Table of Contents

Chapter 1. Solutions to *Lambda Calculi with Types*

Problem 1. (Exercise 3.1.13) *Exercise Statement*

Solution: *Solution!* ■

Problem 2. (Exercise 4.1.20) *Exercise Statement*

Problem 3. (Exercise 4.2.8) *Exercise Statement*

Problem 4. (Exercise 5.1.16) *Exercise Statement*

Chapter 2. Solutions to *Domain-Theoretic Foundations of Functional Programming*

2.1 PCF and its Operational Semantics

Problem 1. (Page 14) Show that the σ with $\Gamma \vdash M : \sigma$ is uniquely determined by Γ and M.

Solution: We prove this by induction on the structure.

- if $M \equiv x$ (variable), then it must be by the variable rule: $\Gamma', x : \sigma\Delta' \vdash x : \sigma$; thus σ must be unique by the definition of the context Γ. ($\Gamma \equiv x_1 : \sigma_1, ..., x_n : \sigma_n$, where x_i are pairwise distinct variables).
- if $M \equiv Z$ (zero), then it must be derived by the zero rule: $\Gamma \vdash Z : \mathbb{N}$; thus its type is unique.
- if $M \equiv (\lambda x : \sigma.M)$, then it must be derived by the abstraction rule: $\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash (\lambda x:\sigma.M):\sigma \to \tau}$. By IH, M and x have unique types τ and σ, respectively. Thus, the type of the abstraction is uniquely determined as $\sigma \to \tau$.
- if $M \equiv (M(N))$, then by the application rule, we would have $\frac{\Gamma \vdash M:\sigma\to\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M(N):\tau}$. By IH, M and N have unique types $\sigma \to \tau$ and σ, respectively. Thus, the type of the application $M(N)$ is uniquely determined as τ.
- Same goes for the other cases (*succ*, *pred*, Y_σ, and ifz).

■

Problem 2. (Page 16) (Lemma 2.1.) The evaluation relation $\Downarrow$ is deterministic, i.e. whenever $M \Downarrow V$ and $M \Downarrow W$ then $V \equiv W$

Solution: We prove this by induction on the structure of the derivation.
Base cases.

- By the rules of the BigStep semantics for PCF, the lemma for the following base cases is trivial:
  - $M \equiv x$, then $x \Downarrow x$. So V and W can only be x; thus, $V \equiv W \equiv x$.
  - $M \equiv \lambda x : \sigma.M$, then $\lambda x : \sigma.M \Downarrow \lambda x : \sigma.M$.
  - $M \equiv \underline{0}$, then $\underline{0} \Downarrow \underline{0}$.

Inductive Steps.

- If $M \equiv succ(M)$, then it must be derived by the rule $\frac{M \Downarrow \underline{n}}{succ(M) \Downarrow \underline{n+1}}$. Then we would have $V \equiv \underline{n+1}$ and $W \equiv \underline{m+1}$ since the successor rule is the only way to derive $succ(M)$. By IH, we know that $\underline{n} = \underline{m}$, thus $\underline{n+1} = \underline{m+1}$, and hence $V = W$.
- If $M \equiv M(N)$. The derivation for $M(N)$ must be of the form $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. A second derivation for $M(N)$ must use the same rule. i.e., $\frac{M \Downarrow \lambda x:\sigma'.E' \quad E'[N/x] \Downarrow W}{M(N) \Downarrow W}$. But then by IH, we would have $\lambda x : \sigma.E \equiv \lambda x : \sigma'.E'$. So $\sigma \equiv \sigma'$ and $E = E'$. Now, we have $E[N/x] \Downarrow V$ and $E[N/x] \Downarrow W$. By the IH on the sub-derivation for $E[N/x]$, we conclude $V \equiv W$.

