MSSC 5931 - Homework 1

Henri Medeiros Dos Reis

September 26, 2022

1. **Quadratic extrapolation of a time series.** We are given a series z up to time t. Using a quadratic model, we want to extrapolate, or predict, $z(t+1)$ based on the three previous elements of the series, $z(t), z(t-1)$, and $z(t-2)$. We'll denote the predicted value of $z(t+1)$ by $\hat{z}(t+1)$. More precisely, you will find $\hat{z}(t+1)$ as follows.

   a) Find the quadratic function $f(\tau) = a_2\tau^2 + a_1\tau + a_0$ which satisfies $f(t) = z(t), f(t-1) = z(t-1)$, and $f(t-2) = z(t-2)$. Then the extrapolated value is given by $\hat{z}(t+1) = f(t+1)$.Show that

   $$\hat{z}(t+1) = c \begin{bmatrix} z(t) \\ z(t-1) \\ z(t-2) \end{bmatrix},$$

   where $c \in \mathbb{R}^{1\times 3}$, and does not depend on t. In other words, the quadratic extrapolator is a linear function. Find c explicitly.

   b) Use the following R code to generate a time series z:

   ```
   t <- seq(1:1000);
   z <- 5*sin(t/10+2) + 0.1*sin(t) + 0.1*sin(2*t-5);
   ```

   Use the quadratic extrapolation method from part (a) to find $\hat{z}(t)$ for $t = 4, ..., 1000$. Find the relative root-mean-square (RMS) error, which is given by

   $$\left(\frac{(1/997)\sum_{j=4}^{1000}(\hat{z}(j) - z(j))^2}{1/997)\sum_{j=4}^{1000} z(j)^2}\right)^{1/2}$$

Solution 1. a)

$$z(t) = f(t) = a_2t^2 + a_1t + a_0$$
$$z(t-1) = f(t-1) = a_2(t-1)^2 + a_1(t-1) + a_0$$
$$z(t-2) = f(t-2) = a_2(t-2)^2 + a_1(t-2) + a_0$$
$$z(t+1) = f(t+1) = a_2(t+1)^2 + a_1(t+1) + a_0$$

Then:

$$a_2(t+1)^2 + a_1(t+1) + a_0 = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_2t^2 + a_1t + a_0 \\ a_2(t-1)^2 + a_1(t-1) + a_0 \\ a_2(t-2)^2 + a_1(t-2) + a_0 \end{bmatrix}$$

$$c_1 + a_2t^2 + c_1a_1t + c_1a_0 + c_2a_2t^2 - 2c_2a_2t + c_2a_2 + c_2a_1t - c_2a_1 + c_2a_0 + c_3a_2t^2$$
$$-4c_3a_2t + 4c_3a_2 + c_3a_1t - 2c_3a_1 + c_3a_0 = a_2t^2 + 2a_2t + a_2 + a_1t + a_1 + a_0$$

Now we can set a system of equations:

$$c_1a_2t^2 + c_2a_2t^2 + c_3a_2t^2 = a_2t^2$$

$$c_1a_1 - 2c_2a_2t + c_2a_1t - 4c_3a_2ta_2t + c_3a_1t = 2a_2t + a_1t$$

$$c_1a_0 + c_2a_2 - c_2a_1 + c_2a_0 + 4c_3a_2 - 2c_3a_1 + c_3a_0 = a_2 + a_1 + a_0$$

Writing as a matrix form gives:

$$\begin{bmatrix} a_2t^2 & a_2t^2 & a_2t^2 & a_2t^2 \\ a_1t & -2a_2t + a_1t & -4a_2t + a_1t & 2a_2t + a_1t \\ a_0 & a_2 - a_1 + a_0 & 4a_2 - 2a_2 + a_0 & a_2 + a_1 + a_0 \end{bmatrix}$$

Which we need to row reduce to echelon form. Let's first divide R^1 by a_2t^2, and R^2 by t:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & -2a_2 + a_1t & -4a_2 + a_1 & 2a_2 + a_1 \\ a_0 & a_2 - a_1 + a_0 & 4a_2 - 2a_2 + a_0 & a_2 + a_1 + a_0 \end{bmatrix}$$

$\Rightarrow R^2 = R^2 - a_1R^1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2a_2 & -4a_2 & 2a_2 \\ a_0 & a_2 - a_1 + a_0 & 4a_2 - 2a_2 + a_0 & a_2 + a_1 + a_0 \end{bmatrix}$$

