3. Seventeen Paradoxes of the Infinite

3.1. A Word about Paradoxes

In this chapter, I aim to convince you that infinity is puzzling. I start with some simple mathematical puzzles before moving on to more exotic paradoxes. I will not attempt to *solve* any of the paradoxes in this chapter; solutions will be considered later.

A word about the term "paradox". Some people take the word to refer to a situation in which a contradiction is true. On this interpretation, there is of course no such thing as a paradox. None of my "paradoxes" would count as true paradoxes, because in fact a true paradox would just be impossible by definition. I find this use of the word uninteresting.

I understand paradoxes, roughly, as situations that are highly puzzling, either because there seem to be compelling arguments for incompatible conclusions about the situation, or because there seems to be a compelling argument for something that seems absurd. A solution to the paradox would be an account that removes the puzzlement. This could be done by explaining why the situation in question cannot arise, by explaining why the seemingly compelling argument (or one of the seemingly compelling arguments) is fallacious, or by somehow making the seemingly absurd conclusion of the argument stop seeming absurd.

3.2. The Arithmetic of Infinity

Suppose there is a number infinity, denoted by "∞". Most people will agree that if you add one to it, you get the same number, infinity. That is:

$$\infty + 1 = \infty \qquad \textit{(Equation 3-1)}$$

If equation 1 is true, then it seems that we should be able to subtract ∞ from both sides of the equation, thus obtaining 1=0.

Here is another way to derive the result. Most people who believe in the number ∞ agree that if you divide a finite number by it, the result is zero:

$$1/\infty = 0 \qquad \textit{(Equation 3-2)}$$

(If you disagree with equation 2, try to think what else could go on the right hand side of the equation. No number other than zero is small enough.) If equation 2 is correct, then it seems that we should be able to multiply both sides of the equation by ∞, thus obtaining:

$$1 = 0 \cdot \infty \qquad \textit{(Equation 3-3)}$$

But in general, $0 \cdot x = 0$, so it seems that we can again obtain the result that 1=0.

More simply put, it is paradoxical that there should be a quantity that is not increased when you add something (greater than zero) to it (as in equation 3-1), or a number such that when you take that number of zeros, you get a quantity larger than zero (as in equation 3-3).

3.3. The Paradox of Geometric Points

According to standard geometry, space is composed of *geometric points*. Every geometrical object – including circles, triangles, lines, planes, spheres, and space itself – consists of a collection of points.

And what is a point? Recall the idea (section 1.3) that any line segment can be divided into two smaller segments. The same is true of two-dimensional objects (for instance, a square can always be cut in half) and three-dimensional objects (for instance, a cube can be cut in half). Well, a *point* is conceived as a part of space so small that it *cannot* be divided; there is no such thing as half of a point.[20] Points are *zero*-dimensional objects, with zero length, zero area, zero volume. This has to be so, since if a point had any size greater than zero, it could be divided in half.

Now, here is something that is generally true about geometrical objects: their sizes (lengths, areas, or volumes) are additive. For instance, if you divide a sphere in half, the volume of the left hemisphere plus the volume of the right hemisphere equals the volume of the sphere. If you divide a line segment in thirds, the length of the left third plus the length of the middle third plus the length of the right third equals the length of the original segment.

Similarly, if a geometrical object is made up of points, one would think, the size of any geometrical object should equal the sum of the sizes of the points that make it up. For instance, the volume of a sphere should equal the sum of the volumes of all the points that make up the sphere. The length of a line segment should equal the sum of the lengths of the points that make it up. And so on.

But how can this be, when each of these points has a size of zero? If one combines many zeroes, the result should be zero, so it would seem that a line segment consisting of points (even if there are infinitely many points) should have a length of zero.[21] If you are tempted now to question whether points really have zero size, consider the alternative: if a point has any length larger than zero, then any line segment would have to have a length of infinity, since there are infinitely many points in any segment.

Furthermore, a one-meter line segment and a two-meter line segment contain the same number of points (infinitely many). Therefore, it appears that we must say that infinity times zero meters equals one meter, and infinity times zero meters also equals two meters:

$$\infty \cdot 0 = 1$$
$$\infty \cdot 0 = 2$$

By the transitivity of equality, 1=2.

Here is a metaphysical "application".[22] Suppose there were a stick made of

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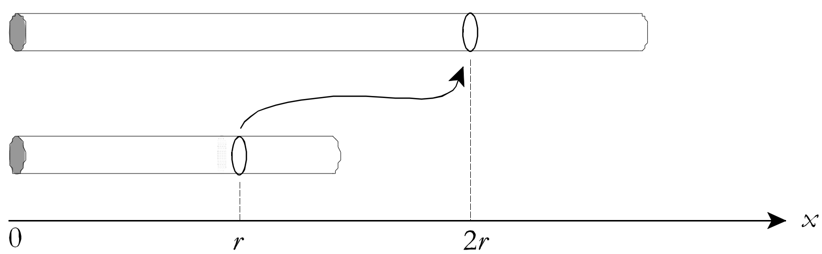
[20]Thus, the first definition from Euclid's (1998) *Elements*, the definitive text of geometry for two millennia, reads, "A point is that which has no part."

[21]This paradox goes back to Aristotle (1941, *On Generation and Corruption*, 316a23-35).