- If $M \equiv pred(M)$, then the rules are $\frac{M \Downarrow \underline{0}}{pred(M) \Downarrow \underline{0}}$ and $\frac{M \Downarrow \underline{n+1}}{pred(M) \Downarrow \underline{n}}$. For the derivation $pred(M) \Downarrow V$, we must have a sub-derivation for $M \Downarrow \underline{x}$ for some numeral $\underline{x}$. Similarly, for $pred(M) \Downarrow W$, we must have a sub-derivation for $M \Downarrow \underline{y}$ for some numeral $\underline{y}$. By the IH on the sub-derivation for M, we can conclude that $\underline{x} \equiv \underline{y} \equiv k$.

  Let's examine k. If $k \equiv \underline{0}$, then both derivations must be $\frac{M \Downarrow \underline{0}}{pred(M) \Downarrow \underline{0}}$. Thus, $V \equiv W \equiv \underline{0}$. Same argument is valid for the case $k \equiv \underline{n+1}$.

- The other cases (Y_σ, and both cases of ifz) can be proved likewise.

■

Problem 3. (Page 16) (Theorem 2.2, Subject Reduction). If $\Gamma \vdash M : \sigma$ and $M \Downarrow V$ then $\Gamma \vdash V : \sigma$.

Solution: We prove this by induction on the structure of the derivation of $M \Downarrow V$.
Base cases.

- If $M \equiv x$, then if $\Gamma \vdash x : \sigma$, then the goal is trivial since $x \Downarrow x$.

  Same goes for the other base cases ($\lambda x : \sigma.M$, $\underline{0}$).

Inductive Steps.

- If $M \equiv succ(M)$, then the only evaluation rule is $\frac{M \Downarrow \underline{n}}{succ(M) \Downarrow \underline{n+1}}$; so $V \equiv \underline{n+1}$. We are given $\Gamma \vdash succ(M) : \sigma$ and by the typing rule for $succ$, we have $\Gamma \vdash M : nat$.

  Now, from $M \Downarrow \underline{n}$ and $\Gamma \vdash M : nat$, by IH, $\Gamma \vdash \underline{n} : nat$; thus, by the typing rule for $succ$, we have $\Gamma \vdash \underline{n+1} : nat$.

- If $M \equiv M(N)$, the evaluation rule is $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. We are given $\Gamma \vdash M(N) : \sigma$ and by the typing rule for application, we have $\Gamma \vdash M : \sigma \to \tau$ and $\Gamma \vdash N : \sigma$. Now, we have $M \Downarrow \lambda x : \sigma.E$ and $\Gamma \vdash M : \sigma \to \tau$ and by IH, we have $\Gamma \vdash \lambda x : \sigma.E : \sigma \to \tau$. The typing rule for abstraction is $\frac{\Gamma, x:\sigma \vdash E:\tau}{\Gamma \vdash \lambda x:\sigma.E:\sigma \to \tau}$, so we have $\Gamma, x : \sigma \vdash E : \tau$. Using the substitution lemma, we can conclude that $\Gamma \vdash E[N/x] : \tau$. Now, from $E[N/x] \Downarrow V$ and $\Gamma \vdash E[N/x] : \tau$, by IH, we have $\Gamma \vdash V : \tau$.

- Same goes for the other cases (both cases of $pred$, Y_σ, and both cases of ifz).

■

Problem 4. (Page 17) $M \Downarrow V$ iff $M \rhd^* V$.

Solution: In order to prove this, we prove the following two lemmas:

(a). if $M \Downarrow V$ then $M \rhd^* V$
(b). if $M \rhd N$ then for all values V, if $N \Downarrow V$, then $M \Downarrow V$.

Then, applying **(b)** iteratively, it follows that:

(c). if $M \rhd^* N$ and $N \Downarrow V$, then $M \Downarrow V$.
and by **(a)** and **(c)**, we can conclude that $M \Downarrow V$ iff $M \rhd^* V$.
Proof of (a). We prove this by induction on the structure of the derivation of $M \Downarrow V$.
Base cases.

