Examining the Risk-Return Relationship
Using the MF2-GARCH Model

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Summary

This thesis investigates whether the multiplicative factor Multi-Frequency Component Generalized Autoregressive Conditional Heteroskedasticity (MF2-GARCH) model of Conrad & Engle (2025) supports earlier findings that the long-term component of market volatility holds more predictive power for market premia than overall volatility.

I create a combined "MF2-GARCH-in-mean" model, placing the components of MF2-GARCH volatility in a risk-return specification similar to those of Maheu & McCurdy (2007) and estimating the parameters for historical data on U.S. market premia. Following Maheu & McCurdy (2007), I test different specifications, varying the choice of components included in the model. To further test the viability of the model, I run several thousand Monte Carlo simulations and record the average biases and standard deviations of the MF2-GARCH-in-mean estimates on this simulated data.

The results show that the long-term component of MF2-GARCH market volatility does better than the short-term component to explain variation in market premia across several specifications and that its coefficients are consistently positive and highly significant. The Monte Carlo simulations show that most parameter estimates of the MF2-GARCH-in-mean model have only moderate bias.

1 Introduction

Merton (1973) introduces the Intertemporal Capital Asset Pricing Model (ICAPM), which provides a theoretical justification for a positive risk-return relationship. Under this model, expected excess market return can be expressed as a linear combination of its own conditional variance and its covariance with a vector of state-variables which predict changes in the future investment opportunity set.[1]:

$$E_{t-1}r_{M,t} - r_{f,t} = \gamma_M \sigma^2_{M,t-1} + \gamma_F \sigma_{MF,t-1}$$

where at time t:

- $r_{M,t}$ is the market return,
- $r_{f,t}$ is the risk free rate,
- $\sigma^2_{M,t-1}$ is the conditional market return variance in the previous period,
- and $\sigma_{MF,t-1}$ is the covariance of market return in the previous period with a vector of state-variables that predict future investment opportunities

γ_M is the expected return that investors demand in return for an additional unit of risk. The second term, $\gamma_F \sigma_{MF,t-1}$, is often described as the "hedging component", as it is the proportion of excess market return which investors demand to hedge against changes in future investment opportunities.

Merton (1980) shows that under certain simplifying assumptions (such as a constant investment opportunity set or myopic preferences), the hedging term can be eliminated and the coefficient on variance (γ_M) represents the entire risk premium. In that case, γ_M is by definition the investors' coefficient of relative risk aversion, which is assumed to be positive. This implies that higher conditional variance should lead to higher expected returns.

Volatility models like Generalized Autoregressive Conditional Heteroskedasticity (GARCH) are commonly used to model this conditional market variance. GARCH

[1]Restated with simplified coefficients for readability

models capture how volatility evolves over time. In a GARCH-in-mean specification, the estimated conditional variance is included directly in the return equation, allowing us to test whether periods of higher estimated volatility are followed by higher returns, as predicted by theory.

Providing evidence for the existence of a positive risk-return relationship has practical implications in asset pricing, asset allocation, portfolio optimization, and policymaking.

Asset pricing is usually done on the basis of the present value of the expected future income that the asset generates. This expected income is often estimated based on the asset's expected return, which in turn is based on the risk with which it is associated. It is taken for granted that riskier assets command higher risk premia and therefore higher expected returns, but without evidence to support this positive relationship, this remains an assumption. Inadequate risk measures which do not accurately capture this relationship reduce the accuracy of expected return estimates and by extension, the estimate of the asset price.

Portfolio optimization methods like mean-variance analysis rely on estimates of expected returns for given levels of risk. If conditional variance forecasts contain information about expected returns, optimization models can make us of this by linking higher volatilities to higher expected equity premia. For an investor looking for a specific return, this means choosing an appropriate volatility level. If this relationship is negative or statistically insignificant, return objectives cannot be linked to volatility forecasts and optimization based on mean-variance analysis is ineffective.

The risk-return relationship is also important for policymakers and regulators trying to maintain financial stability. Risk-weighted asset rules assume that higher-risk assets provide higher expected returns, so evidence of a positive tradeoff supports these approaches. If the market does not reward risk with higher returns, regulators may need to reasses these regulatory approaches. Governments and central banks also monitor equity risk premia as indicators of market sentiment. A rising premium might signal to them that risk aversion has increased, or that market sentiment has worsened. This also relies on the assumption that the risk-return relationship is positive.

Risk aversion and market conditions can change over time, so the factors determining the market risk premium can vary across different ”volatility regimes”. Other variables like liquidity constraints, macroeconomic announcements, or investor sentiment can also influence returns and act as confounding variables when estimating the relationship between market risk and return. As a result, empirical tests often find weak or counterintuitive evidence of a volatility premium at short horizons.

This thesis combines the MF2-GARCH model with a univariate risk-return specification to get a MF2-GARCH-in-mean model. The MF2-GARCH model multiplicatively decomposes overall conditional variance into long-term (persistent) and short-term (transient) components. By separating persistent volatility from transient shocks, this approach aims to achieve more accurate measures of risk and to test whether each volatility component commands a different risk premium (if any). In doing so, the aim is to see whether it is possible to detect the expected positive risk-return relationship in this way.

2 Literature Review

2.1 Background

Early attempts to estimate the risk-return relationship often yielded unexpected results. Many early GARCH-in-mean regressions found a negative or statistically insignificant coefficient on conditional variance, contrary to expectations. French et al. (1987) document this “volatility feedback” effect, where ”unexpected stock market returns are negatively related to the unexpected change in the volatility of stock returns”. Simple regressions of returns on short-horizon variance can fail to detect the expected positive relationship. As a result, researchers explored alternative risk measures and more complex models in an attempt to find a positive slope.

Guo & Whitelaw (2006) explicitly model two components of expected returns. Following the ICAPM, they decompose the expected excess return into a variance term and a hedging demand term (estimated from a vector of state-variables such as the consumption-wealth ratio). They find that once the hedging component is included, the coefficient of variance becomes positive and statistically significant, and the implied coefficient of relative risk aversion is large but reasonable. They show that omitting the hedging demand term biases the estimated risk-return slope downward. When the hedging term is controlled for, the volatility term no longer appears negatively related to returns. They do find, however, that the hedging term is primarily responsible for expected returns.

Kim et al. (2008) estimate the relationship by modeling regime changes in volatility. They derive a model of the equity premium under the assumption that market variance follows a two-state Markov-switching process. In their estimation, they let the volatility feedback effect differ when the variance regime is "high" versus "low." Their results, using historical U.S. returns, show a negative and significant volatility feedback effect on current returns, which in their interpretation implies that higher volatility in the current period predicts a higher equity premium in future periods (a positive risk-return relationship). They find that the risk-return relationship is stronger in high-volatility regimes (crisis periods), and relatively weak in calmer periods.

To find the positive risk-return relationship, Lundblad (2007) takes the approach of greatly expanding the sample size. He uses a very long historical sample of U.S. stock returns (starting in 1836) in his analysis. He finds statistically significant evidence of a positive risk premium when measured over such long horizons. His assertion is that conventional samples which are under 100 years in size may be too short to reliably estimate this relationship. In Monte Carlo simulations, he shows that small sample sizes can produce widely varying estimates of the coefficient of variance. He therefore argues that previous failures to find a positive tradeoff may be due to insufficient sample sizes. He also notes that the level of volatility itself has shifted a lot over history, with very high volatility around periods of market crisis and lower volatility in other periods. This suggests that the risk-return slope may appear different if crises are not accounted for.

What has motivated this thesis is that more recently, several papers have found success when measuring or modeling risk over a longer horizon. One core issue in exploring the risk-return relationship is that volatility is not directly observable and can be modeled in many ways. More recent studies have separated volatility into short- and long-term components (through CGARCH, MIDAS, or other methods) and used longer averaging windows for its calculation. In particular, several papers have reported strong positive risk-return slopes when the risk measure is built to model low-frequency or long-run volatility.