[22]Brentano (1988, 146) uses this to argue against the existence of points.

continuously distributed matter, so that it completely fills a certain region of space. (This is unlike actual sticks, which are almost entirely empty space at the atomic level.) And let's say we have a coordinate system in which the stick extends from x-coordinate 0 to x-coordinate 1 (the y and z coordinates don't matter for the example). It seems that this stick ought to have a part corresponding to each part of the space it occupies – so, for example, for each x-value between 0 and 1, there should be a cross-sectional slice of the stick that occupies just that x-value. Each of these slices would be a single geometric point thick. These are unusual objects, but if you accept the existence of geometric points in general, it seems that you ought to accept that the stick could have such parts.

Now imagine that we cut the stick up into single-point-thick slices, and we rearrange the pieces as follows: for each slice, if the slice starts out with an x-coordinate of r, we move that piece to an x-coordinate of $2r$ (see figure 3-1). The result: the new stick completely fills its space, from x=0 to x=2, just as the old stick completely filled the interval from x=0 to x=1. No x-coordinate in that interval is left empty, and each is occupied by qualitatively the same thing that initially occupied the coordinates in the interval [0,1]. So what we have is a new stick that is just like the old one, only twice as long. We have created more material just by rearranging its parts.[23]



3.4. Infinite Sums

Consider the following infinite collection of numbers:

1, -1, 2, -2, 3, -3, ...

(this includes all the positive and negative integers). What is the sum of all these numbers?

There are many ways to answer this question. First answer: the sum is zero, because we can group the terms to be added as follows:

$$\begin{aligned}&(1 - 1) + (2 - 2) + (3 - 3) + \cdots \\ &\quad = 0 + 0 + 0 + \cdots \\ &\quad = 0.\end{aligned}$$

Second answer: the sum is infinity, because we can group the terms as follows:

[23]This is similar to the Banach-Tarski Paradox, in which a unit sphere is dissected into five pieces, which are then rearranged to form *two* spheres of the same size as the original (Banach and Tarski 1924). The Banach-Tarski Paradox is more amazing, since only finitely many pieces are needed; however, it depends on the controversial Axiom of Choice, and the pieces are not actually describable or imaginable by any human being.

$$
\begin{aligned}
&1 + (-1 + 2) + (-2 + 3) + (-3 + 4) + \cdots \\
&\quad = 1 + 1 + 1 + 1 + \cdots \\
&\quad = \infty.
\end{aligned}
$$

Third answer: the sum is negative infinity, because we can arrange and group the terms as follows:

$$
\begin{aligned}
&-1 + (1 - 2) + (2 - 3) + (3 - 4) + \cdots \\
&\quad = -1 - 1 - 1 - 1 - \cdots \\
&\quad = -\infty.
\end{aligned}
$$

Since one can obtain different sums depending on the order in which one adds the numbers, what is the true sum of these numbers? Is there a "correct" way of ordering the terms?

3.5. Galileo's Paradox

A perfect square is a number that results from multiplying a natural number by itself – for instance, 1, 4, 9, and so on. The numbers that must be multiplied by themselves to obtain the perfect squares are called roots. Thus, 1 is the root of 1, 2 is the root of 4, 3 is the root of 9, and so on. Galileo discovered the following paradox in 1638.[24] Question: in comparing the perfect squares to the roots, which are more numerous? Are there more squares than roots, more roots than squares, or the same number of each? It seems that there must be at least as many squares as roots, because every root has a distinct square.

Second question: in comparing the perfect squares to the natural numbers in general, which are more numerous? Well, of course every perfect square is a natural number, but most of the natural numbers are not perfect squares. Indeed, as one goes through the list of natural numbers, perfect squares become ever rarer. For instance, among the first 100 natural numbers, ten are squares, for a proportion of 0.1. Among the first 10,000 natural numbers, only a hundred are squares, for a proportion of 0.01. Among the first 1,000,000 natural numbers, 1,000 are perfect squares, for a proportion of only 0.001. And so on. As we go on, the ratio of squares to natural numbers approaches ever closer to zero. So it seems that not only are the natural numbers more plentiful than the perfect squares; the perfect squares actually form only an infinitesimal portion of all the natural numbers.

But wait. The set of roots is *identical* to the set of natural numbers, since every natural number can be squared. So it cannot be that there are *fewer* squares than there are *natural numbers*, and also that there are *at least as many* squares as there are *roots*.

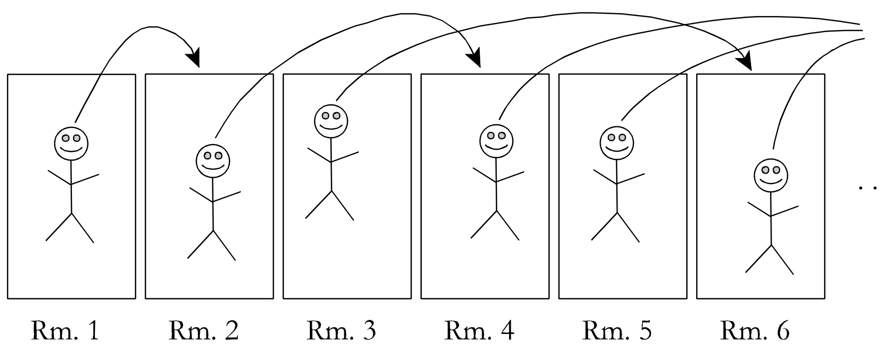
Galileo's conclusion was that infinities cannot be compared in terms of size. If two things are infinite, we cannot say that one is either greater than, less than, or equal to another; we can only say that both are infinite. But this is strange. If x is a quantity, and y is a quantity, how can it be that x and y fail to have any quantitative relation to each other – how can a definite quantity fail to be less than, equal to, or greater than another definite quantity?

[24] Galilei 1914, 31-3. Yes, Galileo discussed this 240 years before Cantor's development of transfinite arithmetic. And no, Cantor did not solve the paradox.