$\Rightarrow R^3 = R^3 - a_0R^1$ and $R^2 = \frac{-1}{2a_2}R^2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & a_2 - a_1 & 4a_2 - 2a_2 & a_2 + a_1 \end{bmatrix}$$

$\Rightarrow R^1 = R^1 - R^2$ and $R^3 = R^3 - (a_2 - a_1)R^2$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2a_2 & 2a_2 \end{bmatrix}$$

$\Rightarrow R^3 = R^3/2a_2$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow R^1 = R^1 + R^3$ and $R^2 = R^2 - 2R^3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, $c = \begin{bmatrix} 3 & -3 & 1 \end{bmatrix}$

b)

```
#define given variables
t <- seq (1:1000) ;
z <- 5 * sin ( t / 10+2) + 0.1 * sin ( t ) + 0.1 * sin (2 *t
   -5) ;
c = matrix(c(3,-3,1), nrow=1)
#create a function that will calculate zhat
zhat <- function(c_param, zt, zt_minus_1, zt_minus_2) {
  zmatrix <- matrix(c(zt,zt_minus_1, zt_minus_2), ncol=1)
  return(c_param%*%zmatrix)
}
#declare empty vector for zhat
zhat_vector<-c(0,0,0)
#compute all zhats
for(i in 4:1000){
  zhat_vector[i]<- zhat(c, z[i-1],z[i-2],z[i-3])
}
#take out the first 4 elements of the arrays, since sum
   starts at j=4
zhat_vector<- tail(zhat_vector, 997)
z<- tail(z, 997)

RMS <- sqrt(((1/997)*sum((zhat_vector-z)^2))/((1/997)*sum((z)
   ^2)))

cat('RMS = ',RMS)
#output => RMS =  0.09724805
```

2. **Price elasticity of demand.** The demand for n different goods is a function of their prices:

$$q = f(p)$$

where p is the price vector, q is the demand vector, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the demand function. The current price and demand are denoted p^* and q^* , respectively. Now suppose there is a small price change δp, so$p = p^* + \delta p$. This induces a change in demand, to $q \approx q^* + \delta q$, where

$$\delta q \approx Df(p^*)\delta p$$

where Df is the derivative or Jacobian of f, with entries

$$Df(p^*)_{ij} = \frac{\delta f_i}{\delta p_j}(p^*)$$

This is usually rewritten in term of the *elasticity matrix* E, with entries

$$E_{ij} = \frac{\delta f_i}{\delta p_j}(p^*)\frac{1/q_i^*}{1/p_i^*},$$

so $E_i j$ gives the relative change in demand for good i per relative change in price j. Defining the vector y of relative demand changes, and the vector x of relative price changes,

$$y_i = \frac{\delta q_i}{q_i^*}, \quad x_j = \frac{\delta p_j}{p_j^*}$$

we have the linear model $y = Ex$.

Here are the questions:

a) What is a reasonable assumption about the diagonal elements E_{ii} of the elasticity matrix?

b) Goods i and j are called substitutes if they provide a similar service or other satisfaction (e.g., train tickets and bus tickets, cake and pie, etc.). They are called complements if they tend to be used together (e.g., automobiles and gasoline, left and right shoes, etc.). For each of these two generic situations, what can you say about E_{ij} and E_{ji}?

c) Suppose the price elasticity of demand matrix for two goods is

$$E = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Describe the *nullspace* of E, and give an interpretation (in one or two sentences). What kind of goods could have such an elasticity matrix?

Solution 2.

3. **Halfspace.** Suppose $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, i.e.:

$$\{x \mid ||x - a|| \ \leq \ ||x - b||\} = \{x \mid c^T x \leq d\}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

Solution 3. Let $x \in \mathbb{R}^n$ be closer to a than to b.

