Homework 1

Problem A: $f(x) = \exp(\exp(x)), \quad x \in [0,1]$

Integrate numerically with Matlab from a=0 to b=1.

Do by pencil and paper with n=4 intervals.

$\Delta x = 0.25 \qquad (x_1^*, x_2^*, x_3^*, x_4^*) = (0.125, 0.375, 0.625, 0.875)$

Write a Matlab program to repeat with n=4 intervals.

Change to n=100. Compare results.

$$0.25\left(e^{e^{0.125}} + e^{e^{0.375}} + e^{e^{0.625}} + e^{e^{0.875}}\right) \approx 6.2194$$

```
%% n=4
a=0;b=1;n=4;
dx = (b-a)/n;
xpts = (a+dx/2:dx:b)';
gpts = exp(exp(xpts));
Exhat = dx*sum(gpts)
% output
%Exhat =
%     6.2194
%% n=100
a=0;b=1;n=100;
dx = (b-a)/n;
xpts = (a+dx/2:dx:b)';
gpts = exp(exp(xpts));
Exhat = dx*sum(gpts)
% output
%Exhat =
%     6.3164
```

Homework 1

Chapter 3: # 1*, 3#, 7, 9, 11.

*Repeat generating 10^4 of these.

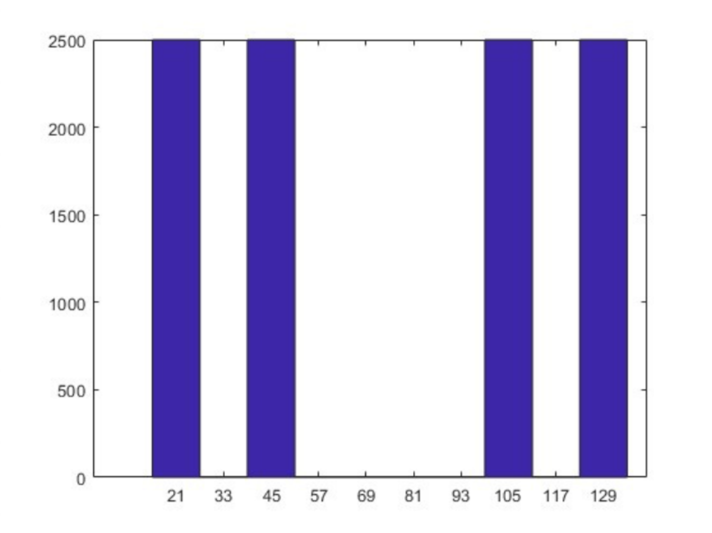
Compute mean, variance, and make a histogram.

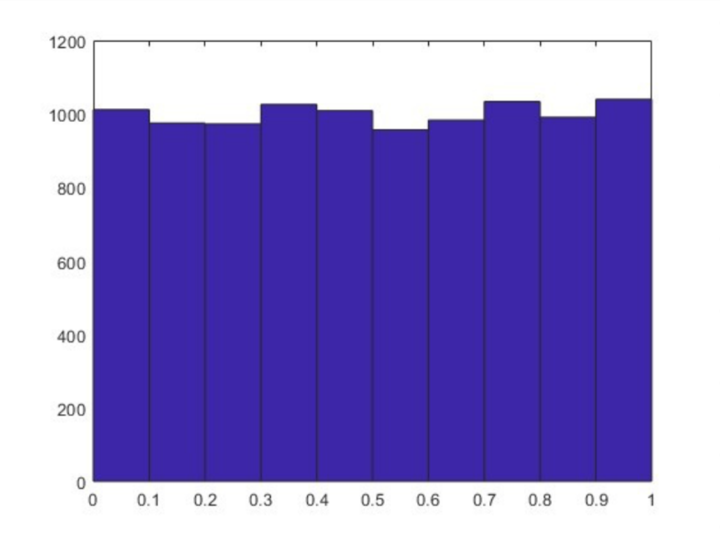
Repeat using Matlab's rand() command.

Compare results from to Matlab's rand().

