

Interactive Theorem Proving (ITP) Course

Web Version

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Part I

Preface



- these slides originate from a course for advanced master students
- it was given by the PROSPER group at KTH in Stockholm in 2017 (see <https://www.kth.se/social/group/interactive-theorem->)
- the course focused on how to use HOL 4
- students taking the course were expected to
 - ▶ know functional programming, esp. SML
 - ▶ understand predicate logic
 - ▶ have some experience with pen and paper proofs
- the course consisted of 9 lectures, which each took 90 minutes
- there were 19 supervised practical sessions, which each took 2 h
- usually there was 1 lecture and 2 practical sessions each week
- students were expected to work about 10 h each week on exercises

- usually, these slides present concepts and some high-level entry points
- often some more details were explained than covered on the slides
- technical details were covered in the practical sessions
- they are provided as they are in the hope that they are useful¹ (there are no guarantees of correctness :-))
- the exercise question-sheets are available as well
- if you have questions, feel free to contact Thomas Tuerk (thomas@tuerk-brechen.de)

¹if you find errors, please contact Thomas Tuerk

Part II

Introduction



- Complex systems almost certainly contain bugs.
- Critical systems (e.g. avionics) need to meet very high standards.
- It is infeasible in practice to achieve such high standards just by testing.
- Debugging via testing suffers from diminishing returns.

**“Program testing can be used to show the presence
of bugs, but never to show their absence!”
— Edsger W. Dijkstra**

- Pentium FDIV bug (1994)
(missing entry in lookup table, \$475 million damage)
- Ariane V explosion (1996)
(integer overflow, \$1 billion prototype destroyed)
- Mars Climate Orbiter (1999)
(destroyed in Mars orbit, mixup of units pound-force and newtons)
- Knight Capital Group Error in Ultra Short Time Trading (2012)
(faulty deployment, repurposing of critical flag, \$440 lost in 45 min on stock exchange)
- ...

Fun to read

<http://www.cs.tau.ac.il/~nachumd/verify/horror.html>

https://en.wikipedia.org/wiki/List_of_software_bugs

- proof can show absence of errors in design
- but proofs talk about a **design**, not a **real system**
- \Rightarrow testing and proving complement each other

**“As far as the laws of mathematics
refer to reality, they are not certain;
and as far as they are certain,
they do not refer to reality.”
— Albert Einstein**

Mathematical Proof

- informal, convince other mathematicians
- checked by community of domain experts
- subtle errors are hard to find
- often provide some new insight about our world
- often short, but require creativity and a brilliant idea

Formal Proof

- formal, rigorously use a logical formalism
- checkable by *stupid* machines
- very reliable
- often contain no new ideas and no amazing insights
- often long, very tedious, but largely trivial

We are interested in formal proofs in this lecture.

Automated vs Manual (Formal) Proof



Fully Manual Proof

- very tedious; one has to grind through many trivial but detailed proofs
- easy to make mistakes
- hard to keep track of all assumptions and preconditions
- hard to maintain, if something changes (see Ariane V)

Automated Proof

- amazing success in certain areas
- but still often infeasible for interesting problems
- hard to get insights in case a proof attempt fails
- even if it works, it is often not that automated
 - ▶ run automated tool for a few days
 - ▶ abort, change command line arguments to use different heuristics
 - ▶ run again and iterate till you find a set of heuristics that prove it fully automatically in a few seconds

- combine strengths of manual and automated proofs
- many different options to combine automated and manual proofs
 - ▶ mainly check existing proofs (e. g. HOL Zero)
 - ▶ user mainly provides lemmata statements, computer searches proofs using previous lemmata and very few hints (e. g. ACL 2)
 - ▶ most systems are somewhere in the middle
- typically the human user
 - ▶ provides insights into the problem
 - ▶ structures the proof
 - ▶ provides main arguments
- typically the computer
 - ▶ checks proof
 - ▶ keeps track of all used assumptions
 - ▶ provides automation to grind through lengthy, but trivial proofs

Typical Interactive Proof Activities



- provide precise definitions of concepts
- state properties of these concepts
- prove these properties
 - ▶ human provides insight and structure
 - ▶ computer does book-keeping and automates simple proofs
- build and use libraries of formal definitions and proofs
 - ▶ formalisations of mathematical theories like
 - ★ lists, sets, bags, ...
 - ★ real numbers
 - ★ probability theory
 - ▶ specifications of real-world artefacts like
 - ★ processors
 - ★ programming languages
 - ★ network protocols
 - ▶ reasoning tools

**There is a strong connection with programming.
Lessons learned in Software Engineering apply.**

- there are many different interactive provers, e. g.
 - ▶ Isabelle/HOL
 - ▶ Coq
 - ▶ PVS
 - ▶ HOL family of provers
 - ▶ ACL2
 - ▶ ...
- important differences
 - ▶ the formalism used
 - ▶ level of trustworthiness
 - ▶ level of automation
 - ▶ libraries
 - ▶ languages for writing proofs
 - ▶ user interface
 - ▶ ...

Which theorem prover is the best one? :-)



- there is no **best** theorem prover
- better question: Which is the **best one for a certain purpose?**
- important points to consider
 - ▶ existing libraries
 - ▶ used logic
 - ▶ level of automation
 - ▶ user interface
 - ▶ importance development speed versus trustworthiness
 - ▶ How familiar are you with the different provers?
 - ▶ Which prover do people in your vicinity use?
 - ▶ your personal preferences
 - ▶ ...

