# Activity 4 Henri

EXERCISE H3: (Lesson 37 of PDE text) Use 2D Finite Differences to solve the following BVP

$$u_{xx} + u_{yy} + 2u = 0, \qquad 0 < x < 1, \ \ 0 < y < 1$$

subject to $u(x, y) = 0$ on the top, left, and ride sides of the square domain with $u(x, y) = \sin(\pi x)$ for $y = 0$ (i.e. the bottom of the square). Use 5 grid points (3 interior points) in each of the $x$ and $y$ directions. Code up your FD method into Matlab and plot the solution. Does your FD solution improve with a finer grid? Experiment with higher levels of discretization.

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Hw 3.) let's start by discretizing $U_{xx} + U_{yy}$ with uniform spacing in $x$ and $y$ $(h = k)$.

then $U_{xx} + U_{yy} = -4U(x,y) + U(x, y+h) + U(x, y-h) + U(x+h, y) + U(x-h, y)$

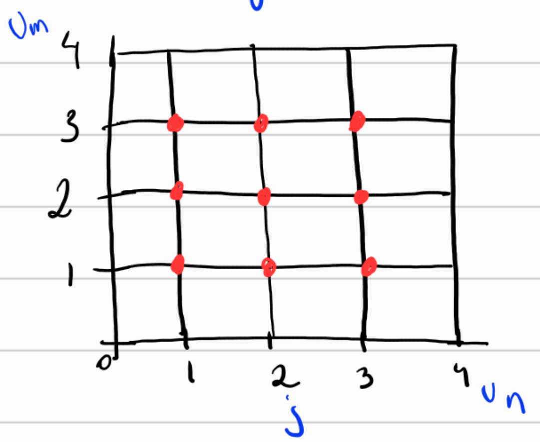
then let

$$U_{ij} = U(x_j, y_i)$$
$$U_{i+1,j} = U(x, y_{i+1}) = U(x, y+h)$$
$$U_{i-1,j} = U(x, y-h)$$
$$U_{i,j+1} = U(x+h, y)$$
$$U_{i,j-1} = U(x-h, y)$$

and let's plug it back to the PDE

$$-4U_{ij} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} + 2U_{ij} = 0$$
$$\Rightarrow \quad -2U_{ij} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} \ U_{i,j-1} = 0$$

and the grid will be




with BCs:

$$U_{mj} = 0 \qquad 1 \le j \le n$$
$$U_{in} = 0 \qquad U_{i0} = 0$$
$$U_{0j} = \sin \pi x_j$$

$i=1,\ j=1$

$$-2u_{11} + u_{21} + \overset{=\sin\pi x_1}{u_{01}} + u_{12} + \overset{=0}{u_{10}} = 0$$

$i=2,\ j=1$

$$-2u_{21} + u_{31} + u_{11} + u_{22} + \overset{=0}{u_{20}} = 0$$

$i=3,\ j=1$

$$-2u_{31} + \overset{=0}{u_{41}} + u_{21} + u_{32} + \overset{=0}{u_{30}} = 0$$

$i=1,\ j=2$

$$-2u_{12} + u_{22} + \overset{=\sin\pi x_2}{u_{02}} + u_{13} + u_{11} = 0$$

$i=2,\ j=2$

$$-2u_{22} + u_{32} + u_{12} + u_{23} + u_{21} = 0$$

$i=3,\ j=2$

$$-2u_{32} + \overset{=0}{u_{42}} + u_{22} + u_{33} + u_{31} = 0$$

$i=1,\ j=3$

$$-2u_{13} + u_{23} + \overset{=\sin\pi x_3}{u_{03}} + \overset{=0}{u_{14}} + u_{12} = 0$$

