Exploring the Dynamics of Waves: An Insight into the Wave Equation

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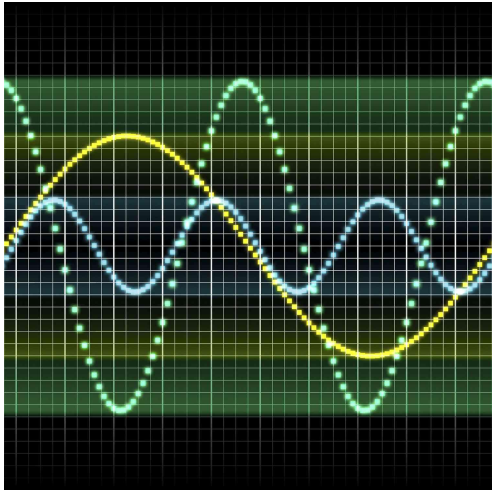
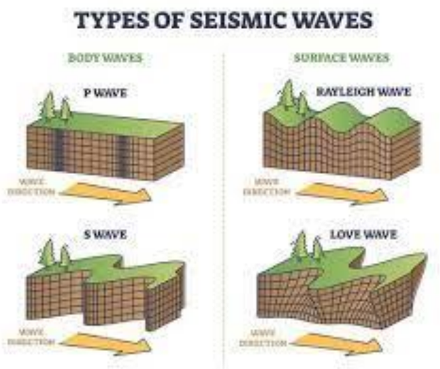
Introduction to Waves

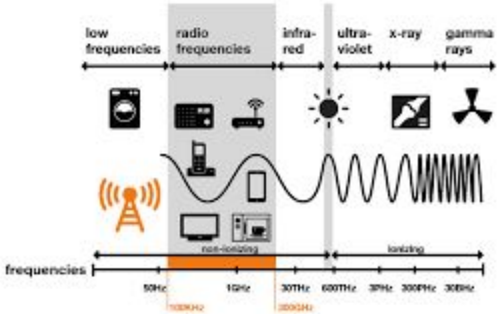
Definition of Waves: Waves are oscillations that transfer energy through space. They are prevalent in nature and can manifest in various forms, from the ripples on a pond to the vibrations of a guitar string.



Importance of Understanding Waves

Applications: Waves play a crucial role in diverse fields such as physics, engineering, acoustics, and beyond.





Overview of the Wave Equation

The wave equation is a fundamental mathematical tool that describes the behavior of waves. It provides a concise framework for understanding how waves propagate. And can be mathematically represented by

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = f$$

One-Dimensional Wave Equation

In its simplest form, the one-dimensional wave equation is expressed as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} 2u + f$$

where u represents the wave function, t is time, x is spatial position, and c is the wave speed, and f is some disturbance.

Discretizing the Wave Equation

To solve the wave equation numerically, we begin by discretizing it. This involves breaking down the continuous wave equation into a finite set of points in both time and space.

$$\frac{1}{(\Delta t)^2}\left(u_{i,j+1} - 2u_{ij} + u_{i,j-1}\right) = \frac{c^2}{(\Delta x)^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right) + f_i$$

$$\Rightarrow \quad u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + f_i$$

Boundary Conditions - Reflect

Boundary conditions are essential for defining the behavior of waves and finding a solution. We'll explore two types: absorbing and reflecting.

Reflecting Boundary: Causes waves to bounce back into the domain, emulating a closed or reflective boundary.

$$u(0, t) = 0 \qquad u(x, t) = 0$$

Boundary Conditions - Absorb

Absorbing Boundary: Allows waves to exit the computational domain without reflecting back, simulating an open boundary.

$$\frac{\partial u}{\partial x}(0,t) = c\frac{\partial u}{\partial t}(0,t) \qquad \frac{\partial u}{\partial x}(x,t) = c\frac{\partial u}{\partial t}(x,t)$$