**In this course we use the HOL theorem prover,
because it is used by the TCS group.**

Part III

HOL 4 History and Architecture



- **Stanford LCF** 1971-72 by Milner et al.
- formalism devised by Dana Scott in 1969
- intended to reason about recursively defined functions
- intended for computer science applications
- strengths
 - ▶ powerful simplification mechanism
 - ▶ support for backward proof
- limitations
 - ▶ proofs need a lot of memory
 - ▶ fixed, hard-coded set of proof commands



Robin Milner
(1934 - 2010)

- Milner worked on improving LCF in Edinburgh
- research assistants
 - ▶ Lockwood Morris
 - ▶ Malcolm Newey
 - ▶ Chris Wadsworth
 - ▶ Mike Gordon
- **Edinburgh LCF** 1979
- introduction of **Meta Language** (ML)
- ML was invented to write proof procedures
- ML became an influential functional programming language
- using ML allowed implementing the **LCF approach**

- implement an abstract datatype **thm** to represent theorems
- semantics of ML ensure that values of type **thm** can only be created using its interface
- interface is very small
 - ▶ predefined theorems are axioms
 - ▶ function with result type theorem are inferences
- interface is carefully designed and checked
 - ▶ size of interface and implementation allow careful checking
 - ▶ one checks that the interface really implements only axioms and inferences that are valid in the used logic
- **However you create a theorem, there is a proof for it.**
- together with similar abstract datatypes for types and terms, this forms the **kernel**

Modus Ponens Example

Inference Rule

$$\frac{\Gamma \vdash a \Rightarrow b \quad \Delta \vdash a}{\Gamma \cup \Delta \vdash b}$$

SML function

```
val MP : thm -> thm -> thm
MP( $\Gamma \vdash a \Rightarrow b$ )( $\Delta \vdash a$ ) = ( $\Gamma \cup \Delta \vdash b$ )
```

- very trustworthy — only the small kernel needs to be trusted
- efficient — no need to store proofs

Easy to extend and automate

However complicated and potentially buggy your code is, if a value of type theorem is produced, it has been created through the small trusted interface. Therefore the statement really holds.

There are now many interactive theorem provers out there that use an approach similar to that of Edinburgh LCF.

- HOL family
 - ▶ HOL theorem prover
 - ▶ HOL Light
 - ▶ HOL Zero
 - ▶ Proof Power
 - ▶ ...
- Isabelle
- Nuprl
- Coq
- ...

- 1979 Edinburgh LCF by Milner, Gordon, et al.
- 1981 Mike Gordon becomes lecturer in Cambridge
- 1985 Cambridge LCF
 - ▶ Larry Paulson and Gérard Huet
 - ▶ implementation of ML compiler
 - ▶ powerful simplifier
 - ▶ various improvements and extensions
- 1988 HOL
 - ▶ Mike Gordon and Keith Hanna
 - ▶ adaption of Cambridge LCF to classical higher order logic
 - ▶ intention: hardware verification
- 1990 HOL90
reimplementation in SML by Konrad Slind at University of Calgary
- 1998 HOL98
implementation in Moscow ML and new library and theory mechanism
- since then HOL Kananaskis releases, called informally **HOL 4**

- **ProofPower**

commercial version of HOL88 by Roger Jones, Rob Arthan et al.

- **HOL Light**

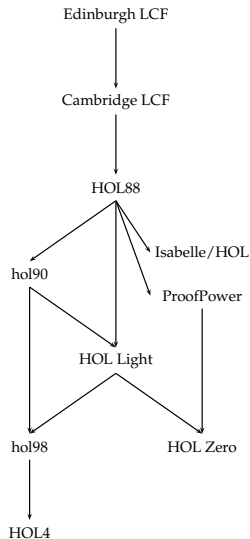
lean CAML / OCaml port by John Harrison

- **HOL Zero**

trustworthy proof checker by Mark Adams

- **Isabelle**

- ▶ 1990 by Larry Paulson
- ▶ meta-theorem prover that supports multiple logics
- ▶ however, mainly HOL used, ZF a little
- ▶ nowadays probably the most widely used HOL system
- ▶ originally designed for software verification



Part IV

HOL's Logic



- the HOL theorem prover uses a version of classical **higher order logic**: classical higher order predicate calculus with terms from the typed lambda calculus (i. e. simple type theory)
- this sounds complicated, but is intuitive for SML programmers
- (S)ML and HOL logic designed to fit each other
- if you understand SML, you understand HOL logic

HOL = functional programming + logic

Ambiguity Warning

The acronym *HOL* refers to both the *HOL interactive theorem prover* and the *HOL logic* used by it. It's also a common abbreviation for *higher order logic* in general.

- SML datatype for types
 - ▶ **Type Variables** ($'a, \alpha, 'b, \beta, \dots$)
Type variables are implicitly universally quantified. Theorems containing type variables hold for all instantiations of these. Proofs using type variables can be seen as proof schemata.
 - ▶ **Atomic Types** (c)
Atomic types denote fixed types. Examples: `num`, `bool`, `unit`
 - ▶ **Compound Types** $((\sigma_1, \dots, \sigma_n)op)$
 op is a **type operator** of arity n and $\sigma_1, \dots, \sigma_n$ **argument types**. Type operators denote operations for constructing types.
Examples: `num list` or `'a # 'b`.
 - ▶ **Function Types** $(\sigma_1 \rightarrow \sigma_2)$
 $\sigma_1 \rightarrow \sigma_2$ is the type of **total** functions from σ_1 to σ_2 .
- types are never empty in HOL, i. e.
for each type at least one value exists
- all HOL functions are total

- SML datatype for terms

- ▶ **Variables** (x, y, \dots)
- ▶ **Constants** (c, \dots)
- ▶ **Function Application** ($f\ a$)
- ▶ **Lambda Abstraction** ($\lambda x. f\ x$ or $\lambda x. fx$)

Lambda abstraction represents anonymous function definition.

The corresponding SML syntax is `fn x => f x`.

- terms have to be well-typed
- same typing rules and same type-inference as in SML take place
- terms very similar to SML expressions
- notice: predicates are functions with return type `bool`, i. e. no distinction between functions and predicates, terms and formulae

HOL term	SML expression	type HOL / SML
0	0	<code>num / int</code>
<code>x:'a</code>	<code>x:'a</code>	variable of type <code>'a</code>
<code>x:bool</code>	<code>x:bool</code>	variable of type <code>bool</code>
<code>x + 5</code>	<code>x + 5</code>	applying function <code>+</code> to <code>x</code> and <code>5</code>
<code>\x. x + 5</code>	<code>fn x => x + 5</code>	anonymous (a. k. a. inline) function of type <code>num -> num</code>
<code>(5, T)</code>	<code>(5, true)</code>	<code>num # bool / int * bool</code>
<code>[5;3;2]++[6]</code>	<code>[5,3,2]@[6]</code>	<code>num list / int list</code>

