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## Chapter 1. Solutions to *Lambda Calculi with Types*

**Problem 1.** (Exercise 3.1.13) \*Exercise Statement\*

**Solution:** \*Solution!\*



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**Problem 2.** (Exercise 4.1.20) \*Exercise Statement\*

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**Problem 3.** (Exercise 4.2.8) \*Exercise Statement\*

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**Problem 4.** (Exercise 5.1.16) \*Exercise Statement\*

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## Chapter 2. Solutions to *Domain-Theoretic Foundations of Functional Programming*

### 2.1 PCF and its Operational Semantics

**Problem 1.** (Page 14) Show that the  $\sigma$  with  $\Gamma \vdash M : \sigma$  is uniquely determined by  $\Gamma$  and  $M$ .

**Solution:** We prove this by induction on the structure.

- if  $M \equiv x$  (variable), then it must be by the variable rule:  $\Gamma', x : \sigma \vdash x : \sigma$ ; thus  $\sigma$  must be unique by the definition of the context  $\Gamma$ . ( $\Gamma \equiv x_1 : \sigma_1, \dots, x_n : \sigma_n$ , where  $x_i$  are pairwise distinct variables).
- if  $M \equiv Z$  (zero), then it must be derived by the zero rule:  $\Gamma \vdash Z : \mathbb{N}$ ; thus its type is unique.
- if  $M \equiv (\lambda x : \sigma. M)$ , then it must be derived by the abstraction rule:  $\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x : \sigma. M) : \sigma \rightarrow \tau}$ . By IH,  $M$  and  $x$  have unique types  $\tau$  and  $\sigma$ , respectively. Thus, the type of the abstraction is uniquely determined as  $\sigma \rightarrow \tau$ .
- if  $M \equiv (M(N))$ , then by the application rule, we would have  $\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M(N) : \tau}$ . By IH,  $M$  and  $N$  have unique types  $\sigma \rightarrow \tau$  and  $\sigma$ , respectively. Thus, the type of the application  $M(N)$  is uniquely determined as  $\tau$ .
- Same goes for the other cases (*succ*, *pred*,  $Y_\sigma$ , and *ifz*).

■

**Problem 2.** (Page 16) (Lemma 2.1.) The evaluation relation  $\Downarrow$  is deterministic, i.e. whenever  $M \Downarrow V$  and  $M \Downarrow W$  then  $V \equiv W$

**Solution:** We prove this by induction on the structure of the derivation.

**Base cases.**

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
  - $M \equiv x$ , then  $x \Downarrow x$ . So  $V$  and  $W$  can only be  $x$ ; thus,  $V \equiv W \equiv x$ .
  - $M \equiv \lambda x : \sigma. M$ , then  $\lambda x : \sigma. M \Downarrow \lambda x : \sigma. M$ .
  - $M \equiv 0$ , then  $0 \Downarrow 0$ .

**Inductive Steps.**

- If  $M \equiv \text{succ}(M)$ , then it must be derived by the rule  $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$ . Then we would have  $V \equiv n+1$  and  $W \equiv m+1$  since the successor rule is the only way to derive  $\text{succ}(M)$ . By IH, we know that  $n = m$ , thus  $n+1 = m+1$ , and hence  $V = W$ .
- If  $M \equiv M(N)$ . The derivation for  $M(N)$  must be of the form  $\frac{M \Downarrow \lambda x : \sigma. E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$ . A second derivation for  $M(N)$  must use the same rule. i.e.,  $\frac{M \Downarrow \lambda x : \sigma'. E' \quad E'[N/x] \Downarrow W}{M(N) \Downarrow W}$ . But then by IH, we would have  $\lambda x : \sigma. E \equiv \lambda x : \sigma'. E'$ . So  $\sigma \equiv \sigma'$  and  $E = E'$ . Now, we have  $E[N/x] \Downarrow V$  and  $E[N/x] \Downarrow W$ . By the IH on the sub-derivation for  $E[N/x]$ , we conclude  $V \equiv W$ .

