Math 4650/MSSC 5650 - Homework 1

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Note: In all problems below, $\langle x, y \rangle$ denotes the standard inner product of two vectors, and $\|x\|$ denotes the Euclidean norm.

Problem 1 (5 pts)**.** Suppose the vector $c \in \mathbb{R}^7$ represents the daily earnings of a company over one week, with c_1 the earnings on Sunday, c_2 the earnings on Monday, and so on, with c_7 the earnings on Saturday. (Negative entries in the earnings vector mean a loss on that day.) Express the following quantities as an inner product $\langle a, c \rangle$. In each case, give the vector a (which can be different for the different quantities, of course).

(a) Wednesday's earnings.

(b) The total earnings over the week.

(c) The average weekend earnings.

(d) The average weekday earnings.

(e) The difference between the average weekend earnings and the average weekday earnings.

Solution 1. .

(a) $\langle a, c \rangle = c_4$ Then $a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\langle a, c \rangle = \sum_{i=1}^{n} c_i$ Then $a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\langle a, c\rangle = \frac{c_1+c_7}{2}$ Then $a = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$

(d) $\langle a, c\rangle = \frac{c_2+c_3+c_4+c_5+c_6}{5}$ Then $a = \begin{bmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}$

(e) $\langle a, c\rangle = \frac{c_1+c_7}{2} - \frac{c_2+c_3+c_4+c_5+c_6}{5}$ Then $a = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{2} \end{bmatrix}$

Problem 2 (5 pts)**.** Suppose $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ with $m \neq n$. Which of the following quantities represent valid mathematical operations between vectors and matrices? If the operation is valid, state whether the result is a scalar, vector, or matrix and give its dimensions.

(a) xy

(b) $x^T y$

(c) xy^T

(d) Ax

(e) yA

(f) $y^T A$

(g) $y^T Ax$

Solution 2. .

(a) Not valid

(b) Not valid

(c) Valid, the result is a $n \times m$ matrix

(d) Valid, the result is a $m \times 1$ vector

(e) Not valid

(f) Valid, the result is a $1 \times n$ vector

(g) Valid, scalar

Problem 3 (5 pts)**.** Given vectors $x, y \in \mathbb{R}^n$, matrices $A, B \in \mathbb{R}^{n\times n}$ and a scalar $\alpha \in \mathbb{R}$, select one choice in each subproblem below. Justify your answer by including an example, where appropriate.

(a) If $\langle x, y \rangle = 0$, then

  (i) at least one of x or y must be 0.

  (ii) both x and y must be 0.

  (iii) both x and y can be nonzero.

(b) If $\alpha x = 0$, then

  (i) one of α or x must be 0

  (ii) both α and x must be 0

  (iii) both α and x can be nonzero

(c) If $Ax = 0$, then

  (i) one of A or x must be 0

  (ii) both A and x must be 0

  (iii) both A and x can be nonzero

(d) If $AB = 0$, then

  (i) one of A or B must be 0

  (ii) both A and B must be 0

  (iii) both A and B can be nonzero

Solution 3. .

(a) Both x and y can be nonzero. Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $x^T y = 0$

(b) One of α or x must be nonzero. $\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$, so the product is only zero for all elements if either x or α equal 0

(c) Both A and x can be nonzero. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) Both A and B can be nonzero. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Problem 4 (MATLAB, 5 pts)**.** Define a vector $x \in \mathbb{R}^5$ and a matrix $A \in \mathbb{R}^{5\times5}$ in MATLAB with the commands:

```
rng(1); %fix random seed -- must include this!
x = randn(5,1);
A = randn(5,5);
```

Now find each of the following quantities using MATLAB commands, such as those included in this cheat sheet. Include in your report the commands you used in each case.

(a) x_2 — the second entry of x

(b) $a_{1,4}$ — the entry of A in the 1st row and 4th column

(c) $[a_1|a_2]$ — the 5×2 matrix consisting of the first two columns of A.

(d) $\|x - Ax\|$ — the distance between x and Ax as measured in Euclidean norm.

(e) The maximum eigenvalue of A.

Solution 4. .