3.6. Hilbert's Hotel

Suppose you run a hotel with an infinite number of rooms. At the moment, all the rooms are filled. A new customer arrives, seeking a room for the night. You tell him that all the rooms are taken. The customer is just about to leave when you say, "Oh, don't worry. I said all the rooms were filled, but I didn't say we couldn't fit you in!" You then tell each guest to move down one room: the guest in room 1 moves to room 2, the guest from room 2 moves to room 3, and so on. This leaves room 1 vacant, which you use to accommodate the new guest.

The next day, all the rooms are still full. But this time, an *infinite* collection of new customers arrives, all wanting separate rooms. "Well, we're all full, but that doesn't mean we can't accommodate you!" you say. This time, you tell each guest to move to room number $2n$, where n is his current room number. Thus, the guest in room 1 moves to room 2, the guest from room 2 moves to room 4, the guest from room 3 moves to room 6, and so on (figure 3-2). This leaves free all the odd-numbered rooms, which you use to accommodate the new guests.[25] This seems paradoxical. If a hotel is full, it ought not to be able to accommodate a new patron, let alone an infinite number of new patrons, without either expelling any of the existing patrons or forcing anyone to double up in a room.



Let us try to make the puzzle sharper. Let R_1 be the number of rooms in the hotel at the beginning of the story. Let G_1 be the number of guests staying in the hotel at the beginning of the story. Since these guests exactly fill the rooms, it seems that $R_1 = G_1$. Now, let G_2 be the infinite number of new guests who arrive at the hotel, seeking accommodations, and let R_2 be the number of rooms existing in the hotel at the end of the story. At the end of the story, when all the new guests have been accommodated, these new guests *together with* the old guests exactly fill up all the rooms. Therefore, it seems that $R_2 = G_1 + G_2$. And therefore, we seem obliged to say one of two things: Either

i. We say that $R_1 = R_2$, in which case $G_1 = G_1 + G_2$, so we can infer that $G_2 = 0$. But we already said that G_2 was infinite; hence, infinity is zero. Or,

ii. We say that $R_2 > R_1$, in which case we have increased the number of rooms in our hotel merely by rearranging the guests. Furthermore, since no new rooms were built, we magically caused the identical set of rooms to constitute a greater number.

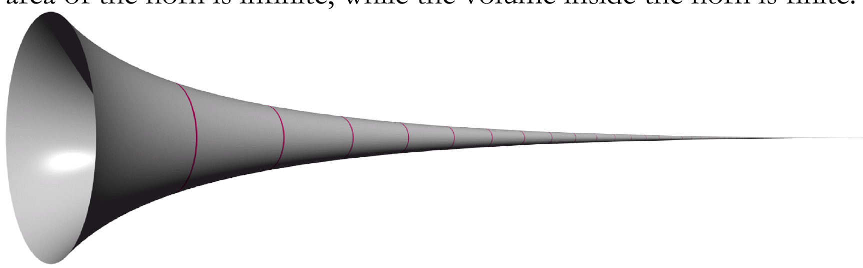
At least one philosopher has taken the strangeness of this scenario to prove that actual

[25] Hilbert first described the scenario in a 1924 lecture (Hilbert 2013, 730). It was subsequently popularized by Gamow (1947, 17). Kragh (2014) traces the entertaining history of the scenario.

infinities cannot exist.[26]

3.7. Gabriel's Horn

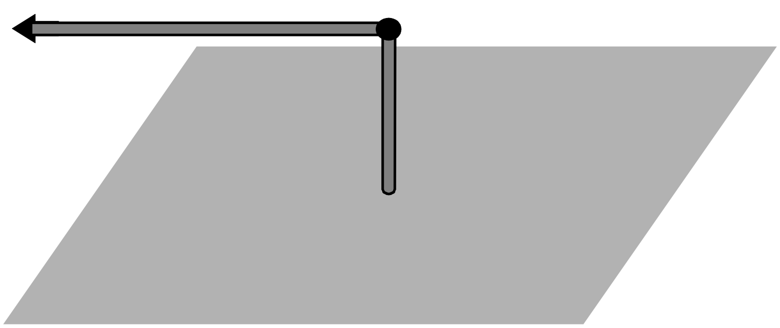
Gabriel's Horn is a geometric figure that has a finite volume but an infinite surface area. To envision the shape, first consider the graph of the function $y = 1/x$, for $x >= 1$ (ignore everything to the left of x=1). Imagine that that shape is rotated about the x axis. The surface traced out is Gabriel's Horn (see figure 3-3). It can be proven that the surface area of the horn is infinite, while the volume inside the horn is finite.



Imagine that you had a physical object with such a shape, and you wanted to paint it. To cover an infinite surface, it would seem that you would need an infinite amount of paint. However, since the volume is finite, you could fill the entire horn with a finite amount of paint, thereby completely covering the inner surface.

3.8. Smullyan's Infinite Rod

Imagine that there is an infinite, flat plane. Everywhere on (and above) the plane, there is a gravitational force pulling downward. There is a finite vertical rod sticking out of the plane, perpendicular to the plane, with a hinge at the top. Attached to the hinge is an infinitely long, perfectly rigid rod (see figure 3-4). The infinite rod is initially oriented perpendicular to the finite rod (parallel to the plane). Assume that the rods and plane are unbendable, unbreakable, and impenetrable.



What happens to the infinite rod in this example? Even though the force of gravity is pulling it downward, and even though it is supported on only one end, the rod cannot tilt downward, because any nonzero amount of downward rotation would require the rod to intersect and break through the plane somewhere. Therefore, the rod will remain suspended parallel to the plane, as if by magic.[27] In modern physics terminology, the rod

[26] Craig 1991.
[27] Smullyan 2008, 246.

undergoes an infinite *net* torque, yet it maintains a constant angular velocity of zero.