Guo & Neely (2007) find that the risk-return relationship is positive and significant when risk is measured by the persistent/long-term component of the Component-GARCH (CGARCH) model (Engle & Lee, 1993). The CGARCH model separates total conditional variance into two additive components. The CGARCH(1,1) specification, for example, would look as follows:

$$r_t = \mu + \epsilon_t$$

$$\sigma_t^2 = q_t + \alpha(\epsilon_{t-1}^2 - \sigma_{t-1}^2) + \beta(\sigma_{t-1}^2 - q_{t-1})$$

$$q_t = \omega + pq_{t-1} + \varphi(\epsilon_{t-1}^2 - \sigma_{t-1}^2)$$

where:

- r_t and μ are the return at time t and its mean, respectively
- $\epsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$ is the return innovation
- σ_t^2 is the total conditional variance
- q_t is the persistent/long-term variance
- $(\epsilon_{t-1}^2 - \sigma_{t-1}^2)$ is the usual GARCH innovation (news)

The MF2-GARCH model, which is at the center of this thesis, similarly separates variance into a short- and long-term component, but does so multiplicatively. They conclude that long-run volatility plays a key role in pricing the equity premium (even though they caution that some of this evidence might be spurious due to difficulty separating shocks).

Similarly, Ghysels et al. (2005) use a mixed data sampling (MIDAS) approach to investigate this relationship, finding that short-term windows of measurement yield insignificant or even negative volatility coefficients, and that increasing the MIDAS window to the medium-term (3-4 months) flips the volatility coefficient to positive and statistically significant, with declining coefficients and model fit for windows larger than 6 months. These results provide evidence for the idea that aggregating volatility information over several months reveals the characteristic risk that investors demand compensation for.

Maheu & McCurdy (2007) propose a parsimonious volatility model that allows different components (short- and long-term) to decay at different rates, using it to estimate the relationship between excess market return and market return volatility. They use a realized volatility (RV) approach, and their volatility components are weighted sums of past RV values. They find that all specifications, whether using levels or logarithms of RV, show a positive risk-return relationship. I use a risk-return specification in this thesis that is similar to their univariate specifications.

2.2 Contribution

In this thesis, I aim to extend the above literature by placing the MF2-GARCH volatility model into a univariate risk-return specification similar to that of Maheu & McCurdy (2007). I employ a "MF2-GARCH-in-mean" model for excess market return. I estimate four risk-return specifications, two where the conditional mean is regressed on only one of the volatility components, one where it is regressed on both, and one where it is regressed on overall conditional variance.

The novelty in this process is in the way that MF2-GARCH models the long-term component of volatility. The MF2-GARCH model takes advantage of the empirical fact that rolling window moving averages of the standardized forecast errors of one-component GARCH models behave counter-cyclically and have predictive power for future standardized forecast errors (Conrad & Engle, 2025). Simple GARCH models do not accurately capture these counter-cyclical movements. MF2-GARCH does so explicitly.

I also test specification options suggested by previous findings.

Christoffersen et al. (2008) show that omitting an intercept in the mean equation can strengthen evidence of a positive risk-return slope. I estimate both proportional (no-intercept) and non-proportional variants of each specification, and following Guo & Neely (2007), I use likelihood-ratio tests to determine whether the intercept adds enough explanatory power to warrant its inclusion.

As touched upon above, there is evidence of volatility "regime-switching" in equity markets, especially during crises. Ghysels et al. (2016) use a "flight to safety" indicator variable to exclude crisis periods from their analysis of the risk-return relationship using the MIDAS approach. They find a significant and positive relationship between return and their MIDAS estimator during the "normal" regime, but a reversal of this relationship in the "crisis" regime.

The intuition is that crises cause investors to move capital to safe haven assets of lower long-term volatility. Following their example, I use a binary dummy variable to control for crisis periods in this thesis. The exact dates of the crisis periods are informed by the recession periods defined by the National Bureau of Economic Research (NBER). I also show the model's parameter estimates without the inclusion of the crisis dummy variable to illustrate the effect of controlling for crisis periods.

I limit the data to a "modern" subsample starting from 1964, as Ghysels et al. (2005) do. As Hetzel (2013) notes, in this period, the U.S. Federal Reserve began actively adjusting short-term interest rates to smooth business-cycle fluctuations (affecting market volatility), whereas their pre-World War Two focus was to back the dollar with gold and prevent speculative credit booms. This is public information which informs investor behavior and therefore affects the equity premium which they demand in return for bearing risk, affecting the market risk-return relationship.

The parameter estimates that I obtain show that the short-term component of MF2-GARCH volatility only has a significant coefficient in limited scenarios, and that it is often negative. On the other hand, the long-term component shows consistent positivity and significance. It is also positively related to excess market return in all cases. In some cases, the crisis period dummy variable also points to components of volatility losing importance during crisis periods, which is consistent with flight to safe haven assets by investors.

Finally, I address estimation bias. To assess the MF2-GARCH-in-mean model's ability to recover "true" parameters, I conduct Monte Carlo simulations of daily returns and MF2-GARCH volatility. For each simulation, I generate $T = 30,240$ days of data (or 120 trading years) and present the sampling distribution of the estimated risk-return coefficients over $R = 1,000$ iterations. This shows the estimators' behavior in finite samples and whether inference might be misleading when the model is fitted to real data.

3 Econometric Model

3.1 The MF2-GARCH Model

Volatility here has two multiplicative components: short- (h_t) and long-term (τ_t)

$$\sigma_t^2 = h_t \tau_t$$

Daily stock returns are written as:

$$r_t = \mu_t + \sigma_t Z_t$$

$$= \mu_t + \sqrt{h_t \tau_t} Z_t$$

where μ is the unconditional mean of returns and Z_t are return innovations (assumptions about these innovations are detailed below).

The short-term component is modeled as a GJR-GARCH(1,1) process:

$$h_t = (1 - \phi) + (\alpha + \gamma 1_{\{r_{t-1}<0\}}) \frac{(r_{t-1} - \mu)^2}{\tau_{t-1}} + \beta h_{t-1} \tag{1}$$

where $\phi = \alpha + \frac{\gamma}{2} + \beta$.

To ensure that this process is covariance stationary, MF2-GARCH relies on the assumption that the parameters satisfy the following inequalities:

- $\alpha > 0$
- $\alpha + \gamma > 0$

- $\beta > 0$

- $\phi = \alpha + \frac{\gamma}{2} + \beta < 1$

and that Z_t:

- is i.i.d.

- has a symmetric density with $E(Z_t) = 0$ and $E(Z_t^2) = 1$

- is such that Z_t^2 has a nondegenerate distribution with $E(Z_t^4) < \infty$

Covariance stationarity is important for this combined model, as it implies that a finite unconditional mean exists, and that the variance of returns is not infinite. While it is not necessary to assume that Z_t follows a standard normal distribution, I do so to generate shocks in the Monte Carlo simulations (Section 4.3), as this satisfies the above assumptions.

Following Engle (2009), Conrad & Engle (2025) define $V_t = \frac{(r_t-\mu)^2}{h_t}$ as the squared "deGARCHed returns". These represent the standardized volatility forecast errors from Equation (1). The long-term component, τ_t, is specified as a multiplicative error model (MEM) equation for the conditional expectation of $V_{t-1}^{(m)}$, the moving average of the standardized errors V_t:

$$\tau_t = \lambda_0 + \lambda_1 V_{t-1}^{(m)} + \lambda_2 \tau_{t-1} \tag{2}$$

$$V_{t-1}^{(m)} = \frac{1}{m} \sum_{j=1}^{m} V_{t-j} = \frac{1}{m} \sum_{j=1}^{m} \frac{(r_{t-j} - \mu)^2}{h_{t-j}} \tag{3}$$

This is because of the empirical fact that a rolling window moving average of the past daily standardized forecast errors of one-component GARCH models has predictive power for future volatility (more specifically, squared returns). The errors are predictable and counter-cyclical, with one-component GARCH models underpredicting volatility in economic recessions and overpredicting it in expansions.

This is the MF2-GARCH-rw-m variant, where rw-m stands for "rolling window of length m". Conrad & Engle (2025) introduce another variant which allows for variable weighting schemes to be applied to different values of V_t. Nonetheless, they find that across various subsamples, the flat weighting scheme of the MF2-GARCH-rw-m is consistently preferred when considering the resulting Bayesian Information Criterion (BIC) values. Allowing for a flexible weighting scheme only increases the standard errors of the parameter estimates. For this reason, I employ MF2-GARCH-rw-m alone.

The MF2-GARCH specification markedly outperforms the nested GJR-Garch, Spline-GARCH, GARCH-MIDAS-RV and log-HAR in out-of-sample forecasts of volatility, especially in the long-term (Conrad & Engle, 2025). This makes it desirable for applications that require long-horizon forward-looking forecasts of volatility.