Which can be discretized to

$$u_{1,j+1} = u_{2,j} + \frac{CFL-1}{CFL+1}(u_{2,j+1} - u_{1,j})$$

$$u_{n,j+1} = u_{n+1,j} + \frac{CFL-1}{CFL+1}(u_{n+1,j+1} - u_{n,j})$$

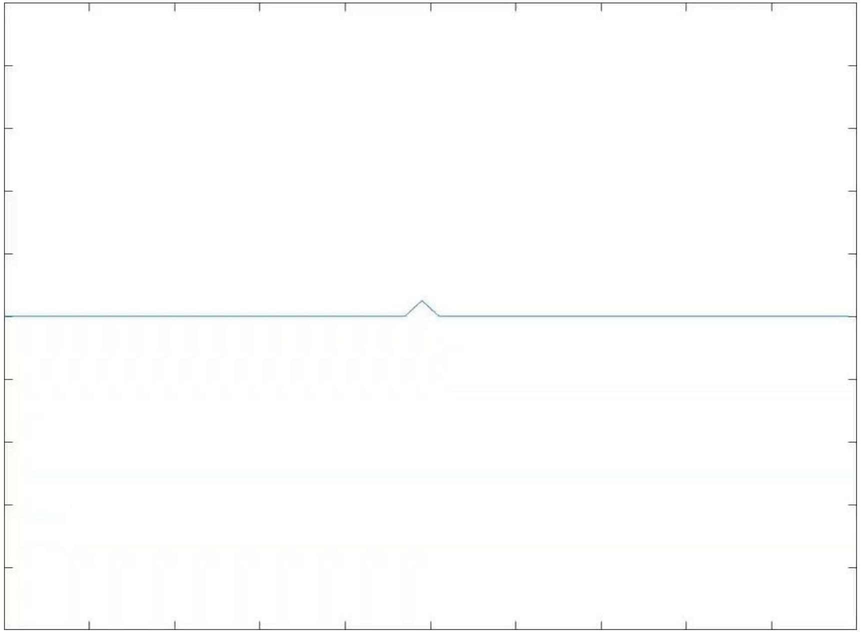
Denote CFL = $\frac{c\Delta t}{\Delta x}$

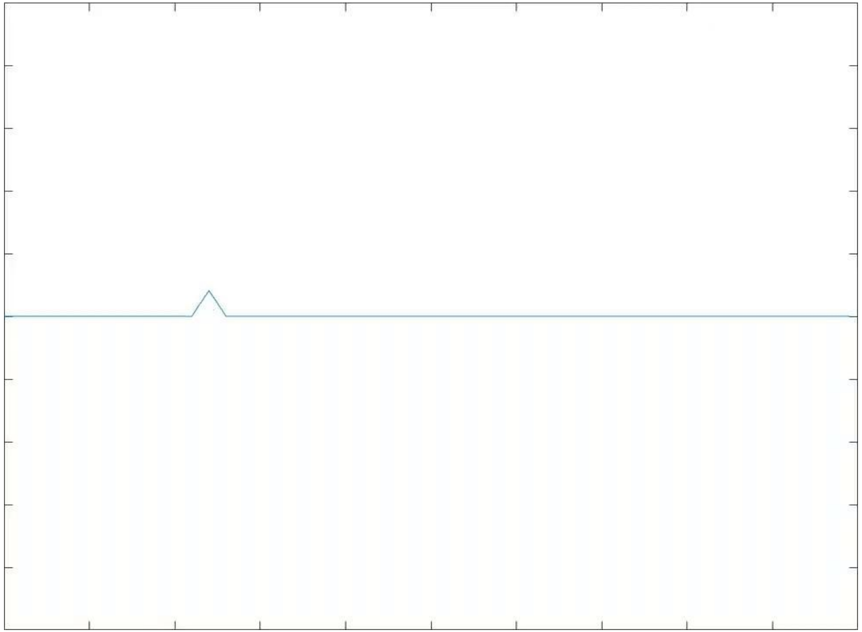
Real-World Examples of Boundary Conditions

Absorbing Boundary Example: In oceanography, absorbing boundaries mimic waves dissipating as they reach the shore, providing accurate simulations of coastal regions.