- All the base cases ($M \equiv x$, $M \equiv \lambda x : \sigma.M$, and $M \equiv \underline{0}$) are trivial by the reflexivity of $\rhd^*$.

Inductive Steps.

- If $M \equiv succ(M)$, then the rule is $\frac{M \Downarrow \underline{n}}{succ(M) \Downarrow \underline{n+1}}$; thus, $V \equiv \underline{n+1}$. By IH on $M \Downarrow \underline{n}$, we have $M \rhd^* \underline{n}$. Now, there are two cases.
  - If $M \equiv \underline{n}$, then $succ(M) \equiv succ(\underline{n}) \equiv \underline{n+1}$. So $M \rhd^* V$.
  - If $M \rhd M_1 \rhd ... \rhd \underline{n}$, then by the congruence rule for *succ*, we would have $succ(M) \rhd succ(M_1) \rhd ... \rhd succ(\underline{n}) \equiv \underline{n+1}$. So, $succ(M) \rhd^* \underline{n+1} \equiv V$.

- If $M \equiv M(N)$, then the rule is $\frac{M \Downarrow \lambda x:\sigma.E \quad E[N/x] \Downarrow V}{M(N) \Downarrow V}$. Then, by IH on $M \Downarrow \lambda x : \sigma.E$, we have $M \rhd^* \lambda x : \sigma.E$, and on $E[N/x] \Downarrow V$, we have $E[N/x] \rhd^* V$.

  If $M \equiv \lambda x : \sigma.E$, then $M(N) \equiv (\lambda x : \sigma.E)(N)$. By small-step rule $(\lambda x : \sigma.E)(N) \rhd E[N/x]$. Since $E[N'/x] \rhd^* V$, we have $M(N) \rhd E[N/x] \rhd^* V$, so $M(N) \rhd^* V$.

  If $M \rhd M_1 \rhd \cdots \rhd \lambda x : \sigma.E$, then by the congruence rule for application $M(N) \rhd M_1(N) \rhd \cdots \rhd (\lambda x : \sigma.E)(N)$. Then $(\lambda x : \sigma.E)(N) \rhd E[N/x]$, and $E[N/x] \rhd^* V$. Then we would have: $M(N) \rhd^* (\lambda x : \sigma.E)(N) \rhd E[N/x] \rhd^* V$. So $M(N) \rhd^* V$.

- The other cases are fairly similar to these.

Proof of (b). We prove this by induction on the structure of the derivation of $M \rhd V$.
Base cases.

- Assume $M \equiv (\lambda x : \sigma.E)(A)$ and $N \equiv E[A/x]$ $(M \rhd N)$. Assume also $N \Downarrow V$, i.e., $E[A/x] \Downarrow V$. We know $\lambda x : \sigma.E \Downarrow \lambda x : \sigma.E$. By the rule: $\frac{\lambda x:\sigma.E \Downarrow \lambda x:\sigma.E \quad E[A/x] \Downarrow V}{(\lambda x:\sigma.E)(A) \Downarrow V}$. So $M \Downarrow V$.

- Assume $M \equiv Y_\sigma(E)$ and $N \equiv E(Y_\sigma(E))$. Assume also $N \Downarrow V$, i.e., $E(Y_\sigma(E)) \Downarrow V$. By the fixpoint rule: $\frac{E(Y_\sigma(E)) \Downarrow V}{Y_\sigma(E) \Downarrow V}$. So $M \Downarrow V$.

- Assume $M \equiv pred(\underline{0})$ and $N \equiv \underline{0}$. Assume also $N \Downarrow V$, i.e., $\underline{0} \Downarrow V$. This means $V \equiv \underline{0}$. We need to show $pred(\underline{0}) \Downarrow \underline{0}$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0}}{pred(\underline{0}) \Downarrow \underline{0}}$.