$$||x-a|| \leq ||x-b||$$

$$||x-a|| = \sqrt{(x-a)^T(x-a}, \; ||x-b|| = \sqrt{(x-b)^T(x-b}$$

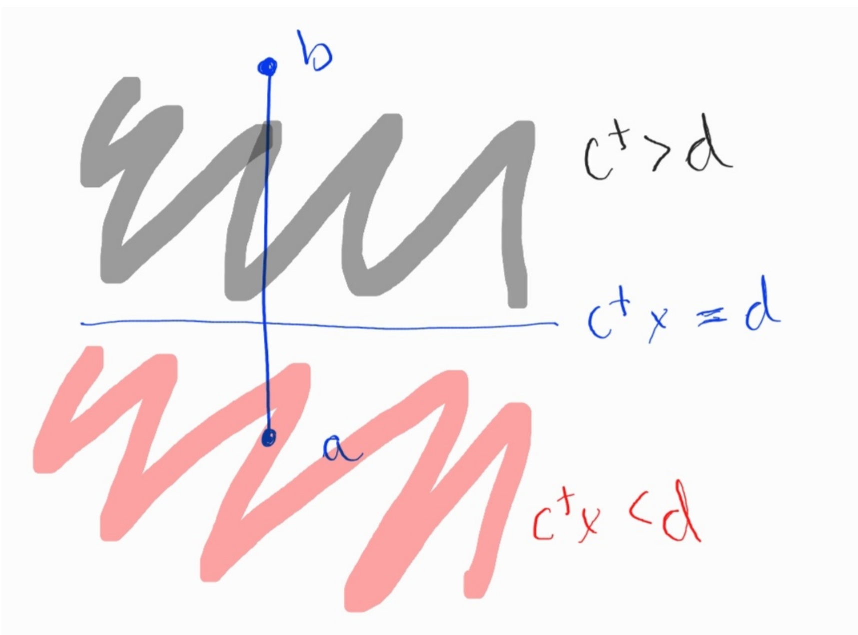
Then $\sqrt{(x-a)^T(x-a} \leq \sqrt{(x-b)^T(x-b}$, let's square both sides, and expand the multiplication.

$$x^Tx - x^Ta - a^Tx + a^Ta \leq x^Tx - x^Tb - b^Tx + b^Tb \Rightarrow$$

$$-2x^Ta + a^Ta \leq -2x^Tb + b^Tb = -2xa^T + 2xb^T \leq b^Tb - a^Ta \Rightarrow$$

$$2x(b-a)^T \leq b^Tb - a^Ta = x(b-a)^T \leq \frac{b^Tb - a^Ta}{2}$$

Let $(b-a) = c$ and $\frac{b^Tb-a^Ta}{2} = d$, then we can rewrite the equation as $c^Tx \leq d$, which shows that any set of points in $\mathbb{R}^n$ that is closer to a than to b has to be a halfspace, where d is orthogonal to c



4. **Temperatures in a multi-core processor** We are concerned with the temperature of a processor at two critical locations. These temperatures, denoted $T = (T_1, T_2)$ (in degrees C), are affine functions of the power dissipated by three processor cores, denoted $P = (P_1, P_2, P_3)$ (in W). We make 4 measurements. In the first, all cores are idling, and dissipate $10W$. In the next three measurements, one of the processors is set to full power, $100W$, and the other two are idling. In each experiment we measure and note the temperatures at the two critical locations.

P_1	P_2	P_3	T_1	T_2
10W	10W	10W	27°	29°
100W	10W	10W	45°	37°
10W	100W	10W	41°	49°
10W	10W	100W	35°	55°

Suppose we operate all cores at the same power, p. How large can we make p, without T_1 or T_2 exceeding 70°?
You must fully explain your reasoning and method, in addition to providing the numerical solution.

Solution 4. The temperature is an affine function of P, $T = AP + b$, where $b \in \mathbb{R}^2$, A is the coefficient matrix, $A \in \mathbb{R}^{2\times 3}$, and P is the power (in W) that give T_1 and T_2.Then we can write the systems as:

$$T = AP + b =$$
$$\begin{bmatrix} 10a_{11} & 10a_{12} & 10a_{13} & b_1 \\ 10a_{21} & 10a_{22} & 10a_{23} & b_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 29 \end{bmatrix},$$
$$\begin{bmatrix} 100a_{11} & 10a_{12} & 10a_{13} & b_1 \\ 100a_{21} & 10a_{22} & 10a_{23} & b_2 \end{bmatrix} = \begin{bmatrix} 45 \\ 37 \end{bmatrix},$$
$$\begin{bmatrix} 10a_{11} & 100a_{12} & 10a_{13} & b_1 \\ 10a_{21} & 100a_{22} & 10a_{23} & b_2 \end{bmatrix} = \begin{bmatrix} 41 \\ 49 \end{bmatrix},$$
$$\begin{bmatrix} 10a_{11} & 10a_{12} & 100a_{13} & b_1 \\ 10a_{21} & 10a_{22} & 100a_{23} & b_2 \end{bmatrix} = \begin{bmatrix} 35 \\ 55 \end{bmatrix}$$

