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Chapter 2. Solutions to *Domain-Theoretic Foundations of Functional Programming*

2.1 PCF and its Operational Semantics

Problem 1. (Page 14) Show that the σ with $\Gamma \vdash M : \sigma$ is uniquely determined by Γ and M.

Solution: We prove this by induction on the structure.

- if $M \equiv x$ (variable), then it must be by the variable rule: $\Gamma', x : \sigma\Delta' \vdash x : \sigma$; thus σ must be unique by the definition of the context Γ. ($\Gamma \equiv x_1 : \sigma_1, ..., x_n : \sigma_n$, where x_i are pairwise distinct variables).
- if $M \equiv Z$ (zero), then it must be derived by the zero rule: $\Gamma \vdash Z : \mathbb{N}$; thus its type is unique.
- if $M \equiv (\lambda x : \sigma.M)$, then it must be derived by the abstraction rule: $\frac{\Gamma,x:\sigma\vdash M:\tau}{\Gamma\vdash(\lambda x:\sigma.M):\sigma\to\tau}$. By IH, M and x have unique types τ and σ, respectively. Thus, the type of the abstraction is uniquely determined as $\sigma \to \tau$.
- if $M \equiv (M(N))$, then by the application rule, we would have $\frac{\Gamma\vdash M:\sigma\to\tau \quad \Gamma\vdash N:\sigma}{\Gamma\vdash M(N):\tau}$. By IH, M and N have unique types $\sigma \to \tau$ and σ, respectively. Thus, the type of the application $M(N)$ is uniquely determined as τ.
- Same goes for the other cases (*succ*, *pred*, Y_σ, and ifz).

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Problem 2. (Page 16) (Lemma 2.1.) The evaluation relation $\Downarrow$ is deterministic, i.e. whenever $M \Downarrow V$ and $M \Downarrow W$ then $V \equiv W$

Solution: We prove this by induction on the structure of the derivation.
Base cases.

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
  - $M \equiv x$, then $x \Downarrow x$. So V and W can only be x; thus, $V \equiv W \equiv x$.
  - $M \equiv \lambda x : \sigma.M$, then $\lambda x : \sigma.M \Downarrow \lambda x : \sigma.M$.
  - $M \equiv \underline{0}$, then $\underline{0} \Downarrow \underline{0}$.

Inductive Steps.

- If $M \equiv succ(M)$, then it must be derived by the rule $\frac{M\Downarrow \underline{n}}{succ(M)\Downarrow\underline{n+1}}$. Then we would have $V \equiv \underline{n+1}$ and $W \equiv \underline{m+1}$ since the successor rule is the only way to derive $succ(M)$. By IH, we know that $\underline{n} = \underline{m}$, thus $\underline{n+1} = \underline{m+1}$, and hence $V = W$.
- If $M \equiv M(N)$. The derivation for $M(N)$ must be of the form $\frac{M\Downarrow\lambda x:\sigma.E \quad E[N/x]\Downarrow V}{M(N)\Downarrow V}$. A second derivation for $M(N)$ must use the same rule. i.e., $\frac{M\Downarrow\lambda x:\sigma'.E' \quad E'[N/x]\Downarrow W}{M(N)\Downarrow W}$. But then by IH, we would have $\lambda x : \sigma.E \equiv \lambda x : \sigma'.E'$. So $\sigma \equiv \sigma'$ and $E = E'$. Now, we have $E[N/x] \Downarrow V$ and $E[N/x] \Downarrow W$. By the IH on the sub-derivation for $E[N/x]$, we conclude $V \equiv W$.

- If $M \equiv \mathit{pred}(M)$, then the rules are $\frac{M \Downarrow \underline{0}}{\mathit{pred}(M) \Downarrow \underline{0}}$ and $\frac{M \Downarrow \underline{n+1}}{\mathit{pred}(M) \Downarrow \underline{n}}$. For the derivation $\mathit{pred}(M) \Downarrow V$, we must have a sub-derivation for $M \Downarrow \underline{x}$ for some numeral $\underline{x}$. Similarly, for $\mathit{pred}(M) \Downarrow W$, we must have a sub-derivation for $M \Downarrow \underline{y}$ for some numeral $\underline{y}$. By the IH on the sub-derivation for M, we can conclude that $\underline{x} \equiv \underline{y} \equiv k$.

