Eksamen MA0001 Brukerkurs i matematikk A

LØSNINGSFORSLAG

1. i.) $f(x) = \frac{\ln x}{x} = \frac{u}{v}$ $\boxed{\begin{matrix} u = \ln x & v = x \\ u' = \frac{1}{x} & v' = 1 \end{matrix}}$

$f'(x) = u' \cdot v + u \cdot v' = \frac{1}{x} \cdot x + \ln x \cdot 1 = \underline{1 + \ln x}$

$\underline{\underline{f'(1)}} = 1 + \ln 1 = 1 + 0 = \underline{\underline{1}}$ (Alt. D)

ii.) $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot (3x)' = \cos(3x) \cdot 3 = \underline{3 \cdot \cos(3x)}$

$\underline{\underline{f'(\frac{\pi}{4})}} = 3 \cdot \cos(\frac{3\pi}{4}) = 3 \cdot (-\frac{1}{2}\sqrt{2}) = \underline{\underline{-\frac{3}{2}\sqrt{2}}}$ (Alt. C)

iii.) $f(x) = \arctan(\sqrt{x})$

$f'(x) = \frac{1}{1+\sqrt{x}^2} \cdot (\sqrt{x})' = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} = \underline{\frac{1}{2(1+x)\sqrt{x}}}$

$\underline{\underline{f'(1)}} = \frac{1}{2 \cdot (1+1) \cdot \sqrt{1}} = \underline{\underline{\frac{1}{4}}}$ (Alt. F)

2. $x^2 + \frac{1}{4}y^2 = 2$

$x = -1, y = 2$ gir: v.s.: $(-1)^2 + \frac{1}{4} \cdot 2^2 = 1 + 1 = 2$
h.s.: 2

v.s. = h.s. ok!
$\Downarrow$
$\underline{\underline{(-1,2) \text{ ligger på kurven}}}$

Implisitt derivasjon på begge sider gir:

$2x + \frac{1}{4} \cdot 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2x + \frac{1}{2}y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{1}{2}y \cdot \frac{dy}{dx} = -2x \; \Big| \cdot \frac{2}{y}$

$\Rightarrow \frac{dy}{dx} = -2x \cdot \frac{2}{y} = -\frac{4x}{y}$

$x = -1, y = 2 \Rightarrow \frac{dy}{dx} = -\frac{4 \cdot (-1)}{2} = 2 = a$

Likningen til tangenten i $(x_1, y_1) = (-1, 2)$ er:

$y - y_1 = a \cdot (x - x_1) \Rightarrow y - 2 = 2 \cdot (x - (-1)) = 2 \cdot (x+1) \Rightarrow \underline{\underline{y}} = 2x + 2 + 2 = \underline{\underline{2x + 4}}$

3. i.) $\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \underline{\underline{0}}$

ii.) $\lim_{x\to\pi} \frac{1+\cos(3x)}{(x-\pi)^2} \left(= \frac{1+\cos(3\pi)}{(\pi-\pi)^2} = \frac{1+(-1)}{(\pi-\pi)^2} = \frac{0}{0} \right)$

$\overset{\text{"}\frac{0}{0}\text{"}}{\underset{L'H}{=}} \lim_{x\to\pi} \frac{-\sin(3x)\cdot 3}{2\cdot(x-\pi)\cdot 1} = \lim_{x\to\pi} \frac{-3\cdot\sin(3x)}{2\cdot(x-\pi)} \left(= \frac{-3\cdot\overbrace{\sin(3\pi)}^{0}}{2\cdot(\pi-\pi)} = \frac{0}{0} \right)$

$\overset{\text{"}\frac{0}{0}\text{"}}{\underset{L'H}{=}} \lim_{x\to\pi} \frac{-3\cdot\cos(3x)\cdot 3}{2\cdot(1-0)} = \lim_{x\to\pi} \frac{-9\cos(3x)}{2} = -\frac{9\cdot\overbrace{\cos(3\pi)}^{-1}}{2} = -\frac{9\cdot(-1)}{2} = \underline{\underline{\frac{9}{2}}}$