- If  $M \equiv \text{pred}(M)$ , then the rules are  $\frac{M \Downarrow 0}{\text{pred}(M) \Downarrow 0}$  and  $\frac{M \Downarrow n+1}{\text{pred}(M) \Downarrow n}$ . For the derivation  $\text{pred}(M) \Downarrow V$ , we must have a sub-derivation for  $M \Downarrow \underline{x}$  for some numeral  $\underline{x}$ . Similarly, for  $\text{pred}(M) \Downarrow W$ , we must have a sub-derivation for  $M \Downarrow \underline{y}$  for some numeral  $\underline{y}$ . By the IH on the sub-derivation for  $M$ , we can conclude that  $\underline{x} \equiv \underline{y} \equiv k$ .

Let's examine  $k$ . If  $k \equiv 0$ , then both derivations must be  $\frac{M \Downarrow 0}{\text{pred}(M) \Downarrow 0}$ . Thus,  $V \equiv W \equiv 0$ . Same argument is valid for the case  $k \equiv n+1$ .

- The other cases ( $Y_\sigma$ , and both cases of  $\text{if}z$ ) can be proved likewise.

■

**Problem 3.** (Page 16) (Theorem 2.2, Subject Reduction). If  $\Gamma \vdash M : \sigma$  and  $M \Downarrow V$  then  $\Gamma \vdash V : \sigma$ .

**Solution:** We prove this by induction on the structure of the derivation of  $M \Downarrow V$ .

**Base cases.**

- If  $M \equiv x$ , then if  $\Gamma \vdash x : \sigma$ , then the goal is trivial since  $x \Downarrow x$ .  
Same goes for the other base cases ( $\lambda x : \sigma. M, 0$ ).

**Inductive Steps.**

- If  $M \equiv \text{succ}(M)$ , then the only evaluation rule is  $\frac{M \Downarrow n}{\text{succ}(M) \Downarrow n+1}$ ; so  $V \equiv n+1$ . We are given  $\Gamma \vdash \text{succ}(M) : \sigma$  and by the typing rule for  $\text{succ}$ , we have  $\Gamma \vdash M : \text{nat}$ .  
Now, from  $M \Downarrow n$  and  $\Gamma \vdash M : \text{nat}$ , by IH,  $\Gamma \vdash n : \text{nat}$ ; thus, by the typing rule for  $\text{succ}$ , we have  $\Gamma \vdash n+1 : \text{nat}$ .
- If  $M \equiv M(N)$ , the evaluation rule is  $\frac{M \Downarrow \lambda x : \sigma. E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$ . We are given  $\Gamma \vdash M(N) : \sigma$  and by the typing rule for application, we have  $\Gamma \vdash M : \sigma \rightarrow \tau$  and  $\Gamma \vdash N : \sigma$ . Now, we have  $M \Downarrow \lambda x : \sigma. E$  and  $\Gamma \vdash M : \sigma \rightarrow \tau$  and by IH, we have  $\Gamma \vdash \lambda x : \sigma. E : \sigma \rightarrow \tau$ . The typing rule for abstraction is  $\frac{\Gamma, x : \sigma \vdash E : \tau}{\Gamma \vdash \lambda x : \sigma. E : \sigma \rightarrow \tau}$ , so we have  $\Gamma, x : \sigma \vdash E : \tau$ . Using the substitution lemma, we can conclude that  $\Gamma \vdash E[N/x] : \tau$ . Now, from  $E[N/x] \Downarrow V$  and  $\Gamma \vdash E[N/x] : \tau$ , by IH, we have  $\Gamma \vdash V : \tau$ .
- Same goes for the other cases (both cases of  $\text{pred}$ ,  $Y_\sigma$ , and both cases of  $\text{if}z$ ).

■

**Problem 4.** (Page 17)  $M \Downarrow V$  iff  $M \triangleright^* V$ .

**Solution:** In order to prove this, we prove the following two lemmas:

- (a). if  $M \Downarrow V$  then  $M \triangleright^* V$
- (b). if  $M \triangleright N$  then for all values  $V$ , if  $N \Downarrow V$ , then  $M \Downarrow V$ .