- in SML, the names of function arguments does not matter (much)
- similarly in HOL, the names of variables used by lambda-abstractions does not matter (much)
- the lambda-expression $\lambda x. t$ is said to **bind** the variables x in term t
- variables that are guarded by a lambda expression are called **bound**
- all other variables are **free**
- Example: x is free and y is bound in $(x = 5) \wedge (\lambda y. (y < x))$
- the names of bound variables are unimportant semantically
- two terms are called **alpha-equivalent** iff they differ only in the names of bound variables
- Example: $\lambda x. x$ and $\lambda y. y$ are alpha-equivalent
- Example: x and y are not alpha-equivalent

- theorems are of the form $\Gamma \vdash p$ where
 - ▶ Γ is a set of hypothesis
 - ▶ p is the conclusion of the theorem
 - ▶ all elements of Γ and p are formulae, i. e. terms of type `bool`
- $\Gamma \vdash p$ records that using Γ the statement p **has been** proved
- notice difference to logic: there it means **can be** proved
- the proof itself is not recorded
- theorems can only be created through a small interface in the **kernel**

- the HOL kernel is hard to explain
 - ▶ for historic reasons some concepts are represented rather complicated
 - ▶ for speed reasons some derivable concepts have been added
- instead consider the HOL Light kernel, which is a cleaned-up version
- there are two predefined constants
 - ▶ $= : 'a \rightarrow 'a \rightarrow \text{bool}$
 - ▶ $@ : ('a \rightarrow \text{bool}) \rightarrow 'a$
- there are two predefined types
 - ▶ `bool`
 - ▶ `ind`
- the meaning of these types and constants is given by inference rules and axioms

$$\frac{}{\vdash t = t} \text{REFL}$$

$$\frac{\begin{array}{l} \Gamma \vdash s = t \\ \Delta \vdash t = u \end{array}}{\Gamma \cup \Delta \vdash s = u} \text{TRANS}$$

$$\frac{\begin{array}{l} \Gamma \vdash s = t \\ \Delta \vdash u = v \\ \text{types fit} \end{array}}{\Gamma \cup \Delta \vdash s(u) = t(v)} \text{COMB}$$

$$\frac{\begin{array}{l} \Gamma \vdash s = t \\ x \text{ not free in } \Gamma \end{array}}{\Gamma \vdash \lambda x. s = \lambda x. t} \text{ABS}$$

$$\frac{}{\vdash (\lambda x. t) x = t} \text{BETA}$$

$$\frac{}{\{p\} \vdash p} \text{ASSUME}$$

$$\frac{\Gamma \vdash p \Leftrightarrow q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \text{EQ_MP}$$

$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{(\Gamma - \{q\}) \cup (\Delta - \{p\}) \vdash p \Leftrightarrow q} \text{DEDUCT_ANTISYM_RULE}$$

$$\frac{\Gamma[x_1, \dots, x_n] \vdash p[x_1, \dots, x_n]}{\Gamma[t_1, \dots, t_n] \vdash p[t_1, \dots, t_n]} \text{INST}$$

$$\frac{\Gamma[\alpha_1, \dots, \alpha_n] \vdash p[\alpha_1, \dots, \alpha_n]}{\Gamma[\gamma_1, \dots, \gamma_n] \vdash p[\gamma_1, \dots, \gamma_n]} \text{INST_TYPE}$$

- 3 axioms needed

ETA_AX	$-(\lambda x. t \ x) = t$
SELECT_AX	$- P \ x \Longrightarrow P((@)P))$
INFINITY_AX	predefined type <code>ind</code> is infinite

- definition principle for constants
 - ▶ constants can be introduced as abbreviations
 - ▶ constraint: no free vars and no new type vars
- definition principle for types
 - ▶ new types can be defined as non-empty subtypes of existing types
- both principles
 - ▶ lead to conservative extensions
 - ▶ preserve consistency

Everything else is derived from this small kernel.

$$\begin{aligned} T &=_{def} (\lambda p. p) = (\lambda p. p) \\ \wedge &=_{def} \lambda p q. (\lambda f. f p q) = (\lambda f. f T T) \\ \implies &=_{def} \lambda p q. (p \wedge q \Leftrightarrow p) \\ \forall &=_{def} \lambda P. (P = \lambda x. T) \\ \exists &=_{def} \lambda P. (\forall q. (\forall x. P(x) \implies q) \implies q) \\ &\dots \end{aligned}$$

- Kernel defines abstract datatypes for types, terms and theorems
- one does not need to look at the internal implementation
- therefore, easy to exchange
- there are at least 3 different kernels for HOL
 - ▶ standard kernel (de Bruijn indices)
 - ▶ experimental kernel (name / type pairs)
 - ▶ OpenTheory kernel (for proof recording)

- HOL theorem prover uses classical higher order logic
- HOL logic is very similar to SML
 - ▶ syntax
 - ▶ type system
 - ▶ type inference
- HOL theorem prover very trustworthy because of LCF approach
 - ▶ there is a small kernel
 - ▶ proofs are not stored explicitly
- you don't need to know the details of the kernel
- usually one works at a much higher level of abstraction

Part V

Basic HOL Usage



- practical issues are discussed in practical sessions
 - ▶ how to install HOL
 - ▶ which key-combinations to use in emacs-mode
 - ▶ detailed signature of libraries and theories
 - ▶ all parameters and options of certain tools
 - ▶ ...
- exercise sheets sometimes
 - ▶ ask to read some documentation
 - ▶ provide examples
 - ▶ list references where to get additional information
- if you have problems, ask me outside lecture
(thomas@tuerk-brechen.de)
- covered only very briefly in lectures

- webpage: <https://hol-theorem-prover.org>
- HOL supports two SML implementations
 - ▶ Moscow ML (<http://mosml.org>)
 - ▶ **PolyML** (<http://www.polym1.org>)
- I recommend using PolyML
- please use emacs with
 - ▶ hol-mode
 - ▶ sml-mode
 - ▶ hol-unicode, if you want to type Unicode
- please install recent revision from git repo or Kananaskis 11 release
- documentation found on HOL webpage and with sources