4. i.) $\int \left(x^2 + \frac{3}{x} + \frac{1}{2}\cos x\right) dx = \underline{\underline{\frac{1}{3}x^3 + 3\ln|x| + \frac{1}{2}\sin x + C}}$

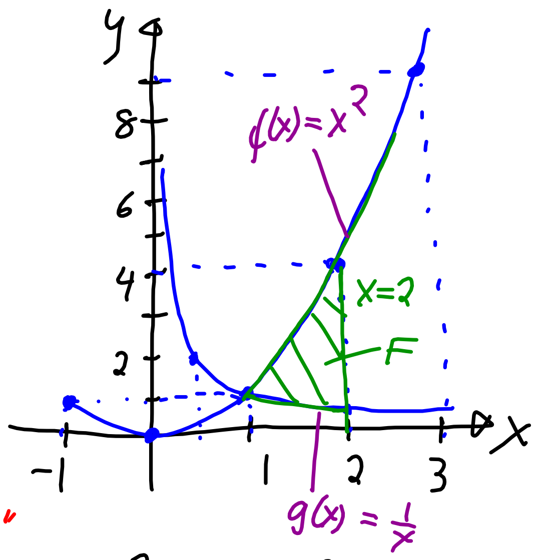
ii.) $\int x^3\cdot\cos(\underbrace{2x^4}_{u})\,dx = \int \cancel{x^3}^{1}\cdot\cos u\cdot\frac{du}{8\cancel{x^3}_{1}} = \frac{1}{8}\int \cos u\,du$

$\boxed{u = 2x^4 \qquad \frac{du}{dx} = 8x^3 \Rightarrow \frac{du}{8x^3} = dx}$

$= \frac{1}{8}\sin u + C = \underline{\underline{\frac{1}{8}\sin(2x^4) + C}}$

5 $f(x) = x^2$

$g(x) = \frac{1}{x}$



Skjæringspunkt:

$f(x)=g(x) \Rightarrow x^2=\frac{1}{x} \quad |\cdot x$

$\Rightarrow x^3=1 \Rightarrow \underline{x=\sqrt[3]{1}=1}$

"største" "minste"

$V=\pi\int_1^2 [f(x)]^2dx-\pi\int_1^2 [g(x)]^2dx=\pi\int_1^2 (x^2)^2dx-\pi\int_1^2 \left(\frac{1}{x}\right)^2dx$

$=\pi\int_1^2\left(x^4-\frac{1}{x^2}\right)dx=\pi\left[\frac{1}{5}x^5+\frac{1}{x}\right]_1^2=\pi\left[\left(\frac{1}{5}\cdot 2^5+\frac{1}{2}\right)-\left(\frac{1}{5}\cdot 1^5+\frac{1}{1}\right)\right]$

$=\pi\cdot\left(\frac{32}{5}+\frac{1}{2}-\frac{1}{5}-1\right)=\pi\cdot\left[\left(\frac{32}{5}-\frac{1}{5}\right)+\left(\frac{1}{2}-1\right)\right]=\pi\cdot\left[\frac{31}{5}-\frac{1}{2}\right]$ (Alt. E)

$=\pi\cdot\left(\frac{31\cdot 2}{5\cdot 2}-\frac{1\cdot 5}{2\cdot 5}\right)=\frac{57}{10}\cdot\pi=\underline{\underline{5{,}7\cdot\pi}}\approx\underline{\underline{17{,}91}}$