  Let's examine k. If $k \equiv \underline{0}$, then both derivations must be $\frac{M \Downarrow \underline{0}}{\mathit{pred}(M) \Downarrow \underline{0}}$. Thus, $V \equiv W \equiv \underline{0}$. Same argument is valid for the case $k \equiv \underline{n+1}$.

- The other cases (Y_σ, and both cases of ifz) can be proved likewise.

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Problem 3. (Page 16) (Theorem 2.2, Subject Reduction). If $\Gamma \vdash M : \sigma$ and $M \Downarrow V$ then $\Gamma \vdash V : \sigma$.

Solution: We prove this by induction on the structure of the derivation of $M \Downarrow V$.
Base cases.

- If $M \equiv x$, then if $\Gamma \vdash x : \sigma$, then the goal is trivial since $x \Downarrow x$.

  Same goes for the other base cases ($\lambda x : \sigma.M$, $\underline{0}$).

Inductive Steps.

- If $M \equiv \mathit{succ}(M)$, then the only evaluation rule is $\frac{M \Downarrow \underline{n}}{\mathit{succ}(M) \Downarrow \underline{n+1}}$; so $V \equiv \underline{n+1}$. We are given $\Gamma \vdash \mathit{succ}(M) : \sigma$ and by the typing rule for succ, we have $\Gamma \vdash M : \mathit{nat}$.

  Now, from $M \Downarrow \underline{n}$ and $\Gamma \vdash M : \mathit{nat}$, by IH, $\Gamma \vdash \underline{n} : \mathit{nat}$; thus, by the typing rule for succ, we have $\Gamma \vdash \underline{n+1} : \mathit{nat}$.

- If $M \equiv M(N)$, the evaluation rule is $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. We are given $\Gamma \vdash M(N) : \sigma$ and by the typing rule for application, we have $\Gamma \vdash M : \sigma \to \tau$ and $\Gamma \vdash N : \sigma$. Now, we have $M \Downarrow \lambda x : \sigma.E$ and $\Gamma \vdash M : \sigma \to \tau$ and by IH, we have $\Gamma \vdash \lambda x : \sigma.E : \sigma \to \tau$. The typing rule for abstraction is $\frac{\Gamma, x:\sigma \vdash E:\tau}{\Gamma \vdash \lambda x:\sigma.E:\sigma \to \tau}$, so we have $\Gamma, x : \sigma \vdash E : \tau$. Using the substitution lemma, we can conclude that $\Gamma \vdash E[N/x] : \tau$. Now, from $E[N/x] \Downarrow V$ and $\Gamma \vdash E[N/x] : \tau$, by IH, we have $\Gamma \vdash V : \tau$.

- Same goes for the other cases (both cases of pred, Y_σ, and both cases of ifz).

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Problem 4. (Page 17) $M \Downarrow V$ iff $M \rhd^* V$.

Solution: In order to prove this, we prove the following two lemmas:

(a). if $M \Downarrow V$ then $M \rhd^* V$
(b). if $M \rhd N$ then for all values V, if $N \Downarrow V$, then $M \Downarrow V$.

Then, applying **(b)** iteratively, it follows that:

(c). if $M \rhd^* N$ and $N \Downarrow V$, then $M \Downarrow V$.
and by **(a)** and **(c)**, we can conclude that $M \Downarrow V$ iff $M \rhd^* V$.
Proof of (a). We prove this by induction on the structure of the derivation of $M \Downarrow V$.
Base cases.

- All the base cases ($M \equiv x$, $M \equiv \lambda x : \sigma.M$, and $M \equiv \underline{0}$) are trivial by the reflexivity of $\rhd^*$.

Inductive Steps.

- If $M \equiv succ(M)$, then the rule is $\frac{M \Downarrow \underline{n}}{succ(M) \Downarrow \underline{n+1}}$; thus, $V \equiv \underline{n+1}$. By IH on $M \Downarrow \underline{n}$, we have $M \rhd^* \underline{n}$. Now, there are two cases.
  - If $M \equiv \underline{n}$, then $succ(M) \equiv succ(\underline{n}) \equiv \underline{n+1}$. So $M \rhd^* V$.
  - If $M \rhd M_1 \rhd ... \rhd \underline{n}$, then by the congruence rule for $succ$, we would have $succ(M) \rhd succ(M_1) \rhd ... \rhd succ(\underline{n}) \equiv \underline{n+1}$. So, $succ(M) \rhd^* \underline{n+1} \equiv V$.

