Lecture 1 - Review of Basic Probability and Introduction to Computational Statistics

Computational Statistics and Applications

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Agenda

1. The birthday problem
2. Review of basic probability
3. Probabilistic approximation by simulation
4. The Monty Hall problem
5. Review of conditional probability

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1. **The birthday problem**
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The birthday problem

Problem Determine the probability p that at least two people in a group of k people will have the same birthday (same day of the same month).

Assumption: each of the 365 days of the year is equally likely to be the birthday of any person in the group, and the birthdays of k people are "unrelated".

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Sample space and Events

- **Probability theory** is a mathematical framework that allows us to quantify, analyze and interpret **random** events or experiments whose outcome is **uncertain**.
- A **random experiment** is a(n) process/activity/experiment/task/manipulation by which we observe one uncertain **outcome** from the pool of possible outcomes which can be identified ahead of time.
- The collection of all possible outcomes of an experiment is called the **sample space** of the experiment, denoted by S or Ω (omega).
- When an experiment T has been performed, we call a set of possible outcomes E **an event associated with T** if the outcome of the experiment satisfied the conditions that specified the event E. Event E is determined by its **favorable outcomes**

$$E = \{\omega \in \Omega : \omega \text{ makes } E \text{ occurs}\} \subset \Omega.$$

Sample space and Events (cont.)

"Probability theory" uses the "language of set". Given an experiment T which has the sample space Ω and events $E, F \subset \Omega$

- $\omega \in \Omega$: **elementary event**,
- Ω: **certain event**,
- $\varnothing$: **impossible event**,
- $E^c = \Omega \setminus E$: **complement** of E, or event "E does not occur",
- $E \cup F$: **union** of events, or event "either E or F or both occur",
- $E \cap F$: **intersection** of events, or event "both E and F occur",
- $E \setminus F$: **different (subtraction)**, or event "E occurs but F does not",
- $E \subset F$: **containment (imply)**, or event "E occurs then so does F",
- $E = F$: event "either both E and F occur or both do not",
- $E \cap F = \varnothing$: E, F **disjoint (mutually exclusive)**, or event "E and F cannot both occur".

Probability

Given an experiment T with the sample space Ω, a function P, which assigns to each event $E \subset \Omega$ a real number $P(E)$, is called a **probability measure** if it satisfies 3 specific axioms:

1. For every event $E \subset \Omega, 0 \leq P(E) \leq 1$.
2. For every infinite sequence of disjoint events $E_1, E_2, \ldots$ $(E_i \cap E_j = \varnothing, \forall i \neq j)$:

   $$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i),$$

   or $P(E_1 \cup E_2 \cup \ldots) = P(E_1) + P(E_2) + \ldots$
3. $P(\Omega) = 1$.

$P(E)$ indicates the **probability** of E and shows how likely the event E is when the outcomes of T is unknown.

Probability (cont.)

General Properties of probability (from three axioms above):

1. $P(E^c) = 1 - P(E)$
2. $P(\varnothing) = 0$
3. If $E_1 \subset E_2$ thì $P(E_1) \leq P(E_2)$ và $P(E_2 \setminus E_1) = P(E_2) - P(E_1)$
4. If $E_1 \cap E_2 = \varnothing$ thì $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
5. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ (**addition law of probability**)
6. $P(E_1 \cup E_2 \cup E_3) =$
   $P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$
7. $P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$ (**union bound**)
8. $P\left(\bigcap_{i=1}^{\infty} E_i\right) \geq 1 - \sum_{i=1}^{\infty} P(E_i^c)$ (**Bonferroni inequality**)

Simple probability model

When the sample space is finite, $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, a probability measure on Ω is specified by assigning a propability to each point of Ω (elementary event) $p_i = P(\omega_i)$

- $p_i \geq 0, i = 1, ..., n,$
- $\sum_{i=1}^{n} p_i = 1,$
- $P(E) = \sum_{w_i \in E} p_i$ for any event $E \subset \Omega$.