- Assume $M \equiv pred(\underline{k+1})$ and $N \equiv \underline{k}$. Assume also $N \Downarrow V$, i.e., $\underline{k} \Downarrow V$. This means $V \equiv \underline{k}$. We need to show $pred(\underline{k+1}) \Downarrow \underline{k}$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1}}{pred(\underline{k+1}) \Downarrow \underline{k}}$.

- Assume $M \equiv ifz(\underline{0}, E_1, E_2)$ and $N \equiv E_1$. Assume also $N \Downarrow V$, i.e., $E_1 \Downarrow V$. We need to show $ifz(\underline{0}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{0} \Downarrow \underline{0} \quad E_1 \Downarrow V}{ifz(\underline{0}, E_1, E_2) \Downarrow V}$.

- Assume $M \equiv ifz(\underline{k+1}, E_1, E_2)$ and $N \equiv E_2$. Assume also $N \Downarrow V$, i.e., $E_2 \Downarrow V$. We need to show $ifz(\underline{k+1}, E_1, E_2) \Downarrow V$. This holds by the rule $\frac{\underline{k+1} \Downarrow \underline{k+1} \quad E_2 \Downarrow V}{ifz(\underline{k+1}, E_1, E_2) \Downarrow V}$.

Inductive Steps.

- $\frac{M_1 \rhd M_2}{succ(M_1) \rhd succ(M_2)}$: $M = succ(M_1), N = succ(M_2)$, where $M_1 \rhd M_2$. Assume $N \Downarrow V$, i.e., $succ(M_2) \Downarrow V$. This implies $V \equiv \underline{k+1}$ and $M_2 \Downarrow \underline{k}$ for some k. By IH on the sub-derivation $M_1 \rhd M_2$: since $M_2 \Downarrow \underline{k}$, it follows that $M_1 \Downarrow \underline{k}$. Then, by the rule $\frac{M_1 \Downarrow \underline{k}}{succ(M_1) \Downarrow \underline{k+1}}$. So $M \equiv succ(M_1) \Downarrow \underline{k+1}$. Since $V \equiv \underline{k+1}$, we have $M \Downarrow V$.

- $\frac{M_1 \rhd M_2}{M_1(A) \rhd M_2(A)}$: $M = M_1(A),\ N = M_2(A)$. Assume $M_2(A) \Downarrow V$. This means $M_2 \Downarrow \lambda x.E$ and $E[A/x] \Downarrow V$. By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \lambda x.E$, then $M_1 \Downarrow \lambda x.E$. Thus, by the rule, $M_1(A) \Downarrow V$.

- $\frac{M_1 \rhd M_2}{pred(M_1) \rhd pred(M_2)}$: $M = pred(M_1),\ N = pred(M_2)$. Assume $pred(M_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and V is $\underline{0}$ (if $k = 0$) or $\underline{k-1}$ (if $k > 0$). By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for *pred*, $pred(M_1) \Downarrow V$.

- $\frac{M_1 \rhd M_2}{ifz(M_1, N_1, N_2) \rhd ifz(M_2, N_1, N_2)}$: $M = ifz(M_1, N_1, N_2),\ N = ifz(M_2, N_1, N_2)$. Assume $ifz(M_2, N_1, N_2) \Downarrow V$. This means $M_2 \Downarrow \underline{k}$ and either $N_1 \Downarrow V$ (if $k = 0$) or $N_2 \Downarrow V$ (if $k > 0$). By IH on $M_1 \rhd M_2$, since $M_2 \Downarrow \underline{k}$, then $M_1 \Downarrow \underline{k}$. Thus, by the rule for ifz, $ifz(M_1, N_1, N_2) \Downarrow V$.

■

Problem 5. (Page 19) Show that the applicative relation $\underset{\sim\sigma}{\subset}$ is a preorder on Prg_σ, i.e. that $\underset{\sim\sigma}{\subset}$ is reflexive and transitive.