Now, let's write all these systems as one, $Cx = d$, where x is a vector containing all entries of A, and all entries of b. And d is a vector of all temperatures.

$$Cx = \begin{bmatrix} 10a_{11} & 10a_{12} & 10a_{13} & 0a_{21} & 0a_{22} & 0a_{23} & 1b_1 & 0b_2 \\ 0a_{11} & 0a_{12} & 0a_{13} & 10a_{21} & 10a_{22} & 10a_{23} & 0b_1 & 1b_2 \\ 100a_{11} & 10a_{12} & 10a_{13} & 0a_{21} & 0a_{22} & 0a_{23} & 1b_1 & 0b_2 \\ 0a_{11} & 0a_{12} & 0a_{13} & 100a_{21} & 10a_{22} & 10a_{23} & 0b_1 & 1b_2 \\ 10a_{11} & 100a_{12} & 10a_{13} & 0a_{21} & 0a_{22} & 0a_{23} & 1b_1 & 0b_2 \\ 0a_{11} & 0a_{12} & 0a_{13} & 10a_{21} & 100a_{22} & 10a_{23} & 0b_1 & 1b_2 \\ 10a_{11} & 10a_{12} & 100a_{13} & 0a_{21} & 0a_{22} & 0a_{23} & 1b_1 & 0b_2 \\ 0a_{11} & 0a_{12} & 0a_{13} & 10a_{21} & 10a_{22} & 100a_{23} & 0b_1 & 1b_2 \end{bmatrix}, d = \begin{bmatrix} 27 \\ 29 \\ 45 \\ 37 \\ 41 \\ 49 \\ 35 \\ 55 \end{bmatrix}$$

$$Cx = d \Rightarrow \begin{bmatrix} 10 & 10 & 10 & 0 & 0 & 0 & 1 & 0 & 27 \\ 0 & 0 & 0 & 10 & 10 & 10 & 0 & 1 & 29 \\ 100 & 10 & 10 & 0 & 0 & 0 & 1 & 0 & 45 \\ 0 & 0 & 0 & 100 & 10 & 10 & 0 & 1 & 37 \\ 10 & 100 & 10 & 0 & 0 & 0 & 1 & 0 & 41 \\ 0 & 0 & 0 & 10 & 100 & 10 & 0 & 1 & 49 \\ 10 & 10 & 100 & 0 & 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 10 & 10 & 100 & 0 & 1 & 55 \end{bmatrix}$$

Row reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{45} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{4}{45} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{4}{45} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{13}{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{203}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 23 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{7}{45} & \frac{4}{45} \\ \frac{4}{45} & \frac{2}{9} & \frac{13}{45} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{203}{9} \\ 23 \end{bmatrix}$$

Now, plugging back to $T = AP + b$, we have$t = \begin{bmatrix} \frac{1}{5}p_1 + \frac{7}{45}p_2 + \frac{4}{45}p_3 + \frac{203}{9} \\ \frac{4}{45}p_1 + \frac{2}{9}p_2 + \frac{13}{45}p_3 + 23 \end{bmatrix}$, since we are operating with $p_1, p_2, p_3 = P$. Then:

$$T_1 = \frac{4}{9}P + \frac{209}{9},\ T_2 = \frac{3}{5}P + 23$$

Finally, we can solve for P, given $T = 70$

$$70 = \frac{4}{9}P + \frac{209}{9} \Rightarrow P = \frac{70 - \frac{203}{9}}{\frac{4}{9}} = 106.73$$

$$70 = \frac{3}{5}P + 23 \Rightarrow P = \frac{70 - 23}{\frac{3}{5}} \approx 78.33$$

Therefore, the maximum P that will not exceed $70°C$ is $P = 78.33W$

5. **Projection matrices.** A matrix $P \in \mathbb{R}^{n\times n}$ is called a projection matrix if $P = P^T$ and $P^2 = P$. (These properties are sometimes called symmetry and idempotency, respectively.)

   a) Show that if P is a projection matrix, then $I - P$ is also a projection matrix.

   b) Suppose $U \in \mathbb{R}^{n\times k}$ has orthonormal columns. Show that UU^T is a projection matrix. (The converse is also true: every projection matrix can be written as UU^T for some matrix U with orthonormal columns; you do not need to prove this.)

   c) Suppose $A \in \mathbb{R}^{n\times k}$ is skinny, and full rank. Show that $A(A^TA)^{-1}A^T$ is a projection matrix.

   d) Given $S \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, the point $\hat{x} \in S$ that is closest to x is called the projection of x onto S. Show that if P is a projection matrix, then $\hat{x} = Px$ is the projection of x onto $range(P)$. (This is the origin of the term "projection matrix.")