6. $y(t)$ = mengden av stoffet i blodet (målt i mg)

$$y(t) = C \cdot e^{-\lambda t} \quad , \; y(0) = 20 \text{ (mg)}$$
$$y(10) = 12 \text{ (mg)}$$

i) $y(0) = 20 \Rightarrow C \cdot \underbrace{e^{-\lambda \cdot 0}}_{=1} = 20 \Rightarrow C = 20$, dvs. $y(t) = 20 \cdot e^{-\lambda t}$

$$y(10) = 12 \Rightarrow 20 \cdot e^{-\lambda \cdot 10} = 12 \;\; \Big| \cdot \frac{1}{20} \Rightarrow e^{-10\lambda} = \frac{12}{20} = 0{,}6$$

$$\Rightarrow -10\lambda = \ln 0{,}6 \Rightarrow \underline{\lambda} = -\frac{\ln 0{,}6}{10} = -\frac{1}{10} \cdot \ln \frac{3}{5} = -\frac{1}{10} \cdot \left(-\ln \frac{5}{3}\right) = \underline{\frac{1}{10} \ln\left(\frac{5}{3}\right)}$$

$$\ln \frac{3}{5} = \ln\left(\left(\frac{5}{3}\right)^{-1}\right) = -\ln\left(\frac{5}{3}\right)$$

Dermed: $\underline{y(t)} = 20 \cdot e^{-\lambda t} = \underline{\underline{20 \cdot e^{-\frac{1}{10}\ln\left(\frac{5}{3}\right) \cdot t}}} \approx \underline{\underline{20 \cdot e^{-0{,}051 \cdot t}}}$

$$\left(= 20 \cdot \left(e^{\ln\left(\frac{5}{3}\right)}\right)^{-\frac{t}{10}} = \underline{20 \cdot \left(\frac{5}{3}\right)^{-\frac{t}{10}}}\right)$$

ii) $y(T_{1/2}) = \frac{1}{2} \cdot y(0) \Rightarrow 20 \cdot e^{-\frac{1}{10}\ln\left(\frac{5}{3}\right) \cdot T_{1/2}} = \frac{1}{2} \cdot 20 \;\; \Big| \cdot \frac{1}{20}$

$$\Rightarrow e^{-\frac{1}{10}\ln\left(\frac{5}{3}\right) \cdot T_{1/2}} = \frac{1}{2} \Rightarrow -\frac{1}{10} \cdot \ln\left(\frac{5}{3}\right) \cdot T_{1/2} = \ln\left(\frac{1}{2}\right) \overset{\ln(2^{-1})}{=} -\ln 2$$

$$\Rightarrow T_{1/2} = \frac{\ln 2}{\frac{1}{10}\ln \frac{5}{3}} = \frac{10 \cdot \ln 2}{\ln \frac{5}{3}} \approx \underline{13{,}57 \approx 13{,}6}$$

Halveringstiden er på ca. 13,6 timer.

7. $f(x) = \frac{1}{2}e^x$

$g(x) = -x + 3x = 2x$

$f(x) = g(x) \Rightarrow \frac{1}{2}e^x = 2x \Rightarrow 2x - \frac{1}{2}e^x = 0$

Setter $h(x) = 2x - \frac{1}{2}e^x$

Vil finne nullpunktet til $h(x)$ ved Newtons metode.

$h'(x) = 2 - \frac{1}{2}e^x$

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)} = x_n - \frac{2x_n - \frac{1}{2}e^{x_n}}{2 - \frac{1}{2}e^{x_n}}$$

$x_0 = 1{,}3$

$$x_1 = x_0 - \frac{2x_0 - \frac{1}{2}e^{x_0}}{2 - \frac{1}{2}e^{x_0}} = 1{,}3 - \frac{2 \cdot 1{,}3 - \frac{1}{2}e^{1{,}3}}{2 - \frac{1}{2}e^{1{,}3}} \approx -3{,}32863$$

$$x_2 = x_1 - \frac{2x_1 - \frac{1}{2}e^{x_1}}{2 \quad \frac{1}{2}e^{x_1}} \approx 0{,}39138$$

$x_3 \approx 0{,}33756$

$x_4 \approx 0{,}35730$

$x_5 \approx 0{,}35740$

$x_6 \approx 0{,}35740$

Likt, så STOPP!

x-verdien til skjæringspunktet, korrekt avrundet til 3 desimaler, er:

$\underline{\underline{x = 0{,}357}}$

8. Uendelig geometrisk rekke:

$$S = a + a\cdot k + ak^2 + \ldots$$

$a = 16$ og $k = 0{,}9$

i) $S_n = a\cdot\frac{1-k^n}{1-k} \Rightarrow \underline{\underline{S_{20}}} = 16\cdot\frac{1-0{,}9^{20}}{\underbrace{1-0{,}9}_{=0{,}1}} = 160\cdot(1-0{,}9^{20}) \approx \underline{\underline{140{,}6}}$

ii) $\underline{\underline{S}} = a\cdot\frac{1}{1-k} = 16\cdot\frac{1}{1-0{,}9} = 16\cdot 10 = \underline{\underline{160}}$

9. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

i) Enklest: $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots$

$u = -x$ gir: $\underline{\underline{e^{-x}}} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = \underline{\underline{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots}}$

Alt:
$f(x) = e^{-x}$ $\quad f(0) = e^{-0} = 1$
$f'(x) = -e^{-x}$ $\quad f'(0) = -e^{0} = -1$
$f''(x) = e^{-x}$ $\quad f''(0) = 1$
$f^{(3)}(x) = -e^{-x}$ $\quad f^{(3)}(0) = -1$

$\left(= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!}}}\right)$

$\underline{\underline{e^{-x}}} = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots$

$= 1 + (-1)\cdot x + \frac{1}{2!}x^2 + \frac{(-1)}{3!}\cdot x^3 + \ldots = \underline{\underline{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots}}$

ii) $\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}{2}$

$\Rightarrow \underline{\underline{\cosh(x)}} = \frac{2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + \ldots}{2} = \underline{\underline{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots}}$

$\left(= \underline{\underline{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}}\right)$

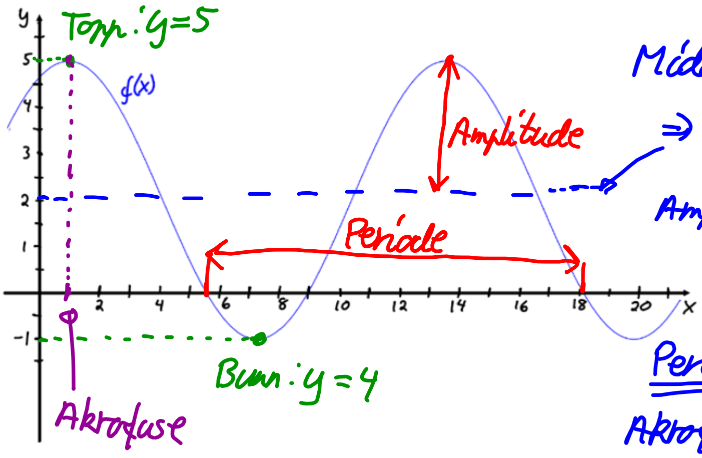
iii) $\cosh(x) \approx T_4(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

$\Rightarrow \underline{\underline{\cosh(1)}} \approx 1 + \frac{1^2}{2!} + \frac{1^4}{4!} \approx 1 + \frac{1}{2} + \frac{1}{24} \approx \frac{24+12+1}{24} = \underline{\underline{\frac{37}{24}}} \approx \underline{\underline{1{,}5417}}$

(Korrekt verdi: $\cosh(1) = \frac{e^1 + e^{-1}}{2} \approx 1{,}5431$)

10.

i.)



$\text{Middelverdi} = \frac{\text{Topp}+\text{Bunn}}{2} = \frac{5+(-1)}{2}$

$\Rightarrow \underline{\text{Middelverdi} = 2}$

$\text{Amplitude} = \frac{\text{Topp}-\text{Bunn}}{2} = \frac{5-(-1)}{2}$

$\Rightarrow \underline{\text{Amplitude} = 3}$

$\underline{\text{Periode}} \approx 18{,}1 - 5{,}6 = \underline{12{,}5}$

Akrofase = x-verdien til første topp

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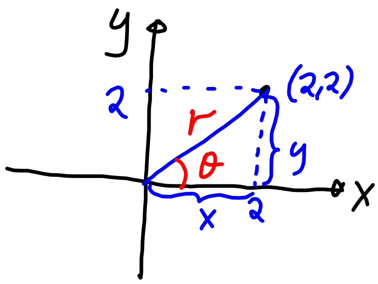
$\underline{\text{Akrofase} = 1}$

ii.) $f(t) = 2\cos(3t) + 2\sin(3t)$



$r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = \sqrt{8}$

$\tan\theta = \frac{y}{x} = \frac{2}{2} \Rightarrow \theta = \frac{\pi}{4}$

(rett kvadrat etter fig.)