Then, applying (b) iteratively, it follows that:

- (c). if  $M \triangleright^* N$  and  $N \Downarrow V$ , then  $M \Downarrow V$ .

and by (a) and (c), we can conclude that  $M \Downarrow V$  iff  $M \triangleright^* V$ .

**Proof of (a).** We prove this by induction on the structure of the derivation of  $M \Downarrow V$ .

**Base cases.**

- All the base cases ( $M \equiv x$ ,  $M \equiv \lambda x : \sigma.M$ , and  $M \equiv \underline{0}$ ) are trivial by the reflexivity of  $\triangleright^*$ .

### Inductive Steps.

- If  $M \equiv \text{succ}(M)$ , then the rule is  $\frac{M \Downarrow \underline{n}}{\text{succ}(M) \Downarrow \underline{n+1}}$ ; thus,  $V \equiv \underline{n+1}$ . By IH on  $M \Downarrow \underline{n}$ , we have  $M \triangleright^* \underline{n}$ . Now, there are two cases.
  - If  $M \equiv \underline{n}$ , then  $\text{succ}(M) \equiv \text{succ}(\underline{n}) \equiv \underline{n+1}$ . So  $M \triangleright^* V$ .
  - If  $M \triangleright M_1 \triangleright \dots \triangleright \underline{n}$ , then by the congruence rule for  $\text{succ}$ , we would have  $\text{succ}(M) \triangleright \text{succ}(M_1) \triangleright \dots \triangleright \text{succ}(\underline{n}) \equiv \underline{n+1}$ . So,  $\text{succ}(M) \triangleright^* \underline{n+1} \equiv V$ .
- If  $M \equiv M(N)$ , then the rule is  $\frac{M \Downarrow \lambda x : \sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$ . Then, by IH on  $M \Downarrow \lambda x : \sigma.E$ , we have  $M \triangleright^* \lambda x : \sigma.E$ , and on  $E[N/x] \Downarrow V$ , we have  $E[N/x] \triangleright^* V$ .
 

If  $M \equiv \lambda x : \sigma.E$ , then  $M(N) \equiv (\lambda x : \sigma.E)(N)$ . By small-step rule  $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$ . Since  $E[N/x] \triangleright^* V$ , we have  $M(N) \triangleright E[N/x] \triangleright^* V$ , so  $M(N) \triangleright^* V$ .

If  $M \triangleright M_1 \triangleright \dots \triangleright \lambda x : \sigma.E$ , then by the congruence rule for application  $M(N) \triangleright M_1(N) \triangleright \dots \triangleright (\lambda x : \sigma.E)(N)$ . Then  $(\lambda x : \sigma.E)(N) \triangleright E[N/x]$ , and  $E[N/x] \triangleright^* V$ . Then we would have:  $M(N) \triangleright^* (\lambda x : \sigma.E)(N) \triangleright E[N/x] \triangleright^* V$ . So  $M(N) \triangleright^* V$ .
- The other cases are fairly similar to these.