3.2 Maheu-McCurdy Univariate Risk-Return Specification

Maheu & McCurdy (2007) introduce a basic risk-return model in which the conditional mean of the excess market return is related to both the conditional variance of market return as well as one, some, or all of the components of variance:

$$\mu_t = \delta_0 + \delta_1 \sigma^2_{t,(q)} + \sigma_{t,(k)} z_t, \qquad z_t \sim N(0,1) \tag{4}$$

where $\sigma_{t,(k)}$ is the (square root of) the conditional variance at time t given by a k-component volatility model and $\sigma^2_{t,(q)}$ represents the components of volatility given by the model. Maheu & McCurdy define $\sigma^2_{t,(q)}$ as a weighted sum of the components and allow their Maximum Likelihood Estimator to determine the appropriate weights.

While they use a realized volatility (RV) approach to model risk in their original work, I use MF2-GARCH volatility and its components instead.

3.3 MF2-GARCH-in-Mean

While the original MF2-GARCH specification treats the mean (μ) unconditionally as a parameter to be estimated, I instead use the conditional mean, μ_t. Here the conditional mean is given by the Maheu-McCurdy specification in Equation (4). The squared demeaned return in Equation (1), $(r_{t-1} - \mu)^2$, becomes:

$$(r_{t-1} - \mu_t)^2 = r_{t-1} - \delta_0 - \delta_1 \sigma^2_{t-1,(q)}$$

and as this thesis uses excess market return, r_{t-1} is the daily market premium for period $t-1$.

In Equation (4), the conditional variance and its components are estimated with MF2-GARCH-rw-m. As in the original paper by Conrad & Engle (2025), I estimate all models for values of m from 20 up to 160 and determine the optimal value for m as the one that minimizes the BIC value.

I investigate if the components of the MF2-GARCH model support the finding that long-term volatility is a better determinant of excess market return.

As h_t represents short-term (daily) volatility, and τ_t persistent, slow-decaying volatility, regressing the market premium on these components should separate and reveal the degree to which a unit change in either component affects the premium that investors demand for bearing additional short- or long-term risk.
For $\sigma^2_{t,(q)}$ I first include just one of the short and long-term components (q=1):

$$\mu_t = \delta_0 + \delta_{1,s} h_t$$

$$\mu_t = \delta_0 + \delta_{1,l} \tau_t$$

Then I include both (q=2):

$$\mu_t = \delta_0 + \delta_{1,s} h_t + \delta_{1,l} \tau_t$$

I also estimate a specification in which the overall conditional variance (as estimated by MF2-GARCH) is the sole regressor:

$$\mu_t = \delta_0 + \delta_1 \sigma_t^2$$

For all four of these specifications, I estimate a no-intercept (proportional) variant, where δ_0 is forced to be zero, and a non-proportional variant, where it is not.

3.4 Controlling for Crisis Periods/Final Specification

Previous attempts to determine the risk-return relationship which account for volatility regime-switching during crisis periods have found that the significance and direction of the relationship varies based on the regime, as mentioned in Section 2.1. Including crisis periods in the sample can thus lead to a breakdown of the linearity of the risk-return relationship.

In addition, Danielsson et al. (2018) use multi-year deviations of volatility from the trend to predict banking crises. Their dummy variables show strong significance, which supports the notion that volatility clustering is more pronounced during crises. This particularly affects the viability of a constant window size (m) for the moving average/long-term volatility component of the MF2-GARCH model.

For these reasons, I include a binary dummy variable in the risk-return specification as an indicator for periods of crisis. The final model thus becomes:

$$\mu_t = \delta_0 + \theta_0 D_t + (\delta_1 + \theta_1 D_t)\sigma^2_{t,(q)} \tag{5}$$

$$h_t = (1-\phi) + (\alpha + \gamma 1_{\{r_{t-1}-\mu_{t-1}<0\}})\frac{(r_{t-1}-\mu_{t-1})^2}{\tau_{t-1}} + \beta h_{t-1} \tag{6}$$

where D_t is the dummy variable, θ_0 and θ_1 are its coefficients, and the long term component τ_t is as previously defined by Equation (2) and (3).

The exact crisis period dates (where $D_t = 1$) are described in Section 4.2 and presented in detail in Table 3. I also repeat the estimation of all specifications without controlling for crisis periods to demonstrate the effect of doing so.

4 Method

4.1 Log-Likelihood Function

Given past information set I_{t-1} and assuming that $r_t|I_{t-1} \sim N(\mu_t, \sigma_t^2)$ gives the t^{th} market premium observation the following likelihood function:

$$f(r_t|I_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} exp\left[-\frac{(r_t-\mu_t)^2}{2\sigma_t^2}\right]$$

Under the model, $\sigma_t^2 = h_t\tau_t$. The total likelihood, denoted as l, is therefore given by:

$$l = \frac{1}{\sqrt{2\pi h_t\tau_t}} \sum_{t=1}^{T} exp\left[-\frac{1}{2}\frac{(r_t-\mu_t)^2}{h_t\tau_t}\right]$$

where T is the sample size, and the total log-likelihood is given by:

$$L = \frac{1}{2}\sum_{t=1}^{T}[ln(2\pi) + ln(h_t\tau_t) + \frac{(r_t-\mu_t)^2}{h_t\tau_t}]$$

This gives the combined risk-return specification given by Equation (5) and (6) the following total log-likelihood function:

$$L = \frac{1}{2}\sum_{t=1}^{T}[ln(2\pi) + ln(h_t\tau_t) + \frac{(r_t-(\delta_0+\theta_0 D_t)-(\delta_1+\theta_1 D_t)\sigma_{t,(q)}^2)^2}{h_t\tau_t}] \tag{7}$$

In my implementation, I minimize the negative log-likelihood function, $-L$.

4.2 Estimation

The full Python code and data files are provided as attachments to the digital version of this thesis, and links to an online repository can be found in Section A.3 of the appendix. The implementation of MF2-GARCH parameter estimation is based on Conrad & Schoelkopf (2025).

I estimate all parameters simultaneously using Quasi-Maximum Likelihood Estimation (QMLE) implemented in Python. Minimization of the negative log-likelihood function is done with the optimize.minimize function of the SciPy package, using the sequential least squares programming (SLSQP) option.

Likelihood ratio tests of the proportional variants against non-proportional counterparts are implemented manually with the SciPy package.

4.3 Monte Carlo Simulation

To test the viability of the MF2-GARCH-in-mean model, I generate simulated data using known parameters, assuming that the model holds, and I estimate the parameters over the simulated data.

I do this for the proportional and non-proportional variant of every specification. For simplicity, no crisis periods are generated and the crisis indicator dummy variable is excluded. The choice of moving average window size (m) is set at $m = 63$, or three trading months, and set at the same value manually during estimation.

For the "true" parameter values, I use averages of the parameter estimates I obtained from fitting the model on real data. The parameter values used in the Monte Carlo simulations are shown in Table 1. The values of τ_t, h_t, and r_t at time $t = 0$ are their average values obtained from fitting the model on real data. The moving average of the daily standardized forecast errors ($V_t^{(m)}$) requires at least m previous data points, and τ_t is a function of $V_{t-1}^{(m)}$. As a result, for all $t < m$, values of τ_t are hard-coded to be equal to the initial τ_0 value, and $V_t^{(m)}$ is simply an average of all past values of V_t.

To avoid dependence on initial value choices in the simulated data, the first 252 data points (one trading year) are discarded as a form of "burn-in". In each simulation, I simulate $T = 30,240$ days of data, the equivalent of 120 trading years. I perform $R = 1,000$ simulations and report the mean and standard deviation of the parameter estimates and mean bias.

This is implemented in Python and Monte Carlo simulation features are included in the code referred to above in Section 4.2. Parameter estimation on simulated data is performed as described in Section 4.2.

5 Empirical Results

5.1 Data

I apply the combined specification given in Equation (5) and (6) to U.S. daily market premium data.

The data are downloaded from CRSP. For excess market return, I use the U.S. market premium from the Fama-French 3 Factor library. The data runs from July of 1926 to April of 2025, inclusive. I limit it to a modern subsample starting in January of 1964 as detailed in Section 5.1.1. The data used is included as an attachment to the digital version of this thesis.

