array Theory

$\downarrow$ $T_1 : A \rightarrow UIF$

$\phi' \in$ UIF Theory

$\downarrow$ $T_2 : UIF \rightarrow EL$

$\phi'' \in$ Equality Logic

There is a SAT algorithm for this!

(Proposed) counting algorithm for arrays

$\phi \in$ Array Theory

$\downarrow$ $T_1' : A \rightarrow UIF$

$\phi' \in$ UIF Theory

$\downarrow$ $T_2' : UIF \rightarrow EL$

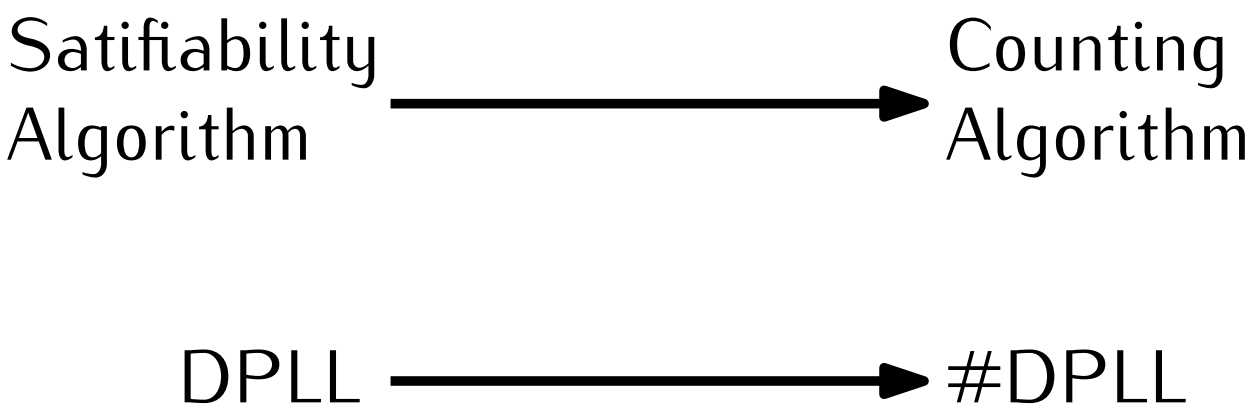
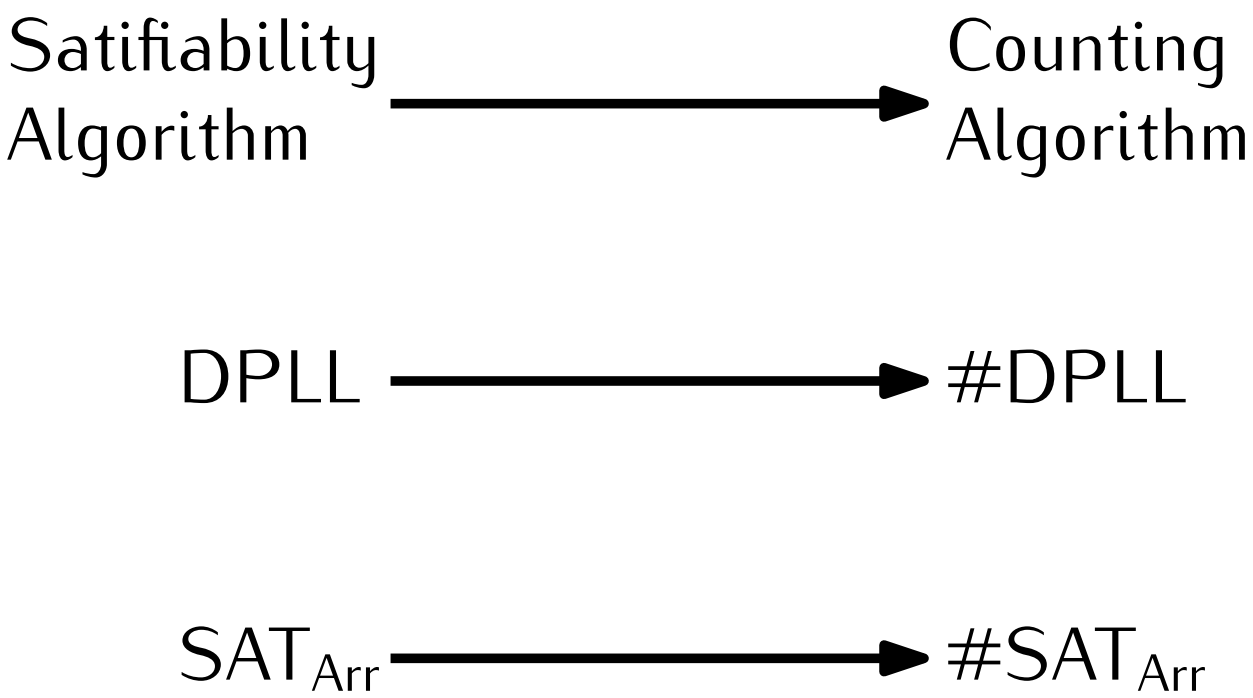
$\phi'' \in$ Equality Logic

$\downarrow$ $T_3' : EL \rightarrow LIA$

$\phi''' \in$ Linear Integer Arithmetic

There is a poly-time counting algorithm for this!

Model Counting Overview



Model Counting Overview