Homework 8

MSSC 6010- Computational Probability

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Question 1. (6.7.4) From book. If $X \sim \chi^2_{10}$, fin the constants a and b so that $\mathbb{P}(a < X < b) = 0.90$ and $\mathbb{P}(X < a) = 0.05$

$$X \sim \chi^2_{10} \Rightarrow f(x) \begin{cases} \frac{1}{\Gamma(5)2^5} x^4 e^{-5} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

because $\sqrt{2\chi^2_n} \sim N(\sqrt{2n-1}, 1), Y = \sqrt{2\chi^2_n} - \sqrt{2n-1} \sim N(0, 1)$, then

$$\mathbb{P}(\sqrt{2\chi^2_{10}} - \sqrt{2n-1} < \sqrt{2a} - \sqrt{10 * 2 - 1}) = 0.05$$
$$\mathbb{P}(Z < \sqrt{2a} - \sqrt{19})$$

```
qnorm(0.05)
```

```
[1] -1.644854
```

$$a = \frac{(-1.644854 + \sqrt{19})^2}{2}$$

```
(a<-(qnorm(0.05)+sqrt(19))**2/2)
```

```
[1] 3.683021
```

```
qchisq(0.05,10)
```

```
[1] 3.940299
```

$$
\begin{aligned}
\mathbb{P}(a < X < b) &= 0.90 \\
\mathbb{P}(\sqrt{2a} - \sqrt{19} < \sqrt{2\chi^2_{10}} - \sqrt{19} < \sqrt{2b} - \sqrt{19}) &= 0.9 \\
\mathbb{P}(\sqrt{2a} - \sqrt{19} < Z < \sqrt{2b} - \sqrt{19}) &= 0.9 \\
\mathbb{P}(Z < \sqrt{2b} - \sqrt{19}) - \mathbb{P}(Z < \sqrt{2a} - \sqrt{19}) &= 0.9 \\
\mathbb{P}(Z < \sqrt{2b} - \sqrt{19}) - 0.05 &= 0.9 \\
\mathbb{P}(Z < \sqrt{2b} - \sqrt{19}) &= 0.95
\end{aligned}
$$

```
b<-(qnorm(0.95)+sqrt(19))**2/2
pchisq(b,10) - pchisq(a,10)
```

```
[1] 0.9059308
```

Question 2. (6.7.11) From book. Given a random sample of size 6 from N(0,σ) calculate

- *(a)* $\mathbb{P}(\frac{\bar{X}}{S} > 2)$ and

$$
\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}
$$

$$
\begin{aligned}
\mathbb{P}(\frac{\bar{X} - 0}{S/\sqrt{n}} > 2) &= \mathbb{P}(\frac{\bar{X}}{S} > 2\sqrt{6}) \\
&= 1 - \mathbb{P}(\frac{\bar{X}}{S} < 2\sqrt{6}) \sim t
\end{aligned}
$$

```
1-pt(2*sqrt(6),5)
```

```
[1] 0.002239216
```

- *(b)* $\mathbb{P}(|\frac{\bar{X}}{S_u}| \leq 2)$.

$$
S_u^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2 \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2
$$

$$
\Rightarrow S_u\sqrt{\frac{n}{n-1}} = S
$$

$$
\begin{aligned}
\mathbb{P}\left(\left|\frac{\bar{X}}{\frac{S_u\sqrt{n/n-1}}{\sqrt{n}}}\right| \leq 4\right) &= \mathbb{P}\left(\left|\frac{\bar{X}}{S_u\sqrt{n-1}}\right| \leq 4\right) \\
&= \mathbb{P}(-4 \leq \frac{\bar{X}}{S_u\sqrt{5}} \leq 4) \\
&= \mathbb{P}(-4\sqrt{5} \leq \frac{\bar{X}}{S_u} \leq 4\sqrt{5}) \\
&= \mathbb{P}(\frac{\bar{X}}{S_u} \leq 4\sqrt{5}) - \mathbb{P}(\frac{\bar{X}}{S_u} \leq -4\sqrt{5})
\end{aligned}
$$

```
pt(4*sqrt(5), 5) - pt(-4*sqrt(5), 5)
```

```
[1] 0.9997089
```

