Algorithm	Graphe Type	Complexity
prim	Undirected + Non-negative Cost	n*n
Dynamic Programing	Undirected + Non-negative Cost	m
Dijkstra	Directed + Non-negative Cost	n*n
Floyd Warshall	Directed + Negative Cost	n*n*n

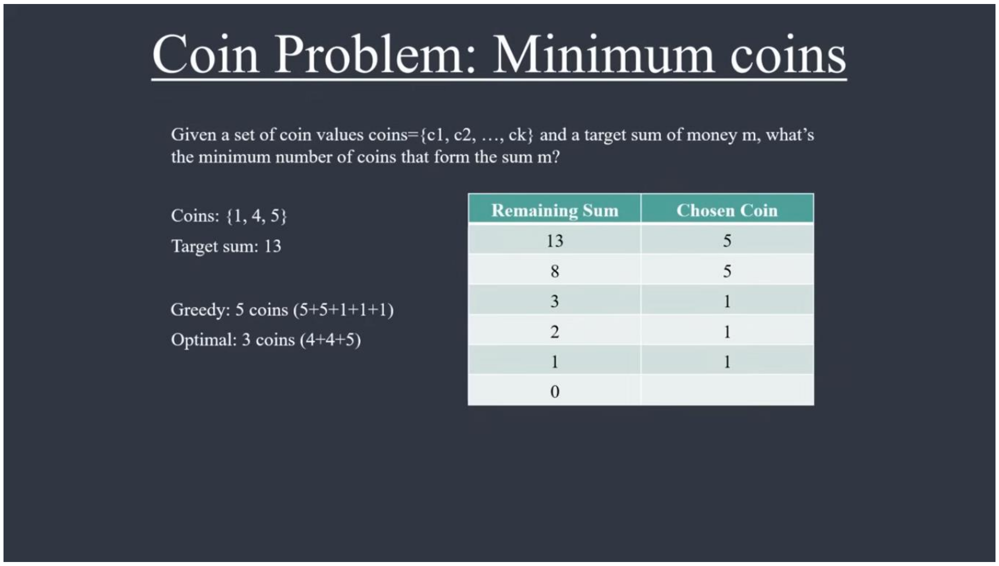
Algorithm	CPLX	The way they work
Prime Cut-edges	n*m	۱–نقطه شروع را که خود مسئله به ما می‌دهد را انتخاب می‌کنیم. ۲–کم هزینه ترین مسیر از این نقطه را انتخاب می‌کنیم. ۳–حال یک شبکه دو نقطه ای داریم با در نظر گرفتن تمامی مسیرها متصل به شبکه, کم هزینه-ترین مسیر را انتخاب می‌کنیم. ۴–شبکه هر مرحله گسترش میابد با در نظر داشتن عدم تشکیل حلقه مرحله ۳ را ادامه می‌دهیم. ۵–تا جایی ادامه می‌دهیم که تمام نود ها داخل شبکه باشند.
Prime Node-update	n*n	۱–نود شروع را صفر بقیه نودها را بی نهایت در نظر میگیریم. ۲–نود شروع به عنوان اولین نود به داخل Node Set می رود. ۳–از نود ابتدایی با توجه به مسیرهای موجود به نودهای مستقیم هزینه هر کدام را اپدیت میکنیم. ۴–حال از میان نودها که هزینه آنها مشخص است, نود با کمترین هزینه را همراه با مسیری که هزینه آن آپدیت شده بود را انتخاب میکنیم و آن را به Node Set انتقال میدهیم و دیگر با آن نود کاری نخواهیم داشت. ۵–این کار را با در نظر گرفتن تشکیل نشدن حلقه تا اضافه کردن اخرین نود به Node Set ادامه میدهیم. پیچیدگی این روش نسبت به روش قبلی در صورت بیشتر بودن یال ها نسبت به نود ها کمتر است.
Dynamic Programming	m	0-Find the topological order of the graph 1-The length of the first node is zero if there is no initial value for it. 2-For the next node the length equals to minimum of the sum of the length of the nodes that have direct path towards the certain node plus the length of the path to current node. 3-For the rest of the nodes we do this process based on topological order to reach the last one.
Critical Path Method	n+m	
Floyd warshall Negative Cost	n*n*n	**Step1: read the question carefully.** **Step2: draw G and $\bar{G}$. There is different between <u>arcs</u> and <u>arrows.</u>** **Step3: <u>only</u> consider $\bar{G}$.** **Step4: find the ways from S to T, one by one- write them down determine $\underline{x}_i$ and δ_i from $i = 1 \ldots n$.** **Step5: when a way toward destination is occupied clear the backward arc and exchange it with $\not\rightarrow k_{ij} \rightarrow$ and also vice versa put $\not\leftarrow k_{ij} \leftarrow$ between two nodes.**

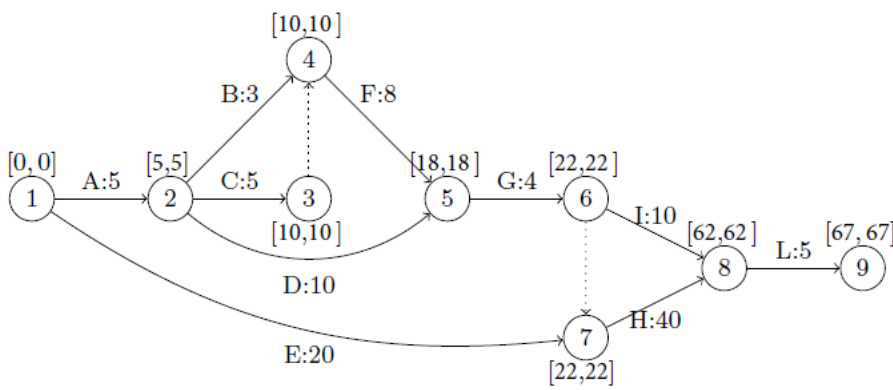
		Step6: consider the ways with $\longrightarrow k_{ij} \longrightarrow$ **as minimum cut.** **Step7: write S*nodes in a set also** $\delta(S^*)$. **Step8: finished.**
Fold Fulkerson	$m*m*k_{max}$	Max{k_{ij}} only intiger
FW + Edmond karp	n*m*m	k_{max} Can be non-integer = arbitrary path
FW + Dinic	n*n*m	k_{max} Can be non-integer = arbitrary path

پس Topological order میتواند پس و پیش باشد و به ترتیب شماره گذاری نود ها نباشد. بعد از مشخص شدن توپولوجیکال اردر به ترتیب آنها مسیر ها را می نویسیم.

- ✓ نوشتن پرنتت آنها را همیشه از یاد میبرم توجهم باید بیشتر شود.
- ✓ Because of negative costs we should use Floyd-Warshall's algorithm $O(n^3)$ and solve the problem or show that the problem is ill-posed.

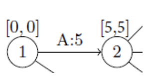
- اگر در روش حل با روش Floyd warshall عدد منفی در هر مرحله از انجام آن روی قطر اصلی ظاهر شود آن مسله قابل حل نیست و کار تمام است.



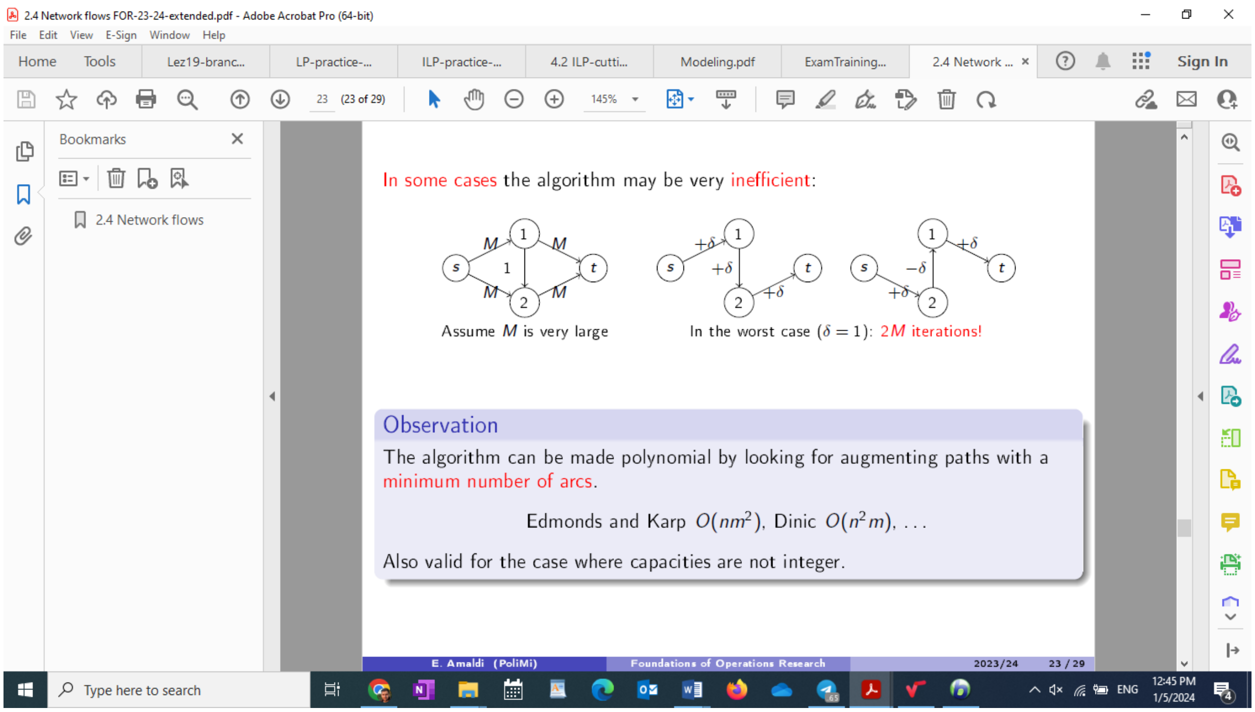


برای درایه اول بیشترین عدد بعد از جمع با مسیر و برای درایه دومی کمترین عدد بعد از کم کردن مسیر را در نظر می گیریم.