- HOL is a collection of SML modules
- starting HOL starts a SML Read-Eval-Print-Loop (REPL) with
 - ▶ some HOL modules loaded
 - ▶ some default modules opened
 - ▶ an input wrapper to help parsing terms called `unquote`
- `unquote` provides special quotes for terms and types
 - ▶ implemented as input filter
 - ▶ `‘my-term‘` becomes `Parse.Term [QUOTE "my-term"]`
 - ▶ `‘:my-type‘` becomes `Parse.Type [QUOTE ":my-type"]`
- main interfaces
 - ▶ **emacs** (used in the course)
 - ▶ vim
 - ▶ bare shell

- `*Script.sml` — HOL proof script file
 - ▶ script files contain definitions and proof scripts
 - ▶ executing them results in HOL searching and checking proofs
 - ▶ this might take very long
 - ▶ resulting theorems are stored in `*Theory.{sml|sig}` files
- `*Theory.{sml|sig}` — HOL theory
 - ▶ auto-generated by corresponding script file
 - ▶ load quickly, because they don't search/check proofs
 - ▶ do not edit theory files
- `*Syntax.{sml|sig}` — syntax libraries
 - ▶ contain syntax related functions
 - ▶ i. e. functions to construct and destruct terms and types
- `*Lib.{sml|sig}` — general libraries
- `*Simps.{sml|sig}` — simplifications
- `selftest.sml` — selftest for current directory

- `bin` — HOL binaries
- `src` — HOL sources
- `examples` — HOL examples
 - ▶ interesting projects by various people
 - ▶ examples owned by their developer
 - ▶ coding style and level of maintenance differ a lot
- `help` — sources for reference manual
 - ▶ after compilation home of reference HTML page
- `Manual` — HOL manuals
 - ▶ Tutorial
 - ▶ Description
 - ▶ Reference (PDF version)
 - ▶ Interaction
 - ▶ Quick (cheat pages)
 - ▶ Style-guide
 - ▶ ...

- HOL supports both Unicode and pure ASCII input and output
- advantages of Unicode compared to ASCII
 - ▶ easier to read (good fonts provided)
 - ▶ no need to learn special ASCII syntax
- disadvantages of Unicode compared to ASCII
 - ▶ harder to type (even with `hol-unicode.el`)
 - ▶ less portable between systems
- whether you like Unicode is highly a matter of personal taste
- HOL's policy
 - ▶ no Unicode in HOL's source directory `src`
 - ▶ Unicode in examples directory `examples` is fine
- I recommend turning Unicode output off initially
 - ▶ this simplifies learning the ASCII syntax
 - ▶ no need for special fonts
 - ▶ it is easier to copy and paste terms from HOL's output

Where to find help?



- reference manual
 - ▶ available as HTML pages, single PDF file and in-system help
- description manual
- Style-guide (still under development)
- HOL webpage (<https://hol-theorem-prover.org>)
- mailing-list hol-info
- `DB.match` and `DB.find`
- `*Theory.sig` and `selftest.sml` files
- ask someone, e. g. me :-) (thomas@tuerk-brechen.de)

Part VI

Forward Proofs



- we already discussed the HOL Logic
- the kernel itself does not even contain basic logic operators
- usually one uses a much higher level of abstraction
 - ▶ many operations and datatypes are defined
 - ▶ high-level derived inference rules are used
- let's now look at this more common abstraction level

Common Terms and Types

	Unicode	ASCII
type vars	α, β, \dots	'a, 'b, ...
type annotated term	term:type	term:type
true	T	T
false	F	F
negation	$\neg b$	~b
conjunction	$b1 \wedge b2$	b1 /\ b2
disjunction	$b1 \vee b2$	b1 \/ b2
implication	$b1 \implies b2$	b1 ==> b2
equivalence	$b1 \iff b2$	b1 <=> b2
disequation	$v1 \neq v2$	v1 <> v2
all-quantification	$\forall x. P\ x$!x. P x
existential quantification	$\exists x. P\ x$?x. P x
Hilbert's choice operator	$@x. P\ x$	@x. P x

There are similar restrictions to constant and variable names as in SML.

HOL specific: don't start variable names with an underscore

- common function syntax
 - ▶ prefix notation, e. g. $SUC\ x$
 - ▶ infix notation, e. g. $x + y$
 - ▶ quantifier notation, e. g. $\forall x. P\ x$ means $(\forall) (\lambda x. P\ x)$
- infix and quantifier notation can be turned into prefix notation
Example: $(+)\ x\ y$ and $\$+\ x\ y$ are the same as $x + y$
- quantifiers of the same type don't need to be repeated
Example: $\forall x\ y. P\ x\ y$ is short for $\forall x. \forall y. P\ x\ y$
- there is special syntax for some functions
Example: $if\ c\ then\ v1\ else\ v2$ is nice syntax for $COND\ c\ v1\ v2$
- associative infix operators are usually right-associative
Example: $b1\ /\ \ b2\ /\ \ b3$ is parsed as $b1\ /\ (b2\ /\ b3)$

Term Parser

Use special quotation provided by `unquote`.

Operator Precedence

It is easy to misjudge the binding strength of certain operators. Therefore use plenty of parenthesis.

Use Syntax Functions

Terms are just SML values of type `term`. You can use syntax functions (usually defined in `*Syntax.sml` files) to create them.

Parser

“:bool”

“T”

“~b”

“... /\ ...”

“... \/ ...”

“... ==> ...”

“... = ...”

“... <=> ...”

“... <> ...”

