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Chapter 1. Solutions to *Lambda Calculi with Types*

Problem 1. (Exercise 3.1.13) *Exercise Statement*

Solution: *Solution!*



Problem 2. (Exercise 4.1.20) *Exercise Statement*

Problem 3. (Exercise 4.2.8) *Exercise Statement*

Problem 4. (Exercise 5.1.16) *Exercise Statement*

Chapter 2. Solutions to *Domain-Theoretic Foundations of Functional Programming*

2.1 PCF and its Operational Semantics

Problem 1. (Page 14) Show that the σ with $\Gamma \vdash M : \sigma$ is uniquely determined by Γ and M .

Solution: We prove this by induction on the structure.

- if $M \equiv x$ (variable), then it must be by the variable rule: $\Gamma', x : \sigma \vdash x : \sigma$; thus σ must be unique by the definition of the context Γ . ($\Gamma \equiv x_1 : \sigma_1, \dots, x_n : \sigma_n$, where x_i are pairwise distinct variables).
- if $M \equiv Z$ (zero), then it must be derived by the zero rule: $\Gamma \vdash Z : \mathbb{N}$; thus its type is unique.
- if $M \equiv (\lambda x : \sigma. M)$, then it must be derived by the abstraction rule: $\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x : \sigma. M) : \sigma \rightarrow \tau}$. By IH, M and x have unique types τ and σ , respectively. Thus, the type of the abstraction is uniquely determined as $\sigma \rightarrow \tau$.
- if $M \equiv (M(N))$, then by the application rule, we would have $\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M(N) : \tau}$. By IH, M and N have unique types $\sigma \rightarrow \tau$ and σ , respectively. Thus, the type of the application $M(N)$ is uniquely determined as τ .
- Same goes for the other cases (*succ*, *pred*, Y_σ , and *ifz*).

■

Problem 2. (Page 16) (Lemma 2.1.) The evaluation relation \Downarrow is deterministic, i.e. whenever $M \Downarrow V$ and $M \Downarrow W$ then $V \equiv W$

Solution: We prove this by induction on the structure of the derivation.

Base cases.

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
 - $M \equiv x$, then $x \Downarrow x$. So V and W can only be x ; thus, $V \equiv W \equiv x$.
 - $M \equiv \lambda x : \sigma. M$, then $\lambda x : \sigma. M \Downarrow \lambda x : \sigma. M$.
 - $M \equiv 0$, then $0 \Downarrow 0$.

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then it must be derived by the rule $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$. Then we would have $V \equiv n+1$ and $W \equiv m+1$ since the successor rule is the only way to derive $\text{succ}(M)$. By IH, we know that $n = m$, thus $n+1 = m+1$, and hence $V = W$.
- If $M \equiv M(N)$. The derivation for $M(N)$ must be of the form $\frac{M \Downarrow \lambda x : \sigma. E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. A second derivation for $M(N)$ must use the same rule. i.e., $\frac{M \Downarrow \lambda x : \sigma'. E' \quad E'[N/x] \Downarrow W}{M(N) \Downarrow W}$. But then by IH, we would have $\lambda x : \sigma. E \equiv \lambda x : \sigma'. E'$. So $\sigma \equiv \sigma'$ and $E = E'$. Now, we have $E[N/x] \Downarrow V$ and $E[N/x] \Downarrow W$. By the IH on the sub-derivation for $E[N/x]$, we conclude $V \equiv W$.

- If $M \equiv \text{pred}(M)$, then the rules are $\frac{M \Downarrow 0}{\text{pred}(M) \Downarrow 0}$ and $\frac{M \Downarrow n+1}{\text{pred}(M) \Downarrow n}$. For the derivation $\text{pred}(M) \Downarrow V$, we must have a sub-derivation for $M \Downarrow \underline{x}$ for some numeral \underline{x} . Similarly, for $\text{pred}(M) \Downarrow W$, we must have a sub-derivation for $M \Downarrow \underline{y}$ for some numeral \underline{y} . By the IH on the sub-derivation for M , we can conclude that $\underline{x} \equiv \underline{y} \equiv k$.

Let's examine k . If $k \equiv 0$, then both derivations must be $\frac{M \Downarrow 0}{\text{pred}(M) \Downarrow 0}$. Thus, $V \equiv W \equiv 0$. Same argument is valid for the case $k \equiv n+1$.

- The other cases (Y_σ , and both cases of $\text{if}z$) can be proved likewise.

■

Problem 3. (Page 16) (Theorem 2.2, Subject Reduction). If $\Gamma \vdash M : \sigma$ and $M \Downarrow V$ then $\Gamma \vdash V : \sigma$.