Slack for activities A B C D E F G H I L



$i=1,\ j=2\ \left[T_{min},T_{max}\right],1:\left[0,0\right]\ ,2:\left[5,5\right]\ ,\sigma\left(A\right)=T_{maxj}-T_{mini}-d_{ij}=5-0-5=0$



In some cases the algorithm may be very inefficient:

Assume M is very large

In the worst case ($\delta = 1$): $2M$ iterations!

Observation

The algorithm can be made polynomial by looking for augmenting paths with a minimum number of arcs.

$$\text{Edmonds and Karp } O(nm^2), \text{ Dinic } O(n^2 m), \ldots$$

Also valid for the case where capacities are not integer.

Complexity

- Since $\delta > 0$, the value φ increases at each iteration (cycle).
- If all k_{ij} are integer, $\underline{x}$ and $\bar{k}_{ij}$ integer and $\delta \geq 1$, then there are at most φ^* increases.
- Since

$$\varphi^* \leq k(\{s\}) \leq m\, k_{\max},$$

where $m = |A|$ and $k_{\max} = \max\{k_{ij} : (i,j) \in A\}$ and each cycle is $O(m)$, the overall complexity is $O(m^2 k_{\max})$.

Definition 7

The size of an instance I, denoted by $|I|$, is the number of bits needed to describe the instance.

Since $\lceil \log_2 i \rceil + 1$ bits are needed to store integer i, $|I| = O(m \log_2 k_{\max})$.

$O(m^2 k_{\max})$ grows exponentially with $|I|$ because $k_{\max} = 2^{\log_2 k_{\max}}$.

Standard Form

6) Standard form of LP

Given the following linear programming problem:

$$\begin{aligned} \max z &= 2x_1 + x_2 \\ x_1 + x_2 &\le 4 \\ 3x_1 - 2x_2 &\ge 9 \\ 2x_1 + x_2 &= 16 \\ x_1 &\ge 5 \\ x_2 &\le 0 \end{aligned}$$

express the problem in standard form.

۱- اولین کار در این مسائل چک کردن علامت متغییرهای اصلی است. در صورت نا متعادل بودن علامت ها باید همه آنها را به صورت متغییرهای مثبت یا Unrestricted تبدیل میکنیم.

۲- چک کردن ماکزیمم یا مینیمم بود تابع هدف گام دوم است

۳- در این گام اگر تابع هدف مینیمم بود کاری نمی کنیم ولی اگر ماکزیمم بود آن را به مینیمم تبدیل کرده و یک منفی به دو طرف Z ضرب میکنیم.

۴- در مرحله بعد با توجه به علامت محدودیت ها سورپلاس یا اسلک ها را به آنها اضافه میکنیم تا نامعادلات به معادله تبدیل گردند.

حل مسئله بالا به دو طریق امکان پذیر است. که یکی از آنها به صورت زیر است و با توجه به این که اینجا فقط از ما خواسته شده که به فرم استاندارد ببریم میتوانیم در روش دوم کانستریت ۴ را در نظر بگیریم ولی متغییر اصلی اول را unrestricted در نظر بگیریم – اگر از ما خواسته شده باشد که این دستگاه را حل کنیم چاره ای نداریم که هم کاری که در روش زیر انجام شده را در نظر بگیریم و پیچیدگی کار ما بیشتر خواهد شد. بهترین روش برای حل این دستگاه استاندارد سازی به روش زیر و حل معادلات استاندارد زیر خواهد بود.

$$\begin{aligned} \min z' &= -2x_1' + x_2' - 10 \\ x_1' - x_2' + x_3 &= -1 \\ 3x_1' + 2x_2' - x_4 &= -6 \\ 2x_1' - x_2' &= 6 \\ x_1', x_2', x_3, x_4 &\ge 0 \end{aligned}$$

Negative number in objective function row

2) Solve the following LP problem with the simplex algorithm:

$$\begin{aligned} \min z = x_1 - 2x_2 & \\ -3x_1 + 3x_2 &\leq 1 \\ +x_1 - x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

What can be deduced about the dual problem? Write the dual problem and solve it graphically to confirm the deduction.

After writing a system in standard form, we should solve it. By considering this fact that all of us know how to change the original problem into standard form the solution are as

<table>
<tr><td>1</td><td>If all the constraints are less or equal</td><td>Variables with positive coefficient are in basic variables column</td><td>Simple solve</td></tr>
<tr><td>2</td><td rowspan="3">If some of the constraints are more or equal</td><td>Max z=f(x,y) $-MR_i$</td><td>Big M</td></tr>
<tr><td>3</td><td>P1: Min R=sum_{i}
P2: delete R row insert OF row and solve it</td><td>Two phase</td></tr>
<tr><td>4</td><td>حل به روش اول – البته اگر امکان پذیر بود</td><td>ضرب منفی یک به طرفین آن ها</td></tr>
<tr><td>5</td><td rowspan="2">If some of constraints are equal</td><td>The RHS of AV at the of phase I should be zero</td><td>Two phase</td></tr>
<tr><td>6</td><td>But all the point should satisfy complementary conditions</td><td>Dual simples method</td></tr>
</table>

9) Duality

Write the dual of the following LP problem:

$$\begin{aligned} \max z = 2x_1 + 3x_2 & \\ 2x_1 + x_2 &\leq 5 \\ 2x_1 - 3x_2 &\leq 3 \\ -4x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Find a lower and an upper bound on the optimum of the problem.

Solution

$$\begin{aligned} \min w = 5y_3 + 3y_4 + 2y_5 \\ 2y_3 + 2y_4 - 4y_5 \geq 2 \\ y_3 - 3y_4 + y_5 \geq 3 \\ y_3, y_4, y_5 \geq 0 \end{aligned}$$

- Any feasible solution of the maximization problem provides a lower bound on the optimum; for example, $x = (0, 0) \Rightarrow z = 0 \leq z^*$
- Any feasible solution of the minimization problem provides an upper bound on the optimum; for example, $y = (3, 0, 0) \Rightarrow w = 15 \geq w^*$

Solution 2

0	1	-2	0	0
1	-3	3	1	0
5	1	-1	0	1

Pivot on element $(1, 2)$

2/3	-1	0	2/3	0
1/3	-1	1	1/3	0
16/3	0	0	1/3	1

The problem is unbounded (first column non positive). Correspondingly, the dual problem is unfeasible. In fact, the dual problem is

$$\begin{aligned} \max w = y_3 + 5y_4 & \\ -3y_3 + y_4 &\leq 1 \\ +3y_3 - y_4 &\leq -2 \\ y_3, y_4 &\leq 0 \end{aligned}$$

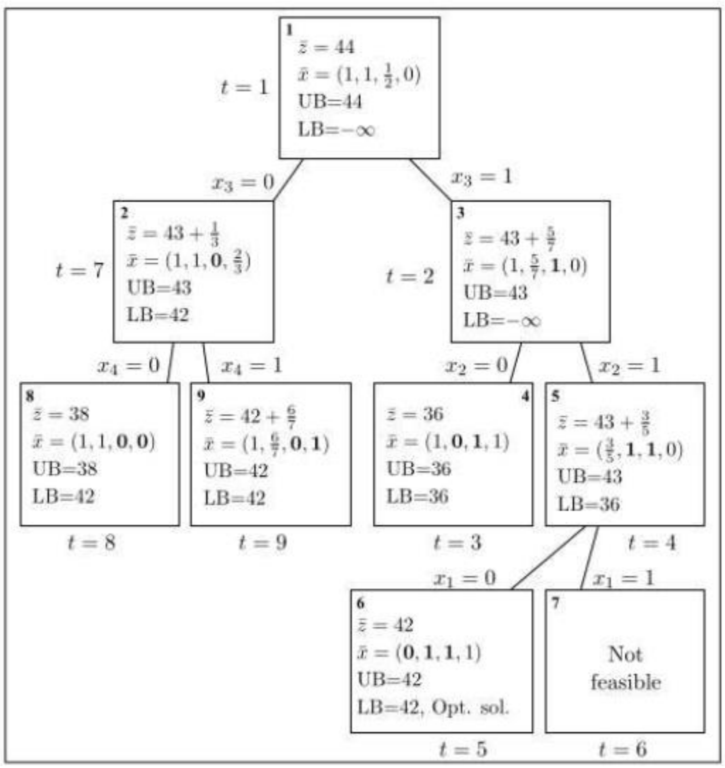
اگر سیستم اصلی ما دارای عدد منفی در سطر اول باشد(باقی بماند) و راهی برای از بین بردن آن نباشد مسئله ما unbounded خواهد شد در نتیجه می توان گفت مسئله dual آن نیز feasible نیست.