Formelsamlingen sier:

For $a, b \in R$, ikke begge lik 0, og $\omega > 0$ gjelder:

$$a\cos(\omega t) + b\sin(\omega t) = r \cdot \cos(\omega t - \theta),$$

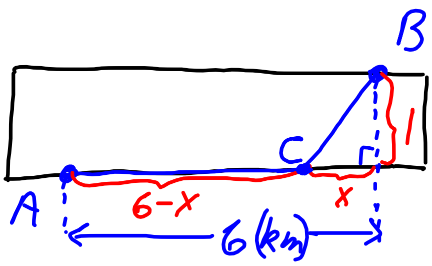
der r og θ er polarkoordinatene til punktet (a, b).

Vi har gitt $a=2, b=2$ og $\omega=3$ og har funnet $r=\sqrt{8}$ og $\theta = \frac{\pi}{4}$.

Dermed: $\underline{\underline{f(t)}} = r\cdot\cos(\omega t - \theta) = \sqrt{8}\cdot\cos\left(3t - \frac{\pi}{4}\right) = \underline{\underline{\sqrt{8}\cdot\cos\left(3\left(t - \frac{\pi}{12}\right)\right)}}$

t_0

11



$BC = \sqrt{x^2+1}$

Kostnader:

Gjennom fjell (B−C):
1 mrd/km

Langs fjellet C−A:
300 mill/km = 0,3 mrd/km

Total kostnad (i mrd.)

$$K(x) = BC \cdot 1 + AC \cdot 0{,}3$$

$$\Rightarrow K(x) = \sqrt{x^2+1} \cdot 1 + (6-x) \cdot 0{,}3$$

Vi vil finne minste kostnad, og setter da $K'(x) = 0$

$$K'(x) = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x - 0{,}3 = 0$$

$$\Rightarrow \frac{x}{\sqrt{x^2+1}} = 0{,}3 \Rightarrow x = 0{,}3 \cdot \sqrt{x^2+1}$$

Kvadrerer

$$\Rightarrow x^2 = 0{,}09 \cdot (x^2+1) \Rightarrow x^2 - 0{,}09x^2 = 0{,}09$$

$$\Rightarrow \frac{91}{100}x^2 = 0{,}09 \Rightarrow x^2 = \frac{0{,}09 \cdot 100}{91} = \frac{9}{91}$$

$$\Rightarrow x = \sqrt{\frac{9}{91}} = \frac{3}{\sqrt{91}}$$

Vi kan vise at $K''(x) > 0$ for $x = \frac{3}{\sqrt{91}}$ for å vise at denne verdien for x gir en minimumsverdi, men det er vel ganske klart at, siden $K(x)$ er en kontinuerlig funksjon, og at vi avgjort vil kunne få større kostnad for x stor og x negativ, så gir ikke dette en største kostnad. Da må $x = \frac{3}{\sqrt{91}}$ gi en minimumskostnad.

Denne minste kostnaden er:

$$K\left(\frac{3}{\sqrt{91}}\right) = \underbrace{\sqrt{\left(\frac{3}{\sqrt{91}}\right)^2 + 1}}_{= \sqrt{\frac{9}{91} + \frac{91}{91}} = \sqrt{\frac{100}{91}} = \frac{10}{\sqrt{91}}} + \left(6 - \frac{3}{\sqrt{91}}\right) \cdot 0{,}3 = 1{,}8 + \frac{10 - 0{,}9}{91} = 1{,}8 + \frac{9{,}1}{\sqrt{91}}$$

$$\approx 2{,}754 \text{ (mrd.)}$$