Question 2. (7.4.14) From book. The following random samples X and Y are drawn from Pois(λ) and Pois(2λ), respectively:

```
x_lambda <- c(4,2,5,7,3,4,3)
y_2lambda <- c(6,10,1,6,3,5,5,4,7,5)
```

- *(a)*Derive the maximum likelihood estimator of λ and calculate its variance.

$$
\begin{aligned}
L(\lambda|x) &= \Pi_{i=1}^{n}[\frac{1}{x_i!}(\lambda t)^{x_i}e^{-\lambda t}] \\
\frac{\partial ln(L(\lambda|x))}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\sum_{i=1}^{n}[ln(\frac{1}{x_i!}(\lambda t)^{x_i}e^{-\lambda t}) \\
&= \frac{\partial}{\partial\lambda}\sum_{i=1}^{n}[ln(\frac{1}{x_i!}) + ln(\lambda t)^{x_i} - \lambda t) \\
&= \sum_{i=1}^{n}[\frac{x_i}{\lambda} - t] \\
&= \frac{1}{\lambda}\sum_{i=1}^{n}x_i - nt
\end{aligned}
$$

and $\frac{\partial(ln(L(\lambda|x)))^2}{\partial\lambda^2} = \frac{-1}{\lambda}\sum_{i=1}^{n}x_i < 0$, since all i observations in poisson are positive integers. Now setting it equal to 0 and solving for λ

$$
0 = \frac{1}{\lambda}\sum_{i=1}^{n}x_i - ntnt\lambda = \sum_{i=1}^{n}x_i\hat{\lambda} = \frac{\sum_{i=1}^{n}x_i}{nt}
$$

then variance

$$\begin{aligned}
Var(\hat{\lambda}) &= Var(\frac{\sum_{i=1}^{n} x_i}{nt}) \\
&= \frac{1}{(nt)^2} Var(\sum_{i=1}^{n} x_i) \\
&= \frac{1}{(nt)^2} \sum_{i=1}^{n} Var(x_i) \\
&= \frac{n}{(nt)^2} Var(x) \\
&= \frac{1}{nt^2} Var(x) \\
&= \frac{\lambda}{nt^2}
\end{aligned}$$

- *(b)* Compute the maximum likelihood estimate of λ and its variance using the two random samples given.

Now, since we are using both variables, they will have different "weights", so that $tn = t_1 n_1 + t_2 n_2$ where we have one t and one n for each distribution, and the sum should add both x and y values.

$$\hat{\lambda} = \frac{\sum_{i=1}^{7} x_i + \sum_{i=1}^{10} y_i}{(n_1 t_1 + n_2 t_2)}$$

```
(sum(x_lambda)+sum(y_2lambda))/
  (1*length(x_lambda)+2*length(y_2lambda))
```

```
[1] 2.962963
```

And variance

$$var(\frac{\sum_{i=1}^{7} x_i + \sum_{i=1}^{10} y_i}{(n_1 t_1 + n_2 t_2)}) = \frac{1}{(n_1 t_1 + n_2 t_2)^2} (\sum_{i=1}^{7} x_i + \sum_{i=1}^{10} y_i)$$

```
1/(1*length(x_lambda)+2*length(y_2lambda))**2*(
  sum(x_lambda)+sum(y_2lambda))
```

```
[1] 0.1097394
```

Question 4. (7.4.31) Consider the density function

$$f(x) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}, 0 < x < 1, 0 < \theta < \infty$$

- *(a)* Derive the maximum likelihood estimator of θ for a random sample of size n.