The excess market return is calculated by subtracting the one-month Treasury bill rate (the risk-free rate) from the value-weighted CRSP market return for NYSE, AMEX, and NASDAQ firms (the market return rate). The resulting values are taken as r_t in the combined MF2-GARCH-in-mean model. The binary crisis indicator variable is added manually (see section 5.1.2 for more details). The data was converted from .csv to .xlsx format for convenience.

Table 2 shows summary statistics for the market premium data. As expected, the premia have a moderately positive mean, a negative skew, a high kurtosis and a weak autocorrelation coefficient.

5.1.1 Sample Dates

While exact dates vary, the mid 1960s mark a clear turning point toward lower volatility in output and inflation in the postwar U.S. economy. They also mark a shift in monetary policy focus. I therefore limit the data to a "modern" subsample starting from January of 1964.

5.1.2 Crisis Periods

A binary dummy variable (D_t) is used to exclude the idiosyncratic effects of crisis periods.

To define the specific dates which mark the beginnings and ends of crisis periods in U.S. markets, I use economic recessions as recognized by the U.S. National Bureau of Economic Research (NBER). This is to ensure that the dummy variable truly captures periods of macroeconomic stress rather than benign clusters of high volatility.

A list of these periods and their corresponding start and end dates is provided in Table 3. The value of the dummy variable is 1 in these periods and 0 otherwise.

5.1.3 Choice of moving average window size

I estimate the model for values of m from 20 to 160 and choose the value which minimizes the BIC. For all specifications, this value is $m = 63$, or around 3 trading months.

5.2 Estimates and Interpretations

5.2.1 Parameter Estimates - Controlling for Crisis Periods

Table 4 reports QMLE parameter estimates from the MF2-GARCH-in-mean model with the inclusion of a binary dummy variable to control for crisis periods. Each panel presents the proportional (no-intercept) and non-proportional (with-intercept) variants of each specification as well as the corresponding log-likelihood and BIC values. Standard errors in parentheses are Bollerslev-Wooldridge robust standard errors. Each panel also shows the value of the likelihood ratio test (LRT) statistic for the proportional variant against the non-proportional variant. A significant LRT statistic means a rejection of the null hypothesis that the proportional variant has sufficient explanatory power.

For all specifications, the moving average window size (m) which minimizes the BIC value is $m = 63$. This is the same optimal value (of three trading months) which Conrad & Engle (2025) find in their original paper. The plots of BIC values against m values are shown in Figure 1.

In the short-term component only specification (Panel A), the estimate of the coefficient of the short-term component ($\widehat{\delta_{1,s}}$) is positive and significant at the 5% level in the proportional variant and negative and significant at the 1% level in the non-proportional variant. The LRT statistic is highly significant, rejecting the proportional variant in favor of the non-proportional one. The estimate of the intercept in the non-proportional variant ($\widehat{\delta_0}$) is positive and significant at the 1% level. The coefficients of the crisis indicator dummy variable are statistically indistinguishable from zero.

In the long-term component only specification (Panel B), the estimate of the coefficient of the long-term component ($\widehat{\delta_{1,l}}$) is positive and significant at the 1% level in both the proportional and non-proportional variants. The LRT fails to reject the null hypothesis, favoring the proportional variant. The estimate of the intercept in the non-proportional variant ($\widehat{\delta_0}$) is positive but statistically insignificant. When interacted with the long-term component in the proportional specification, the crisis indicator dummy variable has a negative coefficient estimate ($\widehat{\theta_{1,l}}$) which is significant at the 1% level. The other coefficients of the crisis indicator dummy variable are statistically indistinguishable from zero.

In the two-component specification (Panel C), where the short- and long-term components are both additively included, $\widehat{\delta_{1,s}}$ is negative and insignificant in both the proportional and non-proportional variants. $\widehat{\delta_{1,l}}$, on the other hand, is positive in both, and significant at the 5% level in the proportional variant and at the 1% level in the non-proportional one. The LRT again fails to reject the null hypothesis, favoring the proportional variant. The estimate of the intercept is positive but insignificant in the non-proportional variant. When interacted with the short-term component in the proportional specification, the crisis indicator dummy variable has a negative coefficient estimate ($\widehat{\theta_{1,s}}$) which is significant at the 5% level. All of the dummy variable's other coefficient estimates are statistically indistinguishable from zero.

Notably, when the risk-return specification uses overall conditional variance (Panel D), the estimate of the coefficient of volatility ($\widehat{\delta_1}$) is significant at the 1% level in the proportional specification. The LRT is significant at the 10% level, however, pointing to the non-proportional variant being preferred. In the non-proportional variant, the coefficient becomes statistically indistinguishable from zero. Nonetheless, the highly significant coefficient estimate in the proportional specification suggests that, by including the long-term component, MF2-GARCH volatility captures some additional information that simpler models do not.

The model with the best fit (according to BIC value) is the proportional long-term component only specification. Summary statistics for volatility and its components under this specification are provided in Table 6. It should be noted, however, that the MF2-GARCH parameter estimates (α, γ, β, λ_0, λ_1, λ_2) are quite similar across all the specifications, and consequently, so are the summary statistics of conditional variance.

5.2.2 Parameter Estimates - Not Controlling for Crisis Periods

Table 5 reports QMLE parameter estimates from the MF2-GARCH-in-mean model without the inclusion of a binary dummy variable to control for crisis periods. It is set up exactly as Table 4 is (*see above section "Parameter Estimates - Controlling for Crisis Periods"*).

For all specifications, the moving average window size (m) which minimizes the BIC value is again $m = 63$ (three trading months). The plots of BIC values against m values are shown in Figure 2.

The magnitudes of the parameter estimates In the short-term component only specification (Panel A) change when the crisis dummy variable is excluded but the levels of significance and signs are similar. The estimate of the coefficient of the short-term component ($\widehat{\delta_{1,s}}$) is positive and significant at the 5% level in the proportional variant and negative and significant at the 5% level in the non-proportional variant. The LRT statistic is again highly significant, rejecting the proportional variant in favor of the non-proportional one. The estimate of the intercept in the non-proportional variant ($\widehat{\delta_0}$) is positive and significant at the 1% level. ($\widehat{\delta_{1,s}}$) is smaller when the crisis dummy is excluded in both variants, and ($\widehat{\delta_0}$) is larger.

In the long-term component only specification (Panel B), the estimate of the coefficient of the long-term component ($\widehat{\delta_{1,l}}$) is positive and significant at the 1% level in the proportional variant as before, but removing the crisis dummy results in ($\widehat{\delta_{1,l}}$) being positive but statistically insignificant in the non-proportional variant. The LRT again fails to reject the null hypothesis, favoring the proportional variant. The estimate of the intercept in the non-proportional variant ($\widehat{\delta_0}$) is positive but statistically insignificant.

In the two-component specification (Panel C), where the short- and long-term components are both additively included, $\widehat{\delta_{1,s}}$ is still negative and insignificant in both the proportional and non-proportional variants. $\widehat{\delta_{1,l}}$, is positive and significant at the 1% level in the proportional variant but positive and insignificant in the non-proportional one. The LRT again fails to reject the null hypothesis, favoring the proportional variant. The estimate of the intercept is positive but insignificant in the non-proportional variant.

In the overall conditional variance specification (Panel D), the estimate of the coefficient of volatility ($\widehat{\delta_1}$) is positive and significant at the 1% level in the proportional specification. In the non-proportional variant, the coefficient becomes statistically indistinguishable from zero. The LRT is significant at the 5% level, favoring the non-proportional variant.

The model with the best fit (according to BIC value) is once again the proportional long-term component only specification. Summary statistics for volatility and its components under this specification are provided in Table 7. The results are very similar of those of the same specification with the crisis dummy variable included. This points to the dynamics of volatility being largely unchanged by the inclusion of the crisis dummy variable.

5.2.3 Interpretation

The BIC values for the model are better (smaller) when the crisis dummy variable is not included. This suggests that the addition of its parameters does not add enough explanatory power to warrant the added complexity. Nonetheless, its inclusion is informative, as we can see that removing it causes the long-term component's coefficient to lose statistical significance in the non-proportional variants of the long-term component only and two-component specifications. Given that the LRT favors the proportional variants of these specifications, and given that the BIC values improve without the crisis dummy variable, I will focus on the results from Table 5, where the dummy is not included.