Solution:
-Relfexivity. We need to show that for any closed PCF term M of type σ, $M \sqsubseteq_\sigma M$.
Base Case. For $M \in Prg_{nat}, M \sqsubseteq_{nat} M$ means that $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow M \Downarrow \underline{n}$. This is trivially true.
Inductive Case. For $M \in Prg_{\sigma\to\tau}, M \sqsubseteq_{\sigma\to\tau} M$ means that $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau M(P)$, which holds by IH.
-Transitivity. We need to show that for any closed PCF terms M, N, K of type σ, if $M \sqsubseteq_\sigma N$ and $N \sqsubseteq_\sigma K$, then $M \sqsubseteq_\sigma K$.
Base Case. For $M, N, K \in Prg_{nat}$, assume $M \sqsubseteq_{nat} N$ and $N \sqsubseteq_{nat} K$. Then, by definition, we have the followings:

- $\forall n \in \mathbb{N}, M \Downarrow \underline{n} \Rightarrow N \Downarrow \underline{n}$
- $\forall n \in \mathbb{N}, N \Downarrow \underline{n} \Rightarrow K \Downarrow \underline{n}$

Thus, if $M \Downarrow \underline{n}$ then $K \Downarrow \underline{n}$, which means $M \sqsubseteq_{nat} K$.
Inductive Case. For $M, N, K \in Prg_{\sigma\to\tau}$, assume $M \sqsubseteq_{\sigma\to\tau} N$ and $N \sqsubseteq_{\sigma\to\tau} K$. Then, by definition, we have the followings:

- $\forall P \in Prg_\sigma, M(P) \sqsubseteq_\tau N(P)$
- $\forall P \in Prg_\sigma, N(P) \sqsubseteq_\tau K(P)$

Thus, we would have $\forall P \in Prg_\sigma, M(P) \sqsubseteq K(P)$. ■

2.2 The Scott Model of PCF

Problem 1. (Page 26) Show that (Scott) continuous functions between predomains are always monotonic.

Solution: Let $f : (A, \sqsubseteq_A) \to (B, \sqsubseteq_B)$ be a Scott continuous function between predomains. For any $x, y \in A$ with $x \leq y$, we have that $X = \{x, y\}$ is a directed subset of A. The supremum of the set X is obviously y. Thus $\bigsqcup X = y$, and since f is continuous, $f(\bigsqcup X) = f(y) = \bigsqcup f(\{x, y\})$. Hence, $f(x) \leq f(y)$. ■

Problem 2. (Page 26) (Theorem 3.3). Let $(A_i | i \in I)$ be a family of predomains. Then their product $\prod_{i \in I} A_i$ is a predomain under the componentwise ordering, and the prodjections $\pi_i : \prod_{i \in I} A_i \to A_i$ are Scott continuous. If, moreover, all A_i are domains then so is their product $\prod_{i \in I} A_i$.

Solution: Let $D = \prod_{i \in I} A_i = \{f : I \to \bigcup_{i \in I} A_i | \forall i \in I, f(i) \in A_i\}$. We need to show that $(D, \sqsubseteq_D)$ is a poset, and every directed subset of D has a least upper bound. Note that the order $\sqsubseteq_D$ is defined as follows:

$$(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I} \quad iff \quad \forall i \in I, d(i) \sqsubseteq_{A_i} d'(i)$$

We now show that $(D, \sqsubseteq_D)$ forms a poset.

- **Reflexivity:** For any $(d_i)_{i \in I} \in D, d_i \sqsubseteq_{A_i} d_i, \forall i \in I$ since each A_i is a poset. Thus $(d_i)_{i \in I} \sqsubseteq_D (d_i)_{i \in I}$
- **Transitivity:** Assume $(d_i)_{i \in I} \sqsubseteq_D (d'_i)_{i \in I}$ and $(d'_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$. And by each A_i being transitive, it follows immediately that $d_i \sqsubseteq_{A_i} d''_i$ for all $i \in I$. Therefore, $(d_i)_{i \in I} \sqsubseteq_D (d''_i)_{i \in I}$.
- **Antisymmetry:** Similar to the previous case, it follows immediately from the fact that each A_i is antisymmetric.