When the sample space is finite and each outcome is equally likely **(equiprobable outcomes)**, a **simple/classical probability model** is derived as

- $p_i = \frac{1}{n}, i = 1, ..., n,$
- $P(E) = \frac{|E|}{|\Omega|}$ for any event $E \subset \Omega$, ($|X|$ is the cardinality of X)
- Thus, finding probability of A reduces to a **counting** problem in which we need to count how many elements are in E and Ω.

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Probabilistic approximation by simulation

To appoximate the probability of event E associated with an experiment T, we can execute an analytical calculation below

- Perform the experiment T N times repetitively (and independently) and count how many times event E occurs, m. Then, $f(E) = \frac{m}{N}$ is defined as the frequency of E.
- When we execute this experiment the large number of times, $f(E) \approx P(E)$.
- Performing this experiment N times repetitively can be implemented by a computer simulation program.

The birthday problem - Theoretically

Without loss of generality, we can call the set of days in one year as

$$\mathcal{Y} = \{1, 2, ..., 365\}.$$

The sample space $\Omega = \{(x_1, x_2, ..., x_k) : x_i \in \mathcal{Y}, i = 1, ..., k\} = \mathcal{Y}^k$ with $|\Omega| = 365^k$.
Putting 2 events:

- A: "at least two people must have the same birthday",
- B: "none of them has the same birthday".

Thus, $A = B^c$ and $B = \{k\text{-permulations of } \mathcal{Y}\}$ with $|B| = P_{365}^k$.
Using the simple probability model, we obtain

$$p = P(A) = 1 - P(B) = 1 - \frac{P_{365}^k}{365^k} = 1 - \frac{365!}{(365-k)!365^k}.$$

The birthday problem - Simulation

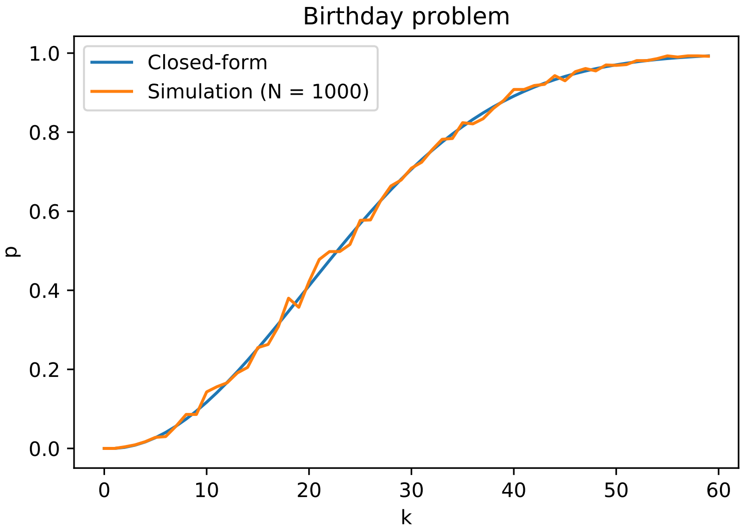
```
def birthday(k):
    return [random.randint(1, 365) for _ in range (k)]

def at_least_2(outcome):
    return len(set(outcome)) < len(outcome)

def relative_frequency(k, N, event):
    m = sum(event(birthday(k)) for _ in range (N))
    return m/N

relative_frequency(50 , 1000, at_least_2)
#0.973
```

