

Homework #3.

Directions. All work is to be done in complete sentences. Assignments must be stapled. Each problem (1., 2., ... not i), ii) ...) must begin on a separate sheet of paper. Please **USE THE BACK**. Please begin each problem with the full statement of the problem. While you are encouraged to work through confusion with your classmates, your work must be written in your own words. The assignment is due by 5:00 pm on Wednesday, November 15, 2023 in Dr. Clough's mailbox **outside** CU 340 in the narrow dark hallway.

1. Consider the bungee jumper using a strong cord.

- i) Write down the equation of motion for the bungee jumper.
- ii) Use the gravitational constant $g = 9.81 \text{ m/s}^2$. Assume a maximum weight of 350 lb, a rope of length 30 m, an initial height of 500 m, and a coefficient of drag of 0.25 kg/m. Show that the units of the equation are balanced.
- iii) Determine the terminal velocity of the jumper.
- iv) Solve the ODE in Matlab using ode45. Plot the velocity as a function of time. Include the graph here.
- v) Is the terminal velocity correct?
- vi) How long does it take the jumper to reach a speed of 80 m/s?

2. Consider the bungee jumper using a stretchable bungee cord.

- i) Write down the equation of motion for the bungee jumper.
- ii) Assume a spring constant of $k = 40 \text{ N/m}$ and a coefficient of damping of 8 N s/m . Show that the units of the equation are balanced.
- iii) Write the equation as a system of ODEs.
- iv) Solve the ODEs in Matlab using ode45. Plot the height above the ground as a function of time. How close to the ground does the jumper get? Include your code.

3. Now assume that we want to position the platform in the tree as close as possible to the ground while always maintaining at least an 8 m height above the ground for the jumper.

- i) How high does the platform need to be? Explain or show how you obtained your solution.
- ii) Now consider that we will also build a bungee jump for kids with maximum weight 75 pounds. How high does the platform need to be? Explain or show how you obtained your solution.

4. Consider the Lorenz system of differential equations. Use $s = 10$, $r = 28$, and $b = 8/3$.

- i.) Determine whether the system of equations has any equilibrium points.
- ii.) Construct matlab code to solve the system on $t = [0, 20]$ using ode45. Check that your code is working correctly by using initial conditions = equilibrium points.
- iii.) Now use $x(0) = y(0) = z(0) = 5$. Generate plots of (t, x) , (t, y) , and (t, z) . Label the axes. Submit graphs and code
- iv.) Plot the phase portrait for the system. Suppose $[t, Y]$ is the output from ode45. Use `plot3(Y(:,1), Y(:,2), Y(:,3))` to generate the plot. Submit plot.
- v.) Repeat 3. and 4. Using an initial value of 5.01, 5.001, 5.0001, and 5.00001.
- vi.) Describe the behavior you are seeing. Do you think the behavior is because of numerical error or the dynamics of the system? What conclusions can you draw about weather prediction?

5. ACTIVITY 9: Stiff differential equations

Consider the differential equation $\frac{dy}{dt} = 10(1 - y), y(0) = \frac{1}{2}$.

This is a linear first order equation. **Solve the problem analytically using an integrating factor. Sketch in the solution to the problem.**

Consider solving this problem using Euler's method. Then we have

$$y_{i+1} = y_i + h \cdot 10(1 - y_i) = y_i(1 - 10h) + 10h.$$

Note that if the method were converging, we would have

$$y_{i+1} \approx y_i = w \text{ say.}$$

So the problem looks like $w = w(1 - 10h) + 10h$,

which is a fixed point problem, $w = g(w)$. **Provide a check that the fixed point is $w = 1$.**

We know that fixed point iteration will converge for $|g'(1)| < 1$. **Determine the values of h for which this inequality is true.**

Solve this differential equation using Euler's method on $t \in [0,2]$ using first an h that is too large, and then an appropriately small h . **Include a plot of both "solutions" here.**

As we discussed earlier, this problem is numerically unstable, when h is “large”. We have here an example of a “stiff” differential equation. There is not a precise mathematical definition of stiff, but here are some statements that “characterize” stiffness:

◦The stability requirement of the problem constrains h , rather than the accuracy (truncation error requirement) constraining h . (For example, in our differential equation we have not set any constraint on the size of the truncation error, but rather you have determined that you need h to be sufficiently small just to obtain a “stable” solution.)

◦A problem is stiff if its solution is composed of some terms that are decaying very rapidly compared to others. (For example, if the solution had an e^{-t} and an e^{-10t} term in it.)