$$
\begin{aligned}
L(\theta|x) &= \Pi_{i=1}^{n}\frac{1}{\theta}x^{\frac{1-\theta}{\theta}} \\
\frac{\partial ln(L(\theta|x))}{\partial\theta} &= \frac{\partial}{\partial\theta}\left(\sum_{i=1}^{n}\left[ln(\frac{1}{\theta}+(\frac{1}{\theta-1})ln(x_i)\right]\right) \\
&= \sum_{i=1}^{n}\left[\frac{-1}{\theta}-\frac{lnx_i}{\theta^2}\right] \\
\Rightarrow^{set=0}\ \frac{-n}{\theta}-\frac{1}{\theta^2}\sum_{i=1}^{n}lnx_i = 0 \\
-\frac{1}{\theta^2}\sum_{i=1}^{n}lnx_i &= \frac{n}{\theta} \\
\hat{\theta} &= \frac{-\sum_{i=1}^{n}lnx_i}{n}
\end{aligned}
$$

Checking second derivative $\frac{\partial^2}{\partial\theta^2}=\sum_{i=1}^{n}\left(\frac{1}{\theta^2}+\frac{2}{\theta^3}lnx_i\right)$, then evaluating at $\hat{\theta}$ gives

$$
\sum_{i=1}^{n}\left(\frac{1}{\left(\frac{-\sum_{i=1}^{n}lnx_i}{n}\right)^2}+\frac{2}{\left(\frac{-\sum_{i=1}^{n}lnx_i}{n}\right)^3}lnx_i\right)=\frac{n^3}{(\sum_{i=1}^{n}lnx_i)^2}-\frac{2n^3}{(\sum_{i=1}^{n}lnx_i)^3}\sum_{i=1}^{n}lnx_i<0
$$

- *(b)* Derive the method of moments estimator of θ for a random sample of size n.

$$
\begin{aligned}
m1=\alpha_1(\theta)=E[X] &= \int_0^1 x\frac{1}{\theta}x^{\frac{1-\theta}{\theta}}dx \\
&= \frac{1}{\theta}\int_0^1 x^{\frac{1}{\theta}}dx \\
&= \frac{1}{\theta}\frac{x^{\frac{1}{\theta}+1}}{\frac{1}{\theta}+1}\Bigg|_0^1 \\
&= \frac{1}{\theta(\frac{1}{\theta})+1} \\
E[X] &= \frac{1}{1+\theta} \\
\hat{\theta} &= \frac{1-E[X]}{E[X]}
\end{aligned}
$$

- *(c)* Show that the maximum likelihood estimator is unbiased.

In order to be unbiased $E[\theta] = \theta$

$$
\begin{aligned}
E[\hat{\theta}] &= E[\frac{-\sum_{i=1}^{n} lnx_i}{n}] \\
&= \frac{-1}{n}\sum_{i=1}^{n} E[lnx_i] \\
&= -E[lnx] \\
&= -\int_0^1 lnx\frac{1}{\theta}x^{\frac{1-\theta}{\theta}} \\
\text{by parts } &\Rightarrow \begin{matrix} u = lnx & dv = \frac{1}{\theta}x^{\frac{1-\theta}{\theta}} \\ du = \frac{1}{x} & v = x^{\frac{1}{\theta}} \end{matrix} \\
E[\hat{\theta}] &= -lnxx^{\frac{1}{\theta}} + \int_0^1 x^{\frac{1}{\theta}-1}dx \\
&= \left[-lnxx^{\frac{1}{\theta}} + \theta x^{\frac{1}{\theta}}\right]_0^1 \\
&= 0 + lim_{x\to 0^+}\frac{lnx}{1/x^{\frac{1}{\theta}}} + \theta \\
&= 0 + \theta
\end{aligned}
$$