#Compare results to numerical integration in Problem A with n=100.

```
%% 1
x_0 = 5;
n = 10;
x_n = zeros(n,1);
for i=1:n
    if i == 1
        x_n(i) = mod(3*x_0,150);
    else
        x_n(i) = mod(3*x_n(i-1),150);
    end
end
%% *
x_0 = 5;
n = 1e4;
x_n_rand = zeros(n,1);
for i=1:n
    if i == 1
        x_n(i) = mod(3*x_0,150);
    else
        x_n(i) = mod(3*x_n(i-1),150);
    end
end
%% mean, var, hist
m = mean(x_n)
v = var(x_n)
hist(x_n)
% output
% m =    75
% v = 2.2502e+03
%% using rand
x_n = rand(1e4,1);
m = mean(x_n)
v = var(x_n)
hist(x_n)
% output
% m =    0.4982
% v = 0.0831
```





3.3-) $\int_0^1 e^{e^x}\,dx$, can't be done analytically

```
%% 3 ------ from 0 to 1
rng('default')
a = 0;b = 1;n = 1e6;
u = rand(n,1);
t = (a+(b-a)*u);
hu = (exp(exp(t)))*(b-a);
Exhat = sum(hu)/n
% output
%Exhat =
%    6.3191
```

Which is the same we got when using numerical integration in Problem A

3.7-) $\int_{-\infty}^{\infty} e^{-x^2}\,dx$, let the integral be equal to some number I

$$\int_{-\infty}^{\infty} e^{-x^2}dx = I \Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = I^2$$

$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx$, then we can change to polar coordinates

$$x^2+y^2 = r^2 \qquad dx\,dy = r\,dr\,d\theta$$

$$\Rightarrow I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{\infty} re^{-r^2}\theta\,dr \Big|_0^{2\pi}$$

$$= \int_0^{\infty} 2\pi r e^{-r^2}\,dr \qquad u = r^2 \quad du = 2r\,dr$$

$$= \int_0^{\infty} \pi e^{-u}\,du = \pi(0-(-1)) \Rightarrow I = \sqrt{\pi} \approx 1.7725$$

```
%% 7 -------- -infty to infty
rng('default')
a = 0;b = 1;n = 1e6;
u = rand(n,1);
%u = exp(u)./(1+exp(u));
g = @(x) exp(-x.^2);  % Given function
hu = g(log(u./(1-u))).*(1./(u.*(1-u))); %g(of transform)*jacobian
Exhat = sum(hu)/n
% output
%Exhat =
%    1.7744
```

9-) $\int_0^\infty \int_0^x e^{-(x+y)}\,dy\,dx = \int_0^\infty \int_0^x e^{-x} e^{-y}\,dy\,dx$

$= \int_0^\infty -e^{-x-x} - (-e^{-x-0})\,dx$

$= \left(\frac{1}{2}e^{-2x} - e^{-x}\right)\Big|_0^\infty = (0-0) - \left(\frac{1}{2} - 1\right) = 1/2$

```
%% 9 ---- two integrals / 0 to infty and 0 to x
rng('default')
n = 1e6;u = rand(n,1);v = rand(n,1);
% Given function * indicator function from hint
g = @(x,y) exp(-(x+y)).*(y<=x);  % Given function * indicator function
hu = g(1./u-1,1./v-1).*(1./u.^2).*(1./v.^2); %g(of transform)*jacobian
Exhat = sum(hu)/n
% output
%Exhat =
%    0.5000
```

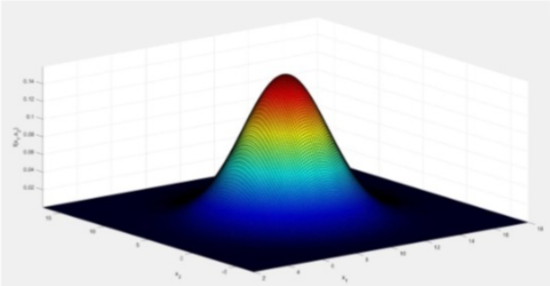
3.11. a-) $\mathrm{Corr}(U, \sqrt{1-U^2})$

b-) $\mathrm{Corr}(U^2, \sqrt{1-U^2})$

```
%% 11
u = rand(n,1);
v = sqrt(1-u.^2);
my_cov = mean(u.*v)-mean(u)*mean(v);
corr = my_cov/(sqrt(var(u)*var(v)))
% output
%corr =
%    -0.9212
```

```
%% b
u = rand(n,1);
v = sqrt(1-u.^2);
my_cov = mean(u.^2.*v)-mean(u.^2)*mean(v);
corr = my_cov/(sqrt(var(u)*var(v)))
% output
%corr =
%     -1.0156
```

Homework 1



Problem B:

Evaluate the double integral

$\iint x_1 f(x_1,x_2)dx_1dx_2$ where $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$

for PDF $f(x_1,x_2)=\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2}\frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2}$.

