Homework 4

MSSC 6010- Computational Probability

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Question 1. (4.4.11) From book. Traffic volume is an important factor for determining the most cost-effective method to surface a road. Suppose that the average number of vehicles passing a certain point on a road is 2 every 30 seconds.

- *(a)* Find the probability that more than 3 cars will pass the point in 30 seconds.

Since we are looking at how many times an event happens in a fixed amount of time, we are looiking at a Poisson Distribution. The event X is when the number of cars that passes every 30 second, and $E[X] = 2 = \lambda$ in Poisson.

$$X \sim \text{Pois}(2), t = \text{ 30 seconds}$$
$$P(X > 3) = 1 - P(X \leq 3)$$

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1-ppois(3,2)
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[1] 0.1428765
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- *(b)* What is the probability that more than 10 cars pass the point in 3 minutes?

$$X \sim \text{Pois}(2(6t)) = X \sim \text{Pois}(12)$$
$$P(x > 10) = 1 - P(X \leq 10)$$

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1-ppois(10,12)
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[1] 0.6527706
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Question 2. (4.4.17) From book. Derive the mean and variance for the discrete uniform distribution. (Hints: $\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$; $\sum_{i=1}^{n} x_i^2 = \frac{n(n+1)(2n+1)}{6}$, when $x = 1, 2, ..., n$.)

$$P(X = x_i | n = n) = \frac{1}{n}, i = 1, 2, ..., n$$

First, let's find the mean

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} x_i (\frac{1}{n}) \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i \\
&= \frac{1}{n} \frac{n(n+1)}{2} \\
&= \frac{n+1}{2}
\end{aligned}$$

Now, the variance

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= \sum_{i=1}^{n} x_i^2 (\frac{1}{n}) - \left(\frac{n+1}{2}\right)^2 \\
&= (\frac{1}{n}) \sum_{i=1}^{n} x_i^2 - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{2} \\
&= \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} \\
&= \frac{n^2 - 1}{12}
\end{aligned}$$

Question 3. (4.4.18) From book. Suppose the percentage of drinks sold from a vending machine are 80 and 20 for soft drinks and bottled water, respectively.

- *(a)* What is the probability that on a randomly selected day the first soft drink is the fourth drink sold?

Drinks are either soft drink or water. Which are mutually exclusive and exhaustive. Which would make it a Bernoulli trial. Since we want the probability of a number of failures before the r^{th} success, we can use Negative Binomial.

$$X \sim NB(r, \pi) \Rightarrow \mathbb{P}(X = 3|1, 0.8)$$

```
dnbinom(x = 3, size = 1, prob = 0.8)
```

```
[1] 0.0064
```

- *(b)* Find the probability that exactly 1 out of 10 drinks sold is a soft drink.

Now, we can use a Binomial distribution to find this probability. Since it does not matter when the success happens, but only that there is one success out of 10 trials.

$$X \sim Bin(n, \pi) \Rightarrow \mathbb{P}(X = 1|10, 0.8)$$

```
dbinom(x = 1, size = 10, prob = 0.8)
```

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[1] 4.096e-06
```

- *(c)* Find the probability that the fifth soft drink is the seventh drink sold

Use Negative Binomial again, were $r = 5$ is the number of successes, and $x = 7 - r = 2$ the number of failures

$$X \sim NB(r, \pi) \Rightarrow \mathbb{P}(X = 2|5, 0.8)$$

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dnbinom(x = 2, size= 5, 0.8)
```

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[1] 0.196608
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- *(d)* Verify empirically that $\mathbb{P}(Bin(n, \pi) \leq r - 1) = 1 - \mathbb{P}(NB(r, \pi) \leq (n - r))$, with $n = 10, \pi = 0.8$, and $r = 4$.

To verify, we just look at the values

```r
n = 10
pi = 0.8
r = 4

sum(dbinom(x = seq(0,r-1), size = n, prob = pi))
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[1] 0.0008643584
```

```r
1-sum(dnbinom(x = seq(0,n-r), size = r, prob = pi))
```

```
[1] 0.0008643584
```

As we can see, those are the same values.