$i=2,\ j=3$

$$-2u_{23} + u_{33} + u_{13} + \overset{=0}{u_{24}} + u_{22} = 0$$

$i=3,\ j=3$

$$-2u_{33} + \overset{=0}{u_{43}} + u_{23} + \overset{=0}{u_{34}} + u_{32} = 0$$

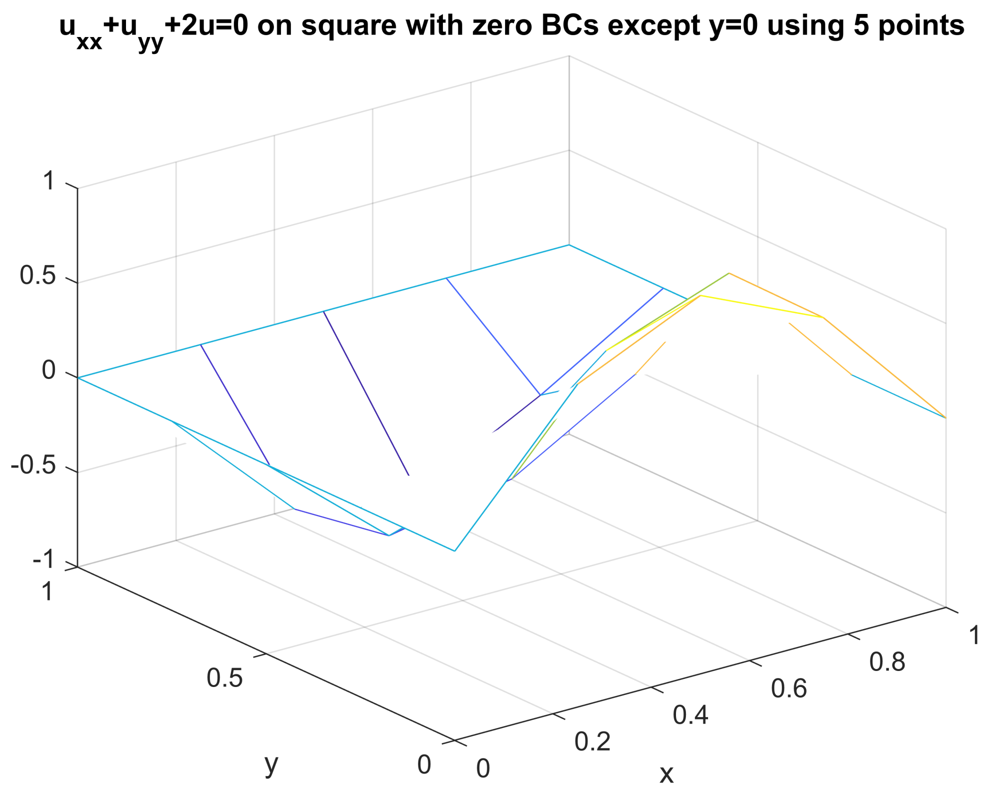
$$\underbrace{\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix} \sin x_1 \\ 0 \\ 0 \\ \sin x_2 \\ 0 \\ 0 \\ \sin x_3 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}}$$

# Exercise H3:

```
clear all; clc; close all;
```

Solution with 5 points:

```
numpts = 5;
[xnew,ynew,sol_new] = FD_2D_func(numpts);


figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['u_{xx}+u_{yy}+2u=0 on square with zero BCs except y=0 using ' num2str(numpts) ' points'
```