Solution: We prove this by induction on the structure of the derivation of $M \Downarrow V$.

Base cases.

- If $M \equiv x$, then if $\Gamma \vdash x : \sigma$, then the goal is trivial since $x \Downarrow x$.
Same goes for the other base cases ($\lambda x : \sigma. M, 0$).

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then the only evaluation rule is $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$; so $V \equiv n+1$. We are given $\Gamma \vdash \text{succ}(M) : \sigma$ and by the typing rule for succ , we have $\Gamma \vdash M : \text{nat}$.
Now, from $M \Downarrow n$ and $\Gamma \vdash M : \text{nat}$, by IH, $\Gamma \vdash n : \text{nat}$; thus, by the typing rule for succ , we have $\Gamma \vdash n+1 : \text{nat}$.
- If $M \equiv M(N)$, the evaluation rule is $\frac{M \Downarrow \lambda x : \sigma. E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. We are given $\Gamma \vdash M(N) : \sigma$ and by the typing rule for application, we have $\Gamma \vdash M : \sigma \rightarrow \tau$ and $\Gamma \vdash N : \sigma$. Now, we have $M \Downarrow \lambda x : \sigma. E$ and $\Gamma \vdash M : \sigma \rightarrow \tau$ and by IH, we have $\Gamma \vdash \lambda x : \sigma. E : \sigma \rightarrow \tau$. The typing rule for abstraction is $\frac{\Gamma, x : \sigma \vdash E : \tau}{\Gamma \vdash \lambda x : \sigma. E : \sigma \rightarrow \tau}$, so we have $\Gamma, x : \sigma \vdash E : \tau$. Using the substitution lemma, we can conclude that $\Gamma \vdash E[N/x] : \tau$. Now, from $E[N/x] \Downarrow V$ and $\Gamma \vdash E[N/x] : \tau$, by IH, we have $\Gamma \vdash V : \tau$.
- Same goes for the other cases (both cases of pred , Y_σ , and both cases of $\text{if}z$).

■

Problem 4. (Page 17) $M \Downarrow V$ iff $M \triangleright^* V$.

Solution: In order to prove this, we prove the following two lemmas:

- (a). if $M \Downarrow V$ then $M \triangleright^* V$
- (b). if $M \triangleright N$ then for all values V , if $N \Downarrow V$, then $M \Downarrow V$.

Then, applying (b) iteratively, it follows that:

- (c). if $M \triangleright^* N$ and $N \Downarrow V$, then $M \Downarrow V$.

and by (a) and (c), we can conclude that $M \Downarrow V$ iff $M \triangleright^* V$.

Proof of (a). We prove this by induction on the structure of the derivation of $M \Downarrow V$.

Base cases.

- All the base cases ($M \equiv x$, $M \equiv \lambda x : \sigma.M$, and $M \equiv \underline{0}$) are trivial by the reflexivity of \triangleright^* .

Inductive Steps.

- If $M \equiv \text{succ}(M)$, then the rule is $\frac{M \Downarrow \underline{n}}{\text{succ}(M) \Downarrow \underline{n+1}}$; thus, $V \equiv \underline{n+1}$. By IH on $M \Downarrow \underline{n}$, we have $M \triangleright^* \underline{n}$. Now, there are two cases.
 - If $M \equiv \underline{n}$, then $\text{succ}(M) \equiv \text{succ}(\underline{n}) \equiv \underline{n+1}$. So $M \triangleright^* V$.
 - If $M \triangleright M_1 \triangleright \dots \triangleright \underline{n}$, then by the congruence rule for succ , we would have $\text{succ}(M) \triangleright \text{succ}(M_1) \triangleright \dots \triangleright \text{succ}(\underline{n}) \equiv \underline{n+1}$. So, $\text{succ}(M) \triangleright^* \underline{n+1} \equiv V$.
- If $M \equiv M(N)$, then the rule is $\frac{M \Downarrow \lambda x : \sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. Then, by IH on $M \Downarrow \lambda x : \sigma.E$, we have $M \triangleright^* \lambda x : \sigma.E$, and on $E[N/x] \Downarrow V$, we have $E[N/x] \triangleright^* V$.

If $M \equiv \lambda x : \sigma.E$, then $M(N) \equiv (\lambda x : \sigma.E)(N)$. By small-step rule $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$. Since $E[N/x] \triangleright^* V$, we have $M(N) \triangleright E[N/x] \triangleright^* V$, so $M(N) \triangleright^* V$.

