Homework 2

MSSC 6010- Computational Probability

Henri Medeiros Dos Reis

September 13, 2023

Question 1. (3.6.23) From book. Assume that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(A \cap C) = 0.2$, $\mathbb{P}(C) = 0.4)$, $\mathbb{P}(B) = 0.4$, $\mathbb{P}(A \cap B \cap C) = 0.1$, $\mathbb{P}(B \cap C) = 0.2$, and $\mathbb{P}(A \cap B) = 0.2)$. Calculate the following probabilities:

- *(a)* $\mathbb{P}(A \cup B \cup C)$

$$\begin{aligned}
\mathbb{P}(A \cup B \cup C) &= \mathbb{P}((A \cup B) \cup C) \\
&= \mathbb{P}[((A) + (B) - A \cap B) \cup C] \\
&= \mathbb{P}(A \cup C) + \mathbb{P}(B \cup C) - A \cap B \cup C \\
&= \mathbb{P}(A) + \mathbb{P}(C) - \mathbb{P}(A \cap C) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cup C) \\
&= 0.5 + 0.4 - 0.2 + 0.4 + 0.4 - 0.2 - \mathbb{P}(A \cap B \cup C) \\
&= 1.3 - \mathbb{P}(A \cap B + C - A \cap B \cap C) \\
&= 1.3 - (0.2 + 0.4 - 0.1) \\
&= 0.8
\end{aligned}$$

- *(b)* $\mathbb{P}(A^c \cap (B \cup C))$

$$\begin{aligned}
\mathbb{P}(A^c \cap (B \cup C)) &= \mathbb{P}(A^c \cap (B + C - B \cap C)) \\
&= \mathbb{P}(A^c \cap B + A^c \cap C - A^c \cap B \cap C)) \\
&= \mathbb{P}((1 - A) \cap B + (1 - A) \cap C - (1 - A) \cap B \cap C)) \\
&= \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(C) - \mathbb{P}(A \cap C) - (\mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C)) \\
&= 0.4 - 0.2 + 0.4 - 0.2 - (0.2 - 0.1) \\
&= 0.3
\end{aligned}$$

- *(c)* $\mathbb{P}((B \cap C)^c \cup (A \cap B)^c)$

$$
\begin{aligned}
\mathbb{P}((B\cap C)^c \cup (A\cap B)^c) &= \mathbb{P}((1-B\cap C)\cup(1-A\cap C)) \\
&= \mathbb{P}((1-B\cap C)+(1-A\cap C)) - \mathbb{P}((1-B\cap C)\cap(1-A\cap C)) \\
&= 1-0.2+1-0.2-\mathbb{P}((1-B\cap C)\cap(1-A\cap C)) \\
&= 1.6-(1-\mathbb{P}(B\cap C\cap 1)-\mathbb{P}(B\cap C\cap 1)+\mathbb{P}(B\cap C\cap A\cap A)) \\
&= 1.6-(1-2\mathbb{P}(B\cap C)+\mathbb{P}(B\cap C\cap A)) \\
&= 1.6-(1-2(0.2)+0.1) \\
&= 0.9
\end{aligned}
$$

- *(d)* $\mathbb{P}(A)-\mathbb{P}(A\cap C)$

$$\mathbb{P}(A)-\mathbb{P}(A\cap C) = 0.5-0.2 = 0.3$$

Question 2: (3.6.25) From book. Verify that $\mathbb{P}(F|E)$ satisfiesr the three axioms of probability.

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(F\cap E)}{\mathbb{P}(E)}$$

1-) $0 \le \mathbb{P}(F|E) \le 1$. Since $\mathbb{P}(F\cap E) \le \mathbb{P}(E)$, because $\mathbb{P}(F\cap E)$ is a subset of $\mathbb{P}(E)$, and $0 \le \mathbb{P}(E) \le 1$. Which gives $1 \ge \mathbb{P}(E) \ge \mathbb{P}(F|E)$. And since $\mathbb{P}(F\cap E)$ and $\mathbb{P}(E)$ are positive values. Then $\frac{\mathbb{P}(F\cap E)}{\mathbb{P}(E)} \ge 0$.