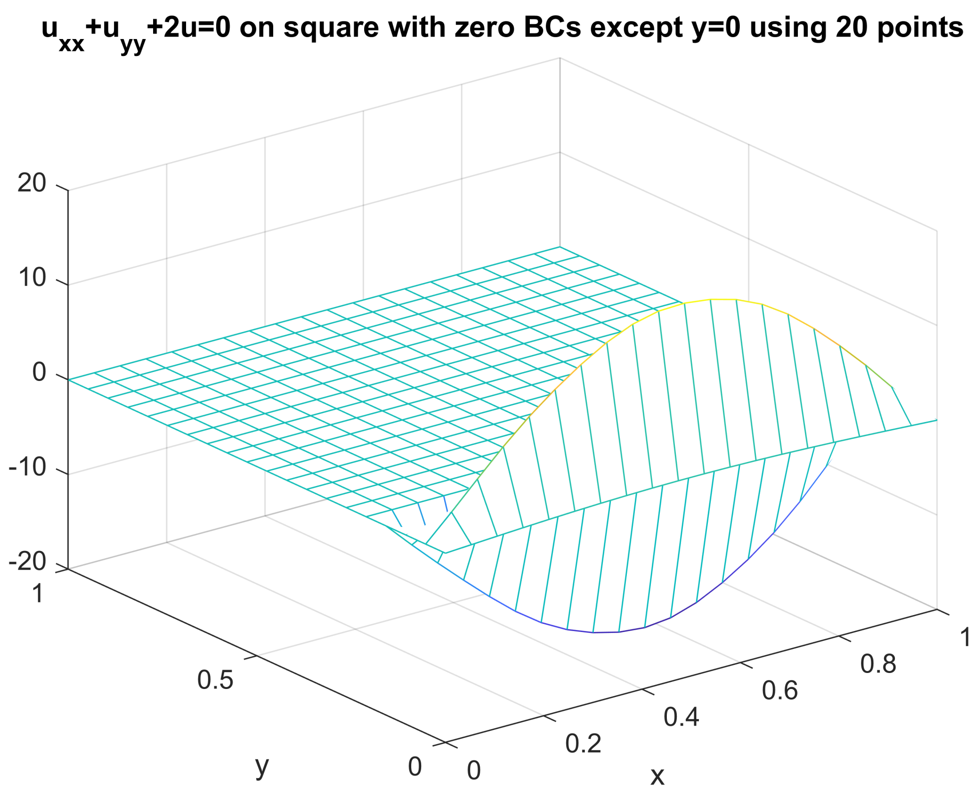



```

```

Then we can see the solution with 20 points

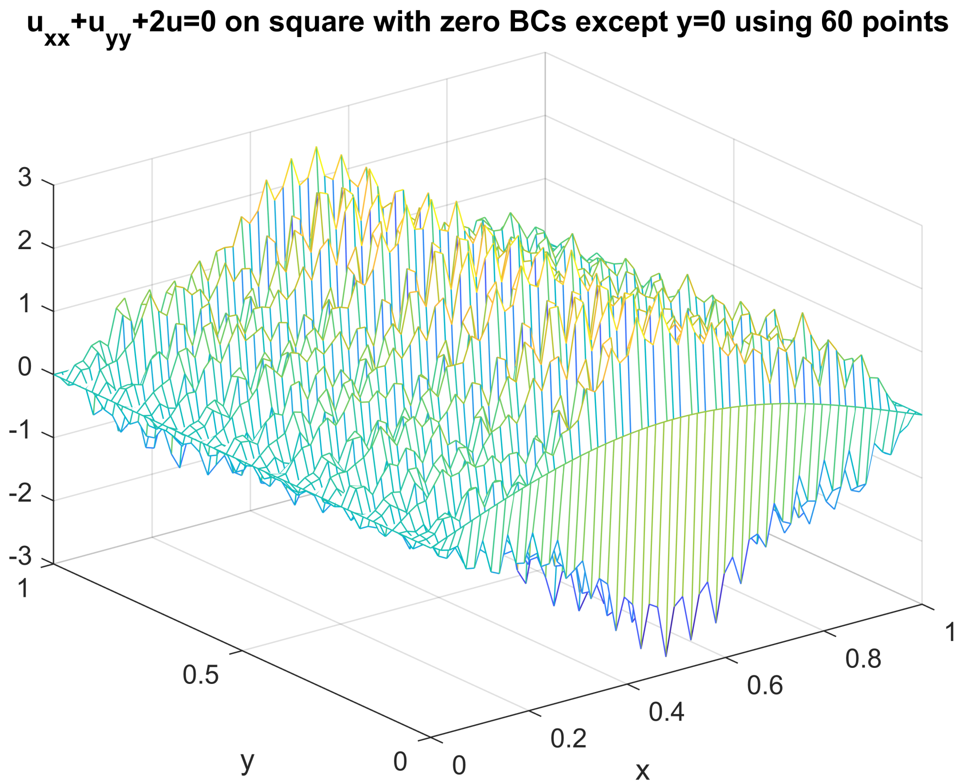
```
%% 20 points
```

```
numpts = 20;
[xnew,ynew,sol_new] = FD_2D_func(numpts);


figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['u_{xx}+u_{yy}+2u=0 on square with zero BCs except y=0 using ' num2str(numpts) ' points
```




And with 60 points

```
%% 60 points
numpts = 60;
[xnew,ynew,sol_new] = FD_2D_func(numpts);


figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['u_{xx}+u_{yy}+2u=0 on square with zero BCs except y=0 using ' num2str(numpts) ' points
```




The solution seems to get unstable with a finer grid. But I did not plot the true solution compared to the numerical solution to be sure of this.

Let's define a function to help us try different point easily:

```
function [x,y,sol] = FD_2D_func(numpts)
N = numpts; % number of grid points in x and y directions, same in both.
m = N - 2; % number of interior points

% --------------------------------------------------------------------------
% Define the grid:
% --------------------------------------------------------------------------
x1 = 0;
xN = 1;
y1 = 0;
yN = 1;
% --------------------------------------------------------------------------
% Determine the stepsize:
% --------------------------------------------------------------------------
h       = (xN-x1)/(N-1);
% --------------------------------------------------------------------------
% We'll just use the same stepsize in x and y
% Build the xvector of grid points:
```

```
% -------------------------------------------------------------------------
x       = x1:h:xN;
y       = y1:h:yN;
% -------------------------------------------------------------------------
% Define the solution matrix and plug in BCs.
% -------------------------------------------------------------------------
sol_FD          = zeros(N,N);
sol_FD(:,1)     = 0; % Enforcing the left side is u=0
sol_FD(:,end)   = 0; % Enforcing right side u=0
sol_FD(end,:)   = 0; % Enforcing the top u=0
sol_FD(1,:)     = sin(pi*x); % Enforcing the bottom BC u(x,0)=sin(pi*x)
% -------------------------------------------------------------------------
% -------------------------------------------------------------------------
% Build the matrix A:
% -------------------------------------------------------------------------
e=1*(ones(m^2-1,1));
ind = 1:m^2-1;
mth_sup_adj = e.*(mod(ind,3)~=0)'; % since every third one should be 0
% Build the main structure of the matrix
A = -2*eye(m^2,m^2) ...  % places -4 on the main diagonal
    + 1*diag(mth_sup_adj,1) ... % Places the adjusted ones on the superdiagonal
    + 1*diag(mth_sup_adj,-1) ... % Places the adjusted ones on the subdiagonal
    + 1*diag(ones(m^2-m,1),m) ... % Places ones on the mth superdiagonal (no adjustments needed
    + 1*diag(ones(m^2-m,1),-m);  % Places ones on the mth subdiagonal (no adjustments needed fo

% -------------------------------------------------------------------------
% Build the RHS vector
% -------------------------------------------------------------------------
RHS_vec     = zeros(m^2,1);
% Adjust the entries that have the BCs in them 1, m+1, 2*m+1, etc.  Note
% that the book uses a different ordering of the EQUATIONS that is not as
% directly compatible with the (:) notation.

for k=1:m
    RHS_vec((k-1)*m + 1)  = -sol_FD(1,k);
end

% -------------------------------------------------------------------------
% Solve for the solution at the interior points
% -------------------------------------------------------------------------
u_interior                      = A\RHS_vec(:);
u_interior                      = reshape(u_interior,m,m);
sol_FD(2:(end-1),2:(end-1))     = u_interior;
sol = sol_FD;
end
```