3.9. Zeno's Paradox

Perhaps the most famous paradox of the infinite is due to the ancient Greek philosopher Zeno of Elea.[28] Zeno was a student of Parmenides, the ancient Greek philosopher who held that change was impossible. Zeno devised his paradoxes of motion to show that – as a special case of the thesis that change is impossible – motion is impossible. Here, I describe two of these paradoxes.[29]

Suppose you drop a ball. What will happen? You may think that the ball will fall to the ground, but actually, this is impossible. Call the ball's starting point in the air "point A", and call the alleged destination on the ground "point B". For the ball to move from point A to point B, it must first travel half the distance. After it does that, it will then have to travel half the remaining distance (bringing it to three quarters of the total distance). Then it must travel half the remaining distance (taking it to seven eighths of the total distance). Then it must travel half the remaining distance . . . and so on. This series continues infinitely. But it is not possible to complete an infinite series. Therefore, the ball can never reach the ground.

Some students who hear of this puzzle draw the conclusion that the ball will get closer and closer to the ground but never *quite* reach it. This is wrong. If the above reasoning is correct, then it can be applied to any object and *any chosen destination point*. Therefore, no object can go anywhere. For example, the ball could not reach the point one foot off the ground, any more than it could reach the ground.

Many students who hear of Zeno's paradox naively assume that the mathematical theory of infinite series somehow solves the paradox, or that calculus somehow solves it.[30] The line of thinking may be something like this: "In a math class, I learned something about infinite series like this. So that must solve the problem." Generally speaking, what students are taught is that the series,

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

converges to 1; that is, as one adds more and more terms, the sum comes arbitrarily close to 1. Mathematicians therefore say that the infinite sum equals 1.

And this would solve the problem, if the problem were to calculate the distance from point A to point B. In that case, the theory of infinite sums will tell us that, if we view the total distance as a sum of these infinitely many smaller parts, this sum is equal to 1, meaning 100% of the original distance. Viewing the distance as a single line segment of a given length, or as an infinite collection of smaller and smaller segments

[28]Sadly, no writings of Zeno survive. We know of his arguments through writers who discussed him, notably Aristotle (1941, *Physics* 233a13, 239b5, 263a4). Benardete (1964) devotes considerable attention to Zeno's paradoxes.

[29]Zeno had a third paradox, but it is too lame to include in the main text. It seems to involve inferring, from the premise that an arrow in flight cannot move any nonzero distance at a single instant of time, that the arrow is at rest at each instant, and thence that the arrow never moves.

[30]If you thought this, don't feel bad; no less a philosopher than Alfred North Whitehead (1929, ii,2,2) commented, "Zeno produces an invalid argument depending on ignorance of the theory of convergent numerical series."

added together, makes no difference to the total length.

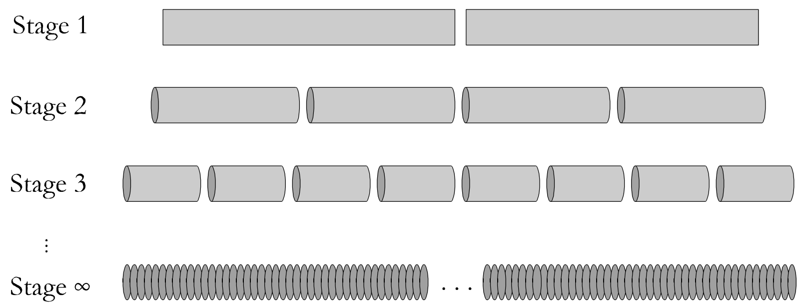
But that was not the problem. The question was not "what is the distance between A and B?", and Zeno is not saying that the distance from A to B is infinite. Zeno is saying that to reach point B, one would have to *complete the infinite series*. But (allegedly), one cannot do such a thing – not because the distance is too long, nor because it will take too much time, but because it is conceptually impossible to *complete* a series that *has no end*. This assumption is in no way challenged by standard modern mathematical treatments of infinite series, nor of any concepts used in calculus. Quite the contrary is in fact the case: standard treatments are designed *precisely* to avoid the assumption that an infinite series can actually be completed.

The version of the paradox that I have just stated uses an *endless* series to argue that the ball can never *complete* its journey. Another variation uses a *beginningless* series to argue that the ball can never *begin* its journey: in order to travel from point A to point B, the ball must first travel half the distance. But *before* it can reach the halfway mark, it must first go one quarter of the distance. But before it does that, it must first go one eighth of the distance. And so on. Because there is an infinite series of preconditions, the ball can never get started. This version of the paradox also has the virtue of forestalling the confused thoughts (a) that the ball can get *close* to the ground even if it can't reach it, and (b) that somehow calculating the infinite sum, $\sum_{n=1}^{\infty}\frac{1}{2^n}$, solves the problem.

That was Zeno's first paradox of motion. The second paradox uses a similar idea. Achilles decides to have a race with a tortoise. Because Achilles is the faster runner, they both agree that the tortoise should get a head start. Achilles starts at point A, while the tortoise starts at point B, which is ahead of A. To overtake the tortoise, Achilles must first get to point B, where the tortoise was at the start of the race. But by the time Achilles reaches B, the tortoise will have advanced to a new place, call it "point C". Now Achilles must run to point C. But by the time he reaches C, the tortoise will have advanced to point D. Then Achilles must run to D... Matters continue in this way forever. To overtake the tortoise, Achilles must first complete this infinite series of catching-up actions. But one cannot complete an infinite series. Therefore, no matter how fast Achilles may be, he can never overtake the tortoise.