(a)

```
%% a
disp(x(2));
% output: 1.1812
```

```
%% b
disp(A(1,4));
% output: -0.2752
```

(b)
```
%% c
disp(A(:,1:2));
% output:
%    -0.5727    -0.8519
%    -0.5587     0.8003
%     0.1784    -1.5094
%    -0.1969     0.8759
%     0.5864    -0.2428
```

```
%% d
disp(norm(x-A*x));
% output: 3.9050
```

(d)
```
%% e
[V,D] = eig(A);
disp(max(diag(abs(D))));
% output: 3.8004
% the actual maximum eigenvalue is -3.8004,
% but we are using the absolute value
```

Problem 5 (MATLAB, 5 pts). Open the provided script `eigencats.m`. The first cell will load a 1246×4096 matrix `X` into memory. Each row of `X` is a vectorized 64×64 pixel image of a cat face. Run the full script to find the "Eigencats", i.e., the top eigenvectors of the data covariance matrix `X`. After running the script, these are stored as columns in the matrix `V`, sorted by size of the corresponding eigenvalue in descending order.

(a) Your first task is to synthesize new random cat faces using the top 50 eigencats according to the following procedure: Let $\hat{V} = [v_1|v_2|...|v_{50}] \in \mathbb{R}^{4096\times 50}$ be the matrix containing the top 50 eigencats, i.e., the first 50 columns of `V`.

1. Generate a random column vector $r \in \mathbb{R}^{50}$ with the command `randn(50,1)`.
2. Form the vector $x = \hat{V}r + \mu$ where μ is the mean vector (stored in the variable `mu`), which represents a random linear combination of the eigencats with the mean added back.
3. Plot the result using the commands

```
imagesc(reshape(x,dim)); colormap(gray); axis image;
```

The image will not always look perfectly like a cat, but it should resemble one!

Rerun the code several times to generate new cats. Select your top three favorites, and give them affectionate names.

(b) Your second task is to compute the distance of three provided images to "cat space", i.e., the subspace spanned by the top 50 eigencats. First, load these images with the command `load test_images`. This will load into memory vectorized verions of the three 64×64 pixel images `x1`, `x2`, and `x3`, pictured below.

Compute the relative distance of each image from "cat space" using the formula:

$$d_{\text{cat}}(y) = \|y - \hat{V}\hat{V}^T y\| / \|y\|$$

where $y = x - \mu$ is the de-meaned image stored as a column vector. What images are close to "cat space", which are far away? Are your results intuitive? Describe in a few sentences how this computation could be used as part of an algorithm for automatically detecting cats in images.

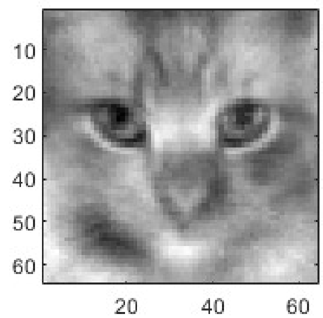
Solution 5. .

(a)

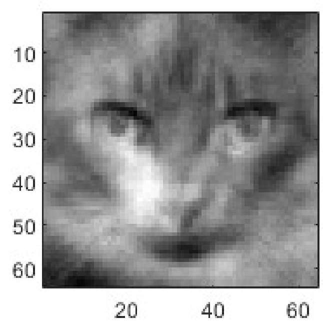
```
%% load dataset
load cat_faces_subset %each row of X is a vectorized 64x64 image
dim = [64,64]; %image dimensions
%% compute eigenvalues & eigenvectors of covariance matrix
mu = mean(X);          %mean image as row vector
Y = X-mu;              %de-mean every image
C = (Y'*Y)/size(Y,1); %data covariance matrix
[V, D] = eig(C);       %V = eigenvectors
eigval = diag(D);      %eigenvalues

%sort eigenvalues in descending order
[eigval,ind] = sort(eigval,'desc');
V = V(:,ind); %reorder eigenvectors as well
mu = mu'; %convert mu to column vector to simplify formulas below

% display top 25 eigencats as images (columns 1-25 of V)
figure(1)
for i=1:25
    subplot(5,5,i)
```

```
    imagesc(reshape(V(:,i),dim)); colormap(gray); axis image
    title(sprintf('eigencat %d',i));
end
%% part(a): Synthesize random cats
%  fill in code here
Vhat = V(:,1:50);
r = randn(50,1);
x = Vhat*r + mu;
imagesc(reshape(x,dim)); colormap(gray); axis image;
```