EXERCISE H0-1: (Lesson 37 of PDE text) Use 2D Finite Differences to solve the following BVP

$$\sin(x)u_{xx} + u_{xy} + 3u = 0, \qquad 0 < x < 1, \ 0 < y < 1$$

subject to $u(x, y) = 0$ on the top, left, and ride sides of the square domain with $u(x, y) = \sin(\pi x)$ for $y = 0$ (i.e. the bottom of the square). Use 6 grid points (4 interior points) in each of the $x$ and $y$ directions. Code up your FD method into Matlab and plot the solution. Does your FD solution improve with a finer grid? Experiment with higher levels of discretization.

let's start by discretizing $u_{xx}$ and $u_{xy}$ with uniform spacing in $x$ and $y$ $(h=k)$.

$$u_{xx} \approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$

and

$$u_{xy} \approx \frac{1}{2h2k}\left\{\left[u(x+h,y+k) - u(x+h,y-k)\right] - \left[u(x-h,y+k) - u(x-h,y-k)\right]\right\}$$

$$= \frac{1}{4h^2}\left\{\left[u(x+h,y+h) - u(x+h,y-h)\right] - \left[u(x-h,y+h) - u(x-h,y-h)\right]\right\}$$

then plugging that to the pde and letting

$u_{ij} = u(x_j, y_i)$

$u_{i,j+1} = u(x+h, y)$

$u_{i,j-1} = u(x-h, y)$

$u_{i+1,j+1} \quad u(x+h, y+h)$

$u_{i-1,j+1} \quad u(x+h, y-h)$

$u_{i+1,j-1} \quad u(x-h, y+h)$

$u_{i-1,j-1} \quad u(x-h, y-h)$

$\sin x \, u_{xx} + u_{xy} + 3u = 0$ gets discretized to

$$\frac{\sin x_j}{h^2}\left(u_{i,j+1} - 2u_{ij} + u_{i,j-1}\right) + \frac{1}{4h^2}\left(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}\right) + 3u_{ij} = 0$$

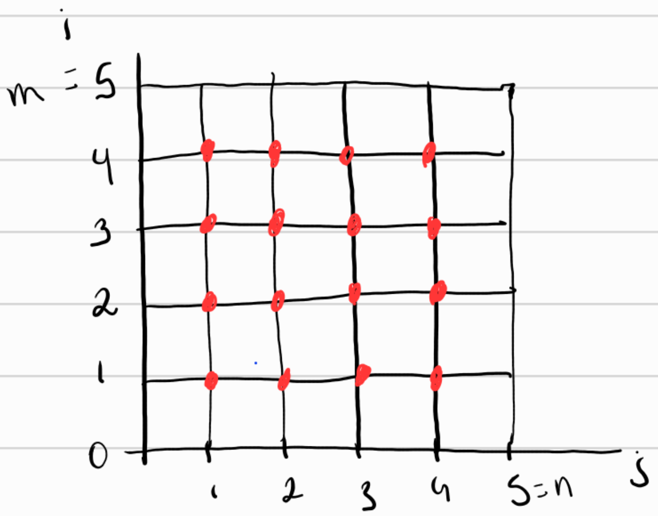
and the 6 point grid will then have

BCs: $u_{mj} = 0 \quad 1 \le j \le n$

$u_{in} = 0 \quad u_{i0} = 0$

$u_{0j} = \sin \pi x_j$

giving




then we have 16 unknowns:

$U_{11}, U_{12}, U_{13}, U_{14}, U_{21}, U_{22}, U_{23}, U_{24}, U_{31}, U_{32}, U_{33}, U_{34}, U_{41}, U_{42}, U_{43}, U_{44}$

$i=1,\ j=1$ ✓

$$\frac{\sin x_1}{h^2}\left(U_{12}-2U_{11}+\overset{=0}{U_{10}}\right)+\frac{1}{4h^2}\left(U_{22}-\overset{=\sin x_2}{U_{02}}-\overset{=0}{U_{20}}+\overset{=0}{U_{00}}\right)+3U_{11}=0$$

$i=2,\ j=1$ ✓

$$\frac{\sin x_1}{h^2}\left(U_{22}-2U_{21}+\overset{=0}{U_{20}}\right)+\frac{1}{4h^2}\left(U_{32}-U_{12}-\overset{=0}{U_{30}}+\overset{=0}{U_{10}}\right)+3U_{21}=0$$

