

# MSSC 6040 - Homework 1

Henri Medeiros Dos Reis

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**Problem 1** (10 pts). The *outer-product* of two vectors  $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^m$  and  $b = \begin{bmatrix} b_1 \\ a_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$  is the  $m \times n$  matrix given by

$$ab^* = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

For example, the outer product of  $a = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$  and  $b = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$  is the  $4 \times 3$  matrix given by

$$ab^* = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 2 & -4 & -10 \\ -3 & 6 & 15 \end{bmatrix}.$$

(a) Express the following matrices as an outer product of two vectors:

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and the matrix  $D \in \mathbb{R}^{m \times n}$  whose  $(i, j)$ th entry is equal to 1 and all other entries are zero, i.e.,  $D_{i,j} = 1$  and  $D_{k,\ell} = 0$  for all  $(k, \ell) \neq (i, j)$ .

**Solution:**

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \end{bmatrix} = \begin{bmatrix} 1(1) & 1(0) & 2(2) & -1(-1) \\ 2(1) & 0(0) & 4(2) & -2(-1) \\ -1(1) & 0(0) & -2(2) & 1(-1) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} \overline{b_1} & \overline{b_2} & \overline{b_3} & \overline{b_4} \end{bmatrix} = \begin{bmatrix} 1(2) & 1(-2) & 1(1) & 1(-1) \\ -1(2) & -1(-2) & -1(1) & -1(-1) \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & \cdots & n \end{bmatrix} = \begin{bmatrix} 1(1) & 1(2) & 1(3) & \cdots & 1(n) \\ 1(1) & 1(2) & 1(3) & \cdots & 1(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1(1) & 1(2) & 1(3) & \cdots & 1(n) \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1(i^{th} \text{entry}) \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1(j^{th} \text{entry}) & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 1(d_{ij}) & \ddots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- (b) Prove that if  $A$  is an  $\ell \times m$  matrix and  $B$  is a  $n \times m$  matrix, then  $C = AB^*$  is equal to the sum of the outer products of the columns of  $A$  with the columns of  $B$ , i.e.,

$$C = \sum_{k=1}^m A_k B_k^*,$$

where  $A_k \in \mathbb{R}^m$  is the  $k$ th column of  $A$  and  $B_k \in \mathbb{R}^n$  is the  $k$ th column of  $B$ .

(Hint: Use the formula for computing the  $(i, j)$ th entry of a matrix-matrix product given in lecture.)

**Solution:**

$$A_{\ell \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \cdots & a_{\ell m} \end{bmatrix}, B_{m \times n}^* = \begin{bmatrix} b_{11}^* & b_{12}^* & \cdots & b_{1m}^* \\ b_{21}^* & b_{22}^* & \cdots & b_{2m}^* \\ \vdots & \vdots & \ddots & \vdots \\ b_{\ell 1}^* & b_{\ell 2}^* & \cdots & b_{\ell m}^* \end{bmatrix}$$

$$C_{\ell \times m} = AB^* \text{ then } C = \sum_{k=1}^m A_k B_k^*$$

$$\text{By definition } C_{ij} = \sum_{k=1}^m A_{ik} B_{kj}^*, \text{ so}$$

$$C = \begin{bmatrix} a_{1m}b_{m1} & \cdots & a_{1m}b_{mk} \\ \vdots & \ddots & \vdots \\ a_{km}b_{m1} & \cdots & a_{km}b_{mk} \end{bmatrix}$$

$$\text{and the definition of an outer product is } C_j = Ar_j = \sum_{k=1}^j a_k$$

$$\text{Therefore } C = \sum_k^m A_k B_k^* = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} = \begin{bmatrix} a_{1m}b_{m1} & \cdots & a_{1m}b_{mk} \\ \vdots & \ddots & \vdots \\ a_{km}b_{m1} & \cdots & a_{km}b_{mk} \end{bmatrix}$$

- (c) Use the above result to compute the matrix product  $AB^*$  where  $A$  and  $B$  are specific  $2 \times 2$  matrices of your choosing. Verify that the matrix product is the same as what you would get using the “conventional” way of doing matrix multiplication using inner products between rows and columns.

**Solution:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 4 & 3 \\ 2 & 4 \end{bmatrix} B^* = \begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left. \begin{aligned} C_1 &= 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix} \\ C_2 &= 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \end{bmatrix} \end{aligned} \right\} = \begin{bmatrix} 10 & 10 \\ 24 & 22 \end{bmatrix}$$

”conventional way”

$$C = \begin{bmatrix} 1(4) + 2(3) & 1(2) + 2(4) \\ 3(4) + 4(3) & 3(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 24 & 22 \end{bmatrix}$$

**Problem 2** (10 pts). T-B 1.3

**Solution:**

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r_{mm} \end{bmatrix} \text{ is upper triangular}$$

$$\text{From 1.8 } e_j = \sum_{i=1}^m Z_{ij} a_{ij} \Rightarrow I = RR^{-1}$$

Then any  $e_j = Rr_j^{-1}$ , since  $e_j$  is a vector from identity. Then:

$$RR^{-1} = R \begin{bmatrix} r_1^{-1} & r_2^{-1} & \cdots & r_m^{-1} \end{bmatrix} = \begin{bmatrix} Rr_1^{-1} & Rr_2^{-1} & \cdots & Rr_m^{-1} \end{bmatrix}$$

Where  $r_i^{-1}$  is a  $m \times 1$  matrix, then  $Rr_j^{-1} = e_j$ , which has only 0s below the  $j^{th}$  entry. Therefore  $R^{-1}$  has to be upper triangular.

**Problem 3** (MATLAB, 5 pts). Use MATLAB's backslash operator `\` to find the vector  $x$  that satisfies the equation:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Verify (in MATLAB) that the vector  $x$  you find is a solution by computing the matrix-vector product  $Ax$  and comparing it with  $b$ . In your submitted pdf include a print-out/screenshot of your code (do not attach your MATLAB script).

**Solution:**

```
A = [1 -1 0 0; 1 1 0 0; 0 0 1 -1; 0 0 1 1]
b = [1; 2 ;3 ;4]

x = A\b %use the \ operator

answer = A*x %make sure the answer is correct
```

Output from code:

```
#x =
    1.5000
    0.5000
    3.5000
    0.5000

#answer =
     1
     2
     3
     4
```