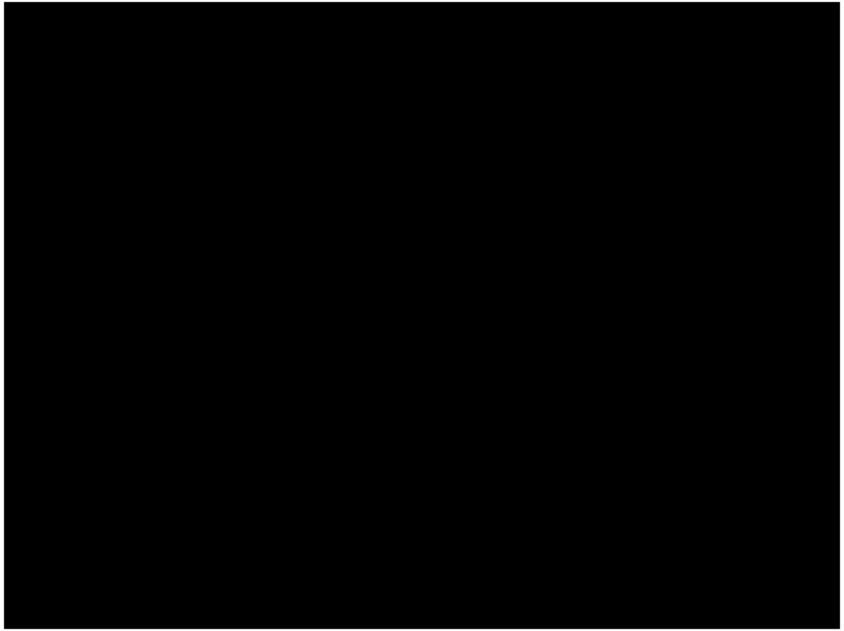
Reflecting Boundary Example: In structural engineering, reflecting boundaries simulate waves bouncing back from structures, aiding in the study of wave interactions with buildings.

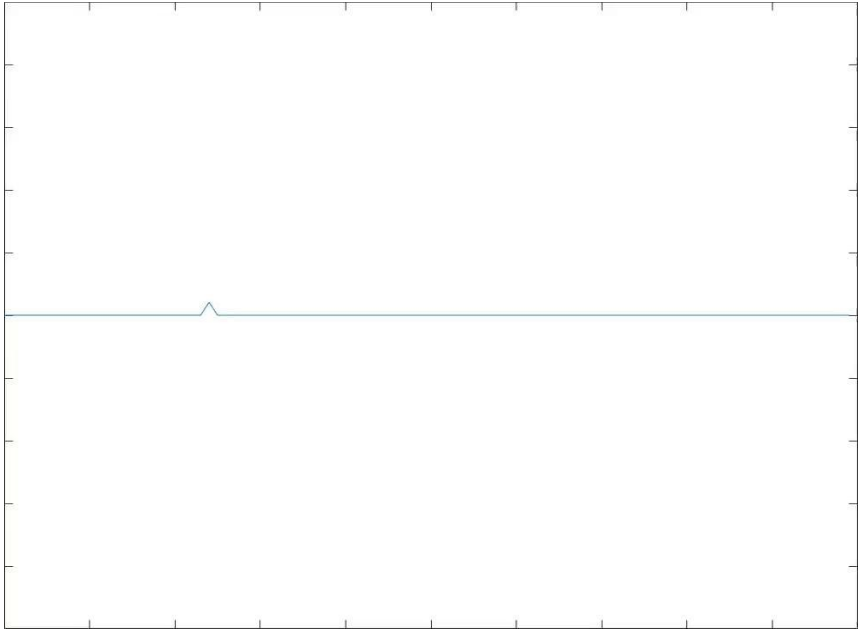
Simulations - 1D Wave Equation - Reflect





Simulations - 1D Wave Equation - Absorb





Extending to 2D

Extend our understanding to two dimensions, where waves propagate in both x and y directions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} + f$$