3.10. The Divided Stick

Contra Zeno, let us now suppose that it is possible to complete an infinite series (it certainly seems that things move!). Now, imagine that there is a stick composed of an infinitely divisible material (forget about atomic theory). We spend half a minute cutting the stick in half. Then we spend the next quarter minute cutting both of those halves in half, so that the original stick has now been cut into quarters. Then we spend an eighth of a minute cutting all of the quarters in half. And so on (see figure 3-5). At the end of one minute, what is left? It appears that the result must be an infinite number of slices of the stick. How thick will each slice be? For any nonzero size you name, there was a point in the series when the pieces were cut smaller than that size. Therefore, the slices have no nonzero size; that is, they have a size of zero. How much matter could each of these slices contain? It seems that the answer must, again, be zero. But it seems that if each slice has zero mass and zero volume, then the total of all of them must have zero mass and zero volume – so the material of the stick has been destroyed.[31]



3.11. Thomson's Lamp

Suppose there is a lamp with an on/off switch. The lamp starts out on. After half a minute, it is switched off. After another quarter minute, it is switched back on. After another eighth of a minute, it is switched off. And so on. At the end of one minute, an infinite number of switchings will have occurred. Will the lamp then be on or off?[32]

This question is puzzling, in the first place, because it seems that the question *should* have an answer – if such a lamp were made, it would be in some state at the end of one minute. But no answer to the question seems right. It doesn't seem correct to say simply "the lamp will be on", nor does it seem correct to say "the lamp will be off". The answer seems to turn on whether infinity is even or odd, but infinity is neither even nor odd. But nor does it seem correct to declare that the lamp will be neither on nor off. (How could this be? If the lamp explodes at the last instant? But surely this isn't the solution.) Nor does it seem correct to say that the lamp's state will be randomly selected – e.g., that it will just have a 50% probability of being on and a 50% probability of being off.

We can also construct an argument against each answer: on the one hand, it seems that the lamp cannot be on at the end of the minute, because it was switched off at the first step of the series, and thereafter it was never turned on again without subsequently being turned off. But by the same token, it cannot be off, because it started out on, and it was never switched off without subsequently being switched back on again.

3.12. The Littlewood-Ross Banker

You are to play a game with a person known as the Banker.[33] Your only goal in this game is to increase your wealth. At the start of the game, there is an infinite pile of one dollar bills. The bills are labeled with the natural numbers: bill #1, bill #2, bill #3, and

[31]This scenario is from Benardete 1964, 184-5. Benardete ascribes to the final pieces an "infinitesimal" width rather than zero width. I stick with zero, since there are no infinitesimal quantities. Cf. Moore 1990, 5; Oppy 2006, 11-12, 66-8. Oppy (67) rejects the infinitesimal approach on the grounds that there is no way of identifying a uniquely correct infinitesimal.

[32]This is from James Thomson (1954), who uses it to argue that it is impossible to complete an infinite set of tasks. Variations appear in Benardete 1964, 23.

[33]What follows in the text is a variant on the paradox first discovered by Littlewood (1986, 26 [originally published 1953]) and later elaborated by Ross (1976, 36-8). For some reason, philosophers tend to falsely credit the paradox to Ross. Barrett and Arntzenius (1999) discuss the decision-theoretic version of the paradox. Cf. Oppy 2006, 15-16.

so on. You start out owning only bill #1; the Banker owns all the rest. The game has infinitely many turns. In the first turn, the Banker offers to let you trade bill #1 for bills #2-10. If you decline, the Banker keeps bills 2-10. If you accept, as presumably you should, you gain $9 (bills #2-10) and the Banker gains $1 (bill #1). In the second turn of the game, the Banker lets you take bills #11-20, which you may keep for yourself, but you must give him your lowest-numbered bill (either bill #1 or bill #2) – netting you a $9 profit. The game continues on in this way, so that at each turn, you are offered the chance to take the next ten bills from the pile for yourself, and give the Banker the lowest numbered bill from your pile. If you accept every time, then after each stage n, you have $9n$ dollars, and the Banker has n dollars. Let us assume, finally, that each turn of the game is played in half the time as the previous one: the first turn takes half a minute, the second turn takes a quarter of a minute, and so on, so that at the end of one minute, all the infinitely many turns have been played.

If you always act to maximize your wealth, taking the $9 profit in each round, how much money will you have at the end of the game? On the one hand, it seems that you should have an infinite amount of money, because you took a $9 net gain, infinitely many times: $9\times\infty = \infty$. But on the other hand, you will have nothing, because at the end of the game, the Banker has bill #1 (which you gave him in the first round), bill #2 (which you gave him in the second round), bill #3 (which you gave him in the third round), and so on. For every n, the Banker has bill #n, because you gave it to him in the nth round. Therefore, in the end the Banker has every bill, so you have nothing. This is odd, because there is no stage of the game at which you lose money; at every moment up until the end of the game, you have some money, getting more and more as the game continues; but at the last instant, as the clock reaches one minute and the game completes, your pile of money vanishes.