$i=3,\ j=1$ ✓

$$\frac{\sin x_1}{h^2}\left(U_{32}-2U_{31}+\overset{=0}{U_{30}}\right)+\frac{1}{4h^2}\left(U_{42}-U_{22}-\overset{=0}{U_{40}}+\overset{=0}{U_{20}}\right)+3U_{31}=0$$

$i=4,\ j=1$ ✓

$$\frac{\sin x_1}{h^2}\left(U_{42}-2U_{41}+\overset{=0}{U_{40}}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{52}}-U_{32}-\overset{=0}{U_{50}}+\overset{=0}{U_{30}}\right)+3U_{41}=0$$

$i=1,\ j=2$ ✓

$$\frac{\sin x_2}{h^2}\left(U_{13}-2U_{12}+U_{11}\right)+\frac{1}{4h^2}\left(U_{23}-\overset{=\sin x_3}{U_{03}}-U_{21}+\overset{=\sin x_1}{U_{01}}\right)+3U_{12}=0$$

$i=2,\ j=2$ ✓

$$\frac{\sin x_2}{h^2}\left(U_{23}-2U_{22}+U_{21}\right)+\frac{1}{4h^2}\left(U_{33}-U_{13}-U_{31}+U_{11}\right)+3U_{22}=0$$

$i=3,\ j=2$ ✓

$$\frac{\sin x_2}{h^2}\left(U_{33}-2U_{32}+U_{31}\right)+\frac{1}{4h^2}\left(U_{43}-U_{23}-U_{41}+U_{21}\right)+3U_{32}=0$$

$i=4,\ j=2$ ✓

$$\frac{\sin x_2}{h^2}\left(U_{43}-2U_{42}+U_{41}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{53}}-U_{33}-\overset{=0}{U_{51}}+U_{31}\right)+3U_{42}=0$$

$i=1,\ j=3$ ✓

$$\frac{\sin x_3}{h^2}\left(U_{14}-2U_{13}+U_{12}\right)+\frac{1}{4h^2}\left(U_{24}-\overset{=\sin x_4}{U_{04}}-U_{22}+\overset{\sin x_2}{U_{02}}\right)+3U_{13}=0$$

$i=2,\ j=3$ ✓

$$\frac{\sin x_3}{h^2}\left(U_{24}-2U_{23}+U_{22}\right)+\frac{1}{4h^2}\left(U_{34}-U_{14}-U_{32}+U_{12}\right)+3U_{23}=0$$

$i=3,\ j=3$

✓ $$\frac{\sin x_3}{h^2}\left(U_{34}-2U_{33}+U_{32}\right)+\frac{1}{4h^2}\left(U_{44}-U_{24}-U_{42}+U_{22}\right)+3U_{33}=0$$

$i=4,\ j=3$

✓ $$\frac{\sin x_3}{h^2}\left(U_{44}-2U_{43}+U_{42}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{54}}-U_{34}-\overset{=0}{U_{52}}+U_{32}\right)+3U_{43}=0$$

$i=1,\ j=4$

✓ $$\frac{\sin x_4}{h^2}\left(\overset{=0}{U_{15}}-2U_{14}+U_{13}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{25}}-\overset{=0}{U_{05}}-U_{23}+\overset{=\sin x_3}{U_{03}}\right)+3U_{14}=0$$

$i=2,\ j=4$

✓ $$\frac{\sin x_4}{h^2}\left(\overset{=0}{U_{25}}-2U_{24}+U_{23}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{35}}-\overset{=0}{U_{15}}-U_{33}+U_{13}\right)+3U_{24}=0$$

$i=3,\ j=4$

✓ $$\frac{\sin x_4}{h^2}\left(\overset{=0}{U_{35}}-2U_{34}+U_{33}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{45}}-\overset{=0}{U_{25}}-U_{43}+U_{23}\right)+3U_{34}=0$$

$i=4,\ j=4$

$$\frac{\sin x_4}{h^2}\left(\overset{=0}{U_{45}}-2U_{44}+U_{43}\right)+\frac{1}{4h^2}\left(\overset{=0}{U_{55}}-\overset{=0}{U_{35}}-\overset{=0}{U_{53}}+U_{33}\right)+3U_{44}=0$$