◦From the above statement, we can see that stiffness also suggests that there are regions of t where h must be excessively small relative to the smoothness of the exact solution. (In the example where the solution has an e^{-t} and an e^{-10t} term in it, we could use a large h for large t , but would need a very small h for very small t , in order accurately capture the solution.)

Neither Euler nor RK, including ode45, are suitable for stiff equations. Instead we introduce the Backward Euler Method, which uses the slope of the solution at the right-hand endpoint (instead of the left as in regular Euler) in Euler’s formula. This gives

$$y_{i+1} = y_i + h f(t_{i+1}, y_{i+1}).$$

The only problem is that we don’t y_{i+1} so how can we possibly plug it in on the RHS? This is an example of an implicit method, rather than the explicit methods we have considered previously.

Consider our example. In this case Backward Euler gives $y_{i+1} = y_i + h 10(1 - y_{i+1})$. **Solve this for y_{i+1} .**

Again, by letting $y_{i+1} \approx y_i = w$, this can be written as a fixed point problem. **What is g in this case?**

Show that this fixed point iteration converges for all $h > 0$.

Write matlab code to solve our example problem using Backward Euler. Run it using the same two values of h that you used previously for regular Euler. **Include your code and the graphs of the solution here.**

Of course, generally it will not be so easy to rewrite the problem so that we can use Backward Euler. So some other strategy will be necessary to solve the problem, for example fixed point iteration or Newton's method.

Backward Euler is a first order method, just like Euler. We can extend Backward Euler to a 2nd order method (just as we did for Euler to get Heun's method for example.) We could then develop the corresponding 2nd order implicit method, which would be suitable for stiff equations. In matlab, ode23s can be used for solving stiff equations (note the s). It's like ode45 in that it compares RK2 and RK3 (rather than RK4 and RK5) to determine the appropriate h .

Another example: $\frac{dy}{dt} = -21y + e^{-t}, y(0) = 0.$

Solve this problem using Euler's method on $t \in [0,2]$ with $h = 0.01$ and again with $h = 0.1$. You should see that the problem is stiff. **Paste the graphs of the solutions here.**

Solve the problem analytically using an integrating factor to obtain the exact solution.

Discuss what you observe about the analytic solution and the numerical solution that suggest the problem is stiff.

Finally, modify your matlab code that uses ode45 to instead use ode23s for this problem. The syntax for ode23s is exactly the same as that for ode45. **Include your code and a graph of the solution here.**

6. Given points (a,s) and (b,t) falling on a straight line, $y = mx + b$, we have the equation

$$x = \frac{b-a}{t-s}(y - s) + a.$$

Write matlab code to solve the heat equation given in class using the shooting method. Use $L = 2$, $w = 4$, $u(0) = 0$, $u(L) = 10$ and $T_{air} = 5$. Graph the solution and verify that it looks like the solution obtained in class.

You will need to write the equation as a system and implement ode45 three different times.

Submit your code and graph.

7. A more realistic ODE describing heat transfer in the rod, that also accounts for radiative transfer, is

$$u'' + w(T_{air} - u) + \sigma(T_{air}^4 - u^4) = 0.$$

Note that this is a nonlinear ODE. Modify the matlab code provided to solve this BVP with the same parameters and BCs. Use $\sigma = 0.01$. Plot the resulting solution.

8. Consider constructing the analytic solution to the BVP $u'' = -wu$, $u(0) = 0$, $u(\pi/2) = 10$.

We'll use the guessing method as done in class.

- i) Start with $w = 1$. What function(s) u satisfies the DE? How many solutions must there be?
- ii) Now solve the ODE with $w = 4$. Recall that the general solution is the linear combination of two linearly independent solutions.
- iii) Impose the boundary conditions to determine the solutions to the BVP.
- iv) Use matlab to plot the solutions.
- v) Now solve the problem numerically using the shooting method. Plot the solutions and compare to iv).

9. Students taking the course for graduate credit only:

- i) Consider the original heat equation ($\sigma=0$) with fixed temperature at $x = L$ (Dirichlet condition) but with a "free" end at $x = 0$. This means that since we are considering steady-state (long time equilibrium), convection must equal conduction at the $x = 0$ end of the rod. Set up the heat balance there to determine the boundary condition at $x = 0$.
- ii) Solve this new BVP using the shooting method. Plot the solution.
- iii) Interpret the solution. Does it make sense?