Syntax Funs

mk_type ("bool", []) or bool

mk_const ("T", bool) or T

mk_neg (
 mk_var ("b", bool))

mk_conj (... , ...)

mk_disj (... , ...)

mk_imp (... , ...)

mk_eq (... , ...)

mk_eq (... , ...)

mk_neg (mk_eq (... , ...))

type of Booleans

term true

negation of

Boolean var b

conjunction

disjunction

implication

equation

equivalence

negated equation

Inference Rules for Equality



$$\frac{}{\vdash t = t} \text{REFL}$$

$$\frac{\Gamma \vdash s = t \quad x \text{ not free in } \Gamma}{\Gamma \vdash \lambda x. s = \lambda x. t} \text{ABS}$$

$$\frac{\begin{array}{l} \Gamma \vdash s = t \\ \Delta \vdash u = v \\ \text{types fit} \end{array}}{\Gamma \cup \Delta \vdash s(u) = t(v)} \text{MK-COMB}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash t = s} \text{GSYM}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash t = u}{\Gamma \cup \Delta \vdash s = u} \text{TRANS}$$

$$\frac{\Gamma \vdash p \Leftrightarrow q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \text{EQ_MP}$$

$$\frac{}{\vdash (\lambda x. t)v = t[v/x]} \text{BETA_CONV}$$

$$\frac{\Gamma[x_1, \dots, x_n] \vdash p[x_1, \dots, x_n]}{\Gamma[t_1, \dots, t_n] \vdash p[t_1, \dots, t_n]} \text{INST}$$

$$\frac{\Gamma[\alpha_1, \dots, \alpha_n] \vdash p[\alpha_1, \dots, \alpha_n]}{\Gamma[\gamma_1, \dots, \gamma_n] \vdash p[\gamma_1, \dots, \gamma_n]} \text{INST_TYPE}$$

Inference Rules for Implication



$$\frac{\Gamma \vdash p \Rightarrow q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \text{ MP, MATCH_MP}$$

$$\frac{\Gamma \vdash p = q}{\Gamma \vdash p \Rightarrow q} \text{ EQ_IMP_RULE}$$
$$\Gamma \vdash q \Rightarrow p$$

$$\frac{\Gamma \vdash p \Rightarrow q \quad \Delta \vdash q \Rightarrow p}{\Gamma \cup \Delta \vdash p = q} \text{ IMP_ANTISYM_RULE}$$

$$\frac{\Gamma \vdash p \Rightarrow q \quad \Delta \vdash q \Rightarrow r}{\Gamma \cup \Delta \vdash p \Rightarrow r} \text{ IMP_TRANS}$$

$$\frac{\Gamma \vdash p}{\Gamma - \{q\} \vdash q \Rightarrow p} \text{ DISCH}$$

$$\frac{\Gamma \vdash q \Rightarrow p}{\Gamma \cup \{q\} \vdash p} \text{ UNDISCH}$$

$$\frac{\Gamma \vdash p \Rightarrow F}{\Gamma \vdash \sim p} \text{ NOT_INTRO}$$

$$\frac{\Gamma \vdash \sim p}{\Gamma \vdash p \Rightarrow F} \text{ NOT_ELIM}$$

$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \wedge q} \text{ CONJ}$$

$$\frac{\Gamma \vdash p \wedge q}{\Gamma \vdash p} \text{ CONJUNCT1}$$

$$\frac{\Gamma \vdash p \wedge q}{\Gamma \vdash q} \text{ CONJUNCT2}$$

$$\frac{\Gamma \vdash p}{\Gamma \vdash p \vee q} \text{ DISJ1}$$

$$\frac{\Gamma \vdash q}{\Gamma \vdash p \vee q} \text{ DISJ2}$$

$$\frac{\begin{array}{l} \Gamma \vdash p \vee q \\ \Delta_1 \cup \{p\} \vdash r \\ \Delta_2 \cup \{q\} \vdash r \end{array}}{\Gamma \cup \Delta_1 \cup \Delta_2 \vdash r} \text{ DISJ_CASES}$$

$$\frac{\Gamma \vdash p \quad x \text{ not free in } \Gamma}{\Gamma \vdash \forall x. p} \text{ GEN}$$

$$\frac{\Gamma \vdash \forall x. p}{\Gamma \vdash p[u/x]} \text{ SPEC}$$

$$\frac{\Gamma \vdash p[u/x]}{\Gamma \vdash \exists x. p} \text{ EXISTS}$$

$$\frac{\begin{array}{c} \Gamma \vdash \exists x. p \\ \Delta \cup \{p[u/x]\} \vdash r \\ u \text{ not free in } \Gamma, \Delta, p \text{ and } r \end{array}}{\Gamma \cup \Delta \vdash r} \text{ CHOOSE}$$

- axioms and inference rules are used to derive theorems
- this method is called **forward proof**
 - ▶ one starts with basic building blocks
 - ▶ one moves step by step forward
 - ▶ finally the theorem one is interested in is derived
- one can also implement own proof tools

Forward Proofs — Example I



Let's prove $\forall p. p \implies p$.

```
val IMP_REFL_THM = let
  val tm1 = 'p:bool';
  val thm1 = ASSUME tm1;
  val thm2 = DISCH tm1 thm1;
in
  GEN tm1 thm2
end
```

```
fun IMP_REFL t =
  SPEC t IMP_REFL_THM;
```

```
> val tm1 = 'p': term
> val thm1 = [p] |- p: thm
> val thm2 = |- p ==> p: thm

> val IMP_REFL_THM =
  |- !p. p ==> p: thm

> val IMP_REFL =
  fn: term -> thm
```

Let's prove $\forall P v. (\exists x. (x = v) \wedge P x) \iff P v$.

```
val tm_v = ''v:'a'';  
val tm_P = ''P:'a -> bool'';  
val tm_lhs = ''?x. (x = v) /\ P x''  
val tm_rhs = mk_comb (tm_P, tm_v);
```

```
val thm1 = let  
  val thm1a = ASSUME tm_rhs;  
  val thm1b =  
    CONJ (REFL tm_v) thm1a;  
  val thm1c =  
    EXISTS (tm_lhs, tm_v) thm1b  
in  
  DISCH tm_rhs thm1c  
end
```

```
> val thm1a = [P v] |- P v: thm  
> val thm1b =  
  [P v] |- (v = v) /\ P v: thm  
> val thm1c =  
  [P v] |- ?x. (x = v) /\ P x  
  
> val thm1 = [] |-  
  P v ==> ?x. (x = v) /\ P x: thm
```

```
val thm2 = let
  val thm2a =
    ASSUME (('u:'a = v) /\ P u)
  val thm2b = AP_TERM tm_P
    (CONJUNCT1 thm2a);
  val thm2c = EQ_MP thm2b
    (CONJUNCT2 thm2a);
  val thm2d =
    CHOOSE (('u:'a',
      ASSUME tm_lhs) thm2c
in
  DISCH tm_lhs thm2d
end
```

```
val thm3 = IMP_ANTISYM_RULE thm2 thm1
val thm4 = GENL [tm_P, tm_v] thm3
```

```
> val thm2a = [(u = v) /\ P u] |-
  (u = v) /\ P u: thm
> val thm2b = [(u = v) /\ P u] |-
  P u <=> P v
> val thm2c = [(u = v) /\ P u] |-
  P v
> val thm2d = [?x. (x = v) /\ P x] |-
  P v

> val thm2 = [] |-
  ?x. (x = v) /\ P x ==> P v

> val thm3 = [] |-
  ?x. (x = v) /\ P x <=> P v
> val thm4 = [] |- !P v.
  ?x. (x = v) /\ P x <=> P v
```

