Homework 3

MSSC 6010- Computational Probability

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September 19, 2023

Question 1. (3.6.39) From book. Let the random variable X be the sum of the numbers on two fair dice. Find an upper bound on $\mathbb{P}(|X-7| \geq 4|)$ using Chebyshev's Inequality as well as the exact probability for $\mathbb{P}(|X-7| \geq 4|)$.

```
X = c(2,3,4,5,6,7,8,9,10,11,12)
X = rbind(X,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))
# Round for the output
X = signif(X,digits=3)
X
```

```
    [,1]   [,2]   [,3]  [,4]  [,5]  [,6]  [,7]  [,8]    [,9]   [,10]   [,11]
X 2.0000 3.0000 4.0000 5.000 6.000 7.000 8.000 9.000 10.0000 11.0000 12.0000
  0.0278 0.0556 0.0833 0.111 0.139 0.167 0.139 0.111  0.0833  0.0556  0.0278
```

```
# Go back to the original
X = c(2,3,4,5,6,7,8,9,10,11,12)
X = rbind(X,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))
```

Then we need μ and σ^2 in order to be able to use Chebyshev's Inequality. We use $\mu = \sum x\mathbb{P}(x)$ and $\sigma^2 = E[X^2] - \mu^2$

```
mu = sum(X[1,]*X[2,])
sigma_2 = sum(X[1,]**2*X[2,])-mu2
print(paste('mu = ', mu, ', sigma2 = ', sigma_2))
```

```
[1] "mu =  7 , sigma^2 =  5.83333333333333"
```

```
# Calculate Chebychev's RHS
my_prob = sigma_2/(4**2)
```

Plugging to Chebychev's Inequality

$$\mathbb{P}(|X-\mu| \geq k|) \leq \frac{\sigma^2}{k^2}$$
$$\mathbb{P}(|X-7| \geq 4|) \leq \frac{5.83333^2}{4^2}$$
$$\mathbb{P}(|X-7| \geq 4|) \leq 0.36458333$$

Now, the exact probability is $\mathbb{P}(X=2)+\mathbb{P}(X=3)+\mathbb{P}(X=11)+\mathbb{P}(X=12)=0.166666$

```
(exact_prob = X[2,1]+X[2,2]+X[2,10]+X[2,11])
```

```
0.1666667
```

```
# Clear environment
rm(list = ls(all=TRUE))
```

Question 2. (3.6.48) From book. Consider the random variable X, which takes the values $1, 2, 3,$ and 4 with probabilities $0.2, 0.3, 0.1,$ and 0.4, respectively. Calculate $E[X]$, $\frac{1}{E[X]}$,$E\left[\frac{1}{X}\right]$,$E[X^2]$, and $E[X]^2$, and check empirically that $E[X]^2 \neq E[X^2]$ and $\frac{1}{E[X]} \neq E\left[\frac{1}{X}\right]$

```
X = c(1,2,3,4)
X = rbind(X, c(0.2,0.3,0.1,0.4))
X
```

```
  [,1] [,2] [,3] [,4]
X  1.0  2.0  3.0  4.0
   0.2  0.3  0.1  0.4
```

Now we can compute all the expectations

```
E_X = sum(X[1,]*X[2,])
one_div_E_X = 1/E_X
E_one_div_X = sum(1/X[1,]*X[2,])
E_X_sqr = sum(X[1,]**2*X[2,])
E_X_qnty_sqr = E_X2
cat("E[X]**2 = ", E_X_qnty_sqr, ", E[X**2] = ",
    E_X_sqr, ", E[X]**2 == E[X**2] is ", E_X_qnty_sqr == E_X_sqr, "\n")
```

```
E[X]**2 =  7.29 , E[X**2] =  8.7 , E[X]**2 == E[X**2] is  FALSE
```

```
cat("And E[1/X] = ", E_one_div_X, ", 1/E[X] = ",
    one_div_E_X, ", E[1/X] == 1/E[X] is ", E_one_div_X==one_div_E_X)
```

```
And E[1/X] =  0.4833333 , 1/E[X] =  0.3703704 , E[1/X] == 1/E[X] is  FALSE
```

Therefore $E[X]^2 \neq E[X^2]$ and $\frac{1}{E[X]} \neq E\left[\frac{1}{X}\right]$.

Question 3. (3.6.50) From book. Find the values of k such that the following functions are probability density functions.