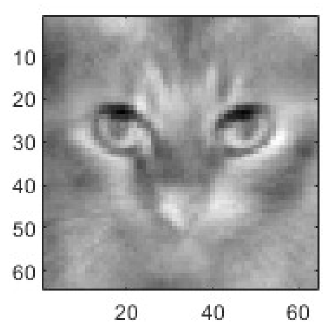
This is Shelly



This is Leonard



This is Penny



(b) .

```
%% part(b): Compute distances onto cat space
%  fill in code here
load test_images.mat

dFromCS = @(y) norm(y-Vhat*Vhat'*y)/norm(y);
dx1 = dFromCS(x1-mu);
dx2 = dFromCS(x2-mu);
dx3 = dFromCS(x3-mu);
disp(dx1);
disp(dx2);
disp(dx3);
% output:
%     0.3515
%     0.4764
%     0.5367
```

The image $x1$ is the closest to "cat space", while image $x3$ is the furthest one. The results are pretty intuitive, since the cat picture is closer to being a cat, while dog is close than a house, but further than a cat, and a house is the furthest of being a cat.

To detect cats in images we would need to set a certain distance from the "cat space" to be considered a cat. Then we can check for slices of the picture and see if that image is smaller than the threshold, which would mean that the image would have cat.

Problem 6 (MSSC, 5pts)**.** Show that $\|x\|_{1/2} := \left(\sum_{i=1}^{n} |x_i|^{1/2}\right)^2$ is NOT a vector norm on $\mathbb{R}^n$ for all $n > 1$. (Hint: Find a pair of vectors x and y in $\mathbb{R}^n$ for which the triangle inequality fails to hold.)

Solution 6. Lets consider the following vectors:

$$x = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Then

$$\|x\|_{1/2} = (\sqrt{0} + \sqrt{2})^2 = 2$$
$$\|y\|_{1/2} = (\sqrt{2} + \sqrt{0})^2 = 2$$
$$\|x + y\|_{1/2} = (\sqrt{2} + \sqrt{2})^2 = (2\sqrt{2})^2 = 8$$

Which does not hold for the triangle inequality $\|x + y\|_{1/2} \leq \|x\|_{1/2} + \|y\|_{1/2}$, since $8 \leq 2 + 2$ is not a true statement.

Problem 7 (MSSC, 5 pts)**.** Let $A \in \mathbb{R}^{n\times n}$ be any symmetric matrix. Define the *Rayleigh quotient* $q_A : \mathbb{R}^n \to \mathbb{R}$ to be the function

$$q_A(x) = \frac{x^T Ax}{x^T x}, \quad x \neq 0$$

Prove that the maximum value of $q_A(x)$ over all $x \neq 0$ is the maximum eigenvalue of A. (Hint: No calculus is needed here. Use the spectral theorem, and consider the invertible change of variables $y = V^T x$ where V is an orthogonal matrix of eigenvectors of V.)

Solution 7. By the Spectral Theorem, we have $A = VDV^T$, where $V \in \mathbb{R}^{nxn}$ is an orthogonal matrix of the eigenvectors of A, and D is $diag(\lambda_1, \cdots, \lambda_n)$, with $\lambda_i \in \mathbb{R}$ being the eigenvalues of A. We can assume the eigenvalues are ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, without losing generality.

Now, if we consider the change of variables $y = V^T x$ we have $Vy = VV^T x \Rightarrow x = Vy$, since V is orthogonal ($VV^T = V^T V = I$). Then

$$\max_{x\neq 0} q_A(x) = \max_{x\neq 0} \frac{x^T Ax}{x^T x} = \max_{y\neq 0} \frac{(Vy)^T AVy}{(Vy)^T Vy} = \max_{y\neq 0} \frac{y^T V^T AVy}{y^T V^T Vy}$$

$$\Rightarrow \max_{y\neq 0} \frac{y^T V^T VDV^T Vy}{y^T y} = \max_{y\neq 0} \frac{y^T Dy}{y^T y} = \max_{y\neq 0} \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2}$$

And because $\lambda_i \leq \lambda_1$ for all $i = 1, 2, \cdots, n$, then $\sum_{i=1}^n \lambda_i y_i^2 \leq \lambda_1 \sum_{i=1}^n y_i^2$, which means that

$$\max_{x\neq 0} q_A(x) = \max_{x\neq 0} \frac{x^T Ax}{x^T x} = \max_{y\neq 0} \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2} \leq \frac{\lambda_1 \sum_{i=1}^n y_i^2}{\sum_{i=1}^n y_i^2} = \lambda_1$$

Therefore $q_A(x) \leq \lambda_1$, where λ_1 is the largest eigenvalue of A.