Part VII

Backward Proofs



- let's prove $\neg A \vee B. A \wedge B \Leftrightarrow B \wedge A$

```
(* Show |- A /\ B ==> B /\ A *)
```

```
val thm1a = ASSUME ``A /\ B``;
```

```
val thm1b = CONJ (CONJUNCT2 thm1a) (CONJUNCT1 thm1a);
```

```
val thm1 = DISCH ``A /\ B`` thm1b
```

```
(* Show |- B /\ A ==> A /\ B *)
```

```
val thm2a = ASSUME ``B /\ A``;
```

```
val thm2b = CONJ (CONJUNCT2 thm2a) (CONJUNCT1 thm2a);
```

```
val thm2 = DISCH ``B /\ A`` thm2b
```

```
(* Combine to get |- A /\ B <=> B /\ A *)
```

```
val thm3 = IMP_ANTISYM_RULE thm1 thm2
```

```
(* Add quantifiers *)
```

```
val thm4 = GENL [``A:bool``, ``B:bool``] thm3
```

- this is how you write down a proof
- for finding a proof it is however often useful to think **backwards**

Motivation II - thinking backwards



- we want to prove
 - ▶ $\neg A \vee B \iff A \wedge B \iff B \wedge A$
- all-quantifiers can easily be added later, so let's get rid of them
 - ▶ $A \wedge B \iff B \wedge A$
- now we have an equivalence, let's show 2 implications
 - ▶ $A \wedge B \implies B \wedge A$
 - ▶ $B \wedge A \implies A \wedge B$
- we have an implication, so we can use the precondition as an assumption
 - ▶ using $A \wedge B$ show $B \wedge A$
 - ▶ $A \wedge B \implies B \wedge A$

- we have a conjunction as assumption, let's split it
 - ▶ using A and B show $B \wedge A$
 - ▶ $A \wedge B \implies B \wedge A$
- we have to show a conjunction, so let's show both parts
 - ▶ using A and B show B
 - ▶ using A and B show A
 - ▶ $A \wedge B \implies B \wedge A$
- the first two proof obligations are trivial
 - ▶ $A \wedge B \implies B \wedge A$
- ...
- we are done

- common practise
 - ▶ think backwards to find proof
 - ▶ write found proof down in forward style
- often switch between backward and forward style within a proof
Example: induction proof
 - ▶ backward step: induct on ...
 - ▶ forward steps: prove base case and induction case
- whether to use forward or backward proofs depend on
 - ▶ support by the interactive theorem prover you use
 - ★ HOL 4 and close family: emphasis on backward proof
 - ★ Isabelle/HOL: emphasis on forward proof
 - ★ Coq : emphasis on backward proof
 - ▶ your way of thinking
 - ▶ the theorem you try to prove

- in HOL
 - ▶ proof tactics / backward proofs used for most user-level proofs
 - ▶ forward proofs used usually for writing automation
- backward proofs are implemented by **tactics** in HOL
 - ▶ decomposition into subgoals implemented in SML
 - ▶ SML datastructures used to keep track of all open subgoals
 - ▶ forward proof used to construct theorems
- to understand backward proofs in HOL we need to look at
 - ▶ **goal** — SML datatype for proof obligations
 - ▶ **goalStack** — library for keeping track of goals
 - ▶ **tactic** — SML type for functions performing backward proofs

- goals represent proof obligations, i. e. theorems we need/want to prove
- the SML type `goal` is an abbreviation for `term list * term`
- the goal `([asm_1, ..., asm_n], c)` records that we need/want to prove the theorem $\{asm_1, \dots, asm_n\} \vdash c$

Example Goals

Goal

`([‘‘A’’, ‘‘B’’, ‘‘A /\ B’’]`
`([‘‘B’’, ‘‘A’’, ‘‘A /\ B’’]`
`([‘‘B /\ A’’, ‘‘A /\ B’’]`
`([], ‘‘(B /\ A) ==> (A /\ B)’’)`

Theorem

$\{A, B\} \vdash A \wedge B$
 $\{A, B\} \vdash A \wedge B$
 $\{B \wedge A\} \vdash A \wedge B$
 $\vdash (B \wedge A) \implies (A \wedge B)$

- the SML type `tactic` is an abbreviation for the type `goal -> goal list * validation`
- `validation` is an abbreviation for `thm list -> thm`
- given a goal, a tactic
 - ▶ decides into which subgoals to decompose the goal
 - ▶ returns this list of subgoals
 - ▶ returns a validation that
 - ★ given a list of theorems for the computed subgoals
 - ★ produces a theorem for the original goal
- special case: empty list of subgoals
 - ▶ the validation (given `[]`) needs to produce a theorem for the goal
- notice: a tactic might be invalid

Tactic Example — CONJ_TAC



$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \wedge q} \text{CONJ} \qquad \frac{t \equiv \text{conj1} \wedge \text{conj2} \quad \text{asl} \vdash \text{conj1} \quad \text{asl} \vdash \text{conj2}}{\text{asl} \vdash t}$$

```
val CONJ_TAC: tactic = fn (asl, t) =>
  let
    val (conj1, conj2) = dest_conj t
  in
    ([ (asl, conj1), (asl, conj2) ],
     fn [th1, th2] => CONJ th1 th2 | _ => raise Match)
  end
handle HOL_ERR _ => raise ERR "CONJ_TAC" ""
```

$$\frac{\Gamma \vdash p \implies q \quad \Delta \vdash q \implies p}{\Gamma \cup \Delta \vdash p = q} \text{IMP_ANTISYM_RULE}$$

$$\frac{t \equiv \text{lhs} = \text{rhs} \quad \text{asl} \vdash \text{lhs} ==> \text{rhs} \quad \text{asl} \vdash \text{rhs} ==> \text{lhs}}{\text{asl} \vdash t}$$

```
val EQ_TAC: tactic = fn (asl, t) =>
  let
    val (lhs, rhs) = dest_eq t
  in
    ([ (asl, mk_imp (lhs, rhs)), (asl, mk_imp (rhs, lhs)) ],
     fn [th1, th2] => IMP_ANTISYM_RULE th1 th2
       | _          => raise Match)
  end
handle HOL_ERR _ => raise ERR "EQ_TAC" ""
```