The problem is unbounded (first column non positive). Correspondingly, the dual problem is unfeasible. In fact, the dual problem is

$$\begin{aligned} \max w = y_3 + 5y_4 & \\ -3y_3 + y_4 &\leq 1 \\ +3y_3 - y_4 &\leq -2 \\ y_3, y_4 &\leq 0 \end{aligned}$$

and its two constraints are incompatible: $-3y_3+y_4 \leq 1$ is equivalent to $3y_3-y_4 \geq -1$, but the second constaint requires that $3y_3 - y_4 \leq -2$.

How to solve ILP with graphical tree



۱-ابتدا با توجه به الگوریتم گریدی سری غیر صعودی را بدست می‌آوریم

۲- سپس مقدار ها را برای xها با توجه به این که شرط ما برقرار باشد را مینویسیم

۳-از متغییری که عدد کسری دارد شروع به حل می کنیم در این مسیر برای باند بالا جز صحیح z را می گذاریم و برای باند پایین منفی بینهایت قرار میدهیم

٤-شروع به حل میکنیم یکی از شاخه ها را گرفته ادامه میدهیم و هر سری باند بالا را اپدیت میکنیم و باند پایین را برابر منی بینهایت قرار می دهیم تا به جایی برسیم که باند بالا و پایین یکی شوند و به یک جواب فیزبل اولیه برسیم

٥-بعد از آن میزان حداقل باند پایین را همین مقدار در نظر میگیریم و حل میکنیم

* جاهایی که باند بالا و پایین جابجا شوند کار تمام است

*در جاهایی که مقدار تابع کمتر از باند پایین باشد کار تمام است

*در جایی که جواب تابع ما عدد صحیح نبوده ولی جز صحیح آن کمتر یا مساوی حد پایینی باشد نیز کار تمام است.

در شکل بالا از ١ شروع میکند به ٢ میرود از آن به ٣ و در اینجا جواب اولیه بدست می آید و حد پایین برای اولین بار ست می شود

به یک مرحله قبل بر میگردیم و با دا شتن حد بالا و پایین ادامه می دهیم- از ٥ به ٦ میرسیم که جواب قابل قبولی نیست و آن را کنار می گذاریم – یه مرحله به قبل بر میکردیم و از ٥ به ٧ میرویم در این مرحله به خد پایین جدیدی می رسیم که از حد قبلی بالا تر است. پس کمترین حد پایین را دوباره با این عدد ست میکنیم.

به ابتدای درخت بر میگردیم از ١ به ٢ میرویم حد پایین ٤٢ و حد بالا ٤٣ پس امیدی هست که به عدد ٤٣ برسیم

از ٢ به ٨ میرویم مقدار تابع کمتر از حد پایین می شود کارش تمام است.

از ٢ به ٩ میرویم با توجه به اینکه بعد از یک کات اقلا ٤٢ به ٤١ تبدیل خواهد شد ادامه دادن این شاخه هم اهمیتی ندارد و کار تمام است

در نهایت ما به یک جواب صحیحح با متغییر های صحیح رسیدم و جواب اپتیمال ما می باشد.

در حل به روش گرافیکی اگر برای دو متغییر سه محدودیت داشته باشیم, نقطه اپتیمال از برخورد هر کدام از دو محدودیت بدست آید اسلک های آنها صفر و اسلک محدودیت دیگر غیر صفر خواهد بود.

Example

$$\begin{array}{llll} \max & z = 8x_1 & +5x_2 & \\ \text{s. t.} & x_1 & +x_2 & \leq 6 \\ & 9x_1 & +5x_2 & \leq 45 \\ & x_1, & x_2 & \geq 0 \text{ integer} \end{array}$$

Optimal tableau:

		x_1	x_2	s_1	s_2
$-z$	-41.25	0	0	-1.25	-0.75
x_1	3.75	1	0	-1.25	0.25
x_2	2.25	0	1	2.25	-0.25

with the fractional optimal basic solution $\underline{x}_B^* = (3.75, 2.25)^T$

Select a row of the optimal tableau (a constraint) whose basic variable has a fractional value:

$$x_1 - 1.25s_1 + 0.25s_2 = 3.75$$

Generate the corresponding Gomory cut: $0.75s_1 + 0.25s_2 \geq 0.75$.

Note: The integer and fractional parts of $a \in \mathbb{R}$ are

$$a = \lfloor a \rfloor + f, \quad \text{with } 0 \leq f < 1,$$

thus we have $-1.25 = -2 + 0.75$ and $0.25 = 0 + 0.25$.

اینجا مسئله برای ماکزیمم داده شده است پس در اولین قدم آن را به مینیمم تبدیل میکنیم با این کار این مسئله را به حالت حل در dual simplex algorithm میبریم و در نهایت فق یک منی به عدد بدست امده برای Z ضرب میکنیم.

یادمان باشد برای حل سیمپلکس با تابلو تابع را اگر ماکزیمم بود بعد از به یکسو بردن ضرایب از منفی ترین ضریب شروع به حل جدول تابلو میکردیم و برای میبیمم از بزرگترین عدد مثبت استفاده میکردیم.

Definition 2

Let

$$(ILP) \quad \begin{array}{ll} z_{ILP} := \max & \underline{c}^T \underline{x} \\ \text{s.t.} & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0}, \underline{x} \in \mathbb{Z}^n \end{array}$$

The problem

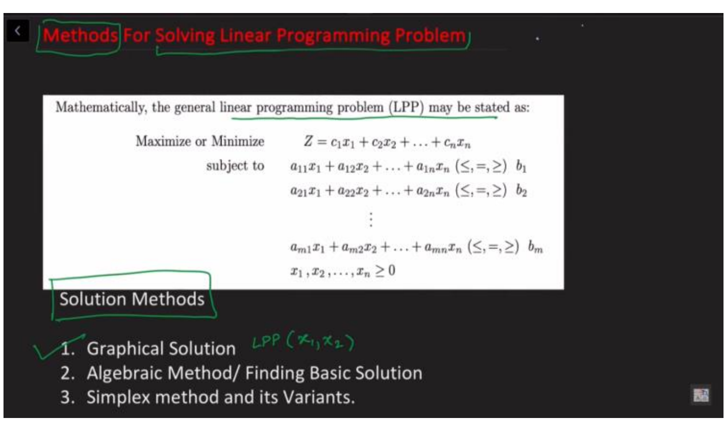
$$(LP) \quad \begin{array}{ll} z_{LP} := \max & \underline{c}^T \underline{x} \\ \text{s.t.} & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq 0 \end{array}$$

is the linear (continuous) relaxation of (ILP).

Property 1

For any ILP with max, we have $z_{ILP} \leq z_{LP}$, i.e., z_{LP} is an upper bound on the optimal value of (ILP).

N.B. For any ILP with min, we have $z_{ILP} \geq z_{LP}$, i.e., z_{LP} is a lower bound on the optimal value of (ILP).



حواستان باشد سود با درآمد فرق دارد.

ماکزیمم یا مینیمم بودن تابع هدف مهم نیست – اگر از زیر ناحیه فیزبل حرکت کنیم- جواب اخرین تماس خط تابع هدف با ناحیه Feasible است.

Example:

$$(P)\quad \begin{array}{lrrrl} \min & 13x_1 & +10x_2 & +6x_3 & \\ \text{s.t.} & 5x_1 & +x_2 & +3x_3 & = 8 \\ & 3x_1 & +x_2 & & = 3 \\ & x1, & x2, & x3 & \geq 0 \end{array} \qquad (D)\quad \begin{array}{lrrl} \max & 8y_1 & +3y_2 & \\ \text{s.t.} & 5y_1 & +3y_2 & \leq 13 \\ & y_1 & +y_2 & \leq 10 \\ & 3y_1 & & \leq 6 \end{array}$$

Verify that the feasible $\underline{x}^* = (1,0,1)^T$ is an optimal, non degenerate solution of (P).

Suppose it is true and derive, via the complementary slackness conditions, the corresponding optimal solution of (D).

