MSSC 5931 - Homework 1

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1. **Some basic properties of eigenvalues.** Show the following:

   a) The eigenvalues of A and A^T are the same.

   b) A is invertible if and only if A does not have a zero eigenvalue.

   c) If the eigenvalues of A are $\lambda_1, \cdots, \lambda_n$ and $A$is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, ..., 1/\lambda_n$.

   d) The eigenvalues of A and $T^{-1}AT$ are the same.

   Hint: you'll need to use the facts that $detA = det(AT)$, $det(AB) = detAdetB$, and, if A is invertible, $detA^{-1} = 1/detA$.

   Solution 1. .

   a) Since $det(A - \lambda I) = 0$ and $det(A) = det(A^T)$ then $det[(A - \lambda I)^T] = det(A^T - \lambda I) = det(A^T - \lambda I)$.
   Therefore, $det(A - \lambda I) = det(A^T - \lambda I)$, which shows that the eigenvalues are the same.

   b) Since $Ax = \lambda x$, then if any eigenvalues are 0, that means $Ax = 0 = \lambda x$ for any $x \neq 0$. Which makes it not full rank, therefore not invertible.

   c) $A = V\Lambda V^T$, then $A^{-1} = (V\Lambda V^T)^{-1} = (V^T)^{-1}\Lambda^{-1}V^T$, and V is unitary.
   Thus $AA^{-1} = V\Lambda V^T V\Lambda^{-1}V^T = V\Lambda\Lambda^{-1}V^T = I$. Finally $\Lambda\Lambda^{-1} = I$.
   Since $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$, and the inverse of a diagonal matrix has to be the inverse of all its entries, then $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{1}{\lambda_n} \end{bmatrix}$

   d) Since $det(A - \lambda I) = 0$, we can say $det(T^{-1}AT - T^-1\lambda IT) = 0$, and by simplifying that equation we can get $det(T^{-1})det(A - \lambda I)det(T) = \frac{1}{det(T)}det(A - \lambda I)det(T) = det(A - \lambda I) = 0$. Therefore the eigenvalues are the same.

2. **Estimating a signal with interference.**

   Solution 2.

3. **Properties of symmetric matrices.** In this problem P and Q are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

   a) If $P \geq 0$ then $P + Q \geq Q$.
   b) If $P \geq Q$ then $-P \geq -Q$.
   c) If $P > 0$ then $P^{-1} > 0$.
   d) If $P \geq Q > 0$ then $P^{-1} \leq Q^{-1}$.
   e) If $P \geq Q$ then $P^2 \geq Q^2$

   Hint: you might find it useful for part (d) to prove $Z \geq I$ implies $Z^{-1} \leq I$.

   Solution 3. .

   a) Since P is greater than 0, anything that we add to it, is going to increase its values.

   $$P \geq 0 \Rightarrow P + Q \geq Q$$

   Simply by adding Q to both sides.

   b) If $A \geq 0$ and $\alpha > 0$, then $\alpha A \geq 0$. Then if $A \geq 0$ and $\alpha < 0$, then $\alpha A \leq 0$. Therefore, if we let $\alpha = -1$, and $P \geq Q$, then $-P \leq -Q$

   c) Since $P > 0$, then $\lambda_1 x^T x > x^T P x > \lambda_n x^T x$ for all x and λ_1 is the biggest eigenvalue and λ_n is the smallest eigenvalue. All the eigenvalues of P are positive. Also, because the eigenvalues of P^{-1} are $\Lambda^{-1} = \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{1}{\lambda_n} \end{bmatrix}$, then the eigenvalues are positive as well.
   Therefore $\frac{1}{\lambda_n} x^T x > x^T P^{-1} x > \frac{1}{\lambda_1} x^T x$, and $P^{-1} > 0$

   d) Let's start by showing that $Z \geq I$ implies $Z^{-1} \leq I$.