$$\begin{bmatrix}
a(x_1) & 0 & 0 & 0 & b(x_2) & c & 0 & 0 & & & & & & & & \\
0 & a(x_1) & 0 & 0 & -c & b(x_2) & c & 0 & & & & & & & & \\
0 & 0 & a(x_1) & 0 & 0 & -c & b(x_2) & c & & & & & \mathbf{0} & & & \\
0 & 0 & 0 & a(x_1) & 0 & 0 & -c & b(x_2) & 0 & & & & & & & \\
b(x_1) & -c & 0 & 0 & a(x_2) & 0 & 0 & 0 & b(x_3) & c & & & & & & \\
c & b(x_1) & -c & 0 & 0 & a(x_2) & 0 & 0 & -c & b(x_3) & c & & & & & \\
 & c & b(x_1) & -c & 0 & 0 & a(x_2) & 0 & 0 & -c & b(x_3) & c & & & & \\
 & & c & b(x_1) & 0 & 0 & 0 & a(x_2) & 0 & 0 & -c & b(x_3) & 0 & & & \\
 & & & 0 & b(x_2) & -c & 0 & 0 & a(x_3) & 0 & 0 & 0 & b(x_4) & c & & \\
 & & & & c & b(x_2) & -c & 0 & 0 & a(x_3) & 0 & 0 & -c & b(x_4) & c & \\
 & & & & & c & b(x_2) & -c & 0 & 0 & a(x_3) & 0 & 0 & -c & b(x_4) & c \\
 & & & & & & c & b(x_2) & 0 & 0 & 0 & a(x_3) & 0 & 0 & -c & b(x_4) \\
 & & \mathbf{0} & & & & & 0 & b(x_3) & -c & 0 & 0 & a(x_4) & 0 & 0 & 0 \\
 & & & & & & & & c & b(x_3) & -c & 0 & 0 & a(x_4) & 0 & \\
 & & & & & & & & & c & b(x_3) & -c & 0 & 0 & a(x_4) & \\
 & & & & & & & & & & c & b(x_3) & 0 & 0 & 0 & a(x_4)
\end{bmatrix}
\begin{bmatrix}
U_{11}\\ U_{21}\\ U_{31}\\ U_{41}\\ U_{12}\\ U_{22}\\ U_{32}\\ U_{42}\\ U_{13}\\ U_{23}\\ U_{33}\\ U_{43}\\ U_{14}\\ U_{24}\\ U_{34}\\ U_{44}
\end{bmatrix}$$

$A$ $\vec{U}$

$$
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16
\end{array}
\begin{bmatrix}
+\frac{\sin x_2}{4h^2} \\ 0 \\ 0 \\ 0 \\ \frac{-\sin x_1 - \sin x_3}{4h^2} \\ 0 \\ 0 \\ 0 \\ -\frac{\sin x_2 - \sin x_4}{4h^2} \\ 0 \\ 0 \\ 0 \\ \frac{-\sin x_3}{4h^2} \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

$\vec{b}$

where $a(x_i) = 3 - \frac{2\sin x_i}{h^2}$

$b(x_i) = \frac{\sin x_i}{h^2}$

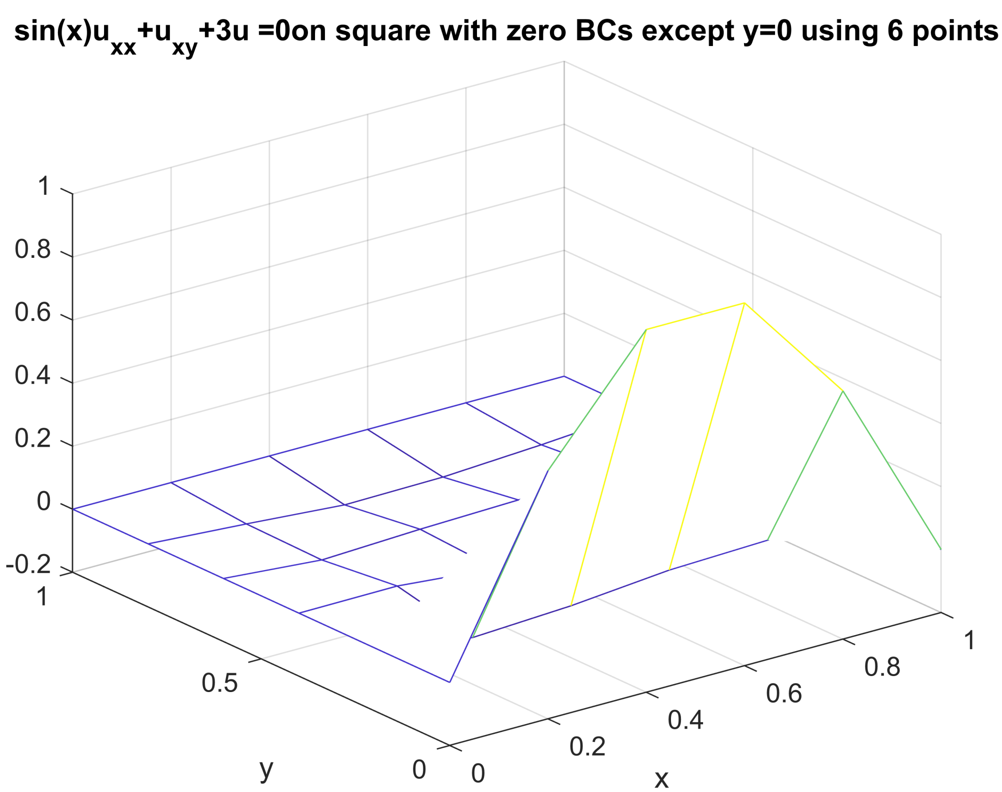
$c = \frac{1}{4h^2}$

# Exercise H3:- 1

```
clear all; clc; close all;
```

Solution with 6 points gives

```
numpts = 6;
[xnew,ynew,sol_new] = FD_2D_func(numpts);


figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['sin(x)u_{xx}+u_{xy}+3u =0' ...
    'on square with zero BCs except y=0 using ' num2str(numpts) ' points'])
```