Since (P) is in standard form, the conditions $y_i^*(\underline{a}_i^T \underline{x}^* - b_i) = 0$ hold for $i = 1, 2$.

Condition $(c_j^T - \underline{y}^{*T} A_j)x_j^* = 0$ is satisfied for $j = 2$ because $x_2^* = 0$.

Since $x_1^*, x_3^* > 0$, we obtain the conditions:

$$5y_1 + 3y_2 = 13, \quad 3y_1 = 6$$

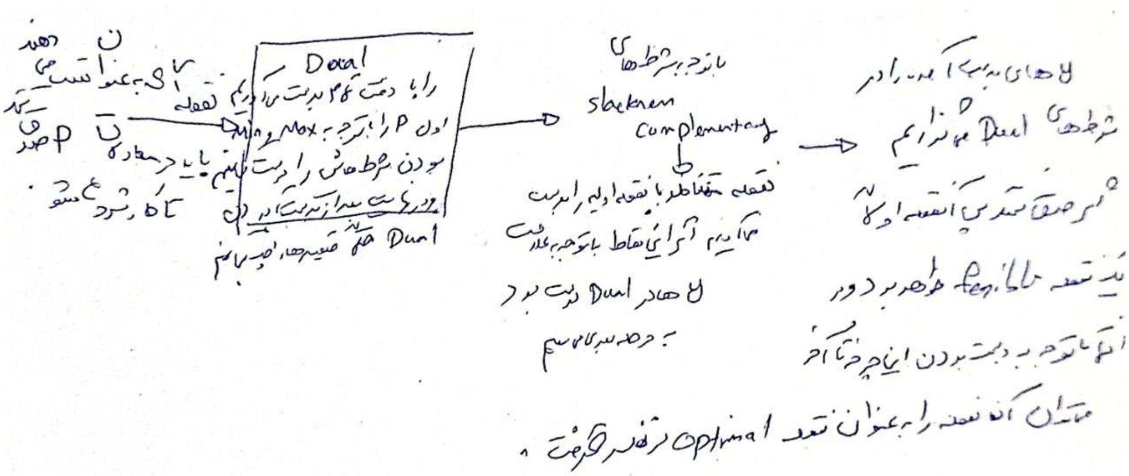
and thus the optimal solution $y_1^* = 2$ and $y_2^* = 1$ of (D) with $\underline{b}^T \underline{y}^* = 19 = \underline{c}^T \underline{x}^*$.

In (P) → $y_i^*(Ax_i - b) = 0$	→	با بررسی این شرط در نقطه داده شده میفهمیم که $y_i = 0 \;\; or \;\; y_i \neq 0$	→	In(D)→ حال شرط دوم را بررسی می کنیم و مقدار دقیق y ها را بدست می آوریم. $(c_i - y_i^* A)x_i^* = 0$	→	برای اینکه بفهمیم این نقاط بهینه هستند یا خیر باید شرط زیر برقرار باشد. $cx^* = y^* b$

X ها را اول کار در P و Y ها را آخر کار در D می سنجیم تا ببینیم اصلا

حل فیزبل ما هستند یا نه. به علامت Xها و Y ها دقت کنید.

در صورت unrestricted بودن میتوانند منفی باشند- برای تست فیزبل بودن یا نبودن چک کردن علامت آن ها الزامیست.



By solving the system, we obtain $\bar{y} = (1, 0, 1)$, which satisfies the dual constraints of D_2. Since $\bar{x}$ is primal feasible and $\bar{y}$ is dual feasible, the primal/dual pair $(\bar{x}, \bar{y})$ satisfies the complementary slackness constraints and, therefore, $\bar{x}$ is an optimal solution of the primal and $\bar{y}$ is an optimal solution of the dual.

To double check, note that the objective function values of the respective problems of the two solutions are equal, namely, we have $c^T x = y^T b$.

Exercise 6

Consider the following linear programming problem:

$$
\begin{aligned}
\max \quad 2x_1 + x_2 \\
-2x_1 - x_2 \leq -1 \\
x_1 - x_2 \leq 3 \\
4x_1 + x_2 \leq 17 \\
x_2 \leq 5 \\
-x_1 + x_2 \leq 4
\end{aligned}
$$

where $x_1, x_2 \geq 0$.

دوآل مثال بالا به این شکل می باشد.

Solution

Part a)

$$
\begin{aligned}
\min \quad -y_1 + 3y_2 + 17y_3 + 5y_4 + 4y_5 \\
-2y_1 + y_2 + 4y_3 - y_5 \geq 2 \\
-y_1 - y_2 + y_3 + y_4 + y_5 \geq 1 \\
y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned}
$$

همان طور که دیده می شود, در مسئله اوریجینال x ها مثبت هستند پس اگر در سوالی به شما نقطه ای بدهند که حداکثر یکی از آنها منفی باشد بدون ورود به مسئله آن نقطه اصلا فیزبل نیست. و در انتها با توجه به اینکه y ها در مسئله دوال مثبت هستند اگر نقاط بدست آمده برای آنها منفی باشد آن جواب برای مسئله دوال آپتیمال نخواهد بود.

We get that $y_2 = -\frac{2}{5}, y_3 = \frac{3}{5}$, which is not a feasible solution to the dual problem and thus (4,1) is not optimal.

Exercise 7

Consider the following problem:

$$\max z = 9x_1 + 8x_2$$

$$\begin{aligned} x_1 - 2x_2 &\leq -1 \\ 4x_1 + 3x_2 &\leq 4 \\ -x_1 + 2x_2 &\leq 3 \\ 2x_1 - x_2 &\leq -4 \end{aligned}$$

Verify if solution $x_1 = -3$, $x_2 = -1$ is optimal. Verify if solution $x_1 = -\frac{5}{3}$, $x_2 = \frac{2}{3}$ is optimal.

در این مثال با توجه به آنکه محدوده ای برای x مشخص نکرده است پس آنها را unrestricted در نظر می گیریم پس نقاط اولیه برای آن می توانند منفی باشند – البته همان طور که میدانید مثبت بود یا unrestricted بودن متغییرها با علامت محدودیت های ما رابطه مستقیم دارد و در نوشتن آنها حتما دقت کنید.

The risk factor represents the maximum fraction of the stock value that can be lost. A risk factor of 0.25 implies that, if stocks are bought for 100 Euro up to 25 Euro can be lost.

Exercise 6

Consider the following linear programming problem:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ -2x_1 - x_2 &\leq -1 \\ x_1 - x_2 &\leq 3 \\ 4x_1 + x_2 &\leq 17 \\ x_2 &\leq 5 \\ -x_1 + x_2 &\leq 4 \end{aligned}$$

where $x_1, x_2 \geq 0$.

a) Write the dual problem of the given problem.

b) Write the equations defining the complementarity slackness for the given problem (Notice that the problem and its dual are in symmetric form).

c) Exploiting the complementarity conditions say whether points (3,5) and (4,1) are optimal.

Solution

Part a)

$$\begin{aligned} \min \quad & -y_1 + 3y_2 + 17y_3 + 5y_4 + 4y_5 \\ & -2y_1 + y_2 + 4y_3 - y_5 \geq 2 \\ & -y_1 - y_2 + y_3 + y_4 + y_5 \geq 1 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Part b)
The equations of complementary slackness are:

$$\begin{aligned} (-2x_1 - x_2 + 1)y_1 &= 0 \\ (x_1 - x_2 - 3)y_2 &= 0 \\ (4x_1 + x_2 - 17)y_3 &= 0 \\ (x_2 - 5)y_4 &= 0 \\ (-x_1 + x_2 - 4)y_5 &= 0 \\ (-2y_1 + y_2 + 4y_3 - y_5 - 2)x_1 &= 0 \\ (-y_1 - y_2 + y_3 + y_4 + y_5 - 1)x_2 &= 0 \end{aligned}$$

Part c)
Point (3,5) is feasible for the primal problem. By using the equations of complementary slackness we have that:

$$\begin{aligned} (-2x_1 - x_2 + 1)y_1 = 0 &\Rightarrow y_1 = 0 \\ (x_1 - x_2 - 3)y_2 = 0 &\Rightarrow y_2 = 0 \\ (-x_1 + x_2 - 4)y_5 = 0 &\Rightarrow y_5 = 0 \\ x_1 > 0 &\Rightarrow 4y_3 = 2 \\ x_2 > 0 &\Rightarrow y_3 + y_4 = 1 \end{aligned}$$

We get that $y_3 = \frac{1}{2}, y_4 = \frac{1}{2}$, which is a feasible solution to the dual problem and thus (3,5) is optimal.

