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Chapter 1. Solutions to *Lambda Calculi with Types*

Problem 1. (Exercise 3.1.13) *Exercise Statement*

Solution: *Solution!* ■

Problem 2. (Exercise 4.1.20) *Exercise Statement*

Problem 3. (Exercise 4.2.8) *Exercise Statement*

Problem 4. (Exercise 5.1.16) :

(i): Define $\neg \equiv \lambda\alpha : *$

Chapter 2. Solutions to *Domain-Theoretic Foundations of Functional Programming*

2.1 PCF and its Operational Semantics

Problem 1. (Page 14) *Problem Statement*

Problem 2. (Page 16) (Lemma 2.1.) The evaluation relation $\Downarrow$ is deterministic, i.e. whenever $M \Downarrow V$ and $M \Downarrow W$ then $V \equiv W$

Solution: We prove this by induction on the structure of the derivation.
Base cases.

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
  - $M \equiv x$, then $x \Downarrow x$. So V and W can only be x; thus, $V \equiv W \equiv x$.
  - $M \equiv \lambda x : \sigma.M$, then $\lambda x : \sigma.M \Downarrow \lambda x : \sigma.M$.
  - $M \equiv \underline{0}$, then $\underline{0} \Downarrow \underline{0}$.

Inductive Steps.

- If $M \equiv \mathit{succ}(M)$, then it must be derived by the rule $\frac{M \Downarrow \underline{n}}{\mathit{succ}(M) \Downarrow \underline{n+1}}$. Then we would have $V \equiv \underline{n+1}$ and $W \equiv \underline{m+1}$ since the successor rule is the only way to derive $\mathit{succ}(M)$. By IH, we know that $\underline{n} = \underline{m}$, thus $\underline{n+1} = \underline{m+1}$, and hence $V = W$.

- If $M \equiv M(N)$. The derivation for $M(N)$ must be of the form $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. A second derivation for $M(N)$ must use the same rule. i.e., $\frac{M \Downarrow \lambda x:\sigma'.E' \quad E'[N/x] \Downarrow W}{M(N) \Downarrow W}$. But then by IH, we would have $\lambda x : \sigma.E \equiv \lambda x : \sigma'.E'$. So $\sigma \equiv \sigma'$ and $E = E'$. Now, we have $E[N/x] \Downarrow V$ and $E[N/x] \Downarrow W$. By the IH on the sub-derivation for $E[N/x]$, we conclude $V \equiv W$.

- If $M \equiv \mathit{pred}(M)$, then the rules are $\frac{M \Downarrow \underline{0}}{\mathit{pred}(M) \Downarrow \underline{0}}$ and $\frac{M \Downarrow \underline{n+1}}{\mathit{pred}(M) \Downarrow \underline{n}}$. For the derivation $\mathit{pred}(M) \Downarrow V$, we must have a sub-derivation for $M \Downarrow \underline{x}$ for some numeral $\underline{x}$. Similarly, for $\mathit{pred}(M) \Downarrow W$, we must have a sub-derivation for $M \Downarrow \underline{y}$ for some numeral $\underline{y}$. By the IH on the sub-derivation for M, we can conclude that $\underline{x} \equiv \underline{y} \equiv k$.

  Let's examine k. If $k \equiv \underline{0}$, then both derivations must be $\frac{M \Downarrow \underline{0}}{\mathit{pred}(M) \Downarrow \underline{0}}$. Thus, $V \equiv W \equiv \underline{0}$. Same argument is valid for the case $k \equiv \underline{n+1}$.

- The other cases (Y_σ, and both cases of ifz) can be proved likewise.

■

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2.2 The Scott Model of PCF

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Problem 4. (Page 30) *Problem Statement*

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2.3 Milner's Context Lemma

Problem 1. (Page 44) *Problem Statement*

2.4 Logical Relations

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