**Proof of (b).** We prove this by induction on the structure of the derivation of  $M \triangleright V$ .

### Base cases.

- Assume  $M \equiv (\lambda x : \sigma.E)(A)$  and  $N \equiv E[A/x]$  ( $M \triangleright N$ ). Assume also  $N \Downarrow V$ , i.e.,  $E[A/x] \Downarrow V$ . We know  $\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E$ . By the rule:  $\frac{\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E \quad E[A/x] \Downarrow V}{(\lambda x : \sigma.E)(A) \Downarrow V}$ . So  $M \Downarrow V$ .
- Assume  $M \equiv Y_\sigma(E)$  and  $N \equiv E(Y_\sigma(E))$ . Assume also  $N \Downarrow V$ , i.e.,  $E(Y_\sigma(E)) \Downarrow V$ . By the fixpoint rule:  $\frac{E(Y_\sigma(E)) \Downarrow V}{Y_\sigma(E) \Downarrow V}$ . So  $M \Downarrow V$ .
- Assume  $M \equiv \text{pred}(\underline{0})$  and  $N \equiv \underline{0}$ . Assume also  $N \Downarrow V$ , i.e.,  $\underline{0} \Downarrow V$ . This means  $V \equiv \underline{0}$ . We need to show  $\text{pred}(\underline{0}) \Downarrow \underline{0}$ . This holds by the rule  $\frac{\underline{0} \Downarrow \underline{0}}{\text{pred}(\underline{0}) \Downarrow \underline{0}}$ .
- Assume  $M \equiv \text{pred}(\underline{k+1})$  and  $N \equiv \underline{k}$ . Assume also  $N \Downarrow V$ , i.e.,  $\underline{k} \Downarrow V$ . This means  $V \equiv \underline{k}$ . We need to show  $\text{pred}(\underline{k+1}) \Downarrow \underline{k}$ . This holds by the rule  $\frac{\underline{k+1} \Downarrow \underline{k+1}}{\text{pred}(\underline{k+1}) \Downarrow \underline{k}}$ .
- Assume  $M \equiv \text{ifz}(\underline{0}, E_1, E_2)$  and  $N \equiv E_1$ . Assume also  $N \Downarrow V$ , i.e.,  $E_1 \Downarrow V$ . We need to show  $\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V$ . This holds by the rule  $\frac{\underline{0} \Downarrow \underline{0} \quad E_1 \Downarrow V}{\text{ifz}(\underline{0}, E_1, E_2) \Downarrow V}$ .
- Assume  $M \equiv \text{ifz}(\underline{k+1}, E_1, E_2)$  and  $N \equiv E_2$ . Assume also  $N \Downarrow V$ , i.e.,  $E_2 \Downarrow V$ . We need to show  $\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V$ . This holds by the rule  $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad E_2 \Downarrow V}{\text{ifz}(\underline{k+1}, E_1, E_2) \Downarrow V}$ .

### Inductive Steps.

- $\frac{M_1 \triangleright M_2}{\text{succ}(M_1) \triangleright \text{succ}(M_2)}$ :  $M = \text{succ}(M_1)$ ,  $N = \text{succ}(M_2)$ , where  $M_1 \triangleright M_2$ . Assume  $N \Downarrow V$ , i.e.,  $\text{succ}(M_2) \Downarrow V$ . This implies  $V \equiv \underline{k+1}$  and  $M_2 \Downarrow \underline{k}$  for some  $k$ . By IH on the sub-derivation  $M_1 \triangleright M_2$ : since  $M_2 \Downarrow \underline{k}$ , it follows that  $M_1 \Downarrow \underline{k}$ . Then, by the rule  $\frac{M_1 \Downarrow \underline{k}}{\text{succ}(M_1) \Downarrow \underline{k+1}}$ . So  $M \equiv \text{succ}(M_1) \Downarrow \underline{k+1}$ . Since  $V \equiv \underline{k+1}$ , we have  $M \Downarrow V$ .

- $\frac{M_1 \triangleright M_2}{M_1(A) \triangleright M_2(A)}$ :  $M = M_1(A)$ ,  $N = M_2(A)$ . Assume  $M_2(A) \Downarrow V$ . This means  $M_2 \Downarrow \lambda x.E$  and  $E[A/x] \Downarrow V$ . By IH on  $M_1 \triangleright M_2$ , since  $M_2 \Downarrow \lambda x.E$ , then  $M_1 \Downarrow \lambda x.E$ . Thus, by the rule,  $M_1(A) \Downarrow V$ .
- $\frac{M_1 \triangleright M_2}{pred(M_1) \triangleright pred(M_2)}$ :  $M = pred(M_1)$ ,  $N = pred(M_2)$ . Assume  $pred(M_2) \Downarrow V$ . This means  $M_2 \Downarrow \underline{k}$  and  $V$  is  $\underline{0}$  (if  $k = 0$ ) or  $\underline{k-1}$  (if  $k > 0$ ). By IH on  $M_1 \triangleright M_2$ , since  $M_2 \Downarrow \underline{k}$ , then  $M_1 \Downarrow \underline{k}$ . Thus, by the rule for  $pred$ ,  $pred(M_1) \Downarrow V$ .
- $\frac{M_1 \triangleright M_2}{ifz(M_1, N_1, N_2) \triangleright ifz(M_2, N_1, N_2)}$ :  $M = ifz(M_1, N_1, N_2)$ ,  $N = ifz(M_2, N_1, N_2)$ . Assume  $ifz(M_2, N_1, N_2) \Downarrow V$ . This means  $M_2 \Downarrow \underline{k}$  and either  $N_1 \Downarrow V$  (if  $k = 0$ ) or  $N_2 \Downarrow V$  (if  $k > 0$ ). By IH on  $M_1 \triangleright M_2$ , since  $M_2 \Downarrow \underline{k}$ , then  $M_1 \Downarrow \underline{k}$ . Thus, by the rule for  $ifz$ ,  $ifz(M_1, N_1, N_2) \Downarrow V$ .