When only the short-term volatility component is included in the conditional mean equation (Panel A), its risk premium behaves differently depending on whether there is an intercept. In the proportional variant, the positivity and high significance of the short-term coefficient suggests that investors demand higher expected returns for bearing transitory fluctuations in return. However, once an intercept is introduced (non-proportional variant), the coefficient becomes negative and significant, and the LRT strongly favors the non-proportional specification. This flip in sign indicates that allowing a baseline premium absorbs much of the average compensation, and what remains implies a discount for short-term fluctuations, possibly reflecting volatility-timing effects. In this case, the negative coefficient may be because of investors who target certain risk levels in their portfolios selling the "market asset" during short-term spikes in volatility in order to rebalance portfolio risk. In any case, the intercept in the non-proportional variant absorbs a positive constant component that was captured by the short-term component in the proportional variant.

By contrast, isolating the long-term volatility component (Panel B) yields a highly significant positive relationship with the premium in the proportional variant and the LRT does not reject the simpler proportional variant, implying that the persistent component of volatilty commands a stable premium. This finding is consistent with the intuition that persistent uncertainty constitutes genuine background risk for investors, who therefore demand a premium that, in this specification, is not absorbed by a constant term. While transient movements captured by the short-term component can be attributed to noise, longer-term patterns might appear to be more characteristic and indicative of the state of the market, influencing the behavior of investors.

With both components included (Panel C), the long-run component again turns out to be the source of compensation. The long-term coefficient retaining positivity and significance and the short-term coefficient losing it entirely in the proportional variant suggests that temporary uncertainty contributes no real premium once persistent uncertainty is accounted for. The LRT favoring the proportional variant indicates that the intercept adds little explanatory power when both volatility horizons are modeled together, which suggests that the average market premium can be taken as arising entirely from the long-term risk component. In this case, we are able to interpret $\widehat{\delta_{1,l}}$ as an estimate of the coefficient of relative risk aversion.

The final specification based on total conditional variance (Panel D) shows a significant positive coefficient in the proportional model, telling us that aggregate volatility matters in determining the premium. However, the LRT again prefers the non-proportional variant, and in the non-proportional variant the estimate of the coefficient becomes insignificant. This and the lower (more desirable) BIC value of the proportional long-term specification both support the assertion that MF2-GARCH's long-term component does better at capturing information which relates directly to the market premium.

According to the BIC values, the proportional long-term only model is the most parsimonious and best-fitting specification. The result that the long-term component alone captures most of the risk-return relationship, and that this result holds even when compared against the more complex two-component or overall variance specifications, is supportive of the idea that long-term uncertainty is the main determinant of the market premium.

In this proportional long-term component specification, we can interpret $\widehat{\delta_{1,l}}$, as the "daily" coefficient of relative risk aversion (CRRA), as explained in Section 1. If we annualize this CRRA (0.049) by multiplying it by 252 (the number of trading days in a year), we get a value of 12.348. This is slightly high compared to the "traditional" benchmark range of 2-10 which is usually estimated by previous research (Elminejad et al., 2025). This is perhaps explained by the focus of this specification on long-horizon risk, which may be viewed by investors as more enduring and characteristic rather than temporary, causing them to appear more averse to risk than is considered reasonable.

What is interesting to note about the LRT statistics and the option of including an intercept is that the regressand in this model is the market premium, meaning that the risk-free rate (which we can consider the constant component of overall market return) is already subtracted and controlled for. A positive and significant intercept, like the one in the non-proportional short-term component only specification (Panel A), can simply mean that the intercept is capturing the average premium, and that the short-term component's inclusion explains deviations from the average premium. If we used the raw market return as a regressand instead, one would expect the intercept to be equal to the risk-free rate.

Finally, the similarity of the underlying MF2-GARCH parameter estimates across all specifications suggests consistent volatility dynamics, meaning that roughly the same information is retained across specification changes, so that they can be reasonably compared.

5.3 Monte Carlo Simulation Results

For each specification, Table 8 reports the true parameters used in data generation, the average bias of the estimates (absolute and percentage value), and the standard deviation of the parameter estimates across the 1,000 iterations.

For γ, β, λ_1, λ_2, $\delta_{1,l}$ and δ_1, the estimates show little bias. The bias does not exceed $\pm 1.50\%$ for any specification for these parameters. This finding provides evidence of the effectiveness of the QMLE process in recovering long-horizon volatility dynamics and their relation to market premia.

However, the simulations also show moderately large biases in the estimation of the short-term component's coefficient estimate ($\delta_{1,s}$) and in the constant terms (α, λ_0, and δ_0). The bias for $\delta_{1,s}$ exceeds 20% in some specifications, suggesting that the model has limited ability to precisely determine the response of market premia to short-horizon volatility fluctuations. The bias in constant terms suggests that sample noise during simulation increase estimation error for these parameters. The model is therefore not very viable for interpretation of the coefficient of the short-term component and the intercepts.

This bias persists with different intialization values and a much longer burn-in periods (where more data points are generated initially and discarded to reduce dependence on initial values), so incorrect initializations are unlikely to be the reason behind this bias. This is also the case with much longer sample sizes (T values), so it is unlikely that this is caused by small sample bias.

While this bias tells us that the model's estimates of the intercepts and short-run volatility dynamics are unreliable, the model still reliably demonstrates the positive and significant relationship between long-run volatility and market premia that this thesis aims to find. The Monte Carlo evidence confirms that the long-term volatility component is accurately identified and that its coefficient can be estimated with minimal bias.

6 Further Research

6.1 Return and Premium Forecasting

One immediate extension is to evaluate the MF2-GARCH-in-mean model as a forecasting tool for returns or equity premia. Given that Conrad & Engle (2025) find MF2-GARCH generates strong out-of-sample volatility forecasts, it is worth exploring whether using the MF2-GARCH long-run volatility component improves return/premium forecasts. Future research could use the MF2-GARCH risk measure in out-of-sample forecasts, testing against earlier models like GJR-GARCH, Spline-GARCH, GARCH-MIDAS-RV, etc. A possible approach would be to simulate multi-step forecasts of expected returns under the MF2-GARCH-in-mean model and compare the forecast errors. In practice, a better forecasting model for returns would be of interest for financial applications, so verifying the predictive power of the MF2-GARCH-in-mean specification warrants further examination.

6.2 Regime-Switching Extensions

Another promising option is to introduce explicit regime or structural breaks into the MF2-GARCH-in-mean model. Allowing the GARCH parameters (α, γ, β, λ_0, λ_1, λ_2) to switch between high-volatility and low-volatility regimes as in two-state Markov-switching models like that of Kim et al. (2008) would achieve this. Doing so might capture changes in investors' risk aversion or volatility dynamics that occur during crises. Similarly, allowing time-varying coefficients on the components of volatility in the risk-return specification to move with economic variables tied to expansion/recession can achieve the same aim. These extensions would build on the evidence that volatility dynamics can vary between regimes. By extending the MF2-GARCH model in this way, one can endogenously change the definition of "long-run" volatility according to the market state, potentially improving fit. This can eliminate the need for the crisis indicator dummy variable.

7 Conclusion

7.1 Limitations

7.1.1 Short-Term Volatility Dynamics

Perhaps the biggest issue in this MF2-GARCH-in-mean model is the bias in the short-term component's parameter estimation. In my estimates the short-run volatility component often shows an insignificant or even negative risk premium. This suggests that high-frequency volatility or transient risk is difficult to identify within the model. Earlier studies have also found that very short measurement windows often yield insignificant or negative coefficients on volatility. As stated earlier, Ghysels et al. (2005) show that estimating volatility over short horizons leads to weak or negative risk-return estimates, and only when the window is lengthened (e.g. 3-6 months) does the volatility coefficient become positive. Similarly, the MF2-GARCH-in-mean short-term component may be capturing noise rather than true risk, which undermines the estimation process.

7.1.2 Crisis Dummy Simplification

The use of a simple binary crisis indicator also has its limitations. By construction the dummy assumes that all of the NBER-defined recessions have the same effect on the market risk premium, which is a strong simplification. In reality, crises can vary widely in severity, duration, and underlying causes. A single binary dummy cannot capture such non-linear or asymmetric effects across different crisis periods. Additionally, the NBER start and end dates mix downturn and recovery months, which might not line up well with market stress. The magnitude and timing of responses by policymakers and changes in investor behavior vary by crisis, so the real adjustment of the market premium to volatility likely changes continuously over time rather than starting and stopping perfectly with the start and end dates of each crisis period. The dummy variable is may therefore mistime how the premium behaves through turbulent periods. More complex approaches (e.g. Markov-switching regimes, mathematically defined recession periods) would allow the data to endogenously identify changes in dynamics rather than forcing a uniform on/off crisis effect, and would likely more accurately model risk premia in recessions.