Therefore, it is an unbiased estimator

Question 2. (7.4.36) From book. Consider the density function

$$f(x) = e^{-(x-\alpha)}, -\infty < \alpha \leq x$$

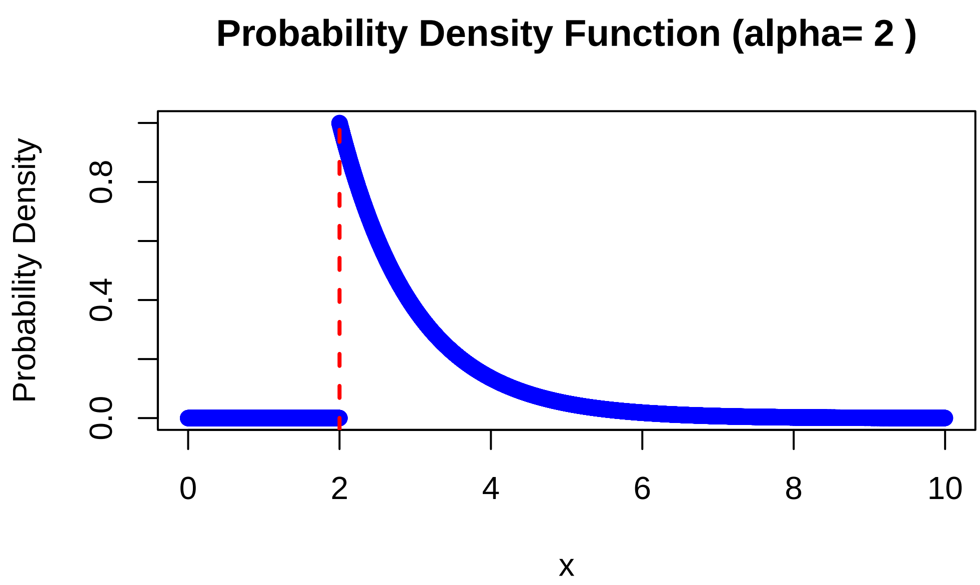
-*(a)* Find the maximum likelihood and method of moments estimators of α

The calculus approach is not going to work for the maximum likelihood, since once we that the natural log followed by the derivative, α is not going to be involved in the equation.

However, since $e^{-(x-\alpha)}$ is maximized when $\alpha = x$, and x changes from α to x then the MLE is $\hat{\alpha} = min(x_i)$

```
# Function to compute the PDF of the given distribution
pdf_distribution <- function(x, alpha) {
  result <- numeric(length(x))

  result[x > alpha] <- exp(-(x[x > alpha] - alpha))
  result[x < alpha] <- 0

  return(result)
}

# Set the parameter 'a'
```

```
alpha <- 2.0

# Generate x values
x_values <- runif(10000, 0, 10)

# Compute the PDF values
pdf_values <- pdf_distribution(x_values, alpha)

# Plot the PDF
plot(x_values, pdf_values, type='p', col='blue', lwd=1,
     main=paste('Probability Density Function (alpha=', alpha, ')'),
     xlim = c(0,10),
     xlab='x', ylab='Probability Density')
valid_indices <- x_values >= alpha
valid <- x_values[valid_indices]
abline(v = min(valid), lty=2, col="red", lwd = 2)
```



And the method of moments

$$
\begin{aligned}
m_1 = E[X] &= \int_\alpha^\infty x e^{-(x-\alpha)} dx \\
&= \left[-xe^{-(x-\alpha)}\right]_\alpha^\infty + \int_\alpha^\infty e^{-(x-\alpha)} dx \\
&= \alpha + \left[-e^{-(x-\alpha)}\right]_\alpha^\infty \\
E[X] &= \alpha + 1 \\
\hat{\alpha} &= E[X] - 1
\end{aligned}
$$

-*(b)* Are both estimators found in (a) asymptotically unbiased?

For the MLE $E[\hat{\alpha}] = min(x_i)$, and as $n \to \infty$, the change of getting x_i close to α goes to 1. Which maximizes the likelihood, then, it is an asymptotically unbiased estimator.

For method of moments

$$
\begin{aligned}
E[\hat{\alpha}] &= E[E[X] - 1] \\
&= E[X] - 1 \\
&= \hat{\alpha}
\end{aligned}
$$

Therefore, this is not an unbiased estimator. Furthermore

$$lim_{n \to \infty} E[X] - 1 = E[X] - 1$$

which means this is an asymptotically unbiased estimator