Discretizing the 2D wave equation

The equation in the previous slide can be discretized to

$$\frac{1}{(\Delta t)^2}\left(u_{ij}^{(n+1)} - 2u_{ij}^{(n)} + u_{ij}^{(n-1)}\right) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1,j}^{(n)} - 2u_{ij}^{(n)} + u_{i-1,j}^{(n)}\right)$$
$$+ \left(\frac{c\Delta t}{\Delta y}\right)^2 \left(u_{i,j+1}^{(n)} - 2u_{ij}^{(n)} + u_{i,j-1}^{(n)}\right)$$

Discretizing the 2D wave equation

And for simplicity, we can let $\Delta x = \Delta y$, then

$$u_{ij}^{(n+1)} = 2u_{ij}^{(n)} - u_{ij}^{(n-1)}$$
$$+ \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1,j}^{(n)} + u_{i-1,j}^{(n)} - 4u_{ij}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right)$$
$$+(\Delta t)^2 f_i^{(n)}$$

Boundary Conditions - Reflect

And the boundary conditions are very similar

$$u(0, y, t) = 0 \qquad u(x_f, y, t) = 0$$
$$u(x, 0, t) = 0 \qquad u(x, y_f, t) = 0$$

Where the subscript f means the final point in that direction

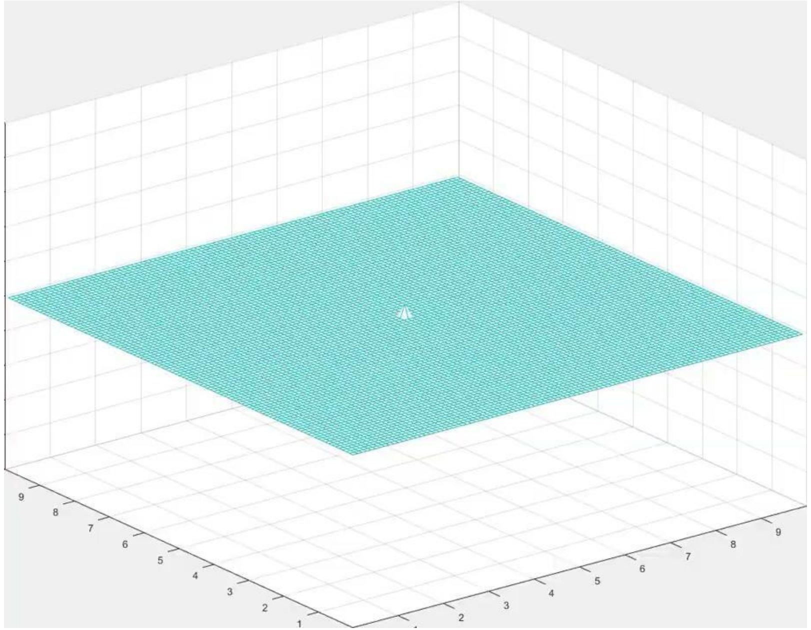
Boundary Conditions - Absorb

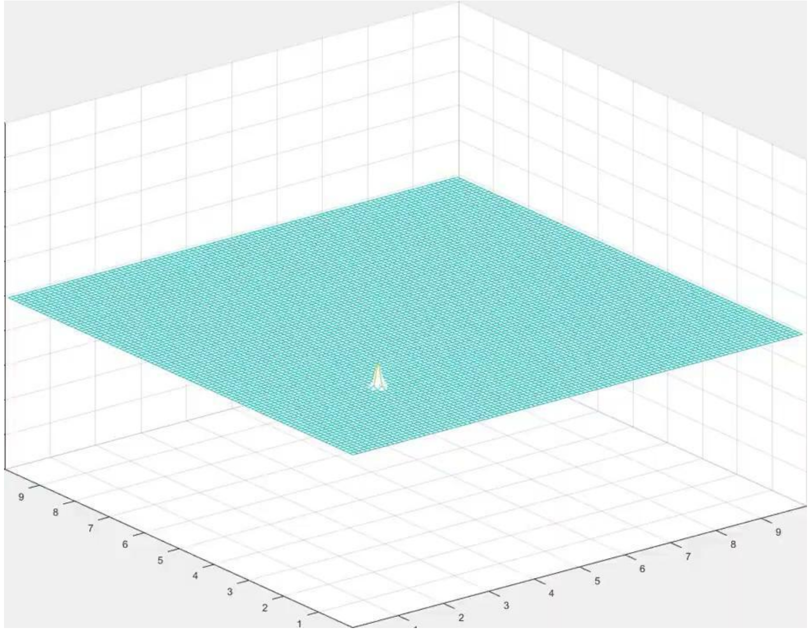
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -c\left.\frac{\partial u}{\partial t}\right|_{x=0} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=x_f} = -c\left.\frac{\partial u}{\partial t}\right|_{x=x_f}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = -c\left.\frac{\partial u}{\partial t}\right|_{y=0} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=y_f} = -c\left.\frac{\partial u}{\partial t}\right|_{y=y_f}$$

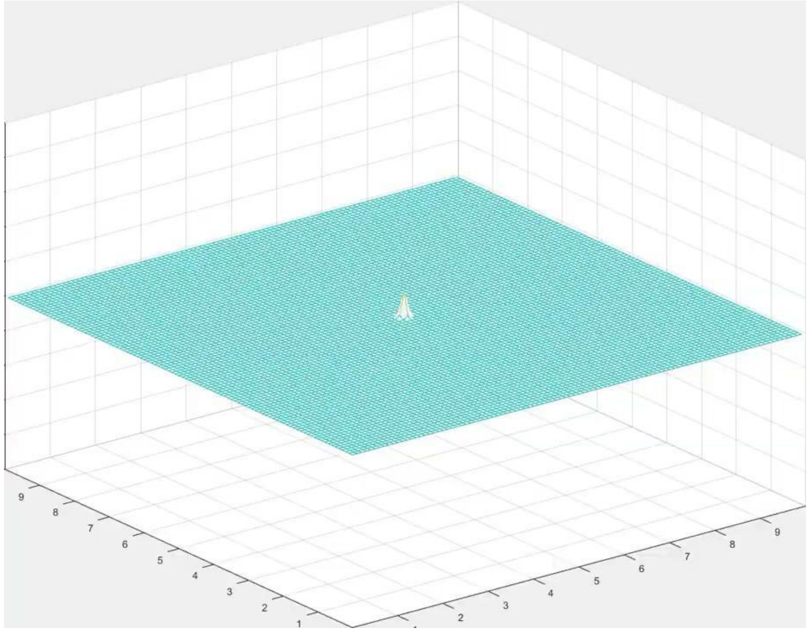
I did not include the discretization, since it follows the exact same form as in 1D, but with the same conditions in y as well.

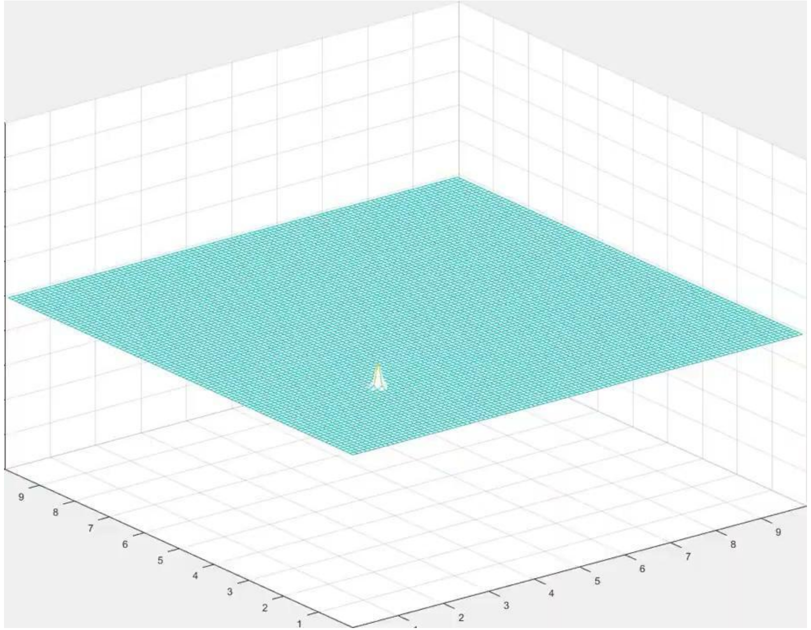
Simulations - 2D Wave Equation - Reflect