Next, imagine that you demand a rematch with the Banker. This time, when he asks you for bill #1, instead of giving it to him, you take bill #10, surreptitiously erase the "0" on the end of its label so that it says "1", and hand it to him. Then you take your bill #1 and append a "0" on the end of its label so that it becomes labeled "10". When he asks you for bill #2, you likewise relabel bill #20 as #2 and hand it to him, then relabel your bill #2 as #20. You continue in this manner through the entire game. At each stage, you effectively switch the labels of your highest- and lowest-numbered bills, give the banker the new lowest-numbered bill, and keep the rest. This time, how much money do you have at the end of the game? You still have the bill that was originally labeled bill #1, although you have relabeled this bill by adding a "0" to its label infinitely many times (in rounds 1, 10, 100, and so on). So it is now labeled with a "1" followed by infinitely many "0"'s, but it is still yours. Similarly for bills 2-9, 11-19, and so on: you took those, and you never gave them away. Each time you were supposed to give one of them away, you surreptitiously gave the banker your highest-numbered bill instead. The bills originally labeled "10", "20", "30", and so on, all went to the banker, though they were relabeled so that they now say "1", "2", "3", and so on. So each of you ends up with an infinite pile of money. This result is surprising, because in this variant, each of you has qualitatively *the same thing at every stage* of the game as you had in the original variant: at the end of round 1, you have bills labeled "2" through "10", and the banker has a bill labeled "1"; at the end of round 2, you have bills labeled "3" through "20", and the banker has bills labeled "1" and "2"; and so on. The two series are qualitatively identical, the only difference between them lying in the bare numerical identity of the bills that each player has at any given stage. Yet the one series gives you an infinite amount of money, whereas the other gives you nothing.

3.13. Benardete's Paradox

A ball is placed at the top of a hill, where, if nothing interferes, it will roll down to the bottom.[34] There is, however, an infinite series of impenetrable walls placed on the hill, any one of which would suffice to stop the ball. Call these walls w_1, w_2, w_3, and so on. w_1 is located halfway down the hill from where the ball is, w_2 is located a quarter of the way down the hill, w_3 is located one eighth of the way down the hill, and so on. (Assume that each wall is half the thickness of the previous one, so that there is room for all of them.) What happens to the ball?

The ball will of course be unable to roll any distance down the hill, since for any distance it might have rolled, there are infinitely many walls that would stop the ball before it got that far. But which wall stops the ball? The ball is not stopped by w_1, since the ball would have been stopped by w_2 before it could reach w_1. Nor is it stopped by w_2, since it would have hit w_3 before it could reach w_2. Nor is it stopped by w_3, since it would have first hit w_4. And so on. For any n, the ball does not hit w_n, since it would have hit w_{n+1} first. Therefore, the ball will simply hold still at the top of the hill, without anything stopping it.

One might propose that there is another object, distinct from all the w_i, which stops the ball. Call this object w_ω. w_ω is the composite object composed of all the walls w_i put together. You might think of this as another impenetrable wall (with a thickness equal to half the distance down the hill), albeit a wall with many gaps in it. Well, whether you want to call w_ω a "wall" or not, it stops the ball from rolling.[35]

Here is a variation of the paradox. Imagine that, instead of an infinite series of walls, there is an infinite series of gods, G_1, G_2, and so on, each of whom is able to create an impenetrable wall at any chosen location. G_1 happens to have the intention of creating an impenetrable wall halfway down the hill, *if and only if* the ball makes it more than one quarter of the way down the hill. G_2, for his part, intends to create an impenetrable wall one quarter of the way down the hill, if and only if the ball makes it more than one eighth of the way down the hill. And so on. In this case, the ball cannot move any distance down the hill. But which wall, created by which god, will stop the ball? For every n, G_n fails to create a wall, since G_{n+1} would have stopped the ball before it reached the point where G_n found it necessary to act. So none of the gods creates a wall. So the ball simply hovers at the top of the hill with nothing holding it back.

In another variation, Benardete asks us to imagine a pile of infinitely many opaque stone slabs. The bottom slab is one inch thick and has the numeral "1" painted on its upper surface. On top of this is a second slab, which is half an inch thick and has the numeral "2" painted on its upper surface. On top of that, a slab one quarter inch thick with a "3" painted on it. And so on. The whole pile is two inches thick. Now, suppose you stand above this pile and look down. You can't see the first slab, since your view of it is blocked by the second slab. Likewise, your view of the second slab is blocked by

[34] The following paradoxes are from Benardete 1964, 236-8, 259-60. Cf. Moore 1990, 4-5; Oppy 2006, 10-11, 16-18.

[35] This is Hawthorne's view (2000, 626). Benardete (1964, 260) entertains similar suggestions. Cf. Laraudogoitia 2003, 126.

the third, your view of the third is blocked by the fourth, and so on. Your view of each slab is blocked, so you can't see any of them. But surely you cannot see through the pile either. So what do you see?

3.14. Laraudogoitia's Marbles

Imagine an infinite series of marbles, arranged in a straight line, in a Newtonian world. The first marble is located at point A. The second marble is located half a meter to the right of A. The third is located a quarter meter to the right of the second marble (so, 75 centimeters to the right of A). The fourth is located an eighth of a meter to the right of the third. And so on. These marbles get smaller and smaller, with each having half the diameter of the last, so that all of them can fit into a one-meter-long space. Assume that every marble is perfectly rigid (so all collisions are perfectly elastic) and all have the same mass. Now, suppose someone pushes the first marble, imparting to it a velocity of 1 meter per second to the right. What will happen?[36]

After half a second, the first marble strikes the second, whereupon the first marble stops and the second starts moving at 1 meter per second to the right. After another quarter second, the second marble runs into the third, whereupon the second marble imparts its momentum to the third. After another eighth of a second, the third marble strikes the fourth, and so on. After one second, every one of the marbles has been struck. However, nothing emerges on the right-hand side of the series. If there were a last marble in the series, that last marble would emerge at the right edge (one meter to the right of point A), moving 1 m/s to the right. But because the series of marbles is infinite, there is no last marble, so there is nothing to emerge at the right. Therefore, the momentum and kinetic energy of the system will simply vanish: all the marbles will have been struck, every one will have transferred its momentum to another marble, and nothing will come out on the right.