   $$Z \geq I \Rightarrow Z - I \geq 0 \Rightarrow x^T (Z - I) x \geq 0$$

   Now, we can use the same work we did in part c): $\lambda_1 x^T x > x^T (Z - I) x > \lambda_n x^T x$ and $\frac{1}{\lambda_n} x^T x > x^T (Z - I)^{-1} x > \frac{1}{\lambda_1} x^T x \Rightarrow \frac{1}{\lambda_n} x^T x > x^T (Z^{-1} - I) x > \frac{1}{\lambda_1} x^T x$. With all values of $\frac{1}{\lambda}$ being between 0 and 1.

   $$x^T (Z^{-1} - I) x \leq 0 \Rightarrow Z^{-1} - I \leq 0 \Rightarrow Z^{-1} \leq I$$

   Now by using the same logic we can show that if $P \geq Q > 0$, then $P - Q \geq 0$. Then $x^T (P - Q) x \geq 0$, and again using part c) again: $\lambda_1 x^T x > x^T (P - Q) x > \lambda_n x^T x$ and $\frac{1}{\lambda_n} x^T x > x^T (P - Q) x > \frac{1}{\lambda_1} x^T x$, with all values of $\frac{1}{\lambda}$ between 1 and 0.

   $$x^T (P - Q)^{-} 1 x \leq 0 \Rightarrow x^T (P^{-1} - Q^{-1}) x \leq 0 \Rightarrow P^{-1} - Q^{-1} \leq 0 \Rightarrow P^{-1} \leq Q^{-1}$$

e) Let $P = -3$ and $Q = -4$, which are just 1×1 matrices. Then:

$$-3 > -4 \text{ But, } (-3)^2 = 9 < 16 = (-4)^2$$

4. **Real modal form.** Generate a matrix A in $\mathbb{R}^{10\times 10}$ using `A=rnorm(10)` . (The entries of A will be drawn from a unit normal distribution.) Find the eigenvalues of A. If by chance they are all real, please generate a new instance of A. Find the real modal form of A, i.e., a matrix S such that $S^{-1}AS$ has the real modal form given in topic 7. Your solution should include a clear explanation of how you will find S, the source code that you use to find S, and some code that checks the results (i.e., computes $S^{-1}AS$ to verify it has the required form).

Solution 4. Let $A = T\Lambda T^{-1}$ be the eigen decomposition of the matrix A, where $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Where we have real and complex values. We can separate the real eigenvalues from the complex ones, such that $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_r, \lambda_{r+1}, \cdots, \lambda_n)$, where the real eigenvalues are from the first until the r^{th} entry, and the remaining are complex.

Then we can write $S^{-1}AS$, where S will have the eigenvectors of A, which will be complex and real as well. So, we can separate those as well.

Then $S = \begin{bmatrix} t_1 & \cdots & t_r & Real(t_{r+1}) & Img(t_{r+1}) & \cdots & Real(t_n) & Img(t_n) \end{bmatrix}$, and since $Ax = \lambda x$, where x is the vector that we need to separate, such that $x = Real(x) + Img(x)$, then $\lambda = \sigma + i\omega$

Therefore $Ax = \lambda x \Rightarrow A(Real(x) + Img(x)) = (\sigma + i\omega)(Real(x) + Img(x)$.