Now, suppose that $X \subseteq D = \prod_{i \in I} A_i$ is a directed subset. define $X_i = \{\pi_i(x) | x \in X\}$, that is, the projection of X to A_i. X_i is directed since X is directed. Moreover, X_i has a least upper bound $\bigsqcup X_i \in A_i$. Define $z \in D$ with $z_i = \bigsqcup X_i$ for each $i \in I$. By construction, it is obvious that z is the least upper bound of X in D. Thus, D is a predomain. ■

Problem 3. (Page 27). Prove that the evaluation map $ev : [A_1 \to A_2] \times A_1 \to A_2$ with $ev(f, a) = f(a)$ is continuous in each argument.

Solution: For the first argument, fix $a \in A_1$ and let $F \subseteq [A_1 \to A_2]$ be a directed set of continuous functions. By Theorem 3.5, we have $\bigsqcup F(a) = g(a) = \bigsqcup_{f \in F} f(a)$. Thus, $ev(\bigsqcup F, a) = g(a) = \bigsqcup_{f \in F} f(a) = \bigsqcup\{ev(f, a) | f \in F\}$.
Now, for the second argument, fix $f \in [A_1 \to A_2]$ and let $X \subseteq A_1$ be a directed set. Because f is continuous, we have $f(\bigsqcup X) = \bigsqcup\{f(x) | x \in X\}$. Thus,

$$ev(f, \bigsqcup X) = f(\bigsqcup X) = \bigsqcup\{f(x) | x \in X\} = \bigsqcup\{ev(f, x) | x \in X\}.$$

Hence, ev is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

Problem 4. (Page 30). Prove that $\Psi : [[D \to D] \to D] \times [D \to D] \to D : (F, f) \mapsto f(F(f))$ is continuous in each argument.

Solution: For the first argument, fix $f \in [D \to D]$ and let $\mathcal{F} \subseteq [[D \to D] \to D]$ be a directed set. We know that $(\bigsqcup \mathcal{F})(f) = \bigsqcup\{F(f)|F \in \mathcal{F}\}$. Thus,

$$\underbrace{f((\bigsqcup \mathcal{F})(f))}_{\Psi(\bigsqcup \mathcal{F}, f)} = f(\bigsqcup\{F(f)|F \in \mathcal{F}\}) = \bigsqcup\{f(F(f))|F \in \mathcal{F}\}$$

For the second argument, fix $F \in [[D \to D] \to D]$ and let $X \subseteq [D \to D]$ be a directed set. We know that $F(\bigsqcup X) = \bigsqcup\{F(f)|f \in X\}$. Thus,

$$f(F(\bigsqcup X)) = f(\bigsqcup\{F(f)|f \in X\}) = \bigsqcup\{f(F(f))|f \in X\}$$

Hence, Ψ is continuous in both arguments and by Lemma 3.4, it is jointly continuous. ■

Problem 5. (Page 33) (β-equality). If, $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$ then

$$[\![\Gamma \vdash (\lambda x : \sigma.M)(N)]\!] = [\![\Gamma \vdash M[N/x]]\!]$$

Solution: $[\![\Gamma \vdash M[N/x]]\!](\vec{d}) = [\![\Gamma, x : \sigma \vdash M]\!](\vec{d}, [\![\Gamma \vdash N]\!](\vec{d}))$ by Lemma 3.15.
Now, for the other side,

$$[\![\Gamma \vdash (\lambda x : \sigma.M)(N)]\!](\vec{d}) = ev([\![\Gamma \vdash \lambda x : \sigma.M]\!](\vec{d}), [\![\Gamma \vdash N]\!](\vec{d}))$$

$$= [\![\Gamma \vdash \lambda x : \sigma.M]\!](\vec{d})([\![\Gamma \vdash N]\!](\vec{d})) = [\![\Gamma, x : \sigma \vdash M]\!](\vec{d}, [\![\Gamma \vdash N]\!](\vec{d}))$$

■

Problem 6. (Page 34) (η-equality). If, $\Gamma \vdash M : \sigma \to \tau$ then

$$[\![\Gamma \vdash \lambda x : \sigma.M(x)]\!] = [\![\Gamma \vdash M]\!]$$

for $x \notin Var(\Gamma)$.