Point (4,1) is feasible for the primal problem. By using the equations of complementary slackness we have that:

$$\begin{aligned} (-2x_1 - x_2 + 1)y_1 = 0 &\Rightarrow y_1 = 0 \\ (x_2 - 5)y_4 = 0 &\Rightarrow y_4 = 0 \\ (-x_1 + x_2 - 4)y_5 = 0 &\Rightarrow y_5 = 0 \\ x_1 > 0 &\Rightarrow y_2 + 4y_3 = 2 \\ x_2 > 0 &\Rightarrow -y_2 + y_3 = 1 \end{aligned}$$

We get that $y_2 = -\frac{2}{5}, y_3 = \frac{3}{5}$, which is not a feasible solution to the dual problem and thus (4,1) is not optimal.

وقتی از روش دو مرحله ای استفاده میکنیم باید در فاز اولی مقدرا تابع هدف برابر صفر گردد در غیر این صئرت جواب فیزبل نداریم.

Exercise 5

Solve the following linear programming problem using the Simplex algorithm with Bland's rule:

$$\begin{array}{ll} \min & 3x_1 + x_2 + x_3 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 6 \\ & x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Solution

We will execute the two-phase simplex method. In Phase I we look for a basic feasible expressed in canonical form or establish that the original LP is infeasible. The auxiliary LP problem is:

$$\begin{array}{ll} \min & w_1 + w_2 \\ \text{s.t.} & 2x_1 + x_2 + x_3 + w_1 = 6 \\ & x_1 + x_2 + 2x_3 + w_2 = 2 \\ & x_1, x_2, x_3, w_1, w_2 \geq 0. \end{array}$$

We express the objective function in terms of the nonbasic variables.

$$w_1 = 6 - 2x_1 - x_2 - x_3$$

$$w_2 = 2 - x_1 - x_2 - 2x_3$$

Therefore,

$$w_1 + w_2 = 8 - 3x_1 - 2x_2 - 3x_3.$$

		x_1	x_2	x_3	w_1	w_2
	-8	-3	-2	-3	0	0
w_1	6	2	1	1	1	0
w_2	2	1	1	2	0	1

Iteration 1: Using Bland's rule x_1 enters the basis. $\theta = \min\{\frac{6}{2}, \frac{2}{1}\} = 2$, thus w_2 exists the basis. The next tableau is

		x_1	x_2	x_3	w_1	w_2
	-2	0	1	3	0	3
w_1	2	0	-1	-3	1	-2
x_1	2	1	1	2	0	1

Since the reduced costs of all the non basic variables are non negative, the corresponding basic feasible solution $[w_1, x_1] = [2, 2]$ and $[x_2, x_3, w_2] = [0, 0, 0]$ with the objective function value of 2 is an optimal solution of the auxiliary problem. Thus the original LP problem is infeasible because the objective function value of the auxiliary problem is strictly positive (and not equal to zero).

Theorem: Complementary slackness conditions

$\underline{x}^* \in X$ and $\underline{y}^* \in Y$ are optimal solutions of, respectively, (P) and (D) if and only if

$$y_i^* \overbrace{(\underline{a}_i^T \underline{x}^* - b_i)}^{\text{slack } s_i \text{ of } i\text{-th constraint of (P)}} = 0, \quad i = 1, \ldots, m$$

$$\underbrace{(c_j^T - \underline{y}^{*T} A_j)}_{\text{slack } s'_j \text{ of } j\text{-th constraint of (D)}} x_j^* = 0, \quad j = 1, \ldots, n$$

where $\underline{a}_i$ denotes the i-th row of A and A_j the j-th column of A.

At optimality, the product of each variable with the corresponding slack variable of the constraint of the relative dual is $= 0$.

3.4.5 Two-phase simplex method

Phase I: Determine an intial basic feasible solution.

Example:

$$\begin{array}{llll} \min & x_1 & +x_3 & \\ \text{s.t.} & x_1+2x_2 & & \leq 5 \\ & & x_2+2x_3 & = 6 \\ & x_1, & x_2, & x_3 \geq 0 \end{array} \Rightarrow \begin{array}{lllll} \min & x_1 & +x_3 & & \\ \text{s.t.} & x_1+2x_2 & & +x_4 & = 5 \\ & & x_2+2x_3 & & = 6 \\ & x_1, & x_2, & x_3, & x_4 \geq 0 \end{array}$$

There is no sub-matrix $I_{2\times 2}$ of A!

Given problem (P): $\begin{array}{ll} \min & z = \underline{c}^T \underline{x} \\ \text{s.t.} & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{array}$ with $\underline{b} \geq \underline{0}$, the auxiliary LP with artificial variables y_i, $i = 1, \ldots, m$, is:

$$(P_A) \quad \begin{array}{ll} \min & v = \sum_{i=1}^{m} y_i \\ \text{s.t.} & A\underline{x} + I\underline{y} = \underline{b} \\ & \underline{x} \geq \underline{0}, \underline{y} \geq \underline{0} \end{array}$$

Obvious initial basic feasible solution: $\underline{y} = \underline{b} \geq \underline{0}$ and $\underline{x} = \underline{0}$.

Example:

$$(P) \quad \begin{array}{lllll} \min & x_1+x_2+10x_3 & & & \\ \text{s.t.} & & x_2 & +4x_3 & = 2 \\ & -2x_1+x_2 & & -6x_3 & = 2 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array} \qquad (P_A) \quad \begin{array}{ll} \min & v = y_1+y_2 \\ \text{s.t.} & x_2+4x_3+y_1 = 2 \\ & -2x_1+x_2-6x_3 +y_2 = 2 \\ & x_1,\ x_2,\ x_3,\ y_1,\ y_2 \geq 0 \end{array}$$

We write $v = y_1 + y_2$ in canonical form by expressing y_1 and y_2 in terms of x_1, x_2 and x_3.

		x_1	x_2	x_3	y_1	y_2
$-v$	-4	2	-2	2	0	0
y_1	2	0	1	4	1	0
y_2	2	-2	1	-6	0	1

		x_1	x_2	x_3	y_1	y_2
$-v$	0	2	0	10	2	0
x_2	2	0	1	4	1	0
y_2	0	-2	0	-10	-1	1

By swapping the roles of y_2 and x_1 (or x_3), we identify an optimal solution of the Phase I problem: $\underline{x}^* = (0, 2, 0)^T$, $\underline{y}^* = (0, 0)^T$, with optimal value $v^* = 0$.

در مثال بالا با توجه به این که مقدار y_2 برابر صفر است میتوان x_2 را بر هر مولفه دیگری از جمله x_1 در متغییرهای بیسیک قرار داد.

BFS= Basic Feasible Solution

مدل هایی که **کانستریت های آنها مساوی** دارند با دو روش قابل حل هستند

۱-با روش دو مرحله ای – در این روش حواسمان باشد که باید در نهایت قسمت سمت راست متغیرهای جعلی صفر شود- اگر نشد جواب فیزبل نداریم

۲-با روش دوآل

Basic Variables	**متغییرهای اصلی**			**متغییرهای کمکی**			**متغییرهای جعلی**			**قسمت سمت راست**
Z	X_1	X_2	....	S_1	S_2	....	R_1	R_2	....	**در روش دو مرحله ای از این متغییرها استفاده میشود- این روش برای کانسترین هایی که بزرگتر مساوی هستند صفر یا مساوی صفر هستند کاربرد دارد-فقط حواسمان باشد که بعد از اجرای فاز اول باید سمت راست z برابر صفر گردد.**
S_i,R_i with positive coefficients										**مقدارهای اپتیمال**

Example

$$
\begin{array}{ll}
\min & 2x_1 + x_2 + 5x_3 \\
\text{s. t.} & x_1 + x_2 + x_3 + x_4 = 4 \\
& x_1 + x_5 = 2 \\
& x_3 + x_6 = 3 \\
& 3x_2 + x_3 + x_7 = 6 \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0
\end{array}
\quad
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}
\quad
b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}
$$

- Choosing columns 4, 5, 6, 7, we have $B = I = B^{-1}$, so $\underline{x}_B = B^{-1}\underline{b} = \underline{b} \geq \underline{0}$, thus we obtain a basic feasible solution.
- Choosing columns 2, 5, 6, 7, we have

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix},\ B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix},\ x_B = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -6 \end{bmatrix}
$$

and thus we obtain an infeasible basic solution.

با کمی توجه بیشتر میتوان دریافت هدف پیدا کردن ماتریس های m*m است که باید دو ویژگی را داشته باشد تا اقلا یک راه حل داشته باشد, یک آن ماتریس یک ماتریس non-singular باشد, دوم اینکه این ماتریس همچنین یک ماتریس full rank باشد و برای اینکه feasible solution داشته باشیم باید بردار x_B هیچگونه عدد منفی نداشته باشد.

