Henri

DIFFERENCES, DERIVATIVES, BOUNDARY CONDITIONS

INSTRUCTIONS: You may work with your classmates but your write-up must be your own. Your write up should be clear and easy to follow with the full problem statement at the beginning of each problem (if not given). Be prepared to be present your work at the start of our next class period.

PROBLEM H2: Consider the BVP

$$-u'' = 1, \quad u(0) = 0, \quad u'(1) = 0.$$

(a) Compute the exact solution by hand.

(b) Find the approximate solution using finite differences. Write out the difference equations and solve the problem using Matlab using n=4, 20, 100 interior nodes. Plot the numerical solution and compare to the true solution found in (a).

(c) How small must your grid size be so that the solution satisfactory? What could this mean?

EXERCISE H0: Use 2D Finite Differences to help you solve the following boundary value problem.

$$u_{xx} + u_{yy} + 2u = 0, \qquad 0 < x < 1, \;\; 0 < y < 1$$

subject to $u(x, y) = 0$ on the top, left, and ride sides of the square domain with $u(x, y) = sin(\pi x)$ for $y = 0$ (i.e. the bottom of the square). Use 5 grid points (3 interior points) in each of the x and y directions and write down the corresponding equations. Next class we'll work on SOLVING these, for now, just work on discretizing the problem.

H2:

a.) $-u'' = 1$, $u(0) = 0$, $u'(1) = 0$

$u'' = -1 \Rightarrow u' = -x + c_1 \Rightarrow u'' = -\frac{x^2}{2} + c_1 x + c_2$

then plugging the boundary conditions gives

$u'(1) = -1 + c_1 = 0 \Rightarrow c_1 = 1$

$u(0) = 0 + 0 + c_2 = 0 \Rightarrow c_2 = 0$

then the solution is $u = \frac{-x^2}{2} + x$

b) first, lets discretize the equation,

let

$$-\left(\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}\right) = 1$$

$$\begin{array}{llllll}
i = 0 & v_0 = 0 & & \text{from} & & BC \\
i = 1 & -v_0 & +2v_1 & -v_2 & = & h^2 \\
i = 2 & -v_1 & +2v_2 & -v_3 & = & h^2 \\
 & \vdots & & & & \\
i = n & -v_{n-1} & +2v_n & -v_{n+1} & = & h^2
\end{array}$$

discretizing $v'(1) = 0$ using backwards difference gives

$0 = v'(1) \approx \frac{v_{n+1} - v_n}{h}$ thus $v_{n+1} = v_n$, then the interior points with boundary conditions are

$$\begin{array}{llllll}
i = 0 & v_0 = 0 & & \text{from} & & BC \\
i = 1 & & +2v_1 & -v_2 & = & h^2 \\
i = 2 & -v_1 & +2v_2 & -v_3 & = & h^2 \\
 & \vdots & & & & \\
i = n & -v_{n-1} & +v_n & & = & h^2
\end{array}$$

which corresponds to the matrix

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & 0 & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} h^2 \\ h^2 \\ \vdots \\ h^2 \end{bmatrix}$$

then we can solve using MATLAB

C o D e

$$U_{xx} + U_{yy} + 2U = 0, \quad 0 < x < 1, \ 0 < y < 1$$

$U(x,y) = 0$ on top, left, right $\quad U(x,y) = \sin(\pi x)$ for $y = 0$

Let's start by discretizing $U_{xx} + U_{yy} + 2U = 0$

$$U_{xx}(x_i, y_j) + U_{yy}(x_i, y_j) + 2U(x_i, y_j) = 0$$

where this is discretized in a plane that is $[0,1]$ and $[0,1]$ and for simplicity, with same size squares of size $h \times h$

then when $i = 0, 1, \dots, n$, and using
$j = 0, 1, \dots, n$

$$U_{xx} + U_{yy} \approx \frac{1}{h^2}\left[U(x+h,y) + U(x-h,y) - 4U(x,y) + U(x,y+h) + U(x,y-h)\right]$$

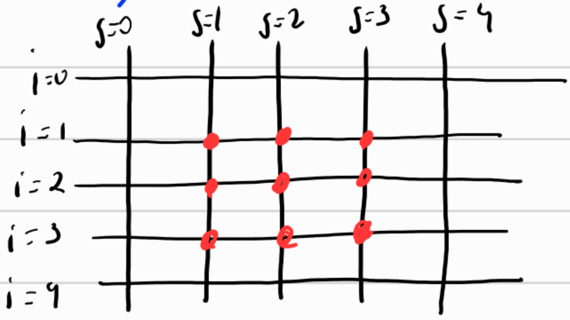
then

$$U_{xx} + U_{yy} + 2U = 0 \approx \frac{1}{h^2}\left[U(x+h,y) + U(x-h,y) - 4U(x,y) + U(x,y+h) + U(x,y-h)\right] + 2U$$

then writting in terms of i gives

$$\frac{1}{h^2}\left[U_{i+1,j} + U_{i-1,j} - 4U_{ij} + U_{i,j+1} + U_{i,j-1}\right] + 2U_{ij} = 0$$

with boundary conditions $U_{0,j} = 0$, $U_{x_f,j} = 0$, $U_{i\,y_f} = 0$
where x_f represents the last point in x direction and y_f, the last point in y direction.
and the last boundary condition is
$U_{i0} = \sin(\pi x_i)$



$i=1,\ j=1$

$$\frac{1}{h^2}\left[u_{21}+u_{01}-4u_{11}+u_{12}+u_{10}\right]+2u_{11}=0$$

$i=1,\ j=2$

$$\frac{1}{h^2}\left[u_{22}+u_{02}-4u_{12}+u_{13}+u_{11}\right]+2u_{12}=0$$

$i=1,\ j=3$

$$\frac{1}{h^2}\left[u_{23}+u_{03}-4u_{13}+u_{14}+u_{12}\right]+2u_{13}=0$$

$i=2,\ j=1$

$$\frac{1}{h^2}\left[u_{31}+u_{11}-4u_{21}+u_{22}+u_{20}\right]+2u_{21}=0$$

$i=2,\ j=2$

$$\frac{1}{h^2}\left[u_{32}+u_{12}-4u_{22}+u_{23}+u_{21}\right]+2u_{22}=0$$

$i=2,\ j=3$

$$\frac{1}{h^2}\left[u_{33}+u_{13}-4u_{23}+u_{24}+u_{22}\right]+2u_{23}=0$$

$i=3,\ j=1$

$$\frac{1}{h^2}\left[u_{41}+u_{21}-4u_{31}+u_{32}+u_{30}\right]+2u_{31}=0$$

$i=3,\ j=2$

$$\frac{1}{h^2}\left[u_{42}+u_{22}-4u_{32}+u_{33}+u_{31}\right]+2u_{32}=0$$

$i=3,\ j=3$

$$\frac{1}{h^2}\left[u_{43}+u_{23}-4u_{33}+u_{34}+u_{32}\right]+2u_{33}=0$$