- If $M \equiv M(N)$, then the rule is $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. Then, by IH on $M \Downarrow \lambda x : \sigma.E$, we have $M \rhd^* \lambda x : \sigma.E$, and on $E[N/x] \Downarrow V$, we have $E[N/x] \rhd^* V$.

  If $M \equiv \lambda x : \sigma.E$, then $M(N) \equiv (\lambda x : \sigma.E)(N)$. By small-step rule $(\lambda x : \sigma.E)(N) \rhd E[N/x]$. Since $E[N'/x] \rhd^* V$, we have $M(N) \rhd E[N/x] \rhd^* V$, so $M(N) \rhd^* V$.

  If $M \rhd M_1 \rhd \cdots \rhd \lambda x : \sigma.E$, then by the congruence rule for application $M(N) \rhd M_1(N) \rhd \cdots \rhd (\lambda x : \sigma.E)(N)$. Then $(\lambda x : \sigma.E)(N) \rhd E[N/x]$, and $E[N/x] \rhd^* V$. Then we would have: $M(N) \rhd^* (\lambda x : \sigma.E)(N) \rhd E[N/x] \rhd^* V$. So $M(N) \rhd^* V$.

- The other cases are fairly similar to these.

Proof of (b). We prove this by induction on the structure of the derivation of $M \rhd V$.
Base cases.

- Assume $M \equiv (\lambda x : \sigma.E)(A)$ and $N \equiv E[A/x]$ $(M \rhd N)$. Assume also $N \Downarrow V$, i.e., $E[A/x] \Downarrow V$. We know $\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E$. By the rule: $\frac{\lambda x:\sigma.E \Downarrow \lambda x:\sigma.E \quad E[A/x] \Downarrow V}{(\lambda x:\sigma.E)(A) \Downarrow V}$. So $M \Downarrow V$.

- Assume $M \equiv Y_\sigma(E)$ and $N \equiv E(Y_\sigma(E))$. Assume also $N \Downarrow V$, i.e., $E(Y_\sigma(E)) \Downarrow V$. By the fixpoint rule: $\frac{E(Y_\sigma(E)) \Downarrow V}{Y_\sigma(E) \Downarrow V}$. So $M \Downarrow V$.

- Assume $M \equiv pred(\underline{0})$ and $N \equiv \underline{0}$. Assume also $N \Downarrow V$, i.e., $\underline{0} \Downarrow V$. This means $V \equiv \underline{0}$. We need to show $pred(\underline{0}) \Downarrow \underline{0}$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0}}{pred(\underline{0}) \Downarrow \underline{0}}$.

- Assume $M \equiv pred(\underline{k+1})$ and $N \equiv \underline{k}$. Assume also $N \Downarrow V$, i.e., $\underline{k} \Downarrow V$. This means $V \equiv \underline{k}$. We need to show $pred(\underline{k+1}) \Downarrow \underline{k}$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1}}{pred(\underline{k+1}) \Downarrow \underline{k}}$.

- Assume $M \equiv ifz(\underline{0}, E_1, E_2)$ and $N \equiv E_1$. Assume also $N \Downarrow V$, i.e., $E_1 \Downarrow V$. We need to show $ifz(\underline{0}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0} \quad E_1 \Downarrow V}{ifz(\underline{0}, E_1, E_2) \Downarrow V}$.

- Assume $M \equiv ifz(\underline{k+1}, E_1, E_2)$ and $N \equiv E_2$. Assume also $N \Downarrow V$, i.e., $E_2 \Downarrow V$. We need to show $ifz(\underline{k+1}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad E_2 \Downarrow V}{ifz(\underline{k+1}, E_1, E_2) \Downarrow V}$.

Inductive Steps.