The birthday problem - Result

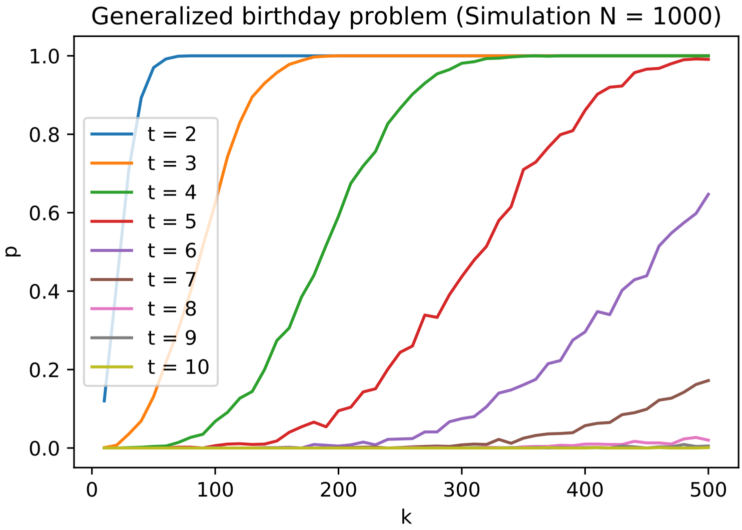


The birthday problem - Extension

An extension Determine the probability p that at least t people in a group of k will have the same birthday (same day of the same month)?

- Theory calculation: obtain a proper result. However, it is hard and nearly impossible to compute for many cases.
- Computer simulation: easy in coding modification, but time and resource-consuming and provides an approximation result only.
  (See the Python codes in Notebook)

The birthday problem - Extension (cont.)



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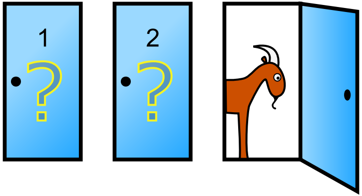
The Monty Hall problem

Monty Hall problem. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say #1, and the host, who knows what's behind the doors, opens another door, say #3, which has a goat. He then says to you, "Do you want to pick door #2?" Is it to your advantage to switch your choice?



(https://en.wikipedia.org/wiki/Monty_Hall_problem)

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Conditional probability

A need of modifying or updating the probabilities of events associated with an experiment T as any addition information of T has been observed

- The addition information of T relates to the occurrence of certain event(s).

The updated probability of event A after we learn that event B has occurred is the conditional probability of A given B, which is denoted by $P(A|B)$ and computed as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ (if } P(B) > 0).$$

- $A \cap B$ means "A in/intersect B",
- Divided by $P(B)$ is to normalize the probability,
- $P(.|B)$ calculates probability "on a new sample space" B, which is completely legal.

Multiplication rule

Multiplication rule

$$P(A \cap B) = P(B)P(A|B) \text{ (if } P(B) > 0),$$

$$P(A \cap B) = P(A)P(B|A) \text{ (if } P(A) > 0).$$

In some experiments, certain conditional probability $P(A|B)$ is relatively easy to compute rather than $P(A \cap B)$.

General multiplication rule Suppose that $A_1, ..., A_n$ are n events such that $P(A_1 \cap ... \cap A_n) > 0$, then

$$P(A_1 \cap ... \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1, A_2) \times ... \times P(A_n|A_1, A_2, ..., A_{n-1}).$$

Law of total probability

$B_1, B_2, ..., B_n$ form a **countably infinite partition** (or **partition**) of Ω if

1. $B_i \cap B_j = \varnothing, \forall i \neq j$,
2. $\Omega = B_1 \cup B_2 \cup ... \cup B_n$.

Law of total probability. Suppose that the events $B_1, B_2, ..., B_n$ form a partition of the sample space Ω and $P(B_i) > 0$ $(i = 1, ..., n)$. Then, for every event A in Ω,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i)P(A|B_i).$$

Especially,

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c).$$

Bayes' theorem

Bayes' theorem (Bayes's rule). Let the events $B_1, B_2, ..., B_n$ form a partition such that $P(B_i) > 0$ $(i = 1, ..., n)$ and let A be an event such that $P(A) > 0$. Then, for all $i = 1, ..., n$,

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}$$

- $P(B_i)$: **prior probability** of B_i,
- $P(B_i|A)$: **posterior probability** of B_i given A,
- $P(A|B_i)$: **likelihood** of A given B_i,
- Note, $P(A)$ does not rely on B_i, thus

$$P(B_i|A) \propto P(B_i)P(A|B_i) \text{ (the symbol } \propto \text{ means "proportional to").}$$

Independent events

Two events $\{A, B\}$ are **independent** (statistically independent, stochastically independent) if

$$P(A \cap B) = P(A) \times P(B).$$

Similarly, $P(A|B) = P(A)$ $(P(B) > 0)$ or $P(B|A) = P(B)$ $(P(A) > 0)$.