If $M \triangleright M_1 \triangleright \dots \triangleright \lambda x : \sigma.E$, then by the congruence rule for application $M(N) \triangleright M_1(N) \triangleright \dots \triangleright (\lambda x : \sigma.E)(N)$. Then $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$, and $E[N/x] \triangleright^* V$. Then we would have: $M(N) \triangleright^* (\lambda x : \sigma.E)(N) \triangleright E[N/x] \triangleright^* V$. So $M(N) \triangleright^* V$.
- The other cases are fairly similar to these.

Proof of (b). We prove this by induction on the structure of the derivation of $M \triangleright V$.

Base cases.

- Assume $M \equiv (\lambda x : \sigma.E)(A)$ and $N \equiv E[A/x]$ ($M \triangleright N$). Assume also $N \Downarrow V$, i.e., $E[A/x] \Downarrow V$. We know $\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E$. By the rule: $\frac{\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E \quad E[A/x] \Downarrow V}{(\lambda x : \sigma.E)(A) \Downarrow V}$. So $M \Downarrow V$.
- Assume $M \equiv Y_\sigma(E)$ and $N \equiv E(Y_\sigma(E))$. Assume also $N \Downarrow V$, i.e., $E(Y_\sigma(E)) \Downarrow V$. By the fixpoint rule: $\frac{E(Y_\sigma(E)) \Downarrow V}{Y_\sigma(E) \Downarrow V}$. So $M \Downarrow V$.
- Assume $M \equiv \text{pred}(\underline{0})$ and $N \equiv \underline{0}$. Assume also $N \Downarrow V$, i.e., $\underline{0} \Downarrow V$. This means $V \equiv \underline{0}$. We need to show $\text{pred}(\underline{0}) \Downarrow \underline{0}$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0}}{\text{pred}(\underline{0}) \Downarrow \underline{0}}$.
- Assume $M \equiv \text{pred}(\underline{k+1})$ and $N \equiv \underline{k}$. Assume also $N \Downarrow V$, i.e., $\underline{k} \Downarrow V$. This means $V \equiv \underline{k}$. We need to show $\text{pred}(\underline{k+1}) \Downarrow \underline{k}$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1}}{\text{pred}(\underline{k+1}) \Downarrow \underline{k}}$.
- Assume $M \equiv \text{ifz}(\underline{0}, E_1, E_2)$ and $N \equiv E_1$. Assume also $N \Downarrow V$, i.e., $E_1 \Downarrow V$. We need to show $\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0} \quad E_1 \Downarrow V}{\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V}$.
- Assume $M \equiv \text{ifz}(\underline{k+1}, E_1, E_2)$ and $N \equiv E_2$. Assume also $N \Downarrow V$, i.e., $E_2 \Downarrow V$. We need to show $\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad E_2 \Downarrow V}{\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V}$.

Inductive Steps.

- $\frac{M_1 \triangleright M_2}{\text{succ}(M_1) \triangleright \text{succ}(M_2)}$: $M = \text{succ}(M_1)$, $N = \text{succ}(M_2)$, where $M_1 \triangleright M_2$. Assume $N \Downarrow V$, i.e., $\text{succ}(M_2) \Downarrow V$. This implies $V \equiv \underline{k+1}$ and $M_2 \Downarrow \underline{k}$ for some k . By IH on the sub-derivation $M_1 \triangleright M_2$: since $M_2 \Downarrow \underline{k}$, it follows that $M_1 \Downarrow \underline{k}$. Then, by the rule $\frac{M_1 \Downarrow \underline{k}}{\text{succ}(M_1) \Downarrow \underline{k+1}}$. So $M \equiv \text{succ}(M_1) \Downarrow \underline{k+1}$. Since $V \equiv \underline{k+1}$, we have $M \Downarrow V$.

- $\frac{M_1 \triangleright M_2}{M_1(A) \triangleright M_2(A)}$: $M = M_1(A)$, $N = M_2(A)$. Assume $M_2(A) \Downarrow V$. This means $M_2 \Downarrow \lambda x.E$ and $E[A/x] \Downarrow V$. By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \lambda x.E$, then $M_1 \Downarrow \lambda x.E$. Thus, by the rule, $M_1(A) \Downarrow V$.
- $\frac{M_1 \triangleright M_2}{pred(M_1) \triangleright pred(M_2)}$: $M = pred(M_1)$, $N = pred(M_2)$. Assume $pred(M_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and V is $\underline{0}$ (if $k = 0$) or $\underline{k-1}$ (if $k > 0$). By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for $pred$, $pred(M_1) \Downarrow V$.
- $\frac{M_1 \triangleright M_2}{ifz(M_1, N_1, N_2) \triangleright ifz(M_2, N_1, N_2)}$: $M = ifz(M_1, N_1, N_2)$, $N = ifz(M_2, N_1, N_2)$. Assume $ifz(M_2, N_1, N_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and either $N_1 \Downarrow V$ (if $k = 0$) or $N_2 \Downarrow V$ (if $k > 0$). By IH on $M_1 \triangleright M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for ifz , $ifz(M_1, N_1, N_2) \Downarrow V$.