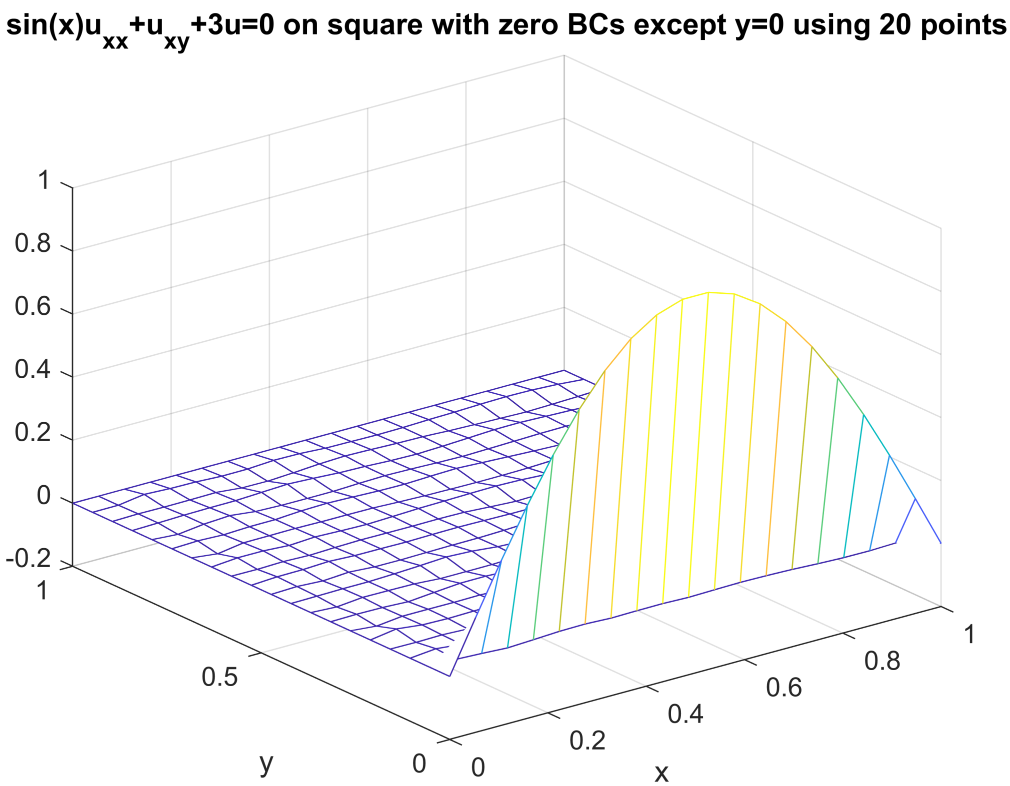



```
```

Then we can see the solution with 20 points

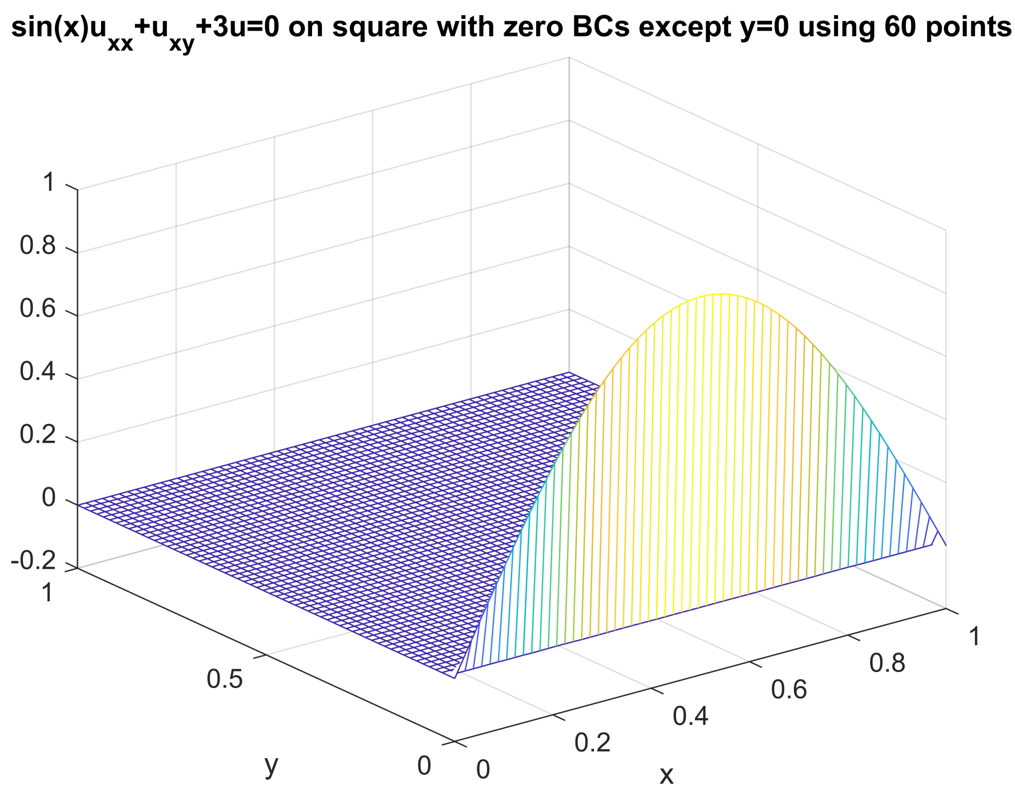
```
%% 20 points
numpts = 20;
[xnew,ynew,sol_new] = FD_2D_func(numpts);
```

```
figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['sin(x)u_{xx}+u_{xy}+3u=0 ' ...
    'on square with zero BCs except y=0 using ' num2str(numpts) ' points'])
```