Number of basic feasible solutions

At most one basic feasible solution for each choice of the $n - m$ non basic variables (to be set to zero) out of the n variables:

$$\#\ \text{basic feasible solutions} \leq \binom{n}{n-m} = \frac{n!}{(n-m)!(n-(n-m))!} = \binom{n}{m}.$$

$$A_{n\times m}\left|\begin{array}{l} \text{Optimal}\ \ B.S \\ BS \qquad\qquad ,n \geq m \\ BFS \leq BS \leq \dfrac{n!}{m!(n-m)!} \end{array}\right.$$

همان طور که می توان دید تعدادی جواب Feasible خواهیم داشت و هدف اصلی ما پیدا کردن آپتیمال ترین جواب با توجه به تابع هدف خواهد بود.

**Finding an optimal solution requires a method that brings us such a matrix or those basic variables. The solution was introduced by George Dantzig as a simplex method for solving LP problems.

In all the models, first, we will try to convert its mathematical form into standard form, and then solve it by methods which will be introduced in the following.

Basic Solution (BS)

Consider a system "AX=b", $(A)_{m\times n}$ (Rank (A)=m) of m equations with n variables (n > m), and Put (n−m) variables to zero and the resulting system $BX_B = b$ solution is called the Basic Solution.

Basic Feasible Solution (BFS)
If the values of all the variables in the basic solution are non-negative, it is called a basic feasible solution (BFS).

infeasible basic solution
A basic solution having at least one variable with a negative value is called an infeasible basic solution.

Degenerate and non-degenerate basic feasible solutions
A BFS with at least one basic variable assuming the value zero is called a degenerate BFS; otherwise it is called a non-degenerate BFS.

Optimal BFS: BFS that optimize the objective function value.

$$x = [x_B, x_N] = [x_1, x_2, s_1, s_2]$$

$$Assume\ \ x_B = [x_1, x_2]\, and\ \ x_N = [s_1, s_2]$$

$$so\ \ s_1 = 0, s_2 = 0$$

$$i)\, x = [1, 2, 0, 0] \triangleright Basic\ \ Feasible\ \ Solution + Non-degenerate$$

$$ii)\, x = [2, -4, 0, 0] \triangleright Basic\ \ Infeasible\ \ Solution$$

$$iii)\, x = [0, 5, 0, 0]\, Basic\ \ Feasible\ \ Solution + degenerate$$

Basic Solution |Part 2| Linear Programming Problem- Degenerate/Non-degenerate Basic Solution

Problem. Write all basic solution to the following system and find optimal solution. Maximize $Z = 2x_1 + 3x_2$

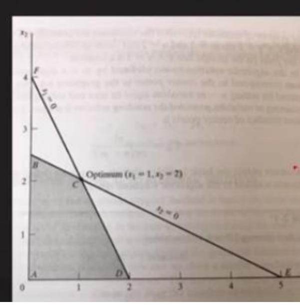
$$\begin{cases} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 5 \end{cases} \text{ and } x_1, x_2 \ge 0$$

Solution. Convert inequalities into equality by adding slack/surplus variable

$$\begin{cases} 2x_1 + x_2 + s_1 = 4 \\ x_1 + 2x_2 + s_2 = 5 \end{cases} \text{ and } x_1, x_2, s_1, s_2 \ge 0$$

Non Basic Variables	Basic Variables	Basic solution	Associated corner point	Feasibility	Objective Value
$(x_1, x_2) = 0$	(s_1, s_2)	(4, 5)	A	Yes	0
$(x_1, s_1) = 0$	(x_2, s_2)	(4, -3)	F	No	-
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, -6)	E	No	-
(s_1, s_2)	(x_1, x_2)	(1, 2)	C	Yes	8 (Optimum)

Graphical and algebraic solution connection



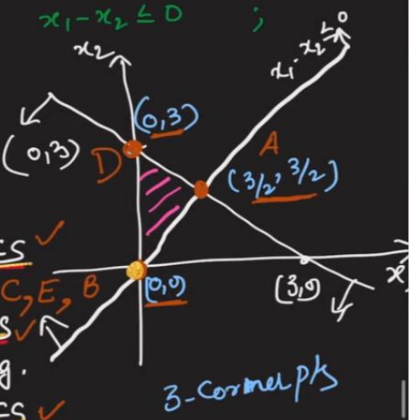
Problem: Find Basic Solution, Identify BFS; degenerate BS; Show correspondence with graphical corner points:

$$\text{Max } Z = 2x_1 + x_2$$
$$\text{s.t. } x_1 + x_2 + x_3 = 3$$
$$x_1 - x_2 + x_4 = 0$$
$$x_i \geq 0;\ i = 1,2,3,4$$

$(A)_{2\times 4}$

Solution: Basic solution

	B.V.	N.B.V.	Solution (x_1, x_2, x_3, x_4)		
1.	x_1, x_2	$x_3 = x_4 = 0$	$A(3/2, 3/2, 0, 0)$ ✓	BFS ✓	Non deg.
2.	x_1, x_3	$x_2 = x_4 = 0$	$B(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓
3.	x_1, x_4	$x_2 = x_3 = 0$	$(3, 0, 0, -3)$ ✗		
4.	x_2, x_3	$x_1 = x_4 = 0$	$C(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓
5.	x_2, x_4	$x_1 = x_3 = 0$	$D(0, 3, 0, 3)$ ✓	BFS	Non deg.
6.	x_3, x_4	$x_1 = x_2 = 0$	$E(0, 0, 3, 0)$ ✓	BFS	deg. BFS ✓



3 vertices $\leq$ 5 = BFS $\leq 6 = {}^nC_m$

Problem. Determine the optimum solution for following LP by enumerating all the basic solution.

Maximize $Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$

s.t.
$$x_1 + 4x_2 - 2x_3 + 8x_4 \leq 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \leq 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$AX = b \qquad {}^6C_2 = 15$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 4 & -2 & 8 & 1 & 0 \\ -1 & 2 & 3 & 4 & 0 & 1 \end{pmatrix}_{2\times 6} \end{array} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Coefficient Matrix

Solution: Introduce slack variables in the constraints.

Cases	B.V.	Non-B.V.	Solution $(x_1,x_2,x_3,x_4,s_1,s_2)$	Value of Z
1	x_1, x_2	$x_3 = x_4 = s_1 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
2	x_1, x_3	$x_2 = x_4 = s_1 = s_2 = 0$	(8, 0, 3, 0, 0, 0)	31 (Optimal)
3	x_1, x_4	$x_2 = x_3 = s_1 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
4	x_1, s_1	$x_2 = x_3 = x_4 = s_2 = 0$	(-1, 0, 0, 0, 3, 0)	Not a BFS ✗ (BS)
5	x_1, s_2	$x_2 = x_3 = x_4 = s_1 = 0$	(2, 0, 0, 0, 0, 3)	4
6	x_2, x_3	$x_1 = x_4 = s_1 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
7	x_2, x_4	$x_1 = x_3 = s_1 = s_2 = 0$	Not a part of BS	Linear dependent columns
8 ✓	x_2, s_1	$x_1 = x_3 = x_4 = s_2 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
9 ✓	x_2, s_2	$x_1 = x_3 = x_4 = s_1 = 0$	(0, 1/2, 0, 0, 0, 0)	-2
10	x_3, x_4	$x_1 = x_2 = s_1 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
11	x_3, s_1	$x_1 = x_2 = x_4 = s_2 = 0$	(0, 0, 1/3, 0, 8/3, 0)	5/3=1.6
12	x_3, s_2	$x_1 = x_2 = x_4 = s_1 = 0$	(0, 0, -1, 0, 0, 4)	Not a BFS
13	x_4, s_1	$x_1 = x_2 = x_3 = s_2 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
14	x_4, s_2	$x_1 = x_2 = x_3 = s_1 = 0$	(0, 0, 0, 1/4, 0, 0)	-1.5
15	s_1, s_2	$x_1 = x_2 = x_3 = x_4 = 0$	(0, 0, 0, 0, 2, 1)	0

14 cases ✗

same corner pt

vertices (corners) $\leq$ BFS $\leq$ BS $\leq {}^nC_m = 15$

$AX = b$

B.V., N.B.V.

$x_2, x_4 \qquad x_1 = x_3 = s_1 = s_2 = 0$

$$B X_B = b \Rightarrow X_B = B^{-1} b$$

$$\begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} = 16 - 16 = 0$$

$$\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 4x_2 + 8x_4 = 2 \\ 2x_2 + 4x_4 = 1 \end{bmatrix}$$

***حواسمان باشد اول مقدار متغییرها را نگاه کنیم اگر منفی داشتند نیازی به پیدا کردن تابع هدف نیست و این جواب در ناحیه امکان پذیر نیست.

Algebraic characterization of the vertices

Consider any $P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\}$ in standard form.

Assumption: $A \in \mathbb{R}^{m \times n}$ is such that $m \leq n$ of rank m (A is of full rank).
Equivalent to assume that there are no "redundant" constraints.

Example

$$\begin{array}{rrrcl} 2x_1 & +x_2 & +x_3 & = & 2 \text{ (I)} \\ x_1 & +x_2 & & = & 1 \text{ (II)} \\ x_1 & & +x_3 & = & 1 \text{ (III)} \\ x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

Since (I) $\equiv$ (II) + (III), then (I) can be dropped.

