MSSC 6040 - Homework 1

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Problem 1 (10 pts)**.** The *outer-product* of two vectors $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^m$ and $b = \begin{bmatrix} b_1 \\ a_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ is the $m \times n$ matrix given by

$$ab^* = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

For example, the outer product of $a = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ is the 4×3 matrix given by

$$ab^* = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 2 & -4 & -10 \\ -3 & 6 & 15 \end{bmatrix}.$$

(a) Express the following matrices as an outer product of two vectors:

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

and the matrix $D \in \mathbb{R}^{m\times n}$ whose (i, j)th entry is equal to 1 and all other entries are zero, i.e., $D_{i,j} = 1$ and $D_{k,\ell} = 0$ for all $(k, \ell) \neq (i, j)$.

Solution:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \end{bmatrix} = \begin{bmatrix} 1(1) & 1(0) & 2(2) & -1(-1) \\ 2(1) & 0(0) & 4(2) & -2(-1) \\ -1(1) & 0(0) & -2(2) & 1(-1) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} \overline{b_1} & \overline{b_2} & \overline{b_3} & \overline{b_4} \end{bmatrix} = \begin{bmatrix} 1(2) & 1(-2) & 1(1) & 1(-1) \\ -1(2) & -1(-2) & -1(1) & -1(-1) \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & \cdots & n \end{bmatrix} = \begin{bmatrix} 1(1) & 1(2) & 1(3) & \cdots & 1(n) \\ 1(1) & 1(2) & 1(3) & \cdots & 1(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1(1) & 1(2) & 1(3) & \cdots & 1(n) \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1(i^{th} entry) \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1(j^{th} entry) & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 1(d_{ij}) & \ddots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

(b) Prove that if A is an $\ell \times m$ matrix and B is a $n \times m$ matrix, then $C = AB^*$ is equal to the sum of the outer products of the columns of A with the columns of B, i.e.,

$$C = \sum_{k=1}^{m} A_k B_k^*,$$

where $A_k \in \mathbb{R}^m$ is the kth column of A and $B_k \in \mathbb{R}^n$ is the kth column of B.
(*Hint: Use the formula for computing the (i, j)th entry of a matrix-matrix product given in lecture.*)

Solution:

$$A_{\ell \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \cdots & a_{\ell m} \end{bmatrix}, B^*_{m \times n} = \begin{bmatrix} b^*_{11} & b^*_{12} & \cdots & b^*_{1m} \\ b^*_{21} & b^*_{22} & \cdots & b^*_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b^*_{\ell 1} & b^*_{\ell 2} & \cdots & b^*_{\ell m} \end{bmatrix}$$

$$C_{\ell\times m} = AB^* \text{ then } C = \sum_{k=1}^{m} A_k B_k^*$$

$$\text{By definition } C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}^*, \text{ so}$$

$$C = \begin{bmatrix} a_{1m}b_{m1} & \cdots & a_{1m}b_{mk} \\ \vdots & \ddots & \vdots \\ a_{km}b_{m1} & \cdots & a_{km}b_{mk} \end{bmatrix}$$

$$\text{and the definition of an outer product is } C_j = Ar_j = \sum_{k=1}^{j} a_k$$

$$\text{Therefore } C = \sum_{k}^{m} A_k B_k^* = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} = \begin{bmatrix} a_{1m}b_{m1} & \cdots & a_{1m}b_{mk} \\ \vdots & \ddots & \vdots \\ a_{km}b_{m1} & \cdots & a_{km}b_{mk} \end{bmatrix}$$

(c) Use the above result to compute the matrix product AB^* where A and B are specific 2×2 matrices of your choosing. Verify that the matrix product is the same as what you would get using the "conventional" way of doing matrix multiplication using inner products between rows and columns.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 4 & 3 \\ 2 & 4 \end{bmatrix} B^* = \begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left.\begin{aligned} C_1 &= 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix} \\ C_2 &= 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \end{bmatrix} \end{aligned}\right\} = \begin{bmatrix} 10 & 10 \\ 24 & 22 \end{bmatrix}$$

"conventional way"

$$C = \begin{bmatrix} 1(4) + 2(3) & 1(2) + 2(4) \\ 3(4) + 4(3) & 3(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 24 & 22 \end{bmatrix}$$

Problem 2 (10 pts)**.** T-B 1.3

Solution:

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r_{mm} \end{bmatrix} \text{ is upper triangular}$$

$$\text{From } 1.8\ e_j = \sum_{i=1}^{m} Z_{ij} a_{ij} => I = RR^{-1}, \text{let } R^{-1} \text{be } Z$$

Then: $R = \left[\vec{R_1} \mid \quad \vec{R_2} \mid \quad \dots \quad \mid \vec{R_n}\right]$, where

$$\vec{R_1} = \begin{bmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{R_2} = \begin{bmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{R_n} = \begin{bmatrix} r_{1n} \\ r_{2n} \\ \vdots \\ r_{nn} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{and} \quad \vec{R_m} = \begin{bmatrix} r_{1m} \\ r_{2m} \\ \vdots \\ r_{mm} \end{bmatrix}, \text{where} \quad n < m$$

So $span\{\vec{R_1}, \vec{R_2}, \cdots, \vec{R_n}\} = \mathbb{R}^n$. Then e_n is snapped by columns 1 through n.

And $e_j = \sum_{i=1}^{n} Z_{ij} r_{ij} = z_1 r_1 + z_2 r_2 + \cdots + z_n r_n$. So when we consider a linear combination where any column bigger than n, then we will have $a_m =_1 r_1 + z_2 r_2 + \cdots + z_n r_n + z_{n+1} r_{n+1} + \cdots + z_m r_m$, where all z_i, where $n < i \leq m$ is going to be zero. Therefore $Z_{in} = 0$ for all values $i > n$. Therefore $Z = R^{-1}$ is upper triangular

Problem 3 (MATLAB, 5 pts)**.** Use MATLAB's backslash operator \ to find the vector x that satisfies the equation:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Verify (in MATLAB) that the vector x you find is a solution by computing the matrix-vector product Ax and comparing it with b. In your submitted pdf include a print-out/screenshot of your code (do not attach your MATLAB script).

Solution:

```
A = [1 -1 0 0; 1 1 0 0; 0 0 1 -1; 0 0 1 1]
b = [1; 2 ;3 ;4]

x = A\b %use the \ operator

answer = A*x %make sure the answer is correct
```

Output from code:

#x =

1.5000
0.5000

3.5000
0.5000
#answer =
1
2
3
4