MATH 4931 - MSSC 5931 Homework 4

1. Some basic properties of eigenvalues. Show the following:

a) The eigenvalues of A and A^T are the same.

b) A is invertible if and only if A does not have a zero eigenvalue.

c) If the eigenvalues of A are $\lambda_1, \dots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \dots, 1/\lambda_n$.

d) The eigenvalues of A and $T^{-1}AT$ are the same.

Hint: you'll need to use the facts that $\det A = \det(A^\mathsf{T})$, $\det(AB) = \det A \det B$, and, if A is invertible, $\det A^{-1} = 1/\det A$.

2. Detecting linear relations. Suppose we have N measurements $y_1, \dots, y_N$ of a vector signal $x_1, \dots, x_N \in \mathbb{R}^n$:

$$y_i = x_i + d_i, \ i = 1, \dots, N.$$

Here d_i is some small measurement or sensor noise. We hypothesize that there is a linear relation among the components of the vector signal x, *i.e.*, there is a nonzero vector q such that $q^\mathsf{T} x_i = 0$, $i = 1, \dots, N$. The geometric interpretation is that all of the vectors x_i lie in the hyperplane $q^\mathsf{T} x = 0$. We will assume that $\|q\| = 1$, which does not affect the linear relation. Even if the x_i's do lie in a hyperplane $q^\mathsf{T} x = 0$, our measurements y_i will not; we will have $q^\mathsf{T} y_i = q^\mathsf{T} d_i$. These numbers are small, assuming the measurement noise is small. So the problem of determing whether or not there is a linear relation among the components of the vectors x_i comes down to finding out whether or not there is a unit-norm vector q such that $q^\mathsf{T} y_i$, $i = 1, \dots, N$, are all small. We can view this problem geometrically as well. Assuming that the x_i's all lie in the hyperplane $q^\mathsf{T} x = 0$, and the d_i's are small, the y_i's will all lie close to the hyperplane. Thus a scatter plot of the y_i's will reveal a sort of flat cloud, concentrated near the hyperplane $q^\mathsf{T} x = 0$. Indeed, for any z and $\|q\| = 1$, $|q^\mathsf{T} z|$ is the distance from the vector z to the hyperplane $q^\mathsf{T} x = 0$. So we seek a vector q, $\|q\| = 1$, such that all the measurements $y_1, \dots, y_N$ lie close to the hyperplane $q^\mathsf{T} x = 0$ (that is, $q^\mathsf{T} y_i$ are all small). How can we determine if there is such a vector, and what is its value? We define the following normalized measure:

$$\rho = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (q^\mathsf{T} y_i)^2} \Bigg/ \sqrt{\frac{1}{N} \sum_{i=1}^{N} \|y_i\|^2}.$$

This measure is simply the ratio between the *root mean square distance* of the vectors to the hyperplane $q^\mathsf{T} x = 0$ and the *root mean square length* of the vectors. If ρ is small, it means that the measurements lie close to the hyperplane $q^\mathsf{T} x = 0$. Obviously, ρ depends on q. Here is the problem: explain how to find the minimum value of ρ over all unit-norm vectors q, and the unit-norm vector q that achieves this minimum, given the data set $y_1, \dots, y_N$.

3. Properties of symmetric matrices. In this problem P and Q are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

a) If $P \geq 0$ then $P + Q \geq Q$.

b) If $P \geq Q$ then $-P \leq -Q$.

c) If $P > 0$ then $P^{-1} > 0$.

d) If $P \geq Q > 0$ then $P^{-1} \leq Q^{-1}$.

e) If $P \geq Q$ then $P^2 \geq Q^2$.

Hint: you might find it useful for part (d) to prove $Z \geq I$ implies $Z^{-1} \leq I$.

4. Real modal form. Generate a matrix A in $\mathbb{R}^{10\times 10}$ using `A=rnorm(10)`. (The entries of A will be drawn from a unit normal distribution.) Find the eigenvalues of A. If by chance they are all real, please generate a new instance of A. Find the real modal form of A, *i.e.*, a matrix S such that $S^{-1}AS$ has the real modal form given in Topic 7 . Your solution should include a clear explanation of how you will find S, the source code that you use to find S, and some code that checks the results (*i.e.*, computes $S^{-1}AS$ to verify it has the required form).

5. Spectral mapping theorem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is analytic, *i.e.*, given by a power series expansion

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots$$

(where $a_i = f^{(i)}(0)/(i!)$). (You can assume that we only consider values of u for which this series converges.) For $A \in \mathbb{R}^{n\times n}$, we define $f(A)$ as

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots$$

(again, we'll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbb{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the *spectral mapping theorem.*

To illustrate this with an example, generate a random 3×3 matrix, for example using `A=rnorm(3)`. Find the eigenvalues of $(I + A)(I - A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various computations.)

6. Square matrices and the SVD.. Let A be an $n \times n$ real matrix. State whether each of the following statements is true or false. Do not give any explanation or show any work.

a) If x is an eigenvector of A, then x is either a left or right singular vector of A

b) If λ is an eigenvalue of A, then $|\lambda|$ is a singular value

c) If A is symmetric, then every singular value of A is also an eigenvalue of A

d) If A is symmetric, then every singular vector of A is also an eigenvector of A

e) If A is symmetric with the following singular value decomposition

$$A = U\Sigma V^T$$

then $U = V$

f) If A is invertible, then

$$\sigma_i \neq 0 \quad \text{for all } i = 1, \ldots, n$$