■

**Problem 5.** (Page 19) Show that the applicative relation  $\sqsubseteq_\sigma$  is a preorder on  $Prg_\sigma$ , i.e. that  $\sqsubseteq_\sigma$  is reflexive and transitive.

**Solution:**

**-Reflexivity.** We need to show that for any closed PCF term  $M$  of type  $\sigma$ ,  $M \sqsubseteq_\sigma M$ .

**Base Case.** For  $M \in Prg_{nat}$ ,  $M \sqsubseteq_{nat} M$  means that  $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow M \Downarrow \underline{n}$ . This is trivially true.

**Inductive Case.** For  $M \in Prg_{\sigma \rightarrow \tau}$ ,  $M \sqsubseteq_{\sigma \rightarrow \tau} M$  means that  $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau M(P)$ , which holds by IH.

**-Transitivity.** We need to show that for any closed PCF terms  $M, N, K$  of type  $\sigma$ , if  $M \sqsubseteq_\sigma N$  and  $N \sqsubseteq_\sigma K$ , then  $M \sqsubseteq_\sigma K$ .

**Base Case.** For  $M, N, K \in Prg_{nat}$ , assume  $M \sqsubseteq_{nat} N$  and  $N \sqsubseteq_{nat} K$ . Then, by definition, we have the followings:

- $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow N \Downarrow \underline{n}$
- $\forall n \in \mathbb{N}, N \Downarrow \underline{n} \Rightarrow K \Downarrow \underline{n}$

Thus, if  $M \Downarrow \underline{n}$  then  $K \Downarrow \underline{n}$ , which means  $M \sqsubseteq_{nat} K$ .

**Inductive Case.** For  $M, N, K \in Prg_{\sigma \rightarrow \tau}$ , assume  $M \sqsubseteq_{\sigma \rightarrow \tau} N$  and  $N \sqsubseteq_{\sigma \rightarrow \tau} K$ . Then, by definition, we have the followings:

- $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau N(P)$
- $\forall P \in Prg_\sigma, N(P) \sqsubseteq_\tau K(P)$

Thus, we would have  $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau K(P)$ .

■

## 2.2 The Scott Model of PCF

**Problem 1.** (Page 26) Show that (Scott) continuous functions between predomains are always monotonic.

**Solution:** Let  $f : (A, \sqsubseteq_A) \rightarrow (B, \sqsubseteq_B)$  be a Scott continuous function between predomains. For any  $x, y \in A$  with  $x \leq y$ , we have that  $X = \{x, y\}$  is a directed subset of  $A$ . The supremum of the set  $X$  is obviously  $y$ . Thus  $\sqcup X = y$ , and since  $f$  is continuous,  $f(\sqcup X) = f(y) = \sqcup f(\{x, y\})$ . Hence,  $f(x) \leq f(y)$ . ■

**Problem 2.** (Page 26) (Theorem 3.3). Let  $(A_i | i \in I)$  be a family of predomains. Then their product  $\prod_{i \in I} A_i$  is a predomain under the componentwise ordering, and the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are Scott continuous. If, moreover, all  $A_i$  are domains then so is their product  $\prod_{i \in I} A_i$ .

**Solution:** Let  $D = \prod_{i \in I} A_i = \{f : I \rightarrow \bigcup_{i \in I} A_i \mid \forall i \in I, f(i) \in A_i\}$ . We need to show that  $(D, \sqsubseteq_D)$  is a poset, and every directed subset of  $D$  has a least upper bound. Note that the order  $\sqsubseteq_D$  is defined as follows:

$$(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I} \text{ iff } \forall i \in I, d(i) \sqsubseteq_{A_i} d'(i)$$

We now show that  $(D, \sqsubseteq_D)$  forms a poset.