Simulations - 2D Wave Equation - Absorb





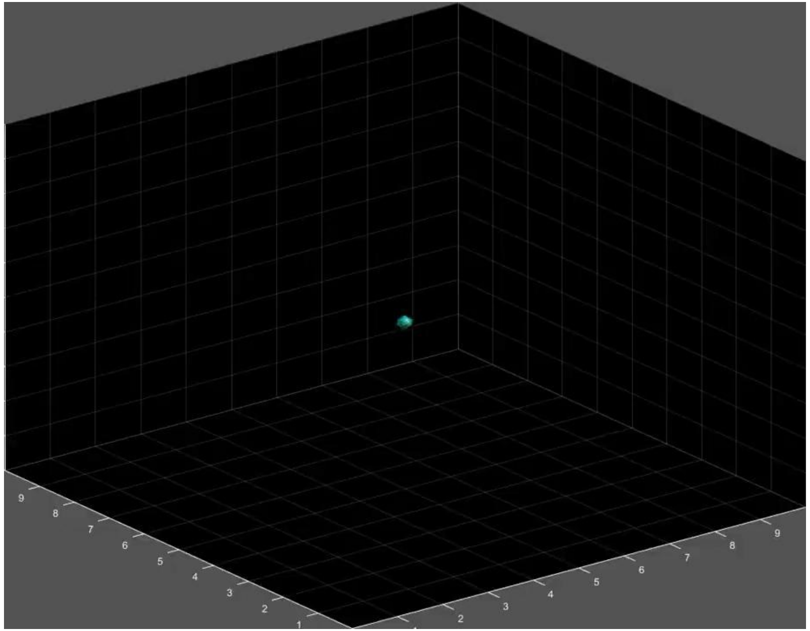
Extending to 3D

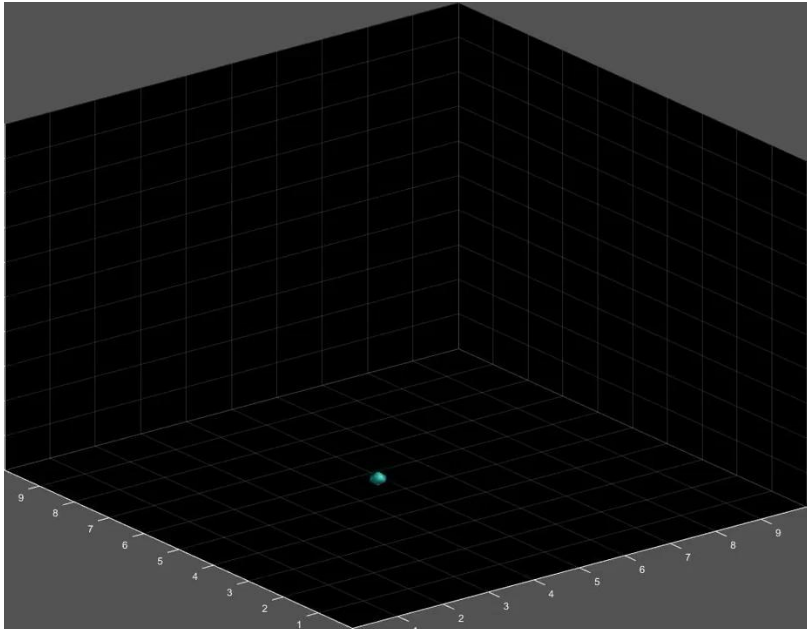
The equation can be extended in a similar fashion to 3D, or even ND space, but it is funny to think how waves would behave in higher dimensions and impossible to visualize.