$\mu_1=10$
$\mu_2=5$
$\sigma_1^2=4$
$\sigma_2^2=9$

a. Use numerical integration by breaking up (x_1,x_2) into rectangles. Use $\Delta x_1 \Delta x_2 \sum_{x_2}\sum_{x_1} x_1 f(x_1,x_2)$.

b. Use stochastic integration $\sum_{i=1}^{k}\frac{g(U_1^i,U_2^i)}{k}$ after transforming (x_1,x_2) to (u_1,u_2).

c. Pencil and paper integration?

a-)

```
%% a
a=-100;
b=200;
n=200;
dx = (b-a)/n;
xpts = (a+dx/2:dx:b)';
ypts = xpts;
mu1 = 10;mu2 = 5; sig1 = 2; sig2 = 3;
f = @(x,y) exp(-((x-mu1).^2)./(2*sig1.^2))./(sqrt(2*pi*sig1.^2))...
    .*exp(-((y-mu2).^2)./(2*sig2.^2))./(sqrt(2*pi*sig2.^2));
gpts = zeros(n,1);
for i=1:n
    gpts(i) = sum(xpts.*f(xpts, ypts(i)));%function
end
Exhat = dx*dx*sum(gpts)
% output
%Exhat =
%     10.0000
```

b.) first I tried using the stochastic integration with the transformation of variables on both variables at the same time, but it was not working properly. But since they are independent, we can integrate it separately, then it worked!

```
%% try 2 b
rng('default')
n = 1e7;
u = rand(n,1);
v = rand(n,1);
mu1 = 10;mu2 = 5; sig1 = 2; sig2 = 3;
% the function and the jacobian are correct
g = @(x) x.*exp(-((x-mu1).^2)./(2*sig1.^2))./(sqrt(2*pi*sig1.^2));
f = @(y) exp(-((y-mu2).^2)./(2*sig2.^2))./(sqrt(2*pi*sig2.^2));
    %.*exp(-((y-mu2).^2)./(2*sig2.^2))./(sqrt(2*pi*sig2.^2));  % Given function
hu = g(log(u./(1-u))).*(1./((u-u.^2))); %g(of transform)*jacobian
hu = g(log(v./(1-v))).*(1./((v-v.^2))); %g(of transform)*jacobian
% this is returning 1, and it should return 10?
Exhat = sum(hu)/n
% output
%Exhat =
%     9.8575
```

c.)

$$\iint x_1 f(x_1, x_2) = \iint x_1 \frac{1}{2\pi\sqrt{\sigma_1\sigma_2}} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} dx_1 x_2$$

$$x = \frac{x_1-\mu_1}{\sigma_1} \qquad y = \frac{x_2-\mu_2}{\sigma_2}$$

$$x_1 = \sigma_1 x + \mu_1$$

then $$\iint \frac{\sigma_1 x + \mu_1}{2\sqrt{\sigma_1\sigma_2}} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy$$

$$= \underbrace{\mu_1 \iint \frac{1}{2\sqrt{\sigma_1\sigma_2}} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy}_{\text{this is a density function for normal}} + \sigma_1 \underbrace{\iint \frac{x}{2\sqrt{\sigma_1\sigma_2}} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy}_{\text{transform to polar}}$$

$r^2 = x^2 + y^2$ $\qquad dx\,dy = r\,dr\,d\theta$

$x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\Rightarrow \quad \mu_1 + \sigma_1 \int_0^{\infty}\int_0^{2\pi} \frac{r\cos\theta}{2\sqrt{\sigma_1\sigma_2}} e^{1/2 r^2} r \, d\theta \, dr$$

$$= \mu_1 + \sigma_1 \int_0^{\infty} \left(\frac{r^2 \sin\theta}{2\sqrt{\sigma_1\sigma_2}} e^{-r^2} \right) \Big|_0^{2\pi} dr$$

$\sin 2\pi = 0$

$\sin 0 = 0$

$$= \mu_1$$