Question 4. (4.4.38) From book. Consider the function $g(x) = (x-a)^2$, where a is a constant and $E[(X-a)^2]$ is finite. Find a so that $E[(X-a)^2] is minimized.

$$E[(X-a)^2] = \sum_{\forall x}(x-a)^2 p(x)$$

To minimize, we take the derivative with respect to a, and set it equal to 0

$$\begin{aligned}
0 &= \frac{d}{da}\sum_{\forall x}(x-a)^2 p(x) \\
&= \sum_{\forall x} -2(x-a)p(x) \\
&= E[-2(x-a)] \\
&= -2E[(x-a)] \\
&= E[(x-a)] \\
&= E[x] - E[a] \\
\Rightarrow E[x] &= E[a], \text{ since a is a constant} \\
a &= E[x]
\end{aligned}$$

Question 5. (4.4.40) From book. If $X \sim Bin(n, \pi)$, use the binomial expansion to find the mean and variance of X. To find the variance, use the second factorial moment $E[X(X-1)]$ and note that $\frac{x}{x!} = \frac{1}{(x-1)!}$ when $x > 0$

First, let's find the mean.

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \left(\frac{n!}{x!(x-n!)} \right) \pi^x (1-\pi)^{n-x} \\
&= \sum_{x=0}^{n} \left(\frac{n!}{(x-1)!(x-n!)} \right) \pi^x (1-\pi)^{n-x} \\
&= n \sum_{x=0}^{n} \left(\frac{(n-1)!}{(x-1)!(x-n!)} \right) \pi^x (1-\pi)^{n-x} \\
&= n \sum_{x=0}^{n} \binom{n-1}{x-1} \pi^x (1-\pi)^{n-x} \\
&= n\pi \sum_{x=0}^{n} \binom{n-1}{x-1} \pi^{x-1} (1-\pi)^{n-x} \\
&= n\pi \sum_{x=0}^{n-1} \binom{n-1}{x} \pi^{x-1+1} (1-\pi)^{n-1-x}
\end{aligned}
$$

Then, we can use the binomial theorem to help the simplification. [1]

$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} \\
(x+y)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k}
\end{aligned}
$$

Which makes it simplifies to

$$
\begin{aligned}
E[X] &= n\pi \sum_{x=0}^{n-1} \binom{n-1}{x} \pi^{x-1+1} (1-\pi)^{n-x+1} \\
&= n\pi(\pi + (1-\pi))^{n-1} \\
&= n\pi(1)^{n-1} \\
&= n\pi
\end{aligned}
$$

Now variance. First lets compute $E[X(X-1)]$

[1] I got stuck on this part and searched how to go from there and found the following website. I used the third answer to the question. https://math.stackexchange.com/questions/3400388/proving-that-the-expectation-of-a-binomial-random-variable-is-np.

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\left(\frac{n!}{x!(x-n!)}\right)\pi^x(1-\pi)^{n-x} \\
&= \sum_{x=0}^{n}\left(\frac{n!}{(x-2)!(x-n!)}\right)\pi^x(1-\pi)^{n-x} \\
&= n(n-1)\sum_{x=0}^{n}\left(\frac{(n-2)!}{(x-2)!(x-n!)}\right)\pi^x(1-\pi)^{n-x} \\
&= n(n-1)\sum_{x=0}^{n}\binom{n-2}{x-2}\pi^x(1-\pi)^{n-x} \\
&= n(n-1)\pi^2\sum_{x=0}^{n}\binom{n-2}{x-2}\pi^{x-2}(1-\pi)^{n-x} \\
&= n(n-1)\pi^2\sum_{x=0}^{n-2}\binom{n-2}{x}\pi^{x-2+2}(1-\pi)^{n-2-x} \\
&= n(n-1)\pi^2(\pi(1-pi))^{n-2}\text{, by binomial theorem} \\
&= n(n-1)\pi^2
\end{aligned}
$$

Then computing the variance gives

$$
\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= E[X^2 - X + X] - E[X]^2 \\
&= E[X(X-1)] + E[X] - E[X]^2 \\
&= E[X(X-1)] + E[X](1 - E[X]) \\
&= n(n-1)\pi^2 + n\pi(1-n\pi) \\
&= (n\pi)^2 - n\pi^2 + n\pi - (n\pi)^2 \\
&= -n\pi^2 + n\pi \\
&= n\pi(1-\pi)
\end{aligned}
$$