Laraudogoitia points out that the laws of Newtonian mechanics are time-reversal-invariant. What this means is that for any sequence of events that is consistent with Newton's laws, the reverse sequence (in which everything happens the same way, except with the opposite directions of motion and opposite order of events) is also consistent with the laws. Thus, if the marble sequence described above is possible, then the reverse sequence must also be possible, in which a system containing infinitely many marbles starts out with zero momentum and zero kinetic energy, and then the marbles spontaneously start moving to the left, each marble transferring a leftward velocity of 1 m/s to the marble to its left. In this reverse sequence, the process ends up with the first marble at point A moving 1 m/s to the left and all the other marbles stationary.

3.15. The Spaceship

Imagine that there is a spaceship that travels in a straight line, starting at a speed of 1 m/s to the right, and accelerates at an increasing rate, so that after one second it is moving at 2 m/s; after another half second, it is moving at 4 m/s; after another quarter of a second it is moving at 8 m/s; and so on.[37] At the end of two seconds, the ship would be traveling at infinite velocity. At this time, *where* would the ship be?

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[36]This case is from Laraudogoitia 1996.

[37]Benardete 1964, 149; Moore 1990, 70-71; Oppy 2006, 12.

The total distance traveled would be infinite. In the first stage, the ship travels at a speed of at least 1 m/s for one second, thus covering a distance of at least one meter. In the second stage, it travels at a speed of at least 2 m/s for a half second, thus covering a distance of at least $(2)(\frac{1}{2}) = 1$ meter. And so on. Since there are infinitely many stages, the total distance is at least $1 + 1 + 1 + \cdots = \infty$.

But there simply is no position infinitely far from the starting point; for any two points in space, there is some (finite) distance between them. The ship would have a definite position, getting farther and farther away, at each instant from the starting time *up to* but not including the final time two seconds later. But at $t = 2$ seconds, there is nowhere the ship could be. Perhaps the ship would go out of existence at that instant, destroyed by its excessive speed.

3.16. The Saint Petersburg Paradox

Imagine that you have the chance to play a simple game. A fair coin is to be flipped some number of times. If it comes up heads on the first flip, you get \$2, and the game ends; otherwise, the coin is flipped again. If it comes up heads on the second flip, you get \$4, and the game ends; otherwise, the coin is flipped again. If it comes up heads on the third flip, you get \$8. And so on. In brief, the coin is flipped until the first time it comes up heads, whereupon you get a payoff of $\$2^n$, where n is the number of times the coin was flipped. The question is: how much is it worth to get a chance to play this game? Or, how much should a rational person be willing to pay to be allowed to play?[38]

There is a standard way of addressing this kind of question: we calculate the expected payoff of the game. This is done by adding up, for every possible outcome of the game, the probability of the outcome occurring multiplied by the payoff you receive if it occurs. The outcomes are listed below (where, for example, "TH" indicates tails on the first flip, followed by heads on the second flip), with their associated probabilities and payoffs:

Outcome	Probability	Payoff
H	1/2	\$2
TH	1/4	\$4
TTH	1/8	\$8
⋮	⋮	⋮

So the expected value of the game is $(1/2)(\$2) + (1/4)(\$4) + (1/8)(\$8) + \cdots = 1 + 1 + 1 + \cdots = \$\infty$. So the fair price for a chance to play the game is $\$\infty$.

Why is this paradoxical? First, it does not seem that the game is worth anything close to an infinite amount of money. Indeed, few people would value it even as high as \$20. Is our intuitive valuation mistaken, or is there something wrong with the reasoning of the preceding paragraph?

Second, it seems almost contradictory to hold that the value of the game is infinite, since if one plays the game, one will never gain an infinite payout. All the possible

[38] This problem was originally published by the Swiss mathematician Daniel Bernoulli in 1738, who attributes it to his *cousin* Nicolas Bernoulli, also a mathematician (not to be confused with Daniel's *brother* Nicolas Bernoulli, who was *also* a mathematician). For the modern English translation, see Bernoulli 1954, 31.

payouts are finite (though they increase without bound), so there is zero chance that one will gain \$∞. Therefore, if one were to pay an infinite amount to play the game, one would be guaranteed to lose money (and an infinite amount, at that!). More generally, it is strange that there should be a variable whose actual value is *known with certainty* to be less than its expected value.

3.17. The Martingale Betting System

You're about to play roulette in a European casino. You like the color red, so you bet that the ball will land on red. The probability of this happening is 18/37 (the wheel has 18 red squares, 18 black squares, and 1 green square), or about 48.6%. So if you play repeatedly, you should win about 48.6% of the time, with the house winning the other 51.4% of the time. In the long run, as everyone knows, the house comes out ahead.

But wait – in spite of the house's advantage, there is a way to guarantee that *you* come out ahead (without cheating). Unfortunately, this system requires that you start with an infinite bankroll, and that there be no betting limits – if these conditions do not apply to you, then don't try this system!

