Homework 6

MSSC 6010- Computational Probability

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Question 1. (5.8.14) From book. Given the joint density function

$$f(x, y) = 6x, \quad 0 < x < y < 1$$

find the $E[Y|X]$ that is the regression line resulting from regressing Y on X.

First let's compute the marginals

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy = \int_x^1 6xdy = 6x(1 - x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = \int_0^y 6xdx = 3y^2$$

then the conditional pdf

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{6x}{6x(1 - x)} = \frac{1}{1 - x}, 0 < x < y < 1$$

$$E[Y|X] = \int_0^1 y(\frac{1}{1 - x})dy = \frac{y^2}{2(1 - x)}\Bigg|_0^1 = \frac{1}{2(1 - x)}$$

Question 2. (5.8.17) From book. If $f(x, y) = e^{-(x+y)}$, $x > 0$ and $y > 0$, find $\mathbb{P}(X+3 > Y|X > \frac{1}{3})$.

From conditional probability, we get

$$\mathbb{P}(X + 3 > Y|X > \frac{1}{3}) = \frac{\mathbb{P}(X + 3 > Y, X > \frac{1}{3})}{\mathbb{P}(X > \frac{1}{3})}$$

Let's first compute $\mathbb{P}(X > \frac{1}{3})$

$$f_X(x) = \int_0^{\infty} e^{-(x+y)}dy = e^{-x}\int_0^{\infty} e^{-y}dy = e^{-x}$$

$$\mathbb{P}(X > \frac{1}{3}) = \int_{\frac{1}{3}}^{\infty} e^{-x}dx = e^{\frac{-1}{3}}$$

Now $\mathbb{P}(X + 3 > Y, X > \frac{1}{3})$

$$\begin{aligned}
\mathbb{P}(X + 3 > Y, X > \frac{1}{3}) &= \int_{\frac{1}{3}}^{\infty} \int_{0}^{x+3} e^{-(x+y)}dydx \\
&= \int_{\frac{1}{3}}^{\infty} e^{-x} \int_{0}^{x+3} e^{-y}dydx \\
&= \int_{\frac{1}{3}}^{\infty} -e^{-x}e^{-(x+3)} + e^{-x}dx \\
&= \int_{\frac{1}{3}}^{\infty} -e^{-(2x+3)} + e^{-x}dx \\
&= \left(\frac{e^{-(2x+3)}}{2} - e^{-x}\right)\Big|_{\frac{1}{3}}^{\infty} \\
&= -\frac{e^{-(2/3+3)}}{2} + e^{-1/3}
\end{aligned}$$

Plugging back to the conditional probability, we get

$$\begin{aligned}
\mathbb{P}(X + 3 > Y | X > \frac{1}{3}) &= \frac{\mathbb{P}(X + 3 > Y, X > \frac{1}{3})}{\mathbb{P}(X > \frac{1}{3})} \\
&= \frac{-\frac{e^{-(2/3+3)}}{2} + e^{-1/3}}{e^{-1/3}} \\
&= 0.982163
\end{aligned}$$

Question 3. (5.8.26) From book.Given the joint density function $f_{X,Y}(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1,$

- *(a)* Show that $f_{X,Y}(x, y) \geq 0$ for all x and and that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1.$

The smallest $f_{X,Y}(x,y) $ can ever be is whenever x and y are the smallest it can ever be, since it is addition. Therefore since $x \geq 0$ and $y \geq 0$, $f_{X,Y}(x, y)$ is the smallest value at $x = y = 0$ then $f_{X,Y}(x, y) \geq 0$

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy &= \int_{0}^{1}\int_{0}^{1} x + ydxdy \\
&= \int_{0}^{1} (\frac{1}{2} + y)dy \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

- *(b)* Find the cumulative distribution function.

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(r,s)dsdr \\
&= \int_{0}^{x}\int_{0}^{y} r + sdsdr \\
&= \int_{0}^{x} (rs + \frac{s^2}{2})\Big|_0^y dr \\
&= \int_{0}^{x} (ry + \frac{y^2}{2})dr \\
&= \frac{yx^2}{2} + \frac{y^2x}{2}
\end{aligned}$$

- *(c)* Find the marginal means of X and Y.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 x + ydy = x + \frac{1}{2} \\
E[X] &= \int_0^1 x(x+\frac{1}{2}) = (\frac{x^3}{3} + \frac{x^2}{4})|_0^1 = \frac{1}{3} + \frac{1}{4} \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 x + ydx = y + \frac{1}{2} \\
E[Y] &= \int_0^1 y(y+\frac{1}{2}) = (\frac{y^3}{3} + \frac{y^2}{4})|_0^1 = \frac{1}{3} + \frac{1}{4}
\end{aligned}$$

- *(d)* Find the marginal variances of X and Y.

$$\begin{aligned}
E[X^2] &= \int_0^1 x^2(x+\frac{1}{2}) = (\frac{x^4}{4} + \frac{x^3}{6})|_0^1 = \frac{1}{4} + \frac{1}{6} \\
Var(X) &= \frac{1}{4} + \frac{1}{6} - (\frac{1}{3} + \frac{1}{4})^2 = \frac{11}{144} \\
E[Y^2] &= \int_0^1 y^2(y+\frac{1}{2}) = (\frac{y^4}{4} + \frac{y^3}{6})|_0^1 = \frac{1}{4} + \frac{1}{6} \\
Var(Y) &= \frac{1}{4} + \frac{1}{6} - (\frac{1}{3} + \frac{1}{4})^2 = \frac{11}{144}
\end{aligned}$$

Question 4. (5.8.31) From book. Let X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} Ky & -2 \le x \le 2, 1 \le y \le x^2 \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

- *(a)* Find K such that $f_{X,Y}(x,y)$ is a valid pdf.