2-) $\mathbb{P}(\Omega|E) = 1$. $\frac{\mathbb{P}(\Omega\cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E)}{\mathbb{P}(E)} = 1$. This is because $\mathbb{P}(\Omega\cap E) = \mathbb{P}(E)$, since Ω is the whole space, and E is the only limiting probability.

3-) For any sequence of mutually exclusive events we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i|E\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} \frac{(F_i\cap E)}{(E)}\right) = \mathbb{P}\left(\frac{\bigcup_{i=1}^{\infty}(F_i\cap E)}{E}\right)$$

If all F_i are mutually exclusive, then it will intersect E in different parts. Therefore $F_i\cap E$and $F_j\cap E$ are also mutually exclusive for all $i\neq j$. Then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}(F_i\cap E)\right) = \sum_{i=1}^{\infty}\mathbb{P}(F_i\cap E)$$

Thus

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i|E\right) = \frac{\sum_{i=1}^{\infty}\mathbb{P}(F_i\cap E)}{\mathbb{P}(E)} = \sum_{i=1}^{\infty}\frac{\mathbb{P}(F_i\cap E)}{\mathbb{P}(E)} = \sum_{i=1}^{\infty}\mathbb{P}(F_i|E)$$

Question 3: (3.6.33) From book. A salesman in a department store receives household appliances from three suppliers I,I,III. From previous experience, the salesman knows that 2%, 1%, and 3% of the appliances from suppliers I,II,III, respectively, are defective. The salesman sells 35% of the appliances from supplier I, 25% from supplier II, and 40% from supplier III. If an appliance randomly selected is defective, find the probability that it comes from supplier III.

$$\begin{array}{ccc} \mathbb{P}(d|I) = 0.02 & \mathbb{P}(d|II) = 0.01 & \mathbb{P}(d|III) = 0.03 \\ \mathbb{P}(I) = 0.65 & \mathbb{P}(II) = 0.25 & \mathbb{P}(III) = 0.4 \end{array}$$

We need to find $\mathbb{P}(III|d) = \frac{\mathbb{P}(III|d)}{\mathbb{P}(d)}$, let's start by finding $\mathbb{P}(d)$, which is the probability of defective and supplier I, or defective and supplier II, or defective and supplier III. Let S denote the supplier.

$$\mathbb{P}(d|I) = \frac{\mathbb{P}(d \cap S)}{\mathbb{P}(S)} \Rightarrow \mathbb{P}(d \cap S) = \mathbb{P}(d|S)\mathbb{P}(S)$$

Then since these events are mutually exclusive,

$$\begin{aligned} \mathbb{P}(d) &= \mathbb{P}(d|I)\mathbb{P}(I) + \mathbb{P}(d|II)\mathbb{P}(II) + \mathbb{P}(d|III)\mathbb{P}(III) \\ &= (0.02)(0.35) + (0.01)(0.25) + (0.03)(0.4) \\ &= 0.0215 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(III|d) &= \frac{\mathbb{P}(III \cap d)}{\mathbb{P}(d)} \\ &= \frac{\mathbb{P}(d|III)\mathbb{P}(III)}{\mathbb{P}(d)} \\ &= \frac{(0.03)(0.4)}{0.0215} \\ &= 0.55814 \end{aligned}$$

Question 4: (3.6.37) From book. John and Peter play a game with a coin such that $\mathbb{P}(head) = p$. The game consists of tossing a coin twice. John wins if the same result is obtained in the two tosses, and Peter wins if the two results are different.

- *(a)* At what value of p is neither of them favored by the game?

Let $\mathbb{P}(J)$ denote the probability of John win and $\mathbb{P}(P)$ denote the probability of Peter win. For John to win, the result is either Head, Head, or Tail, Tail. So $\mathbb{P}(J) = p^2 + (1-p)^2$, and $\mathbb{P}(P) = 1 - \mathbb{P}(J)$.