■

Problem 5. (Page 19) Show that the applicative relation \sqsubseteq_σ is a preorder on Prg_σ , i.e. that \sqsubseteq_σ is reflexive and transitive.

Solution:

-Reflexivity. We need to show that for any closed PCF term M of type σ , $M \sqsubseteq_\sigma M$.

Base Case. For $M \in Prg_{nat}$, $M \sqsubseteq_{nat} M$ means that $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow M \Downarrow \underline{n}$. This is trivially true.

Inductive Case. For $M \in Prg_{\sigma \rightarrow \tau}$, $M \sqsubseteq_{\sigma \rightarrow \tau} M$ means that $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau M(P)$, which holds by IH.

-Transitivity. We need to show that for any closed PCF terms M, N, K of type σ , if $M \sqsubseteq_\sigma N$ and $N \sqsubseteq_\sigma K$, then $M \sqsubseteq_\sigma K$.

Base Case. For $M, N, K \in Prg_{nat}$, assume $M \sqsubseteq_{nat} N$ and $N \sqsubseteq_{nat} K$. Then, by definition, we have the followings:

- $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow N \Downarrow \underline{n}$
- $\forall n \in \mathbb{N}, N \Downarrow \underline{n} \Rightarrow K \Downarrow \underline{n}$

Thus, if $M \Downarrow \underline{n}$ then $K \Downarrow \underline{n}$, which means $M \sqsubseteq_{nat} K$.

Inductive Case. For $M, N, K \in Prg_{\sigma \rightarrow \tau}$, assume $M \sqsubseteq_{\sigma \rightarrow \tau} N$ and $N \sqsubseteq_{\sigma \rightarrow \tau} K$. Then, by definition, we have the followings:

- $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau N(P)$
- $\forall P \in Prg_\sigma, N(P) \sqsubseteq_\tau K(P)$

Thus, we would have $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau K(P)$.

■

2.2 The Scott Model of PCF

Problem 1. (Page 26) Show that (Scott) continuous functions between predomains are always monotonic.

Solution: Let $f : (A, \sqsubseteq_A) \rightarrow (B, \sqsubseteq_B)$ be a Scott continuous function between predomains. For any $x, y \in A$ with $x \leq y$, we have that $X = \{x, y\}$ is a directed subset of A . The supremum of the set X is obviously y . Thus $\sqcup X = y$, and since f is continuous, $f(\sqcup X) = f(y) = \sqcup f(\{x, y\})$. Hence, $f(x) \leq f(y)$. ■

Problem 2. (Page 26) (Theorem 3.3). Let $(A_i | i \in I)$ be a family of predomains. Then their product $\prod_{i \in I} A_i$ is a predomain under the componentwise ordering, and the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are Scott continuous. If, moreover, all A_i are domains then so is their product $\prod_{i \in I} A_i$.

Solution: Let $D = \prod_{i \in I} A_i = \{f : I \rightarrow \bigcup_{i \in I} A_i \mid \forall i \in I, f(i) \in A_i\}$. We need to show that (D, \sqsubseteq_D) is a poset, and every directed subset of D has a least upper bound. Note that the order \sqsubseteq_D is defined as follows:

$$(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I} \text{ iff } \forall i \in I, d(i) \sqsubseteq_{A_i} d'(i)$$

We now show that (D, \sqsubseteq_D) forms a poset.

- **Reflexivity:** For any $(d_i)_{i \in I} \in D$, $d_i \sqsubseteq_{A_i} d_i, \forall i \in I$ since each A_i is a poset. Thus $(d_i)_{i \in I} \sqsubseteq_D (d_i)_{i \in I}$.
- **Transitivity:** Assume $(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I}$ and $(d'_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$. And by each A_i being transitive, it follows immediately that $d_i \sqsubseteq_{A_i} d''_i$ for all $i \in I$. Therefore, $(d_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$.
- **Antisymmetry:** Similar to the previous case, it follows immediately from the fact that each A_i is antisymmetric.