- If $m = n$, there is a unique solution of $A\underline{x} = \underline{b}$. $(\underline{x} = A^{-1}\underline{b})$.
- If $m < n$, there are ∞ solutions of $A\underline{x} = \underline{b}$ (the system has $n - m$ degrees of freedom, $n - m$ variables can be fixed arbitrarily). By fixing them to 0, we get a vertex.

A full rank matrix, also known as a full-rank matrix or a matrix of full rank, is a matrix in which all its rows and columns are linearly independent. In other words, the rows (or columns) cannot be expressed as linear combinations of each other. The rank of a matrix is the maximum number of linearly independent rows or columns it contains.
Let's consider an example to illustrate a full rank matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To determine if this matrix is of full rank, we can perform row reduction or use other methods to find its rank. However, in this case, it's evident that the rows are linearly independent.
Let's check if the rows are linearly independent by setting up a linear combination:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0$$

Now, solving the system of equations:

$$c_1 + 3c_2 = 0$$
$$2c_1 + 4c_2 = 0$$

This system has a unique solution: $c_1 = 0, c_2 = 0$. Therefore, the only way to make the linear combination equal to the zero vector is by setting both coefficients (c_1, c_2) to zero. This implies that the rows are linearly independent.
Since the matrix has two rows, and both rows are linearly independent, the matrix is of full rank.
In general, for an $m \times n$ matrix, if $m \leq n$, the matrix is of full rank if and only if all its rows are linearly independent. If $m > n$, the matrix is of full rank if and only if all its columns are linearly independent.

Non-singuar matrix[1]

*****Solve a problem about cooking cake and so on.

Decision variables:

$$x_j = \text{ number of devices } D_j \text{ produced, for } j = 1, 2, 3$$

Objective function:

$$\max\ z = 1.6x_1 + 1x_2 + 2x_3$$

Constraints: Production capacity limit for each phase

$$\begin{array}{rcll} 80x_1 + 70x_2 + 120x_3 & \leq & 30\,000 & \text{(assembly)} \\ 70x_1 + 90x_2 + 20x_3 & \leq & 25\,000 & \text{(refinement)} \\ 40x_1 + 30x_2 + 20x_3 & \leq & 18\,000 & \text{(quality control)} \end{array}$$

Non-negative variables:

$$x_1, x_2, x_3 \geq 0 \qquad \text{may be fractional (real) values}$$

[1] Any matrix that has inverse

1.10 Problems

1. Your firm produces two products, Thyristors (T) and Lozenges (L), that compete for the scarce resources of your distribution system. For the next planning period, your distribution system has available 6,000 person-hours. Proper distribution of each T requires 3 hours and each L requires 2 hours. The profit contributions per unit are 40 and 30 for T and L, respectively. Product line considerations dictate that at least 1 T must be sold for each 2 L's.

   (a) Draw the feasible region and draw the profit line that passes through the optimum point.
   (b) By simple common sense arguments, what is the optimal solution?

1.6 Unbounded Formulations

If we forget to include the labor constraint and the constraint on the production of Cosmos, then an unlimited amount of profit is possible by producing a large number of Cosmos. This is illustrated here:

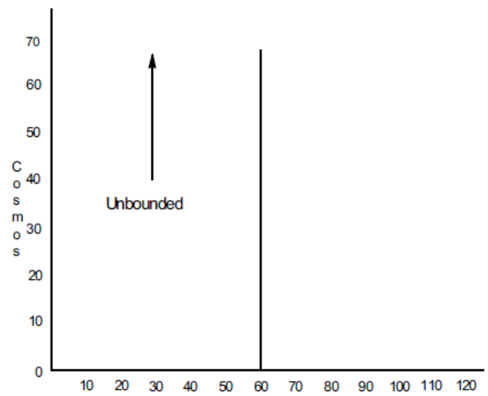
```
MAX = 20 * A + 30 * C;
A <= 60;
```

This generates an error window with the message:

```
UNBOUNDED SOLUTION
```

There is nothing to prevent C from being infinitely large. The feasible region is illustrated in Figure 1.7. In larger problems, there are typically several unbounded variables and it is not as easy to identify the manner in which the unboundedness arises.

Figure 1.7 Graph of Unbounded Formulation



My answer is

$$x_T, x_L = number\ \ of\ \ the\ \ two\ \ productions$$

$$Objective\ \ Function(profit) = Max\ \ 40x_T + 30x_L$$

$$Constraints = \left| \begin{array}{l} 3x_T + 2x_L \le 6000 \\ x_T \ge 2x_L \end{array} \right.$$

$$non-negative\ \ variables = x_T \ge 0, x_L \ge 0$$

It is better to use x_i symbols I belongs to {1,2,…}

	XT	XL		RHS
Objective	40	30	->	MAX
Row1	3	2	<=	6000
Row2	1	-2	>=	0
Lower Bound	0	0		
Upper Bound	INF	INF		

Solution in a software application

*** Phase II --- Start ***

Basis	X1	X2	s3	s4	RHS
s3	3	2	1	0	6000
s4	-1	2	0	1	0
Obj.	40	30	0	0	0

Variable to be made basic -> X1
Ratios: RHS/Column X1 -> { 2000 - }
Variable out of the basic set -> s3

*** Phase II --- Iteration 1 ***

Basis	X1	X2	s3	s4	RHS
X1	1	2/3	1/3	0	2000
s4	0	8/3	1/3	1	2000
Obj.	0	10/3	-40/3	0	80000

Variable to be made basic -> X2
Ratios: RHS/Column X2 -> { 3000 750 }
Variable out of the basic set -> s4

*** Phase II --- Iteration 2 ***

Basis	X1	X2	s3	s4	RHS
X1	1	0	0.25	-0.25	1500
X2	0	1	1/8	3/8	750
Obj.	0	0	-13.75	-1.25	82500

>> Optimal solution FOUND
>> Maximum = 82500

*** RESULTS - VARIABLES ***

Variable	Value	Obj. Cost	Reduced Cost
X1	1500	40	0
X2	750	30	0

*** RESULTS - CONSTRAINTS ***

Constraint	Value	RHS	Dual Price
Row1	6000	6000	13.75
Row2	0	0	-1.25

The Volkswagen Company produces two products, the Bug and the SuperBug, which share production facilities. Raw materials costs are $600 per car for the Bug and $750 per car for the SuperBug. The Bug requires 4 hours in the foundry/forge area per car; whereas, the SuperBug, because it uses newer more advanced dies, requires only 2 hours in the foundry/forge. The Bug requires 2 hours per car in the assembly plant; whereas, the SuperBug, because it is a more complicated car, requires 3 hours per car in the assembly plant. The available daily capacities in the two areas are 160 hours in the foundry/forge and 180 hours in the assembly plant. Note, if there are multiple machines, the total hours available per day may be greater than 24. The selling price of the Bug at the factory door is $4800. It is $5250 for the SuperBug. It is safe to assume whatever number of cars are produced by this factory can be sold.

(a) Write the linear program formulation of this problem.

(b) The above description implies the capacities of the two departments (foundry/forge and assembly) are sunk costs. Reformulate the LP under the conditions that each hour of foundry/forge time cost $90; whereas, each hour of assembly time cost $60. The capacities remain as before. Unused capacity has no charge.

Profit = Revenue – costs

$$x_i = number\ \ of\ \ the\ \ productions\ \ i \in \{1,2\}$$

$$Objective\ \ Function(profit) = Max\ \left(4800-600\right)x_1 + \left(5250-750\right)x_2$$

$$Constraints = \left| \begin{array}{l} 4x_1 + 2x_2 \le 160 \\ 2x_1 + 3x_2 \le 180 \end{array} \right.$$

$$non-negative\ \ variables = x_1 \ge 0, x_2 \ge 0$$

24 will be used for what?????????????

	X1	X2		RHS
Objective	4200	4500	->	MAX
Row1	4	2	<=	160
Row2	2	3	<=	180
Lower Bound	0	0		
Upper Bound	INF	INF		
Type	CONT	CONT		

*** Phase II --- Start ***

Basis	X1	X2	s3	s4	RHS
s3	4	2	1	0	160
s4	2	3	0	1	180
Obj.	4200	4500	0	0	0

Variable to be made basic -> X2
Ratios: RHS/Column X2 -> { 80 60 }
Variable out of the basic set -> s4

*** Phase II --- Iteration 1 ***

Basis	X1	X2	s3	s4	RHS
s3	8/3	0	1	-2/3	40
X2	2/3	1	0	1/3	60
Obj.	1200	0	0	-1500	270000

Variable to be made basic -> X1
Ratios: RHS/Column X1 -> { 15 90 }
Variable out of the basic set -> s3

*** Phase II --- Iteration 2 ***

Basis	X1	X2	s3	s4	RHS
X1	1	0	3/8	-0.25	15
X2	0	1	-0.25	0.5	50
Obj.	0	0	-450	-1200	288000

>> Optimal solution FOUND
>> Maximum = 288000

*** RESULTS - VARIABLES ***

Variable	Value	Obj. Cost	Reduced Cost
X1	15	4200	0

X2	50	4500	0

*** RESULTS - CONSTRAINTS ***

Constraint	Value	RHS	Dual Price
Row1	160	160	450
Row2	180	180	1200

Model 4: portfolio selection

در این مدل سوالات درصد برگشت سرمایه میدهند و چون هدف انتخاب سبد سرمایه است در **تابع هدف** درصد برگشت سرمایه در میزان سرمایه در آن بازار به همراه این نکته که باید مشخص کنیم این مورد انتخاب میشود یا نه که با ضرب یک متغییر که مقدار آن صفر یا یک است در هنگام انتخاب آن را یک و در صورت عدم انتخاب آن را صفر میکنیم تا اثرش از بین برود.