First, bet \$1 on red. If you win, walk away with your \$1 profit. If you lose, play again, but this time stake \$2 on red. Again, if you win, walk away with your net profit of \$1 (the \$2 payout, minus the \$1 loss from the first bet). If you lose the second bet, place a third bet, this time staking \$4. If you win, walk away with your \$1 profit (\$4 from the third bet, minus the \$2 loss from the second bet and the \$1 loss from the first bet). And so on. Simply continue doubling down until you win, at which point you can walk away with your \$1 profit. Eventually, the ball will land on red, so you are guaranteed to come out ahead.[39]

So you have a guaranteed way to win \$1, provided that you start with \$∞. This may not sound terribly impressive. But if this works, there is also a way to extract an *infinite* amount of money from the casino. First, apply the betting strategy as described above, until you win your dollar of profit. Then, instead of walking away, simply *start over* with the \$1 bet on red again. Follow the strategy again, doubling down every time you lose, until you win a second dollar. Then start over, applying the strategy until you win a third dollar. And so on.

Something seems to have gone wrong here. Given the house's built-in advantage (the odds in each bet favor the house, 19-18), the house should come out ahead in the long run. Yet the system we have just described seems to guarantee, over the long run, unlimited losses for the house.

Now, you may worry about the infinite time it will presumably take to collect your desired infinite winnings. For the impatient, imagine the following further stipulations, all of which appear jointly logically consistent: in your first implementation of the betting strategy, your first spin of the roulette wheel takes one minute, the second spin (if there is one) takes half a minute, the third spin (if there is one) takes a quarter minute, and so on. Thus, the first application of the betting strategy, netting you your first \$1 profit, is guaranteed to be complete within two minutes.

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[39] It is *logically possible*, consistent with the stipulations of the scenario, that you get stuck with an infinite series of black and green; however, the probability of this happening is zero.

Now, the second time you implement the betting strategy, assume that the first spin takes a *half* minute, the second spin (if there is one) takes a quarter minute, and so on. So you are guaranteed to win your second dollar within *one* minute.

In your third application of the betting strategy, the first spin take a quarter minute, the second takes an eighth of a minute, and so on. So you win your third dollar within a half minute.

And so on. All told, then, you complete infinitely many iterations of the betting strategy, netting an infinite profit, within four minutes.[40]

3.18. The Delayed Heaven Paradox

One final decision-theoretic puzzle. You have just died and gone to meet your Maker. After death, there are three destinations to which souls may be assigned:

A. People who lived sufficiently good lives are assigned to Heaven, where they enjoy a high positive level of wellbeing – let's say, 100 utils – every day for eternity. (Note: a "util" is a hypothetical unit of happiness or desire-satisfaction.)
B. People who lived sufficiently bad lives are assigned to Hell, where they suffer a similarly high *negative* welfare level – say, -100 utils – every day for eternity. (Negative utils represent unhappiness or desire-frustration.)
C. People who lived so-so lives are assigned to Limbo, where their existence is completely neutral (0 utils), every day for eternity.

Assume that the suffering of one day in hell exactly counterbalances the enjoyment of a day in heaven, so that a day in heaven plus a day in hell is exactly as good as two days in Limbo.

When you arrive at the Pearly Gates, eager (or apprehensive?) to learn your fate, God informs you that your life was not good enough to earn eternity in Heaven, nor yet bad enough to earn eternity in Hell. In such circumstances, he says, you would normally be assigned to eternity in Limbo. However, as a special offer just for you, you may instead choose to take one day in Hell, followed by *two* days in Heaven, followed by the rest of eternity in Limbo. If you decline, you go straight to Limbo. It's a good deal, so you accept.

You spend your day in Hell – where, we presume, you are forced to grade stacks of philosophy essays by C students – then return to the Pearly Gates. "Whew", you say to God. "That really sucked. Now I'm ready for some Heaven!"

"Of course", says God. "You may now take your two days in Heaven, followed by an eternity in Limbo. *Or* . . . I have another special offer, just for you! You may spend another day in Hell, after which I will let you have *four* days in Heaven, followed by the rest of eternity in Limbo. What say you?"

Again, it's a good deal – the goodness of two more days in heaven more than makes up for the badness of one extra day in Hell. So you accept, spend another day grading papers, then come back to see God.

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[40]Here is the math: the total time taken is $\sum_{i=0}^{\infty}\left(\frac{1}{2^i}\sum_{j=0}^{\infty}\frac{1}{2^j}\right)=\sum_{i=0}^{\infty}\left(\frac{1}{2^i}\cdot 2\right)=2\cdot\sum_{i=0}^{\infty}\frac{1}{2^i}=2\cdot 2=4$.

Once again, he offers you a special deal: spend a third day in Hell, then you can have *six* days in Heaven, followed by Limbo.

Everyone can see where this is going. Each time you return from a day in Hell, God offers to let you take another day in Hell, thereby earning an extra two days in Heaven. And yes, you always have to take your day in Hell before you can take any days in Heaven. Each time, if you are rational, you accept. The result: you spend eternity grading papers. How can it be that, by making the prudent choice every time, you wind up with the worst possible outcome?

3.19. Conclusion

The conclusion to be drawn from this chapter is similar to that of the last chapter: we need an account of when infinite series are possible or impossible. It is natural to try to resolve some of the preceding paradoxes by declaring that the infinite series postulated in the paradoxes are impossible. If you are tempted to say this, remember that not all of the paradoxes depended on the assumption that an infinite series can be completed. Zeno's paradoxes start from precisely the opposite premise, that an infinite series *cannot* be completed. So if one denies in general that infinite series can be completed, one avoids some of the paradoxes, but at the price of joining sides with Zeno.

What we would like to be able to say is that *some* infinite series cannot be completed, while others can; that, for example, the Littlewood-Ross Banker and Thomson Lamp are of the uncompletable kind; and that the Zeno series are of the completable kind. But we can't just *say* that. We need an account of why that is the case. We need a theory of when infinities are impossible.