Since $1 \leq y \leq x^2$, then $x^2 \geq 1$, so $x \geq 1$ or $x \leq -1$, then combining with the original constraint we have $1 \leq x \leq 2$ and $-2 \leq x \leq -1$ so

$$\begin{aligned}
1 &= \int_1^2 \int_1^{x^2} Kydydx + \int_{-2}^{-1} \int_1^{x^2} Kydydx \\
&= K \int_1^2 (\frac{x^4}{2} - \frac{1}{2})dx + K \int_{-2}^{-1} (\frac{x^4}{2} - \frac{1}{2})dx \\
&= K(\frac{x^5}{10} - \frac{x^2}{4})|_1^2 + K(\frac{x^5}{10} - \frac{x^2}{4})|_{-2}^{-1} \\
&= K(\frac{62}{10} - \frac{2}{10}) \\
K &= \frac{10}{62}
\end{aligned}$$

- *(b)* Find the marginal densities of X and Y

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\
&= \int_1^{x^2} \frac{10}{62} ydy \\
&= \frac{10}{62}(\frac{x^4}{2} - \frac{1}{2}) \\
&= \begin{cases} \frac{10}{62}(\frac{x^4}{2} - \frac{1}{2}) & \text{if } -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Since $1 \leq y \leq x^2$, then $1 \leq (y)^{1/2} \leq x$, then $-2 \leq -\sqrt{y} \leq x$, and $x \leq \sqrt{y} \leq 2$, so

$$\begin{aligned}
f_Y(y) &= \int_{-2}^{-\sqrt{y}} \frac{10}{62} ydx + \int_{\sqrt{y}}^{2} \frac{10}{62} ydx \\
&= \frac{10}{62} y(-\sqrt{y} + 2 + 2 - \sqrt{y}) \\
&= \begin{cases} \frac{10}{62} y(-2\sqrt{y} + 4) & \text{if } -2 \leq -\sqrt{y} \leq -1 \text{ or } 1 \leq \sqrt{y} \leq 2 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \frac{10}{62} y(-2\sqrt{y} + 4) & \text{if } 1 \leq y \leq 4 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

- *(c)* Find $\mathbb{P}(Y > \frac{3}{2} | X < \frac{1}{2})$

Since y values are $\frac{3}{2} \leq y \leq x^2$, and x values are $-2 \leq y \leq -1$, since $-1 < \frac{1}{2}$. Then

$$\begin{aligned}
\mathbb{P}(Y > \frac{3}{2}|X < \frac{1}{2}) &= \frac{\int_{-2}^{-\sqrt{3/2}} \int_{3/2}^{x^2} \frac{10}{62} y dy dx}{\int_{-2}^{-1} \frac{10}{62}(x^4/4 - 1/2)dx} \\
&= \frac{\int_{-2}^{-\sqrt{3/2}} \int_{3/2}^{x^2} y dy dx}{\int_{-2}^{-1}(x^4/4 - 1/2)dx} \\
&= \frac{\int_{-2}^{-\sqrt{3/2}} (y^2/2)\Big|_{\frac{3}{2}}^{x^2} dx}{(\frac{-1}{10} + \frac{1}{2}) - (\frac{-2^5}{10} + 1)} \\
&= \frac{\left(\frac{x^5}{10} - \frac{(3/2)^2}{2}x\right)\Big|_{-2}^{-\sqrt{3/2}}}{26/10} \\
&= 0.789335
\end{aligned}$$

Question 5. (5.8.32) From book. An engineer has designed a new diesel motor that is used in a prototype earth mover. The prototype's diesel consumption in gallons per mile C follows the equation $C = 3 + 2X + \frac{3}{2}Y$, where X is a speed coefficient and Y is the quality diesel coefficient. Suppose the joint density for X and Y is $f_{X,Y}(x, y) = ky, 0 \leq x \leq 2, 0 \leq y \leq x$.

- *(a)* find k so that $f_{X,Y}(x, y)$ is a valid density function.

$$\begin{aligned}
1 &= \int_0^2 \int_0^x ky dy dx \\
&= k \int_0^2 \frac{x^2}{2} dx \\
&= k\frac{8}{6} \\
k &= \frac{3}{4}
\end{aligned}$$

- *(b)* Are X and Y independent?

$$f_X(x) = \int_0^x \frac{3}{4} y dy = \frac{3}{8}x^2$$

$$f_Y(y) = \int_y^2 \frac{3}{4} y dy = \frac{6}{4}y - \frac{3}{4}y^2$$

Since $f_{X,Y}(x, y) \neq f_X(x) f_Y(y) = \frac{3}{8}x^2(\frac{6}{4}y - \frac{3}{4}y^2)$, then they are not independent

- *(c)* Find the mean diesel consumption for the prototype.

Since the diesel consumption $C = 3 + 2X + \frac{3}{2}Y$, we need to find $E[C]$

$$\begin{aligned}
E[3 + 2X + \frac{3}{2}Y] &= E[3] + 2E[X] + \frac{3}{2}E[Y] \\
&= 3 + 2\int_0^2 x\frac{3}{8}x^2 dx + \frac{3}{2}\int_0^2 y(\frac{6}{4}y - \frac{3}{4}y^2)dy \\
&= 3 + 2\frac{3}{8}(\frac{2^4}{4}) + \frac{3}{2}(\frac{6}{4}\frac{2^3}{3} - \frac{3}{4}\frac{2^4}{4}) \\
&= 6 + \frac{3}{2} \\
&= \frac{15}{2}
\end{aligned}$$