- the `proofManagerLib` keeps track of open goals
- it uses `goalStack` internally
- important commands
 - ▶ `g` — set up new goal
 - ▶ `e` — expand a tactic
 - ▶ `p` — print the current status
 - ▶ `top_thm` — get the proved thm at the end

Tactic Proof Example I



Previous Goalstack

–

User Action

g ' !A B. A /\ B <=> B /\ A ' ;

New Goalstack

Initial goal:

!A B. A /\ B <=> B /\ A

: proof

Tactic Proof Example II



Previous Goalstack

Initial goal:

$!A \ B. \ A \ /\ \ B \ \Leftrightarrow \ B \ /\ \ A$

: proof

User Action

e GEN_TAC;

e GEN_TAC;

New Goalstack

$A \ /\ \ B \ \Leftrightarrow \ B \ /\ \ A$

: proof

Tactic Proof Example III



Previous Goalstack

$A \wedge B \Leftrightarrow B \wedge A$

: proof

User Action

e EQ_TAC;

New Goalstack

$B \wedge A \Rightarrow A \wedge B$

$A \wedge B \Rightarrow B \wedge A$

: proof

Tactic Proof Example IV



Previous Goalstack

$B \wedge A \implies A \wedge B$

$A \wedge B \implies B \wedge A$: proof

User Action

e STRIP_TAC;

New Goalstack

$B \wedge A$

0. A

1. B

Tactic Proof Example V



Previous Goalstack

B /\ A

0. A

1. B

User Action

e CONJ_TAC;

New Goalstack

A

0. A

1. B

B

0. A

1. B

Tactic Proof Example VI



Previous Goalstack

A

-
- 0. A
 - 1. B

B

-
- 0. A
 - 1. B

User Action

```
e (ACCEPT_TAC (ASSUME 'B:bool'));  
e (ACCEPT_TAC (ASSUME 'A:bool'));
```

New Goalstack

$B \wedge A \implies A \wedge B$

: proof

Previous Goalstack

$B \wedge A \implies A \wedge B$

: proof

User Action

```
e STRIP_TAC;  
e (ASM_REWRITE_TAC[]);
```

New Goalstack

Initial goal proved.

```
|- !A B. A /\ B <=> B /\ A:  
  proof
```

Previous Goalstack

Initial goal proved.

```
|- !A B. A /\ B <=> B /\ A:  
  proof
```

User Action

```
val thm = top_thm();
```

Result

```
val thm =  
  |- !A B. A /\ B <=> B /\ A:  
  thm
```

Combined Tactic

```
val thm = prove (''!A B. A /\ B <=> B /\ A'',
  GEN_TAC >> GEN_TAC >>
  EQ_TAC >| [
    STRIP_TAC >>
    STRIP_TAC >| [
      ACCEPT_TAC (ASSUME 'B:bool'),
      ACCEPT_TAC (ASSUME 'A:bool')
    ],
    STRIP_TAC >>
    ASM_REWRITE_TAC[]
  ]);
```

Result

```
val thm =
  |- !A B. A /\ B <=> B /\ A:
  thm
```


Cleaned-up Tactic

```
val thm = prove (''!A B. A /\ B <=> B /\ A'',  
  REPEAT GEN_TAC >>  
  EQ_TAC >> (  
    REPEAT STRIP_TAC >>  
    ASM_REWRITE_TAC []  
  ));
```

Result

```
val thm =  
  |- !A B. A /\ B <=> B /\ A:  
  thm
```

- in HOL most user-level proofs are tactic-based
 - ▶ automation often written in forward style
 - ▶ low-level, basic proofs written in forward style
 - ▶ nearly everything else is written in backward (tactic) style
- there are **many** different tactics
- in the lecture only the most basic ones will be discussed
- **you need to learn about tactics on your own**
 - ▶ good starting point: Quick manual
 - ▶ learning finer points takes a lot of time
 - ▶ exercises require you to read up on tactics
- often there are many ways to prove a statement, which tactics to use depends on
 - ▶ personal way of thinking
 - ▶ personal style and preferences
 - ▶ maintainability, clarity, elegance, robustness
 - ▶ ...

Part VIII

Basic Tactics



- originally tactics were written all in capital letters with underscores
Example: `ALL_TAC`
- since 2010 more and more tactics have overloaded lower-case syntax
Example: `all_tac`
- sometimes, the lower-case version is shortened
Example: `REPEAT`, `rpt`
- sometimes, there is special syntax
Example: `THEN`, `\`, `>>`
- which one to use is mostly a matter of personal taste
 - ▶ all-capital names are hard to read and type
 - ▶ however, not for all tactics there are lower-case versions
 - ▶ mixed lower- and upper-case tactics are even harder to read
 - ▶ often shortened lower-case name is not *speaking*

In the lecture we will use mostly the old-style names.