- $\frac{M_1 \rhd M_2}{succ(M_1) \rhd succ(M_2)}$: $M = succ(M_1), N = succ(M_2)$, where $M_1 \rhd M_2$. Assume $N \Downarrow V$, i.e., $succ(M_2) \Downarrow V$. This implies $V \equiv \underline{k+1}$ and $M_2 \Downarrow \underline{k}$ for some k. By IH on the sub-derivation $M_1 \rhd M_2$: since $M_2 \Downarrow \underline{k}$, it follows that $M_1 \Downarrow \underline{k}$. Then, by the rule $\frac{M_1 \Downarrow \underline{k}}{succ(M_1) \Downarrow \underline{k+1}}$. So $M \equiv succ(M_1) \Downarrow \underline{k+1}$. Since $V \equiv \underline{k+1}$, we have $M \Downarrow V$.

- $\frac{M_1 \rhd M_2}{M_1(A) \rhd M_2(A)}$: $M = M_1(A)$, $N = M_2(A)$. Assume $M_2(A) \Downarrow V$. This means $M_2 \Downarrow \lambda x.E$ and $E[A/x] \Downarrow V$. By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \lambda x.E$, then $M_1 \Downarrow \lambda x.E$. Thus, by the rule, $M_1(A) \Downarrow V$.

- $\frac{M_1 \rhd M_2}{pred(M_1) \rhd pred(M_2)}$: $M = pred(M_1)$, $N = pred(M_2)$. Assume $pred(M_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and V is $\underline{0}$ (if $k = 0$) or $\underline{k-1}$ (if $k > 0$). By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for *pred*, $pred(M_1) \Downarrow V$.

- $\frac{M_1 \rhd M_2}{ifz(M_1, N_1, N_2) \rhd ifz(M_2, N_1, N_2)}$: $M = ifz(M_1, N_1, N_2)$, $N = ifz(M_2, N_1, N_2)$. Assume $ifz(M_2, N_1, N_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and either $N_1 \Downarrow V$ (if $k = 0$) or $N_2 \Downarrow V$ (if $k > 0$). By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for ifz, $ifz(M_1, N_1, N_2) \Downarrow V$.

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Problem 5. (Page 19) Show that the applicative relation $\underset{\sim\sigma}{\subset}$ is a preorder on Prg_σ, i.e. that $\underset{\sim\sigma}{\subset}$ is reflexive and transitive.

Solution:
-Relfexivity. We need to show that for any closed PCF term M of type σ, $M \sqsubseteq_\sigma M$.
Base Case. For $M \in Prg_{nat}, M \sqsubseteq_{nat} M$ means that $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow M \Downarrow \underline{n}$. This is trivially true.
Inductive Case. For $M \in Prg_{\sigma\to\tau}, M \sqsubseteq_{\sigma\to\tau} M$ means that $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau M(P)$, which holds by IH.
-Transitivity. We need to show that for any closed PCF terms M, N, K of type σ, if $M \sqsubseteq_\sigma N$ and $N \sqsubseteq_\sigma K$, then $M \sqsubseteq_\sigma K$.
Base Case. For $M, N, K \in Prg_{nat}$, assume $M \sqsubseteq_{nat} N$ and $N \sqsubseteq_{nat} K$. Then, by definition, we have the followings:

- $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow N \Downarrow \underline{n}$
- $\forall n \in \mathbb{N}, N \Downarrow \underline{n} \Rightarrow K \Downarrow \underline{n}$

Thus, if $M \Downarrow \underline{n}$ then $K \Downarrow \underline{n}$, which means $M \sqsubseteq_{nat} K$.
Inductive Case. For $M, N, K \in Prg_{\sigma\to\tau}$, assume $M \sqsubseteq_{\sigma\to\tau} N$ and $N \sqsubseteq_{\sigma\to\tau} K$. Then, by definition, we have the followings:

- $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau N(P)$
- $\forall P \in Prg_\sigma, N(P) \sqsubseteq_\tau K(P)$

Thus, we would have $\forall P \in Prg_\sigma, M(P) \sqsubseteq K(P)$. ■

2.2 The Scott Model of PCF

Problem 1. (Page 26) *Problem Statement*

Problem 2. (Page 26) *Problem Statement*

Problem 3. (Page 27) *Problem Statement*

Problem 4. (Page 30) *Problem Statement*

Problem 5. (Page 33) *Problem Statement*

Problem 6. (Page 34) *Problem Statement*

2.3 Milner's Context Lemma

Problem 1. (Page 44) *Problem Statement*

2.4 Logical Relations

Problem 1. (Page 52) *Problem Statement*

Problem 2. (Page 54) *Problem Statement*