$$\Rightarrow A[Real(x) \;\; Img(x)] = [Real(x) \;\; Img(x)] \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\text{and } S^{-1}AS = diag\left(\Lambda_r, \begin{bmatrix} \sigma_{r+1} & \omega_{r+1} \\ -\omega_{r+1} & \sigma_{r+1} \end{bmatrix}, \cdots, \begin{bmatrix} \sigma_n & \omega_n \\ -\omega_n & \sigma_n \end{bmatrix}\right)$$

```
set.seed(1)#set seed for consistent results
A=matrix(rnorm(100),nr=10)
eigDec <- eigen(A)

eigDec$values
#------------Output---------
#[1] -2.6820626+0.000000i -1.0060561+2.268741i
   -1.0060561-2.268741i
#[4]  1.5343031+0.938029i  1.5343031-0.938029i
   0.7624088+1.153254i
#[7]  0.7624088-1.153254i -0.3368476+1.107465i
   -0.3368476-1.107465i
#[10] -0.0569953+0.000000i
```

```
#we can see that the eigenvalues 1,10 are real
#So lets arrange them

S <- matrix(0,10,10)
S[,1] <- eigDec$vectors[,1] #the first real eigenvector
S[,2] <- eigDec$vectors[,10] #the last real eigenvector
# now we can start to populate S with the complex
   eigenvectors, by separating their
# real and complex part, by pairs, since they are always two
   that are the same.
S[,3] <- Re(eigDec$vectors[,2])
S[,4] <- Im(eigDec$vectors[,2])
S[,5] <- Re(eigDec$vectors[,4])
S[,6] <- Im(eigDec$vectors[,4])
S[,7] <- Re(eigDec$vectors[,6])
S[,8] <- Im(eigDec$vectors[,6])
S[,9] <- Re(eigDec$vectors[,8])
S[,10] <- Im(eigDec$vectors[,8])

solution <- solve(S)%*%A%*%S
round(solution)
#------------Output---------
#      [,1] [,2]  [,3]  [,4]  [,5] [,6]  [,7] [,8]  [,9] [,10]
#[1,] -3+0i 0+0i  0+0i  0+0i  0+0i 0+0i  0+0i 0+0i  0+0i 0+0i
#[2,]  0+0i 0+0i  0+0i  0+0i  0+0i 0+0i  0+0i 0+0i  0+0i 0+0i
#[3,]  0+0i 0+0i -1+0i  2+0i  0+0i 0+0i  0+0i 0+0i  0+0i 0+0i
#[4,]  0+0i 0+0i -2+0i -1+0i  0+0i 0+0i  0+0i 0+0i  0+0i 0+0i
#[5,]  0+0i 0+0i  0+0i  0+0i  2+0i 1+0i  0+0i 0+0i  0+0i 0+0i
#[6,]  0+0i 0+0i  0+0i  0+0i -1+0i 2+0i  0+0i 0+0i  0+0i 0+0i
#[7,]  0+0i 0+0i  0+0i  0+0i  0+0i 0+0i  1+0i 1+0i  0+0i 0+0i
#[8,]  0+0i 0+0i  0+0i  0+0i  0+0i 0+0i -1+0i 1+0i  0+0i 0+0i
#[9,]  0+0i 0+0i  0+0i  0+0i  0+0i 0+0i  0+0i 0+0i  0+0i 1+0i
#[10,] 0+0i 0+0i  0+0i  0+0i  0+0i 0+0i  0+0i 0+0i -1+0i 0+0i
```

As we can see in the output, the matrix has the proposed form. Also, all the entries of the matrix have a complex part, however, all those values are 0.

5. **Spectral mapping theorem.** Suppose $f : R \to R$ is analytic, i.e., given by a power series expansion

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots$$

where $a_i = f^{(i)}(0)/(i!))$. (You can assume that we only consider values of u for which this series converges.) For $A \in \mathbb{R}^{n\times n}$, we define $f(A)$ as

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots$$

(again, we'll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbb{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the spectral mapping theorem.

To illustrate this with an example, generate a random 3×3 matrix, for example using `A=rnorm(3)`. Find the eigenvalues of $(I+A)(I-A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various computations.)

Solution 5.

$$f(A)v = a_0 Iv + a_1 Av + a_2 A^2 v + \cdots$$

And since $\lambda v = Av$ and $\lambda^k v = A^k v$ for $k = 1, 2, \cdots$, then

$$f(A)v = a_0 Iv + a_1 \lambda v + a_2 \lambda^2 v + \cdots = f(\lambda)v$$

```
# Compute the random 3x3 matrix
A = matrix(rnorm(9), ncol=3)
# Get the eigenvalue decomposition A
lambda = eigen(A)
# Create the matrix  asked by the problem (I+A)(I-A)^{-1}
fA = (diag(3)+A)%*%solve(diag(3)-A)
# Get the eigen decomposition of the created matrix
eigFA = eigen(fA)
# Then by using the work we showed above, where f(A)v  =  f(
   lambda)v
lambda_fA_spectral_map <- (1+lambda$values)/((1-lambda$values
   ))

#print both values, so that we can compare
eigFA$values
lambda_fA_spectral_map

#---------------Output----------------
#> eigFA$values
#[1] -3.8549872+0.0000000i  0.0199289+0.9189822i
   0.0199289-0.9189822i
#> lambda_fA_spectral_map
#[1] -3.8549872+0.0000000i  0.0199289+0.9189822i
   0.0199289-0.9189822i
```

