# Math 4650/MSSC 5650 - Homework 4

Henri Medeiros Dos Reis

Spring 2023

**Problem 1** (5 pts)**.** For each of the following matrices, determine whether it is positive/negative semi-definite/definite or indefinite. Justify your answer.

(i) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -2 & 1 \\ 3 & -2 & 1 & 0 \\ 4 & 1 & 0 & -1 \end{bmatrix}$

**Solution 1.** .

(i) This matrix is positive definite because it has eigenvalues equal to $\lambda_1 \approx 2.61803$ and $\lambda_2 \approx 0.381966$

(ii) This matrix is negative definite because it has eigenvalues equal to $\lambda_1 \approx -2.61803$ and $\lambda_2 \approx -0.381966$

(iii) This matrix is indefinite because it has eigenvalues equal to $\lambda_1 \approx 5.74166, \lambda_2 \approx -5.12311, \lambda_3 \approx 3.12311$, and $\lambda_4 \approx -1.74166$

**Problem 2** (5 pts)**.** Prove that $f(x_1, x_2) = x_1 x_2$ has a saddle point at $(x_1, x_2) = (0, 0)$.

**Solution 2.**

$$\nabla f(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(x_1, x_2) = (0, 0)$ is the only critical point. Then let's look at the Hessian

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \nabla^2 f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues of $\nabla^2 f(0, 0)$ are $\lambda = -1$ and $\lambda = 1$. Which means that the Hessian matrix is indefinite.

Therefore, the point $(0, 0)$ is a saddle point.

**Problem 3** (15 pts)**.** For each of the following functions, find all the critical points and classify them according to whether they are strict/non-strict global/local minimum/maximum points or saddle points:

(i) $f(x_1, x_2) = (4x_1^2 - x_2)^2$

(ii) $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$

**Solution 3.** (i)

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2(4x_1^2 - x_2)(8x_1) \\ 2(4x_1^2 - x_2)(-1) \end{bmatrix} = \begin{bmatrix} 64x_1^2 - 16x_1x_2 \\ -8x_1^2 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 16x_1(4x_1^2 - x_2) & = 0 \\ -2(x_1^2 - x_2) & = 0 \end{cases}$$

Then either $x_1 = 0$ or $4x_1^2 - x_2 = 0$. If $x_1 = 0$, then $-x_2 = 0 \Rightarrow x_2 = 0$, which means that there is a critical point (0,0).

If $4x_1^2 - x_2 = 0$, then $x_2 = 4x_1^2$ and all points of the form $(x_1, 4x_1^2)$ for all $x_1 \in \mathbb{R}$ are critical points, and that also includes $0, 0$. Then, let's compute the Hessian

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 192x_1^2 - x_2 & -16x_1 \\ -16x_1 & 2 \end{bmatrix}$$

And the Hessian at the critical points

$$\nabla^2 f(x_1, 4x_1^2) = \begin{bmatrix} 128x_1^2 & -16x_1 \\ -16x_1 & 2 \end{bmatrix}$$

To classify this matrix, we need to analyze the eigenvalues, so let's compute the eigenvalues by using the characteristic equation

$$\begin{aligned} 0 &= (128x_1^2 - \lambda)(2 - \lambda) - 256x_1^2 \\ &= -128x_1^2\lambda - 2\lambda + \lambda^2 \\ &= (-128x_1^2 - 2)\lambda + \lambda^2 \\ \Rightarrow \lambda &= \frac{-(-128x_1^2 - 2) \pm \sqrt{(-128x_1^2 - 2)^2 - 4(1)(0)}}{2} \\ &= \frac{128x_1^2 + 2 \pm (128x_1^2 + 2)}{2} \\ \Rightarrow \lambda_1 &= \frac{128x_1^2 + 2 + (128x_1^2 + 2)}{2} = 128x_1 + 2 \\ \lambda_2 &= \frac{128x_1^2 + 2 - (128x_1^2 + 2)}{2} = 0 \end{aligned}$$

Since $x_1^2 \geq 0$, then both eigenvalues are greater or equal to 0, then $\nabla^2 f(x_1, 4x_1^2) \succeq 0$.

Which leaves 2 possibilities, either the critical points are local minimum, or saddle points.

But we know $f(x_1, x_2) = (4x_1^2 - x_2)^2 \geq 0$ and $f(x_1, 4x_1^2) = 0$.