- *(a)* $f(x) = \frac{kx^4}{5}, 0 < x < 1$

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \int_0^1 \frac{kx^4}{5} dx \\
&= \frac{kx^5}{25}\Bigg|_0^1 \\
&= k(\frac{x^5}{25})\Bigg|_0^1 \\
&= k(\frac{1}{25} - 0) \\
1 &= \frac{k}{25} \\
k &= 25
\end{aligned}$$

Question 4. (3.6.52) From book. Consider an experiment where two dice are rolled. Let the random variable X equal the sum of the two dice and the random variable Y be the difference of the two dice.

```
# Clear environment
rm(list = ls(all=TRUE))
```

First let's define X and Y

```
X = c(2,3,4,5,6,7,8,9,10,11,12)
X = rbind(X,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))
Y = c(-5,-4,-3,-2,-1,0,1,2,3,4,5)
Y = rbind(Y,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))

# Round for the output
X = signif(X,digits=3)
X
```

```
    [,1]   [,2]   [,3]  [,4]  [,5]  [,6]  [,7]  [,8]    [,9]   [,10]   [,11]
X 2.0000 3.0000 4.0000 5.000 6.000 7.000 8.000 9.000 10.0000 11.0000 12.0000
  0.0278 0.0556 0.0833 0.111 0.139 0.167 0.139 0.111  0.0833  0.0556  0.0278
```

```
# Go back to the original
X = c(2,3,4,5,6,7,8,9,10,11,12)
X = rbind(X,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))


Y = signif(Y,digits=3)
Y
```

```
     [,1]    [,2]    [,3]   [,4]   [,5]  [,6]  [,7]  [,8]   [,9]  [,10]  [,11]
Y -5.0000 -4.0000 -3.0000 -2.000 -1.000 0.000 1.000 2.000 3.0000 4.0000 5.0000
   0.0278  0.0556  0.0833  0.111  0.139 0.167 0.139 0.111 0.0833 0.0556 0.0278
```

```
# Go back to the original
Y = c(-5,-4,-3,-2,-1,0,1,2,3,4,5)
Y = rbind(Y,c(1/36,2/36,3/36,4/36,5/36,6/36,5/36,4/36,3/36,2/36,1/36))
```

- *(a)* Find the mean of X

```
mu = sum(X[1,]*X[2,])
cat("mean = ", mu)
```

```
mean =  7
```

- *(b)* Find the variance of X

```
sigma_sqr = sum(X[1,]**2*X[2,])-mu2
cat("variance = ", sigma_sqr)
```

```
variance =  5.833333
```

- *(c)* Find the skewness of X

```
skewness = sum((X[1,]-mu)**3*X[2,])/(sqrt(sigma_sqr))**3
cat("skewness = ", skewness)
```

```
skewness =  0
```

- *(d)* Find the mean of Y

```
mu_y = sum(Y[1,]*Y[2,])
cat("mean_y = ", mu_y)
```

```
mean_y =  0
```

- *(e)* Find the variance of Y

```
sigma_sqr_y = sum(Y[1,]**2*Y[2,])-mu_y2
cat("variance_y = ", sigma_sqr_y)
```

```
variance_y =  5.833333
```

- *(f)* Find the skewness of Y

```
skewness_y = sum((Y[1,]-mu_y)**3*Y[2,])/(sqrt(sigma_sqr_y))**3
cat("skewness_y = ", skewness_y)
```

```
skewness_y =  0
```

Question 5. (3.6.58) From book. Consider the probability density function

$$f(x) = \frac{1}{36} x e^{\frac{-x}{6}}, x > 0$$

Derive the moment generating function, and calculate the mean and the variance.

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tX} \left(\frac{1}{36} x e^{\frac{-x}{6}} \right) dx, x > 0 \\
&= \frac{1}{36} \int_0^{\infty} \left(x e^{-x(\frac{1}{6} - t)} \right) dx \\
&= \frac{1}{36} \left[x \left(\frac{1}{t - \frac{1}{6}} e^{-x(\frac{1}{6} - t)} \right) - \int_0^{\infty} \left(\frac{1}{t - \frac{1}{6}} e^{-x(\frac{1}{6} - t)} \right) dx \right] \\
&= \frac{1}{36} \left[x \left(\frac{1}{t - \frac{1}{6}} e^{-x(\frac{1}{6} - t)} \right) - \left(\frac{1}{(t - \frac{1}{6})^2} e^{-x(\frac{1}{6} - t)} \right) \right] \Bigg|_0^{\infty} \\
&= \lim_{b \to \infty} \frac{1}{36} \left[x \left(\frac{1}{t - \frac{1}{6}} e^{-x(\frac{1}{6} - t)} \right) - \left(\frac{1}{(t - \frac{1}{6})^2} e^{-x(\frac{1}{6} - t)} \right) \right] \Bigg|_0^{b} \\
&= \frac{1}{36} \lim_{b \to \infty} \left[b \left(\frac{1}{t - \frac{1}{6}} e^{-b(\frac{1}{6} - t)} \right) - \left(\frac{1}{(t - \frac{1}{6})^2} e^{-b(\frac{1}{6} - t)} \right) - \left(0 - \left(\frac{1}{(t - \frac{1}{6})^2} e^0 \right) \right) \right] \\
&= \frac{1}{36} \lim_{b \to \infty} \left[b \left(\frac{1}{t - \frac{1}{6}} e^{-b(\frac{1}{6} - t)} \right) - \left(\frac{1}{(t - \frac{1}{6})^2} e^{-b(\frac{1}{6} - t)} \right) - \left(-\frac{1}{(t - \frac{1}{6})^2} \right) \right] \\
&= \frac{1}{36} \left(\frac{1}{(t - \frac{1}{6})^2} \right) + \lim_{b \to \infty} \left[b \left(\frac{1}{t - \frac{1}{6}} e^{-b(\frac{1}{6} - t)} \right) - \left(\frac{1}{(t - \frac{1}{6})^2} e^{-b(\frac{1}{6} - t)} \right) \right] \\
M_X(t) &= \frac{1}{36} \left(\frac{1}{(t - \frac{1}{6})^2} \right)
\end{aligned}$$