7.2 Conclusion

This thesis introduces the MF2-GARCH volatility decomposition to a univariate risk-return model, explicitly separating short-run and long-run volatility in the conditional mean. This approach reveals that only the long-run volatility component carries a significant risk premium. Across many specifications the coefficient on the long-run component is statistically significant and positive, whereas the short-run component is frequently statistically insignificant or negative. This implies that persistent volatility (rather than temporary fluctuations) is what investors demand compensation for. Monte Carlo simulations support this finding by showing the long-run risk premium is identified reliably, while the short-run premium is not. The results support the idea that long-run uncertainty is a main determinant of the equity premium.

These results are consistent with recent literature that makes use of persistent volatility to arrive at the same positive relationmship. The use of MF2-GARCH is novel in the risk-premium context, and it explicitly makes use of the counter-cyclical behavior of forecast errors to model the smooth volatility component.

Practically, the results have important implications for investment, risk management and policymaking. A positive long-horizon risk-return relationship means that achieving higher returns over many years requires bearing greater long-run volatility, as standard asset pricing theory would predict. This suggests that long-term investors (such as pension funds) should focus on and hedge against persistent volatility changes over short-run volatility fluctuations. Portfolio management strategies that adjust exposure to assets based on short-term volatility measures may be ineffective. In risk management, making use of MF2-GARCH components for value at risk models or stress tests could improve the forecasting of tail events if those events are driven by changes in long-run volatility. Finally, policymakers concerned with financial stability should note that their policies that affect long-term volatility are likely to have sustained effects on market premia.

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Appendix

A.1 - Figures

Figure 1: This figure shows the Bayesian Information Criterion (BIC) against moving-average window size m for all MF2-GARCH-in-mean specifications. Left column: proportional variants, right column: non-proportional. The rows from top to bottom show: short-term component, long-term component, two-component, and overall conditional variance specifications.

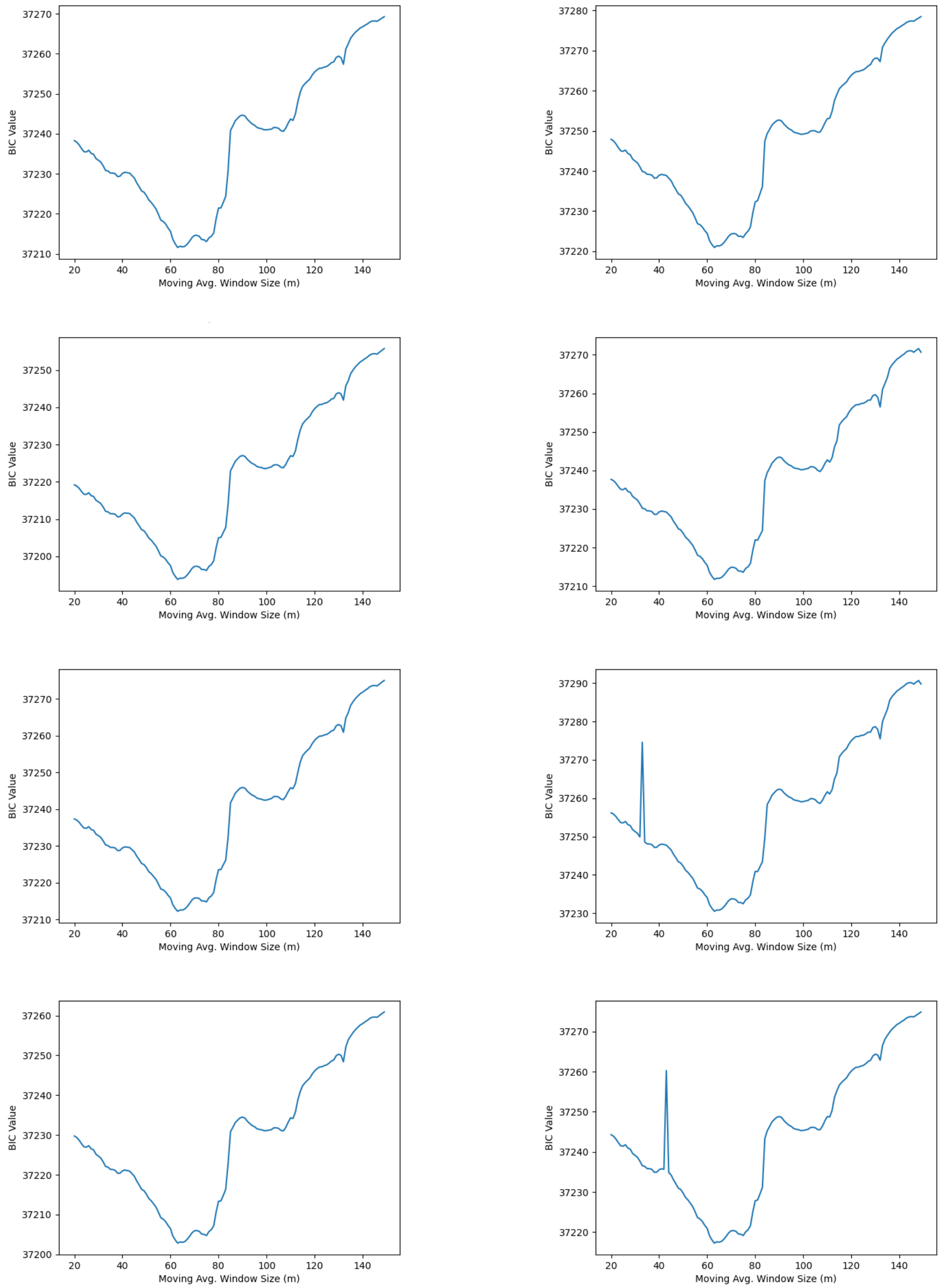
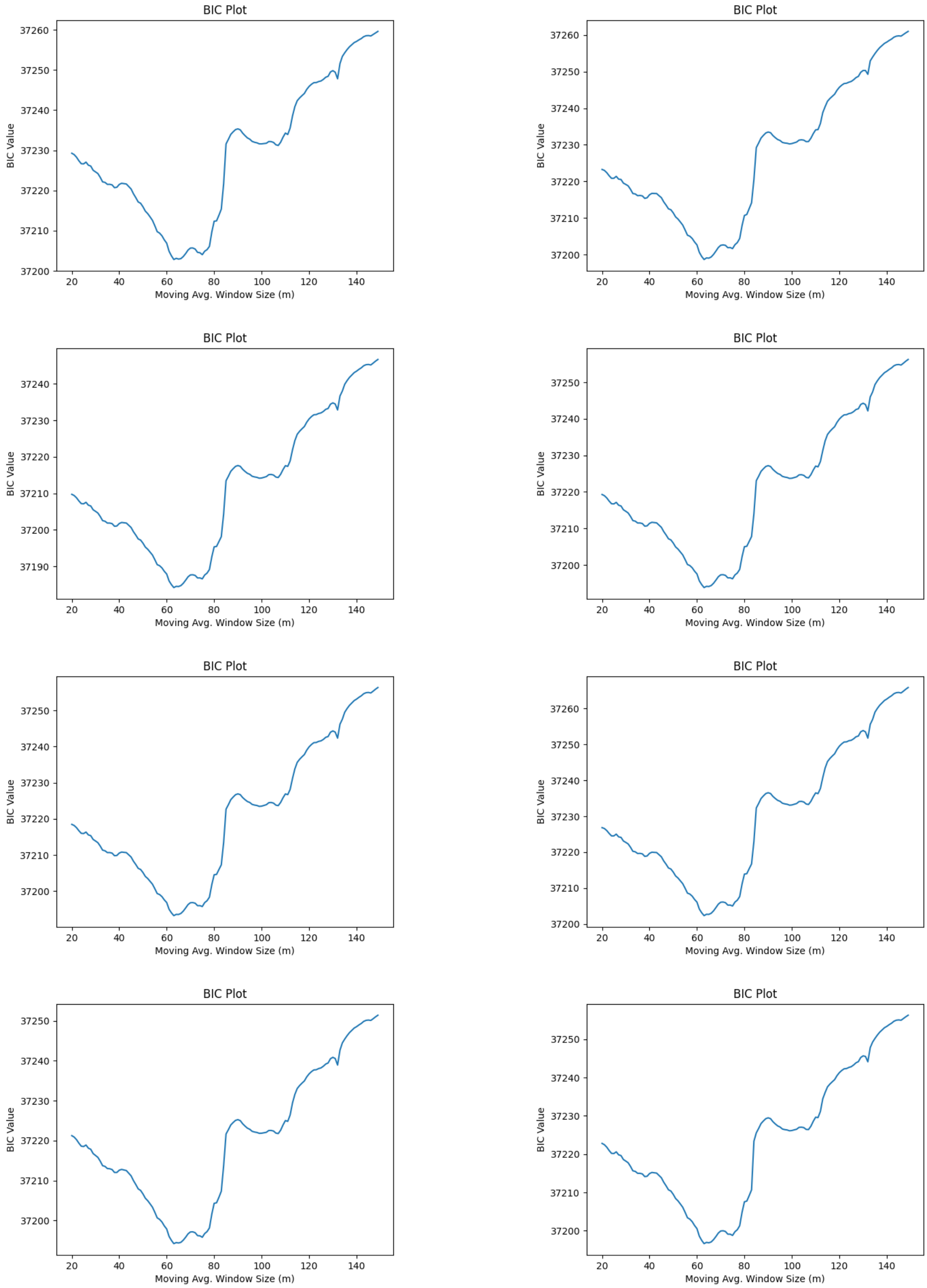