Therefore, only simulations in 3D, since we can plot those.

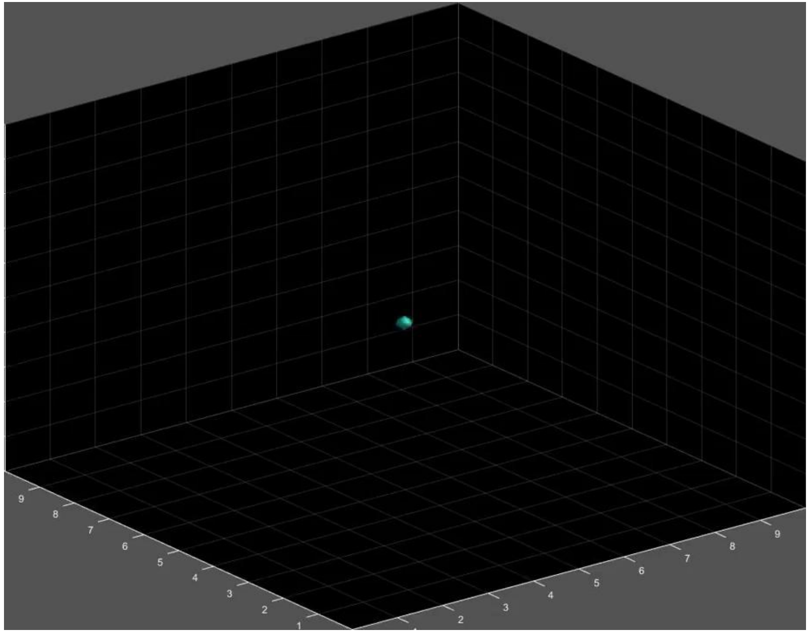
Math was also omitted, same idea with more superscripts and subscripts.

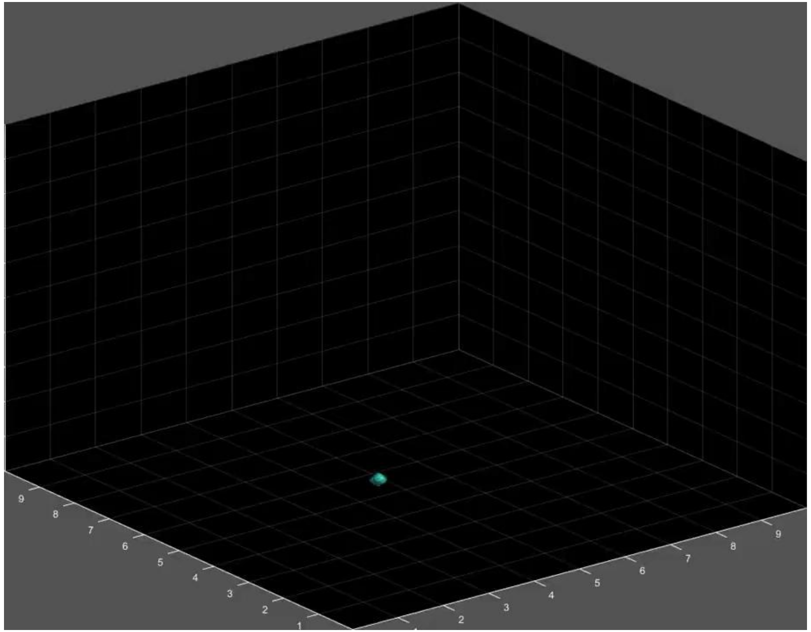
Simulations - 3D Wave Equation - Reflect





Simulations - 3D Wave Equation - Absorb





Sources

- https://core.ac.uk/download/pdf/158320152.pdf. Discretization of the Mur's boundary condition
- https://www.mathworks.com/matlabcentral/fileexchange/115205-wave3d. Plot 3D in Matlab
- Textbook for this class

Thank you!
Questions?