Since

$$\lim_{b\to\infty}\left[b\left(\frac{1}{t-\frac{1}{6}}e^{-b(\frac{1}{6}-t)}\right)-\left(\frac{1}{(t-\frac{1}{6})^2}e^{-b(\frac{1}{6}-t)}\right)\right]=0$$

$$\lim_{b\to\infty}\left[b\left(\frac{1}{t-\frac{1}{6}}e^{-b(\frac{1}{6}-t)}\right)\right]-\lim_{b\to\infty}\left[\left(\frac{1}{(t-\frac{1}{6})^2}e^{-b(\frac{1}{6}-t)}\right)\right]=$$

$$\lim_{b\to\infty}\left[b\left(\frac{1}{t-\frac{1}{6}}e^{-b(\frac{1}{6}-t)}\right)\right]-\left[\frac{1}{\infty}\right]=$$

$$\lim_{b\to\infty}\left[b\left(\frac{1}{t-\frac{1}{6}}e^{-b(\frac{1}{6}-t)}\right)\right]=$$

$$\frac{1}{t-\frac{1}{6}}\lim_{b\to\infty}\left[b\left(e^{-b(\frac{1}{6}-t)}\right)\right]=$$

$$\frac{1}{t-\frac{1}{6}}\lim_{b\to\infty}\left[\frac{b}{e^{-b(\frac{1}{6}-t)}}\right]=$$

$$\frac{1}{t-\frac{1}{6}}\lim_{b\to\infty}\left[\frac{1}{(-(\frac{1}{6}-t)e^{-b(\frac{1}{6}-t)}}\right]=0 \text{ by L'Hopital}$$

$$\frac{1}{\infty}=0$$

Finally, we can compute the mean and variance.

The mean is the first derivative evaluated at $t=0$

$$\begin{aligned}
E[X]&=\frac{d}{dt}\frac{1}{36(t-1/6)^2}\\
&=\frac{-2}{36}(t-\frac{1}{6})^{-3}\\
&=\frac{-2}{36}(-\frac{1}{6})^{-3}\\
&=12
\end{aligned}$$

And the variance is the second moment, which is the second derivative evaluated at 0, minus $E[X]^2$

$$\begin{aligned}
var(X)&=E[X^2]-E[X]^2\\
&=\frac{6}{36}(t-\frac{1}{6})^{-4}-12^2\\
&=\frac{6}{36}(\frac{1}{6})^{-4}-12^2\\
&=72
\end{aligned}$$

Question 6. (3.6.60) From book. Prove that if a and b are real-valued constants, then

- *(1)*$M_{X+a}(t) = E[e^{(X+a)t}] = e^{at}M_X(t)$.

$$\begin{aligned}
M_{X+a}(t) &= E[e^{(X+a)t}] \\
&= \int_{-\infty}^{\infty} e^{(X+a)t} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{Xt+at} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{Xt}e^{at} f(x)dx \text{ ,and } e^{at} \text{ is a constant} \\
&= e^{at}\int_{-\infty}^{\infty} e^{Xt} f(x)dx \\
&= e^{at}M_X(t)
\end{aligned}$$

- *(2)*$M_{bX}(t) = E[e^{bXt}] = M_X(bt)$.

$$\begin{aligned}
M_{bX}(t) &= E[e^{bXt}] \\
&= \int_{-\infty}^{\infty} e^{bXt} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{(bt)X} f(x)dx \\
&= M_X(bt)
\end{aligned}$$

- *(3)*$M_{\frac{X+a}{b}}(t) = E[e^{(\frac{X+a}{b}t)}] = e^{\frac{a}{b}t}M_X(\frac{t}{b})$.

$$\begin{aligned}
M_{\frac{X+a}{b}}(t) &= E[e^{(\frac{X+a}{b})}] \\
&= \int_{-\infty}^{\infty} e^{(\frac{X+a}{b})} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{\frac{Xt}{b}}e^{\frac{at}{b}} f(x)dx \\
&= e^{\frac{at}{b}}\int_{-\infty}^{\infty} e^{X\frac{t}{b}} f(x)dx \\
&= e^{\frac{at}{b}}M_X(\frac{t}{b})
\end{aligned}$$