Therefore, $(x_1, 4x_1^2)$ is a non-strict global minimizer for any $x_1 \in \mathbb{R}$

(ii)

$$\nabla f(x_1, x_2) = \begin{bmatrix} 6x_1x_2 \\ 6x_2^2 - 12x_2 + 3x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6x_1x_2 & = 0 \\ 6x_2^2 - 12x_2 + 3x_1^2 & = 0 \end{cases}$$

From the first equation either $x_1 = 0$ or $x_2 = 0$

If $x_1 = 0$ then

$$6x_2^2 - 12x_2 = x_2^2 - 2x_2 = x_2(x_2 - 2) = 0$$
$$\Rightarrow x_2 = 0, x_2 = 2$$

If $x_2 = 0$, then

$$3x_1^2 = 0 \Rightarrow x_1 = 0$$

So the critical points are $(0, 0)$ and $(0, 2)$, the Hessian is

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_2 & 6x_1 \\ 6x_1 & 12x_2 - 12 \end{bmatrix}$$

$$\nabla^2 f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -12 \end{bmatrix} \preceq 0$$

Then, $(0, 0)$ is either a local maximizer or a saddle point. If we look at any small neighborhood of $(0, 0), f$ increases in the positive $x_1$ direction and decreases in the positive $x_2$ direction, therefore $(x_1, x_2) = (0, 0)$ is a saddle point. Now, let's look at the Hessian at other point.

$$\nabla^2 f(0, 2) = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \succ 0$$

Therefore, $(x_1, x_2) = (0, 2)$ is a local minimum.

**Problem 4** (MSSC, 5 pts). Let $y_1, y_2, ..., y_m$ be a collection of $m$ vectors in $\mathbb{R}^n$. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{m} \|x - y_i\|^2.$$

Use the optimality conditions to find a global minimizer $x^*$ of $f$. Is the global minimizer unique? Justify your answer.

**Solution 4.**

$$\begin{aligned}
f(x) &= \sum_{i=1}^{m} \|x - y_i\|^2 \\
&= \sum_{i=1}^{m} (x^T x - x^T y_i + y_i^T y_i) \\
&= \sum_{i=1}^{m} (\|x\|^2 - 2x^T y_i + \|y_i\|^2 \\
&= m\|x\|^2 - 2\sum_{i=1}^{m} x^T y_i + \sum_{i=1}^{m} \|y_i\|^2
\end{aligned}$$

Then, we take the gradient of f and set it equal to 0, then solve for x

$$\nabla f(x) = 2mx - 2\sum_{i=1}^{m} y_i$$

$$\begin{aligned}
\Rightarrow 2(mx - \sum_{i=1}^{m} y_i) &= 0 \\
mx &= \sum_{i=1}^{m} y_i \\
x &= \frac{1}{m}\sum_{i=1}^{m} y_i
\end{aligned}$$

So $x = \frac{1}{m}\sum_{i=1}^{m} y_i$ is the only critical point.

The Hessian of f, $\nabla^2 f(x) = 2mI_{n\times n}$, and since m is the size of the collection of vectors $y_i$, then $m > 0$, which means that $\nabla^2 f(x) \succ 0$.

Therefore, f is strictly convex, and $x = \frac{1}{m}\sum_{i=1}^{m} y_i$ is the strict global minimizer.

**Problem 5** (MSSC, 5 pts). *True or False:* Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and $x^* \in \mathbb{R}^2$ is a critical point of $f$. If $\nabla^2 f(x^*) \succeq 0$ then $x^*$ is a local minimizer of $f$. (If true, prove it; if false, provide a counter-example).

**Solution 5.** False.

Let's consider $f(x_1, x_2) = 2x_1^2 - 4x_2^4$, which is twice continuously differentiable function. Then when we take the gradient and set it equal to 0 we have

$$\nabla f(x_1, x_2) = \begin{bmatrix} 4x_1 \\ 16x_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 0 \text{ and } x_2 = 0$$

Then, the only critical point $x^* = (x_1, x_2) = (0, 0)$. And the Hessian is

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 48x_2^2 \end{bmatrix}$$

And the Hessian evaluated at $(0, 0)$ is

$$\nabla^2 f(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$$

Hence, either the critical point is a local min or a saddle point. Which in this case is a saddle point. Because $f(0, x_2) = -4x_2^2 \leq 0$

Therefore, $x^* = (0, 0)$ is a critical point, the Hessian at $x^*$ is positive semi definite. But, $x^*$ is not a local minimizer, which makes the statement false.