And with 60 points

```
%% 60 points
numpts = 60;
[xnew,ynew,sol_new] = FD_2D_func(numpts);


figure;
mesh(xnew,ynew,sol_new);
xlabel('x')
ylabel('y')
title(['sin(x)u_{xx}+u_{xy}+3u=0 ' ...
    'on square with zero BCs except y=0 using ' num2str(numpts) ' points'])
```




The solution seems to get better with a finer grid. But I did not plot the true solution compared to the numerical solution to be sure of this.

Let's define a function to help us try different point easily:

```
function [x,y,sol] = FD_2D_func(numpts)
N = numpts; % number of grid points in x and y directions, same in both.
m = N - 2; % number of interior points

% ----------------------------------------------------------------
% Define the grid:
% ----------------------------------------------------------------
x1 = 0;
xN = 1;
y1 = 0;
yN = 1;
% ----------------------------------------------------------------
% Determine the stepsize:
% ----------------------------------------------------------------
h       = (xN-x1)/(N-1);

%-----------------------------------------------------------------
% Helper functions
```

```
%--------------------------------------------------------------------------
a_fun = @(x) 3-2*(sin(x))/h.^2;
b_fun = @(x) sin(x)/h.^2;
c = 1/(4*h.^2);
% ------------------------------------------------------------------------
% We'll just use the same stepsize in x and y
% Build the xvector of grid points:
% ------------------------------------------------------------------------
x       = x1:h:xN;
y       = y1:h:yN;
% ------------------------------------------------------------------------
% Define the solution matrix and plug in BCs.
% ------------------------------------------------------------------------
sol_FD         = zeros(N,N);
sol_FD(:,1)    = 0; % Enforcing the left side is u=0
sol_FD(:,end)  = 0; % Enforcing right side u=0
sol_FD(end,:)  = 0; % Enforcing the top u=0
sol_FD(1,:)    = sin(pi*x); % Enforcing the bottom BC u(x,0)=sin(pi*x)
% ------------------------------------------------------------------------
% ------------------------------------------------------------------------
% Build the matrix A:
% ------------------------------------------------------------------------
ind = 1:m^2-3;
inner_supsub_diag = -c.*(mod(ind,4)~=1)';
ind = 1:m^2-5;
outer_supsub_diag = c.*(mod(ind,4)~=0)';
x_rep = repmat(x(2:end-1), m, 1);
x_rep = x_rep(:);
% Build the main structure of the matrix
A = a_fun(x_rep).*eye(m^2,m^2) ...  % main diagonal
    + 1*diag(inner_supsub_diag,3) ... % yellow on notes
    + 1*diag(inner_supsub_diag,-3) ... % yellow
    + 1*diag(outer_supsub_diag,5) ... % light blue
    + 1*diag(outer_supsub_diag,-5) ... % light blue
    + diag(b_fun(x_rep(1:m^2-m)).*ones(m^2-m,1),m) ... % dark blue
    + diag(b_fun(x_rep(m+1:m^2)).*ones(m^2-m,1),-m);  % red

% ------------------------------------------------------------------------
% Build the RHS vector
% ------------------------------------------------------------------------
RHS_vec     = zeros(m^2,1);
% Adjust the entries that have the BCs in them 1, m+1, 2*m+1, etc.  Note
% that the book uses a different ordering of the EQUATIONS that is not as
% directly compatible with the (:) notation.

for k=1:m
    RHS_vec((k-1)*m + 1)  = -((sin(k)-sin(k+1))./4);
end

% ------------------------------------------------------------------------
```

```
% Solve for the solution at the interior points
% ----------------------------------------------------------------------------
u_interior                          = A\RHS_vec(:);
u_interior                          = reshape(u_interior,m,m);
sol_FD(2:(end-1),2:(end-1))         = u_interior;
sol = sol_FD;
end
```