Solution:

$$\forall \vec{d} \in [\![\Gamma]\!], d' \in D_\sigma, [\![\Gamma \vdash \lambda x : \sigma.M(x)]\!](\vec{d})(d') = [\![\Gamma, x : \sigma \vdash M(x)]\!](\vec{d}, d')$$

$$= ev([\![\Gamma, x : \sigma \vdash M]\!](\vec{d}, d'), \underbrace{[\![\Gamma, x : \sigma \vdash x]\!](\vec{d}, d')}_{\pi_x(\vec{d}, d') = d'}) =^\star [\![\Gamma \vdash M]\!](\vec{d})(d')$$

For the last equation ($\star$), because $x \notin Var(\Gamma)$ and $\Gamma \vdash M$, we have

$$[\![\Gamma, x : \sigma \vdash M]\!](\vec{d}, d') = [\![\Gamma \vdash M]\!](\vec{d})$$

. ■

2.3 Milner's Context Lemma

Problem 1. (Page 44) Prove that $\leq_\sigma$ is closed under suprema of directed sets. That is, if $X \subseteq D_\sigma \times D_\sigma$ is directed and $X \subseteq \leq_\sigma$, meaning that for every $(x, y) \in X, x \leq_\sigma y$, then $\bigsqcup X \in \leq_\sigma$.

Solution: Let $X = \{(x_k, y_k) | k \in K\}$ be a directed subset of $D_\sigma \times D_\sigma$ such that for all $k \in K, (x_k, y_k) \in \leq_\sigma$. This means for each $k \in K$, we have $\forall P \in Prg_\sigma, y_k R_\sigma P \implies x_k R_\sigma P (\star)$.
Let $x = \bigsqcup_{k \in K} x_k$ and $y = \bigsqcup_{k \in K} y_k$ (Note that $\bigsqcup X = (x, y)$). We need to show $x \leq_\sigma y$. That is, $\forall P \in Prg_\sigma y R_\sigma P \implies x R_\sigma P$.
Let $P \in Prg_\sigma$ be an arbitrary closed PCF term and that $y R_\sigma P$. Now, for each $k \in K$, we have $y_k \sqsubseteq y$ and by Lemma 4.2(1), we get $y_k R_\sigma P$ for all $k \in K$. Using $(\star)$, we have $x_k R_\sigma P$ for each $k \in K$.
By Lemma 4.2(2), $R_\sigma P$ is closed under directed suprema and $\{x_k | k \in K\}$ is a directed subset of D_σ whose elements are all in $R_\sigma P$. So, their suprema x must also be in $R_\sigma P$, meaning that $x R_\sigma P$. ■

2.4 Logical Relations

Problem 1. (Page 52) (Theorem 7.2 When M is Variable) Let R be a logical relation of arity W on the Scott model of PCF. Then for λ-terms $x_1 : \sigma_1, ..., x_n : \sigma_n \vdash x_k : \sigma_k$ for some k and $d_j \in R_{\sigma_j}$ for $j = 1, ..., n$ it holds that

$$\underline{\lambda} i \in W. [\![x_1 : \sigma_1, ..., x_n : \sigma_n \vdash x_k]\!] (\vec{d}(i)) \in R_{\sigma_k}$$

Solution: By premise, $d_k \in R_{\sigma_k}$. By definition, the goal reduces to $d_k \in R_{\sigma_k}$! ■

Problem 2. (Page 54) *Problem Statement*