- **Reflexivity:** For any  $(d_i)_{i \in I} \in D$ ,  $d_i \sqsubseteq_{A_i} d_i, \forall i \in I$  since each  $A_i$  is a poset. Thus  $(d_i)_{i \in I} \sqsubseteq_D (d_i)_{i \in I}$ .
- **Transitivity:** Assume  $(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I}$  and  $(d'_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$ . And by each  $A_i$  being transitive, it follows immediately that  $d_i \sqsubseteq_{A_i} d''_i$  for all  $i \in I$ . Therefore,  $(d_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$ .
- **Antisymmetry:** Similar to the previous case, it follows immediately from the fact that each  $A_i$  is antisymmetric.

Now, suppose that  $X \subseteq D = \prod_{i \in I} A_i$  is a directed subset. define  $X_i = \{\pi_i(x) \mid x \in X\}$ , that is, the projection of  $X$  to  $A_i$ .  $X_i$  is directed since  $X$  is directed. Moreover,  $X_i$  has a least upper bound  $\sqcup X_i \in A_i$ . Define  $z \in D$  with  $z_i = \sqcup X_i$  for each  $i \in I$ . By construction, it is obvious that  $z$  is the least upper bound of  $X$  in  $D$ . Thus,  $D$  is a predomain. ■

**Problem 3.** (Page 27). Prove that the evaluation map  $ev : [A_1 \rightarrow A_2] \times A_1 \rightarrow A_2$  with  $ev(f, a) = f(a)$  is continuous in each argument.

**Solution:** For the first argument, fix  $a \in A_1$  and let  $F \subseteq [A_1 \rightarrow A_2]$  be a directed set of continuous functions. By Theorem 3.5, we have  $\sqcup F(a) = g(a) = \sqcup_{f \in F} f(a)$ . Thus,  $ev(\sqcup F, a) = g(a) = \sqcup_{f \in F} f(a) = \sqcup \{ev(f, a) \mid f \in F\}$ .

Now, for the second argument, fix  $f \in [A_1 \rightarrow A_2]$  and let  $X \subseteq A_1$  be a directed set. Because  $f$  is continuous, we have  $f(\sqcup X) = \sqcup \{f(x) \mid x \in X\}$ . Thus,

$$ev(f, \sqcup X) = f(\sqcup X) = \sqcup \{f(x) \mid x \in X\} = \sqcup \{ev(f, x) \mid x \in X\}.$$

Hence,  $ev$  is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

**Problem 4.** (Page 30). Prove that  $\Psi : [[D \rightarrow D] \rightarrow D] \times [D \rightarrow D] \rightarrow D : (F, f) \mapsto f(F(f))$  is continuous in each argument.

**Solution:** For the first argument, fix  $f \in [D \rightarrow D]$  and let  $\mathcal{F} \subseteq [[D \rightarrow D] \rightarrow D]$  be a directed set. We know that  $(\sqcup \mathcal{F})(f) = \sqcup \{F(f) \mid F \in \mathcal{F}\}$ . Thus,

$$\underbrace{f((\sqcup \mathcal{F})(f))}_{\Psi(\sqcup \mathcal{F}, f)} = f(\sqcup \{F(f) \mid F \in \mathcal{F}\}) = \sqcup \{f(F(f)) \mid F \in \mathcal{F}\}$$

For the second argument, fix  $F \in [[D \rightarrow D] \rightarrow D]$  and let  $X \subseteq [D \rightarrow D]$  be a directed set. We know that  $F(\sqcup X) = \sqcup \{F(f) \mid f \in X\}$ . Thus,

$$f(F(\sqcup X)) = f(\sqcup \{F(f) \mid f \in X\}) = \sqcup \{f(F(f)) \mid f \in X\}$$

Hence,  $\Psi$  is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

**Problem 5.** (Page 33) ( $\beta$ -equality). If,  $\Gamma, x : \sigma \vdash M : \tau$  and  $\Gamma \vdash N : \sigma$  then

$$\llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket = \llbracket \Gamma \vdash M[N/x] \rrbracket$$