Theorem. If two events $\{A, B\}$ are independent, then the others $\{A^c, B\}, \{A, B^c\}, \{A^c, B^c\}$ are also independent.

Three events $\{A, B, C\}$ are (mutually) independent if all of the pairwise $\{A, B\}, \{A, C\}, \{B, C\}$ are also independent respectively, and

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C).$$

Statistical procedure “independent replication”

The events $\{A_1, A_2, \ldots\}$ are independent if, for every non-empty and countable subcollection $\{B_1, B_2, \ldots, B_k\}$ of these events,

$$P\left(\bigcap_{i=1}^{k} B_i\right) = \prod_{i=1}^{k} P(B_i).$$

- From right to left: to demonstrate/prove the independence,
- From left to right: to calculate the simple probability based on the indenpendence.

Statistical procedure “independent replication”: performing the experiment T repetitively and independently the large number of times, suppose that A_i is the “associated with the i-th replicate” event,

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i).$$

Conditional independence

Given an event C with $P(C) > 0$, we say that the events $\{A_1, A_2, ...\}$ are **conditionally independent** given C if, for every non-empty and countable subcollection $\{B_1, B_2, ..., B_k\}$ of these events,

$$P\left(\bigcap_{i=1}^{k} B_i | C\right) = \prod_{i=1}^{k} P(B_i|C).$$

Two events $\{A, B\}$ are conditionally independent given C if and only if

$$P(A \cap B|C) = P(A|C) \times P(B|C).$$

Similarly,

$$P(A|B, C) = P(A|C) \ (P(B|C) > 0) \text{ or } P(B|A, C) = P(B|C) \ (P(A|C) > 0).$$

Probabilistic approximation by simulation

To approximate the independent probability of event B given event A associated with an experiment T, we can execute an analytical calculation below

- Perform the experiment T N times repetitively (and independently), count how many times event A occurs, m, and B occurs also whenever A occurs, p. Then, $f(B|A) = \frac{p}{m}$ is defined as the frequency of B given A.
- When we execute this experiment the large number of times, $f(B|A) \approx P(B|A)$.
- Performing this experiment N times repetitively can be implemented by a computer simulation program.

The Monty Hall problem - Theoretically

Such the final stratergy is as above (player picked door #1, the host opened #3), since all of them share the same type of result.

Consider the event A_i, indicating that "the car is behind door number $\#i$" $(1 \leq i \leq 3)$ and event B_j which is "the host opens the door number $\#j$" $(1 \leq j \leq 3)$. Player picked door number #1, then

- $P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$.
- $P(B_3|A_1) = \frac{1}{2}$ (Player picked door #1, which contains the car, so the host can open both of the others #2, #3).
- $P(B_3|A_2) = 1$ (Player picked door #1, the car is in #2, so the host only can open #3).
- $P(B_3|A_3) = 0$ (the car is in #3 so #3 could not be opened).

The Monty Hall problem - Theoretically (cont.)