Model 5: assignment

Statement *The production process in a factory requires 3 parallel independent tasks.*

Each task requires exactly one machine, and each machine can perform at most one task at a time. To save time, the tasks are performed in parallel on different machines.

The cost required to perform each task on each machine is reported in the following table:

	M_1	M_2	M_3
T_1	2	6	3
T_2	8	4	9
T_3	5	7	8

The factory managers want to assign the tasks to the machines so as to minimize the total cost.

Model 4: portfolio selection

Statement *An insurance company must select the investments to perform out of a given set of possible assets (stocks, bonds, etc. . .).*

The total budget is 600 *Keuros, and the capital required by each asset is:*

Asset	A	B	C	D	E	F	G	H
Capital	150	150	60	100	125	100	50	80

Each asset refers to a given sector and nation, and has an expected return on investment (ROI):

Asset	A	B	C	D	E	F	G	H
Nation	Germany	Italy	USA	Italy	Italy	France	Italy	UK
Sector	autom.	autom.	ICT	ICT	real estate	real estate	short bond	long bond
ROI	11%	9%	13%	10%	8%	7%	3%	5%

The internal rules impose:

- *to select at most* 5 *investements (to avoid excessive fragmentation)*
- *to select at most* 3 *investments in Italy, at most* 3 *abroad (for geographical diversification)*
- *to select a bond if an ICT investement is selected (to limit risk)*

Model 8: multi-period production

Statement *A factory produces a single product, at a unitary production cost which changes month after month as follows:*

	Jan	Feb	Mar	Apr	May	Jun
Prod. cost (euros/unit)	20	25	30	40	50	60

The factory can produce a different maximum amount in each month:

	Jan	Feb	Mar	Apr	May	Jun
Prod. capacity (units)	1 500	2 000	2 200	3 000	2 700	2 500

The products are sold in blocks at the end of each month; the demand and the price are known in advance:

	Jan	Feb	Mar	Apr	May	Jun
Demand (units)	1 100	1 500	1 800	1 600	2 300	2 500
Price (euros/unit)	180	180	250	270	300	320

The products are first stored in a temporary depot for free; at the end of the month, the unsold ones are moved to a larger depot which can store 3 000 *units at* 2 *euros/unit per month.*

Determine the production plan in order to maximize the total profit (total revenue minus total production and storage costs.

Model

- x_i = number of products for month $i \in \{1, \ldots, 6\}$
- y_i = number of products stored at the beginning of month $i \in \{1, \ldots, 7\}$ (i. e., after selling the products and moving the unsold ones to the depot)

$$
\begin{aligned}
\max f(x) = &(180\ 1100 + 180\ 1500 + 250\ 1800 + && (8a)\\
&270\ 1600 + 300\ 2300 + 320\ 2500) - && (8b)\\
&-(20x_1 + 25x_2 + 30x_3 + 40x_4 + 50x_5 + 60x_6) + && (8c)\\
&-2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) && (8d)\\
&x_1 \leq 1500 && (8e)\\
&x_2 \leq 2000 && (8f)\\
&x_3 \leq 2200 && (8g)\\
&x_4 \leq 3000 && (8h)\\
&x_5 \leq 2700 && (8i)\\
&x_6 \leq 2500 && (8j)\\
&y_i \leq 3000 \quad i \in \{1, \dots, 6\} && (8k)\\
&y_1 = y_7 = 0 && (8l)\\
&y_i + x_i - d_i = y_{i+1} \quad i \in \{1, \dots, 6\} && (8m)\\
&x_i \geq 0 \quad i \in \{1, \dots, 6\} && (8n)\\
&y_i \geq 0 \quad i \in \{1, \dots, 7\} && (8o)
\end{aligned}
$$

انبار اخر هر ماه تولید, شارژ می‌شود پس اول ماه اول انبار خالی اشت

و اول ماه چهارم موجودی انبار برابر با تعداد باقیمانده در انبار از ماه سوم بعلاوه تعداد تولید شده در انتهای ماه سوم منهای محصولات تحویل داده شده به متقاضیان, که همان لحظه ورود کالای اولی شده به انبار به آنها تحویل کیگردد ولی تقاضایشان را اول ماه سوم داده اند, خواهد بود.

Exercise 1 - A catering service must organize two coffee-breaks, for different customers. They have already prepared 30 cookies, whose price is 1.5 Euros each, and 50 fruit tarts, whose price is 1 Euro each. The catering service is free to mix the two kinds of desserts, without exceeding the budget of the two customers, which is 30 Euros for the first one and 40 Euros for the second. In addition, the first customer wants the cookies to be at least twice as many as the tarts, whereas the second wants a balanced distribution, with a difference not larger than 5 units between cookies and tarts.

Give a mathematical programming formulation for the problem of maximizing the revenue of the catering service (**without solving it**).

Exercise 1 - The following is one of the possible models. The decision variables x_{c1}, x_{c2}, x_{t1}, x_{t2} determine the number of cookies and tarts sold to the first and second customer. They are all integer and nonnegative. The objective is the total revenue of the catering service:

$$\max f = 1.5[\text{Euros/cookie}] \cdot (x_{c1} + x_{c2})\,[\text{cookies}] + 1.0[\text{Euros/tart}] \cdot (x_{t1} + x_{t2})\,[\text{tarts}]$$

The cookies and tarts sold should not exceed the available number.

$$\begin{aligned} x_{c1}[\text{cookie}] + x_{c2}[\text{cookie}] &\leq 30[\text{cookie}] \\ x_{t1}[\text{tarts}] + x_{t2}[\text{tarts}] &\leq 50[\text{tarts}] \end{aligned}$$

The money required from each customer should not exceed the corresponding budget.

$$\begin{aligned} 1.5[\text{Euros/cookie}] \cdot x_{c1}[\text{cookies}] + 1.0[\text{Euros/tart}] \cdot x_{t1}[\text{tarts}] &\leq 30[\text{Euros}] \\ 1.5[\text{Euros/cookie}] \cdot x_{c2}[\text{cookies}] + 1.0[\text{Euros/tart}] \cdot x_{t2}[\text{tarts}] &\leq 40[\text{Euros}] \end{aligned}$$

For the first customer, the cookies should be at least twice as many as the tarts; for the second one, both possible differences between the two numbers (cookies minus tarts and tarts minus cookies) should not exceed 5.

$$\begin{aligned} x_{c1} &\geq 2x_{t1} \\ x_{c2} - x_{t2} &\leq 5 \\ x_{t2} - x_{c2} &\leq 5 \end{aligned}$$

All variables should be nonnegative integers.

$$x_{c1}, x_{c2}, x_{t1}, x_{t2} \in \mathbb{N}$$

Exercise 1 - The following is a possible model. The decision variables are the amount of product transported from eac dept to each selling point (in tons). The objective function is the total transportation cost, in Euros. The amount transported out of each depot must not exceed the available amount. The amount transported to each selling point must not be lower than the required amounts. Of course, all amounts are nonnegative.

$$\begin{aligned} \max f = {} & 5[Euros/ton] x_{A1}[tons] + 7[Euros/ton] x_{B1}[tons] + \\ & 8[Euros/ton] x_{A2}[tons] + 9[Euros/ton] x_{B2}[tons] + \\ & 11[Euros/ton] x_{A3}[tons] + 6[Euros/ton] x_{B3}[tons] \end{aligned}$$

$$\begin{aligned} x_{A1}[tons] + x_{A2}[tons] + x_{A3}[tons] &\leq 70[tons] \\ x_{B1}[tons] + x_{B2}[tons] + x_{B3}[tons] &\leq 170[tons] \\ x_{A1}[tons] + x_{B1}[tons] &\geq 60[tons] \\ x_{A2}[tons] + x_{B2}[tons] &\geq 60[tons] \\ x_{A3}[tons] + x_{B3}[tons] &\geq 80[tons] \\ x_{A1}, x_{A2}, x_{A3}, x_{B1}, x_{B2}, x_{B3} &\geq 0 \end{aligned}$$