Figure 2: BIC Plots. *See Figure 1 notes.* This figure shows the same BIC plots for the model when the crisis indicator dummy variable is not included.



A.2 - Tables

Table 1: Parameter Values for Monte Carlo Simulations

α	γ	β	λ_0	λ_1	λ_2	δ_0	$\delta_{1,s}$	$\delta_{1,l}$	δ_1
Short-term component									
Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	-	0.027	-	-
Non-Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	0.033	-0.003	-	-
Long-term component									
Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	-	-	0.049	-
Non-Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	0.003	-	0.045	-
Both components (additive)									
Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	-	-0.005	0.054	-
Non-Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	0.008	-0.008	0.046	-
Overall conditional variance (multiplicative)									
Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	-	-	-	0.042
Non-Proportional									
0.006	0.160	0.842	0.011	0.085	0.902	0.020	-	-	0.023

Notes: This table presents the "true" parameter values I used in Monte Carlo simulations of daily market premium data. The MF2-GARCH-in-mean model is fitted $R = 1,000$ times on these simulated samples, each of size $T = 15,120$. This is repeated for the proportional and non-proportional variant of every specification. These values were chosen based on estimates from real data *(see Table 4 below)*.

Table 2: Summary Statistics - Daily Market Premia

	mean	sd	skew	kurtosis	min	max	AC(1)
r_t	0.028	1.028	-0.487	15.564	-17.440	11.360	0.016

Notes: This table shows summary statistics for the U.S. daily market premium data. The data runs from January 1964 to April 2025. The columns present the mean, standard deviation (sd), skewness, kurtosis, minimum (min), maximum (max) and the first-order autocorrelation coefficient (AC(1)).

Table 3: NBER Recession Periods

Start date	End date	Remarks
December 1969	November 1970	-
November 1973	March 1975	1973 oil crisis and stagflation
January 1980	July 1980	Volcker recession I
July 1981	November 1982	Volcker recession II
July 1990	March 1991	-
March 2001	November 2001	Dot-com bubble
December 2007	June 2009	Global financial crisis
February 2020	April 2020	COVID-19 pandemic

Notes: This table shows the start and end dates of recession periods defined by the U.S. National Bureau of Economic Research (NBER) which fall within the sample period on which the MF2-GARCH-in-mean model is estimated. The dummy variable used to control for periods of crisis is given the value 1 for the above periods and 0 otherwise.

Table 4: Combined Specification Parameter Estimates (Controlling for Crisis Periods)

α	γ	β	λ_0	λ_1	λ_2	δ_0	$\delta_{1,s}$	$\delta_{1,l}$	δ_1	θ_0	$\theta_{1,s}$	$\theta_{1,l}$	θ_1	LLF	BIC
Panel A: Short-term component															
Proportional															
0.007	0.158***	0.841***	0.012	0.086	0.900***	-	0.027**	-	-	-	-0.013	-	-	-18567	37212
(0.014)	(0.019)	(0.018)	(0.020)	(0.150)	(0.173)		(0.011)				(0.020)				
Non-Proportional															
0.005*	0.166***	0.845***	0.011***	0.081***	0.907***	0.033***	-0.003***	-	-	0.009	-0.009	-	-	-18562	37221
(0.003)	(0.015)	(0.012)	(0.002)	(0.018)	(0.018)	(0.007)	(0.001)			(0.008)	(0.009)				
LRT: 9.928***															
Panel B: Long-term component															
Proportional															
0.006**	0.160***	0.842***	0.011***	0.085***	0.902***	-	-	0.049***	-	-	-	-0.002***	-	-18558	**37194**
(0.003)	(0.015)	(0.013)	(0.003)	(0.018)	(0.020)			(0.009)				(0.001)			
Non-Proportional															
0.006*	0.161***	0.843***	0.011*	0.080***	0.907***	0.003	-	0.045***	-	-0.054	-	0.049	-	-18558	37212
(0.004)	(0.015)	(0.014)	(0.006)	(0.030)	(0.036)	(0.003)		(0.017)		(0.060)		(0.049)			
LRT: 1.358															
Panel C: Both components (additive)															
Proportional															
0.006	0.161***	0.844***	0.012*	0.085**	0.901***	-	-0.005	0.054**	-	-	-0.012**	0.009	-	-18558	37212
(0.013)	(0.015)	(0.016)	(0.006)	(0.035)	(0.042)		(0.021)	(0.023)			(0.006)	(0.012)			
Non-Proportional															
0.006	0.162***	0.844***	0.011	0.081**	0.906***	0.008	-0.008	0.046***	-	-0.050	-0.002	0.048	-	-18557	37230
(0.003)	(0.016)	(0.014)	(0.007)	(0.036)	(0.043)	(0.006)	(0.007)	(0.012)		(0.067)	(0.002)	(0.041)			
LRT: 0.990															
Panel D: Overall conditional variance (multiplicative)															
Proportional															
0.007	0.158***	0.839***	0.011**	0.080***	0.907***	-	-	-	0.042***	-	-	-	-0.014*	-18563	37203
(0.009)	(0.016)	(0.014)	(0.005)	(0.031)	(0.035)				(0.010)				(0.008)		
Non-Proportional															
0.006	0.162***	0.841***	0.011	0.080	0.908***	0.020	-	-	0.023	-0.005	-	-	-0.004	-18560	37217
(0.012)	(0.025)	(0.018)	(0.016)	(0.084)	(0.101)	(0.059)			(0.083)	(0.016)			(0.022)		
LRT: 4.850*															

Notes: This table shows the results of the QMLE parameter estimates for the MF2-GARCH-in-mean model. The numbers in parentheses are Bollerslev-Wooldridge robust standard errors. *** ,** and * indicate significance at the 1%, 5% and 10% level. Each panel shows the results for a different specification, depending on which components of volatility are included. Each panel shows the proportional (no intercept) and non-proportional variant. The likelihood ratio test (LRT) statistic for the proportional variant against the non-proportional variant is also shown at the bottom of each panel. All specifications are estimated using daily U.S. market premium data for the period January 1964 to April 2025 inclusive. The moving average window size (m) which minmizes the Bayesian Information Criterion for all specifications is m=63.