Using Bayes' rule, we have the probability that the player wins the car if staying at door #1 is

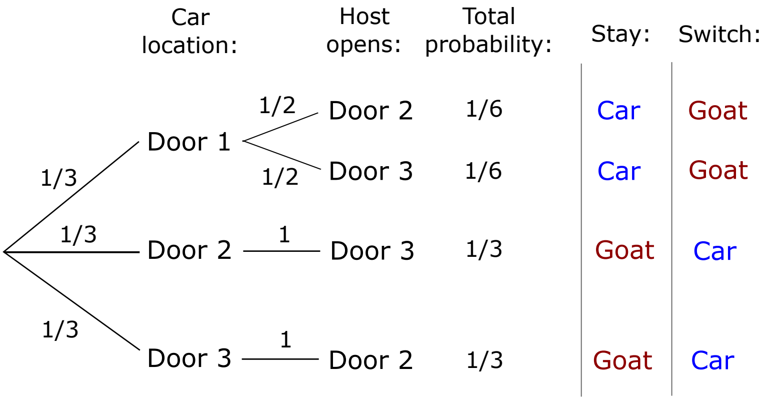
$$\begin{aligned} P(A_1|B_3) &= \frac{P(A_1)P(B_3|A_1)}{P(A_1)P(B_3|A_1) + P(A_2)P(B_3|A_2) + P(A_3)P(B_3|A_3)} \\ &= \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{3}\frac{1}{2} + \frac{1}{3}1 + \frac{1}{3}0} = \frac{1}{3} \end{aligned}$$

and the probability that player wins by switching is

$$\begin{aligned} P(A_2|B_3) &= \frac{P(A_2)P(B_3|A_2)}{P(A_1)P(B_3|A_1) + P(A_2)P(B_3|A_2) + P(A_3)P(B_3|A_3)} \\ &= \frac{\frac{1}{3}1}{\frac{1}{3}\frac{1}{2} + \frac{1}{3}1 + \frac{1}{3}0} = \frac{2}{3} \end{aligned}$$

In conclusion, it is never to the contestant's disadvantage to switch, as the conditional probability of winning by switching is twice as much as staying.

The Monty Hall problem - Tree diagram



(https://en.wikipedia.org/wiki/Monty_Hall_problem#Conditional_probability_by_direct_calculation)

The Monty Hall problem - Simulation

```
def Monty_Hall(doors={"#1", "#2", "#3"}):
    car_door = random.choice(list(doors))
    choice_door = random.choice(list(doors))
    open_door = random.choice(list(doors - {choice_door,
        car_door}))
    op_door = random.choice(list(doors - {choice_door,
        open_door}))

    return car_door == choice_door, car_door == op_door
```

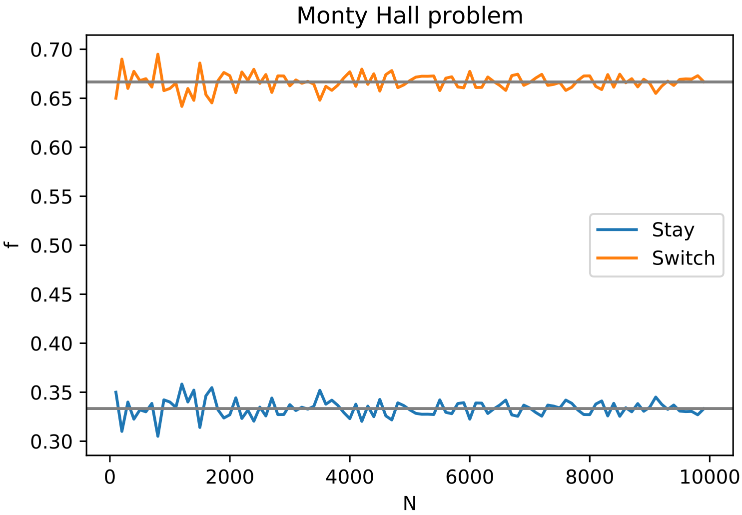
The Monty Hall problem - Simulation (cont.)

```
N = 10000
results = [Monty_Hall() for _ in range(N)]

stay_freq = sum([v for v, _ in results])/N
switch_freq = sum([v for _, v in results])/N

print(stay_freq, switch_freq)
#0.3317 0.6683
```

The Monty Hall problem - Simulation (cont.)



References

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Chapter 1-2. H. Pishro-Nik. "*Introduction to probability, statistics, and random processes*", available at `https://www.probabilitycourse.com`. Kappa Research LLC, 2014.