Now, suppose that $X \subseteq D = \prod_{i \in I} A_i$ is a directed subset. define $X_i = \{\pi_i(x) \mid x \in X\}$, that is, the projection of X to A_i . X_i is directed since X is directed. Moreover, X_i has a least upper bound $\sqcup X_i \in A_i$. Define $z \in D$ with $z_i = \sqcup X_i$ for each $i \in I$. By construction, it is obvious that z is the least upper bound of X in D . Thus, D is a predomain. ■

Problem 3. (Page 27). Prove that the evaluation map $ev : [A_1 \rightarrow A_2] \times A_1 \rightarrow A_2$ with $ev(f, a) = f(a)$ is continuous in each argument.

Solution: For the first argument, fix $a \in A_1$ and let $F \subseteq [A_1 \rightarrow A_2]$ be a directed set of continuous functions. By Theorem 3.5, we have $\sqcup F(a) = g(a) = \sqcup_{f \in F} f(a)$. Thus, $ev(\sqcup F, a) = g(a) = \sqcup_{f \in F} f(a) = \sqcup \{ev(f, a) \mid f \in F\}$.

Now, for the second argument, fix $f \in [A_1 \rightarrow A_2]$ and let $X \subseteq A_1$ be a directed set. Because f is continuous, we have $f(\sqcup X) = \sqcup \{f(x) \mid x \in X\}$. Thus,

$$ev(f, \sqcup X) = f(\sqcup X) = \sqcup \{f(x) \mid x \in X\} = \sqcup \{ev(f, x) \mid x \in X\}.$$

Hence, ev is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

Problem 4. (Page 30). Prove that $\Psi : [[D \rightarrow D] \rightarrow D] \times [D \rightarrow D] \rightarrow D : (F, f) \mapsto f(F(f))$ is continuous in each argument.

Solution: For the first argument, fix $f \in [D \rightarrow D]$ and let $\mathcal{F} \subseteq [[D \rightarrow D] \rightarrow D]$ be a directed set. We know that $(\sqcup \mathcal{F})(f) = \sqcup \{F(f) \mid F \in \mathcal{F}\}$. Thus,

$$\underbrace{f((\sqcup \mathcal{F})(f))}_{\Psi(\sqcup \mathcal{F}, f)} = f(\sqcup \{F(f) \mid F \in \mathcal{F}\}) = \sqcup \{f(F(f)) \mid F \in \mathcal{F}\}$$

For the second argument, fix $F \in [[D \rightarrow D] \rightarrow D]$ and let $X \subseteq [D \rightarrow D]$ be a directed set. We know that $F(\sqcup X) = \sqcup \{F(f) \mid f \in X\}$. Thus,

$$f(F(\sqcup X)) = f(\sqcup \{F(f) \mid f \in X\}) = \sqcup \{f(F(f)) \mid f \in X\}$$

Hence, Ψ is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

Problem 5. (Page 33) (β -equality). If, $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$ then

$$\llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket = \llbracket \Gamma \vdash M[N/x] \rrbracket$$

Solution: $\llbracket \Gamma \vdash M[N/x] \rrbracket(\vec{d}) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d}))$ by Lemma 3.15. Now, for the other side,

$$\begin{aligned} \llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket(\vec{d}) &= ev(\llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d}), \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \\ &= \llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d})(\llbracket \Gamma \vdash N \rrbracket(\vec{d})) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \end{aligned}$$
■

Problem 6. (Page 34) (η -equality). If, $\Gamma \vdash M : \sigma \rightarrow \tau$ then

$$\llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket = \llbracket \Gamma \vdash M \rrbracket$$

for $x \notin \text{Var}(\Gamma)$.

Solution:

$$\begin{aligned} \forall \vec{d} \in \llbracket \Gamma \rrbracket, d' \in D_\sigma, \llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket(\vec{d})(d') &= \llbracket \Gamma, x : \sigma \vdash M(x) \rrbracket(\vec{d}, d') \\ &= ev(\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d'), \underbrace{\llbracket \Gamma, x : \sigma \vdash x \rrbracket(\vec{d}, d')}_{\pi_x(\vec{d}, d')=d'}) =^* \llbracket \Gamma \vdash M \rrbracket(\vec{d})(d') \end{aligned}$$

For the last equation $(*)$, because $x \notin \text{Var}(\Gamma)$ and $\Gamma \vdash M$, we have

$$\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d') = \llbracket \Gamma \vdash M \rrbracket(\vec{d})$$

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■

2.3 Milner's Context Lemma

Problem 1. (Page 44) *Problem Statement*

2.4 Logical Relations

Problem 1. (Page 52) *Problem Statement*

Problem 2. (Page 54) *Problem Statement*