Solution 5.

(a) To show that a matrix is a projection matrix, the matrix $(I-P)=(I-P)^T$ and $(I-P)^2 = I-P$. $(I-P)^T = I^T - P^T$, since $I^T = I$ and because P is a projection matrix, then $P^T = P$. Finally $(I-P)^T = (I-P)$. And $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$. Therefore $(I-P)$ is a projection matrix.

(b) Because columns of U are orthonormal, $U^TU = I$. First, $UU^T = (UU^T)^T = U^TU = UU^T$. Second, $(UU^T)^2 = (UU^T)(UU^T) = U(U^TU)U^T$, where $U^TU = I$, then $(UU^T)^2 = UU^T$. Therefore UU^T is a projection matrix.

(c) First,

$$\begin{gathered}(A(A^TA)^{-1}A^T)^T = A^T((A^TA)^{-1})^T(A^T)^T \Rightarrow \\ A^T((A^TA)^T)^{-1}A = A^T(AA^T)^{-1}A \Rightarrow \\ A(A^TA)^{-1}A^T = (A(A^TA)^{-1}A^T)^T\end{gathered}$$

Second,

$$\begin{gathered}(A(A^TA)^{-1}A^T)^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \Rightarrow \\ A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T\end{gathered}$$

Since $B^{-1}B = I$, then $(A^TA)^{-1}A^TA = I$. Then

$$(A(A^TA)^{-1}A^T)^2 = A(A^TA)^{-1}A^T$$

Therefore $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) We need to show that Px is a projection of x onto $range(P)$. then $\hat{x} - P$ has to be orthogonal to any vector in $range(P)$. Hence $P^T(\hat{x} - Px) = 0$, since P is a projection matrix, then $P^T = P$ and $P^2 = P$.

$$P^T(\hat{x} - Px) = P(x - Px) = Px - P^2x = P_x - P_x = 0$$

Therefore $\hat{x} = Px$ is the projection of x onto $range(P)$

6. **Single sensor failure detection and identification.** We have $y = Ax$, where $A \in \mathbb{R}^{m\times n}$ is known, and $x \in \mathbb{R}^n$ is to be found. Unfortunately, up to one sensor may have failed (but you don't know which one has failed, or even whether any has failed). You are given $\tilde{y}$ and not y, where $\tilde{Y}$ is the same as y in all entries except, possibly, one (say, the k_{th} entry). If all sensors are operating correctly, we have $y = \tilde{Y}$. If the k_{th} sensor fails, we have $\tilde{y}_i = y_i$ for all $i \neq k$. The file **one_bad_sensor.Rdata**, available on the course web site, defines A and $\tilde{y}$ (as A and ytilde). Determine which sensor has failed (or if no sensors have failed). You must explain your method, and submit your code. For this exercise, you can use the R code **qr(cbind(F,g))$rank == qr(F)$rank** to check if $g \in range(F)$. (We will see later a much better way to check if $g \in range(F)$.)

Solution 6. Let A_test_i be the A matrix with the ith row removed. Let y_test_i be he vector y, with the ith entry removed. If the sensor has a failure, then $y \notin Range(A)$ but if we remove the sensor that is failing then all the other y_test_i $\in Range(A)$. Then we have can write the test in R, and expecting **qr(cbind(F,g))$rank == qr(F)$rank +1** to give true for all $g \in Range(F)$. The code and results:

```
#given variables
A
ytilde


#run 15 times, and remove i from A and ytilde and test it
for(i in 1:15){
  y_test_i <- ytilde[-i]
  A_test_i <- A[-i,]
  cat(i = , i, , result:  ,
      qr(cbind(A_test_i,y_test_i))$rank == qr(A)$rank+1, \n)
}
##Output
#i =  1 , result:   TRUE
#i =  2 , result:   TRUE
#i =  3 , result:   TRUE
#i =  4 , result:   TRUE
#i =  5 , result:   TRUE
#i =  6 , result:   TRUE
#i =  7 , result:   TRUE
#i =  8 , result:   TRUE
#i =  9 , result:   TRUE
#i =  10 , result:   TRUE
#i =  11 , result:   FALSE
#i =  12 , result:   TRUE
#i =  13 , result:   TRUE
#i =  14 , result:   TRUE
#i =  15 , result:   TRUE
```

Therefore, the sensor 11 is the one that is failing