Table 5: Combined Specification Parameter Estimates (**Not** Controlling for Crisis Periods)

α	γ	β	λ_0	λ_1	λ_2	δ_0	$\delta_{1,s}$	$\delta_{1,l}$	δ_1	LLF	BIC
Panel A: Short-term component											
Proportional											
0.007*	0.158***	0.840***	0.012	0.086	0.901***	-	0.024**	-	-	-18568	37203
(0.004)	(0.015)	(0.013)	(0.011)	(0.066)	(0.079)		(0.010)				
Non-Proportional											
0.006***	0.163***	0.843***	0.011***	0.089***	0.898***	0.043***	-0.013**	-	-	-18561	37199
(0.002)	(0.015)	(0.014)	(0.003)	(0.027)	(0.029)	(0.007)	(0.005)				
LRT: 13.806***											
Panel B: Long-term component											
Proportional											
0.006**	0.160***	0.842***	0.011***	0.084***	0.902***	-	-	0.049***	-	-18558	37184
(0.003)	(0.015)	(0.013)	(0.003)	(0.021)	(0.022)			(0.010)			
Non-Proportional											
0.006	0.160***	0.843***	0.011	0.085	0.902**	0.003	-	0.046	-	-18558	37194
(0.033)	(0.021)	(0.020)	(0.020)	(0.329)	(0.386)	(0.009)		(0.038)			
LRT: 0.034											
Panel C: Both components (additive)											
Proportional											
0.006*	0.161***	0.843***	0.012***	0.085***	0.901***	-	-0.008	0.057***	-	-18558	37193
(0.003)	(0.015)	(0.013)	(0.003)	(0.023)	(0.024)		(0.008)	(0.015)			
Non-Proportional											
0.005	0.162***	0.843***	0.012	0.088	0.898***	0.013	-0.013	0.046	-	-18558	37202
(0.033)	(0.021)	(0.030)	(0.039)	(0.139)	(0.188)	(0.138)	(0.104)	(0.332)			
LRT: 0.542											
Panel D: Overall conditional variance (multiplicative)											
Proportional											
0.007	0.158***	0.839***	0.010***	0.079***	0.909***	-	-	-	0.037***	-18563	37194
(0.005)	(0.015)	(0.013)	(0.003)	(0.023)	(0.024)				(0.007)		
Non-Proportional											
0.007	0.159***	0.840***	0.011**	0.083**	0.904***	0.022	-	-	0.019	-18560	37197
(0.013)	(0.016)	(0.022)	(0.005)	(0.034)	(0.039)	(0.015)			(0.034)		
LRT: 7.182**											

Notes: *See Table 4 notes.* This table show parameter estimates when the crisis dummy variable is not included.

Table 6: Summary Statistics - With Crisis Control Dummy

(Proportional LT Component Specification)

	mean	min	max	AC(1)
σ_t^2	1.051	0.122	60.992	0.94736
h_t	1.186	0.474	69.731	0.98251
τ_t	0.830	0.236	3.810	0.99988

Notes: This table shows summary statistics for the daily conditional variance and its components as estimated by MF2-GARCH in the proportional (no intercept) long-term component specification when controlling for crisis periods. The columns show the mean, maximum (max), minimum (min)and the first-order autocorrelation coefficient (AR(1)).

Table 7: Summary Statistics - Without Crisis Control Dummy

(Proportional LT Component Specification)

	mean	min	max	AC(1)
σ_t^2	1.051	0.122	60.969	0.94738
h_t	1.186	0.474	69.720	0.92859
τ_t	0.830	0.236	3.793	0.99988

Notes: *See Table 6 notes.* This table presents the summary statistics for the proportional long-term component specification without controlling for crisis periods.

Table 8: Monte Carlo Parameter Estimates & Standard Deviations

	α	γ	β	λ_0	λ_1	λ_2	δ_0	$\delta_{1,s}$	$\delta_{1,l}$	δ_1
Short-term component										
Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	-	0.02700	-	-
Bias	-0.00116	-0.00128	0.00198	0.00082	0.00119	-0.00226	-	-0.00011	-	-
(%)	(-19.33%)	(-0.80%)	(0.24%)	(7.45%)	(1.40%)	(-0.25%)	-	(-0.41%)	-	-
S.d.	0.00353	0.00684	0.00675	0.00601	0.03777	0.04460	-	0.00468	-	-
Non-Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	0.03300	-0.00300	-	-
Bias	-0.00115	-0.00072	0.00122	0.00086	0.00147	-0.00259	-0.00155	0.00150	-	-
(%)	(-19.17%)	(-0.45%)	(0.14%)	(7.82%)	(1.73%)	(-0.29%)	(-4.70%)	(-50.00%)	-	-
S.d.	0.00356	0.00691	0.00711	0.00603	0.03789	0.04476	0.01192	0.01303	-	-
Long-term component										
Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	-	-	0.04900	-
Bias	-0.00116	-0.00070	0.00139	0.00071	0.00052	-0.00145	-	-	-0.00010	-
(%)	(-19.33%)	(-0.44%)	(0.17%)	(6.45%)	(0.61%)	(-0.16%)	-	-	(-0.20%)	-
S.d.	0.00353	0.00693	0.00678	0.00578	0.03593	0.04253	-	-	0.00636	-
Non-Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	0.00300	-	0.04500	-
Bias	-0.00116	-0.00070	0.00140	0.00073	0.00075	-0.00170	-0.00035	-	0.00022	-
(%)	(-19.33%)	(-0.44%)	(0.17%)	(6.64%)	(0.88%)	(-0.19%)	(-11.67%)	-	(0.49%)	-
S.d.	0.00355	0.00692	0.00679	0.00577	0.03660	0.04314	0.01413	-	0.01833	-
Both components (additive)										
Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	-	-0.00500	0.05400	-
Bias	-0.00116	-0.00064	0.00124	0.00066	0.00050	-0.00138	-	0.00057	-0.00079	-
(%)	(-19.33%)	(-0.40%)	(0.15%)	(6.00%)	(0.59%)	(-0.15%)	-	(-11.40%)	(-1.46%)	-
S.d.	0.00354	0.00692	0.00702	0.00563	0.03599	0.04239	-	0.01071	0.01272	-
Non-Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	0.00800	-0.00800	0.04600	-
Bias	-0.00115	-0.00061	0.00108	0.00068	0.00054	-0.00145	-0.00166	0.00156	0.00010	-
(%)	(-19.17%)	(-0.38%)	(0.13%)	(6.18%)	(0.64%)	(-0.16%)	(-20.75%)	(-19.50%)	(0.22%)	-
S.d.	0.00356	0.00693	0.00710	0.00568	0.03638	0.04281	0.01735	0.01309	0.01863	-
Overall conditional variance (multiplicative)										
Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	-	-	-	0.04200
Bias	-0.00119	-0.00144	0.00211	0.00077	0.00095	-0.00195	-	-	-	-0.00006
(%)	(-19.83%)	(-0.90%)	(0.25%)	(7.00%)	(1.12%)	(-0.22%)	-	-	-	(-0.14%)
S.d.	0.00350	0.00682	0.00672	0.00594	0.03701	0.04378	-	-	-	0.00610
Non-Proportional										
True	0.00600	0.16000	0.84200	0.01100	0.08500	0.90200	0.02000	-	-	0.02300
Bias	-0.00117	-0.00109	0.00176	0.00081	0.00116	-0.00221	-0.00018	-	-	0.00011
(%)	(-19.50%)	(-0.68%)	(0.21%)	(7.36%)	(1.36%)	(-0.25%)	(-0.90%)	-	-	(0.48%)
S.d.	0.00353	0.00686	0.00689	0.00600	0.03745	0.04429	0.00846	-	-	0.01140

Notes: This table presents the results of MF2-GARCH-in-mean QMLE parameter estimation on data generated by Monte Carlo simulations of daily market premia. Each specification was fitted on a simulated sample of size $T = 30,240$ and this was repeated $R = 1,000$ times. The table shows the true parameter values (True), the average bias of the parameter estimates (value and percent), and the standard deviation (S.d.) of the parameter estimates across the 1,000 simulations.

A.3 - Code

The full Python code and data files are provided as attachments to the digital version of this thesis. The code can also be found online at: `https://github.com/HNash/MF2-GARCH-in-Mean`. The code is authored by me except where explicitly stated in the comments of the code files. I took the MATLAB files from Conrad & Schoelkopf (2025), which were originally written for QMLE of MF2-GARCH parameters, translated them to Python, and extended them in order to estimate the full MF2-GARCH-in-mean model (*see files "estimation.py" and "stderr.py"*).

A.4 - Notes on the use of Artificial Intillegence

Aside from direct references to previous research, the contents of this thesis are all written in my own words.

Large Language Models (LLMs) such as Chat-GPT and Perplexity were used to:

- Review the factual accuracy, logical consistency and clarity/precision of writing in this thesis
- Review the code attached to the digital version of this thesis and linked in Section A.3 of the appendix for mistakes
- Provide suggestions on topics that should be explored in this thesis