In order to be a fair game, $\mathbb{P}(J)$ need to equal $\mathbb{P}(P)$. And each one should win 50% of the times.

$$
\begin{aligned}
p^2 + (1-p)^2 &= \frac{1}{2} \\
p^2 + 1 - 2p + p^2 &= \frac{1}{2} \\
2p^2 - 2p + \frac{1}{2} &= 0 \\
4p^2 - 4p + 1 &= 0 \\
(2p-1)^2 &= 0 \\
p &= \frac{1}{2}
\end{aligned}
$$

Therefore, for $p = \frac{1}{2}$ neither of them is favored.

- *(b)* If p is different from your answer in (a), who is favored?

Since for any $p \neq \frac{1}{2}$, $(2p-1)^2 > 0$, then the probability of John winning increases.

Question 5: (3.6.41) From book. Two independently wealthy philatelists, Alvin and Bob, are interested in buying rare stamps at a private auction. For each stamp up for auction, given that the previous bid did no win, Alvin or Bob wins on their i^{th} bid with probability p. Assume that Alvin always makes the first bid.

- *(a)* Find the probability that Alvin wins the first auction.

Let $\mathbb{P}(A)$ denote probability of Alvin wins, $\mathbb{P}(S_i)$ denote the probability of stamp being bought in bid i.

$\mathbb{P}(S_i)+$ (stamp not sold in the $i-1$ bids before)$*p$, since they are independent, we can just multiply.

$$\mathbb{P}(S_i) = (1-p)^{i-1}p$$

Then, since Alvin is making the first bid, and the events S_i and S_j for $i \neq j$ are mutually exclusive, we have the following

$$
\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(S_1 \cup S_3 \cup \dots) \\
&= \mathbb{P}(S_1) + \mathbb{P}(S_3) + \dots \\
&= p + (1-p)^2 p + (1-p)^4 p + (1-p)^6 p + \dots \\
&= p(1 + (1-p)^2 + (1-p)^4 + (1-p)^6 + \dots) \text{ which converges to} \\
&= p\left(\frac{1}{1-(1-p)^2}\right)
\end{aligned}
$$

Question 6: (3.6.44) From book. Consider tossing three fair coins. The eight possible outcomes are:

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.$$

Define X as the random variable "number of heads showing when three coins are tossed." Obtain the mean and the variance of X. Simulate tossing three fair coins with $10,000$ times. Compute the simulated mean and variance of X. Are the simulated values within 2% of the theoretical answers?

$$\begin{array}{ccccc} X = & 0, & 1, & 2, & 3 \\ \mathbb{P}(x) = & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Then the expected value is

$$\begin{aligned} E[X] &= \sum_x x\mathbb{P}(x) \\ &= 0(\frac{1}{8}) + 1(\frac{3}{8}) + 2(\frac{3}{8}) + 3(\frac{1}{8}) \\ &= 1.5 \end{aligned}$$

and the variance is

$$\begin{aligned} var(X) &= E[X^2] - \mu^2 \\ &= 0^2(\frac{1}{8}) + 1^2(\frac{3}{8}) + 2^2(\frac{3}{8}) + 3^2(\frac{1}{8}) - 1.5^2 \\ &= 0.75 \end{aligned}$$

Now, let's simulate it in R

```
library(dplyr)
```

```
set.seed(42)
n <- 10000
outcomes <- sample(0:3, n, replace=TRUE, prob=c(1/8, 3/8, 3/8, 1/8))
x <- outcomes

mean_x <- mean(x)
var_x <- var(x)
mean_2p = between(mean_x, 1.5-0.02*1.5, 1.5+0.02*1.5)
var_2p =  between(var_x, 0.75-0.02*0.75, 0.75+0.02*0.75)
if(mean_2p)
  print("Mean value is within 2% of the theoretical answer.")
```

```
[1] "Mean value is within 2% of the theoretical answer."
```

```r
if(var_2p)
  print("Variance value is within 2% of the theoretical answer.")
```

```
[1] "Variance value is within 2% of the theoretical answer."
```

As we can see, the simulated values are within 2% of the theoretical answers.