6. **Square matrices and the SVD.** Let A be an $n \times n$ real matrix. State whether each of the following statements is true or false. Do not give any explanation or show any work.

a) If x is an eigenvector of A, then x is either a left or right singular vector of A

b) If λ is an eigenvalue of A, then $|\lambda|$ is a singular value

c) If A is symmetric, then every singular value of A is also an eigenvalue of A

d) If A is symmetric, then every singular vector of A is also an eigenvector of A

e) If A is symmetric with the following singular value decomposition

$$A = U\Sigma V^T$$

then $U = V$

f) If A is invertible, then

$$\sigma_i \neq 0 \text{ for all } i = 1, \cdots, n$$

Solution 6. Used R code to check.

(a) F

```
A <- matrix ( rnorm (4) , nr =2)
eigenA <- eigen(A)
svdA <- svd(A)

eigenA$vectors
svdA$u
svdA$v
#-----------------Output------------------
#> eigenA$vectors
 #             [,1]       [,2]
#[1,]   0.9821933 0.3634659
#[2,]  -0.1878734 0.9316075
#> svdA$u
#             [,1]       [,2]
#[1,]  -0.9857810 0.1680354
#[2,]   0.1680354 0.9857810
#> svdA$v
#             [,1]        [,2]
#[1,]  -0.9240912 -0.3821721
#[2,]   0.3821721 -0.9240912
```

(b) F

```
A <- matrix ( rnorm (4) , nr =2)
eigenA <- eigen(A)
svdA <- svd(A)

eigenA$values
svdA$d
#-----------------Output------------------
#> eigenA$values
#[1]  1.218409 -0.124271
#> svdA$d
#[1] 1.2437377 0.1217402
```

(c) F

```
A <- matrix ( rnorm (4) , nr =2)
symetricMatrix<- A%*%t(A)


eigenA <- eigen(A)
svdA <- svd(A)

eigenA$values
svdA$d
#-----------------Output------------------
#> eigenA$values
#[1] -2.340315  1.486775
#> svdA$d
#[1] 2.742518 1.268733
```

(d) F

```
A <- matrix ( rnorm (4) , nr =2)
symetricMatrix<- A%*%t(A)


eigenA <- eigen(A)
svdA <- svd(A)

eigenA$vectors
svdA$u
svdA$v
#-----------------Output------------------
#> eigenA$vectors
```

```
#            [,1]       [,2]
#[1,] 0.4373587  0.7270010
#[2,] 0.8992871 -0.6866364
#> svdA$u
#            [,1]       [,2]
#[1,] 0.1278487  0.9917937
#[2,] 0.9917937 -0.1278487
#> svdA$v
#             [,1]       [,2]
#[1,] -0.8825755  0.4701707
#[2,] -0.4701707 -0.8825755
```

(e) F

```
A <- matrix ( rnorm (4) , nr =2)
symetricMatrix<- A%*%t(A)


eigenA <- eigen(A)
svdA <- svd(A)

svdA$u==svdA$v
#-----------------Output------------------
#> svdA$u==svdA$v
#       [,1]  [,2]
#[1,] FALSE FALSE
#[2,] FALSE FALSE
```

(f) T

```
A <- matrix ( rnorm (100) , nr =10)

eigenA <- eigen(A)

solve(diag(eigenA$values))

svdA <- svd(A)

svdA$d
```

In this case I run a lot of times, and it gave the correct answer every time. In this case we can see that the fact that Λ is invertible, makes this matrix full rank. Which means that we will have a nonzero singular value for all σ. Since the number of nonzero singular values of A equals the rank of A.