**Solution:**  $\llbracket \Gamma \vdash M[N/x] \rrbracket(\vec{d}) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d}))$  by Lemma 3.15. Now, for the other side,

$$\begin{aligned} \llbracket \Gamma \vdash (\lambda x : \sigma. M)(N) \rrbracket(\vec{d}) &= ev(\llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d}), \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \\ &= \llbracket \Gamma \vdash \lambda x : \sigma. M \rrbracket(\vec{d})(\llbracket \Gamma \vdash N \rrbracket(\vec{d})) = \llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, \llbracket \Gamma \vdash N \rrbracket(\vec{d})) \end{aligned}$$
■

**Problem 6.** (Page 34) ( $\eta$ -equality). If,  $\Gamma \vdash M : \sigma \rightarrow \tau$  then

$$\llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket = \llbracket \Gamma \vdash M \rrbracket$$

for  $x \notin \text{Var}(\Gamma)$ .

**Solution:**

$$\begin{aligned} \forall \vec{d} \in \llbracket \Gamma \rrbracket, d' \in D_\sigma, \llbracket \Gamma \vdash \lambda x : \sigma. M(x) \rrbracket(\vec{d})(d') &= \llbracket \Gamma, x : \sigma \vdash M(x) \rrbracket(\vec{d}, d') \\ &= ev(\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d'), \underbrace{\llbracket \Gamma, x : \sigma \vdash x \rrbracket(\vec{d}, d')}_{\pi_x(\vec{d}, d')=d'}) =^* \llbracket \Gamma \vdash M \rrbracket(\vec{d})(d') \end{aligned}$$

For the last equation  $(*)$ , because  $x \notin \text{Var}(\Gamma)$  and  $\Gamma \vdash M$ , we have

$$\llbracket \Gamma, x : \sigma \vdash M \rrbracket(\vec{d}, d') = \llbracket \Gamma \vdash M \rrbracket(\vec{d})$$

.

■



## 2.3 Milner's Context Lemma

**Problem 1.** (Page 44) Prove that  $\leq_\sigma$  is closed under suprema of directed sets. That is, if  $X \subseteq D_\sigma \times D_\sigma$  is directed and  $X \sqsubseteq \leq_\sigma$ , meaning that for every  $(x, y) \in X, x \leq_\sigma y$ , then  $\sqcup X \in \leq_\sigma$ .

**Solution:** Let  $X = \{(x_k, y_k) | k \in K\}$  be a directed subset of  $D_\sigma \times D_\sigma$  such that for all  $k \in K, (x_k, y_k) \in \leq_\sigma$ . This means for each  $k \in K$ , we have  $\forall P \in \text{Prg}_\sigma, y_k R_\sigma P \implies x_k R_\sigma P(\star)$ . Let  $x = \sqcup_{k \in K} x_k$  and  $y = \sqcup_{k \in K} y_k$  (Note that  $\sqcup X = (x, y)$ ). We need to show  $x \leq_\sigma y$ . That is,  $\forall P \in \text{Prg}_\sigma, y R_\sigma P \implies x R_\sigma P$ .

Let  $P \in \text{Prg}_\sigma$  be an arbitrary closed PCF term and that  $y R_\sigma P$ . Now, for each  $k \in K$ , we have  $y_k \sqsubseteq y$  and by Lemma 4.2(1), we get  $y_k R_\sigma P$  for all  $k \in K$ . Using  $(\star)$ , we have  $x_k R_\sigma P$  for each  $k \in K$ .

By Lemma 4.2(2),  $R_\sigma P$  is closed under directed suprema and  $\{x_k | k \in K\}$  is a directed subset of  $D_\sigma$  whose elements are all in  $R_\sigma P$ . So, their suprema  $x$  must also be in  $R_\sigma P$ , meaning that  $x R_\sigma P$ . ■

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## 2.4 Logical Relations

**Problem 1.** (Page 52) \*Problem Statement\*

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**Problem 2.** (Page 54) \*Problem Statement\*

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