Some Basic Tactics



GEN_TAC	remove outermost all-quantifier
DISCH_TAC	move antecedent of goal into assumptions
CONJ_TAC	splits conjunctive goal
STRIP_TAC	splits on outermost connective (combination of GEN_TAC, CONJ_TAC, DISCH_TAC, ...)
DISJ1_TAC	selects left disjunct
DISJ2_TAC	selects right disjunct
EQ_TAC	reduce Boolean equality to implications
ASSUME_TAC thm	add theorem to list of assumptions
EXISTS_TAC term	provide witness for existential goal

- tacticals are SML functions that combine tactics to form new tactics
- common workflow
 - ▶ develop large tactic interactively
 - ▶ using `goalStack` and editor support to execute tactics one by one
 - ▶ combine tactics manually with tacticals to create larger tactics
 - ▶ finally end up with one large tactic that solves your goal
 - ▶ use `prove` or `store_thm` instead of `goalStack`
- make sure to **clearly mark proof structure** by e. g.
 - ▶ use indentation
 - ▶ use parentheses
 - ▶ use appropriate connectives
 - ▶ ...
- goalStack commands like `e` or `g` should not appear in your final proof

Some Basic Tacticals



tac1 >> tac2	THEN, \\ THENL	applies tactics in sequence applies list of tactics to subgoals
tac > tacL	THEN1	applies tac2 to the first subgoal of tac1
tac1 >- tac2	rpt	repeats tac until it fails
REPEAT tac		
NTAC n tac	reverse	apply tac n times reverses the order of subgoals
REVERSE tac		
tac1 ORELSE tac2		applies tac1 only if tac2 fails
TRY tac		do nothing if tac fails
ALL_TAC	all_tac	do nothing
NO_TAC		fail

- (equational) rewriting is at the core of HOL's automation
- we will discuss it in detail later
- details complex, but basic usage is straightforward
 - ▶ given a theorem `rewr_thm` of form $\vdash P\ x = Q\ x$ and a term `t`
 - ▶ rewriting `t` with `rewr_thm` means
 - ▶ replacing each occurrence of a term $P\ c$ for some `c` with $Q\ c$ in `t`
- **warning:** rewriting may loop

Example: rewriting with theorem $\vdash X \iff (X \wedge T)$

<code>REWRITE_TAC</code> thms	rewrite goal using equations found in given list of theorems
<code>ASM_REWRITE_TAC</code> thms	in addition use assumptions
<code>ONCE_REWRITE_TAC</code> thms	rewrite once in goal using equations
<code>ONCE_ASM_REWRITE_TAC</code> thms	rewrite once using assumptions

Case-Split and Induction Tactics



<code>Induct_on 'term'</code>	induct on term
<code>Induct</code>	induct on all-quantifier
<code>Cases_on 'term'</code>	case-split on term
<code>Cases</code>	case-split on all-quantifier
<code>MATCH_MP_TAC</code> thm	apply rule
<code>IRULE_TAC</code> thm	generalised apply rule

POP_ASSUM thm-tac

use and remove first assumption
common usage **POP_ASSUM MP_TAC**

PAT_ASSUM term thm-tac

also **PAT_X_ASSUM** term thm-tac

use (and remove) first
assumption matching pattern

WEAKEN_TAC term-pred

removes first assumption
satisfying predicate

- decision procedures try to solve the current goal completely
- they either succeed or fail
- no partial progress
- decision procedures vital for automation

TAUT_TAC	propositional logic tautology checker
DECIDE_TAC	linear arithmetic for num
METIS_TAC thms	first order prover
numLib.ARITH_TAC	Presburger arithmetic
intLib.ARITH_TAC	uses Omega test

- it is vital to structure your proofs well
 - ▶ improved maintainability
 - ▶ improved readability
 - ▶ improved reusability
 - ▶ saves time in medium-run
- therefore, use many small lemmata
- also, use many explicit subgoals

'term-frag' <code>by</code> tac	show term with tac and add it to assumptions
'term-frag' <code>suffices_by</code> tac	show it suffices to prove term

- notice that `by` and `suffices_by` take **term fragments**
- term fragments are also called **term quotations**
- they represent (partially) unparsed terms
- parsing takes place during execution of tactic in context of goal
- this helps to avoid type annotations
- however, this means syntax errors show late as well
- the library `Q` defines many tactics using term fragments

- here many tactics are presented in a very short amount of time
- there are many, many more important tactics out there
- few people can learn a programming language just by reading manuals
- similar few people can learn HOL just by reading and listening
- you should write your own proofs and play around with these tactics
- solving the exercises is highly recommended
(and actually required if you want credits for this course)

- we want to prove $!1. \text{LENGTH} (\text{APPEND } 1 \ 1) = 2 * \text{LENGTH } 1$
- first step: set up goal on `goalStack`
- at same time start writing proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘,
```

Actions

- run `g ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘`
- this is done by hol-mode
- move cursor inside term and press `M-h g`
(menu-entry HOL - Goalstack - New goal)

Current Goal

`!1. LENGTH (1 ++ 1) = 2 * LENGTH 1`

- the outermost connective is an all-quantifier
- let's get rid of it via `GEN_TAC`

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ``!1. LENGTH (1 ++ 1) = 2 * LENGTH 1``,  
  GEN_TAC
```

Actions

- run `e GEN_TAC`
- this is done by hol-mode
- mark line with `GEN_TAC` and press `M-h e`
(menu-entry `HOL - Goalstack - Apply tactic`)

Current Goal

`LENGTH (1 ++ 1) = 2 * LENGTH 1`

- `LENGTH` of `APPEND` can be simplified
- let's search an appropriate lemma with `DB.match`

Actions

- run `DB.print_match [] 'LENGTH (_ ++ _)'`
- this is done via hol-mode
- press M-h m and enter term pattern
(menu-entry HOL - Misc - DB match)
- this finds the theorem `listTheory.LENGTH_APPEND`
`| - !11 12. LENGTH (11 ++ 12) = LENGTH 11 + LENGTH 12`

Current Goal

`LENGTH (1 ++ 1) = 2 * LENGTH 1`

- let's rewrite with found theorem `listTheory.LENGTH_APPEND`

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘,  
  GEN_TAC >>  
  REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- connect the new tactic with tactical `>>` (`THEN`)
- use hol-mode to expand the new tactic

Current Goal

$\text{LENGTH } 1 + \text{LENGTH } 1 = 2 * \text{LENGTH } 1$

- let's search a theorem for simplifying $2 * \text{LENGTH } 1$
- prepare for extending the previous rewrite tactic

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘,  
  GEN_TAC >>  
  REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- `DB.match` finds theorem `arithmeticTheory.TIMES2`
- press M-h b and undo last tactic expansion
(menu-entry HOL - Goalstack - Back up)

Current Goal

`LENGTH (1 ++ 1) = 2 * LENGTH 1`

- extend the previous rewrite tactic
- finish proof

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘,  
  GEN_TAC >>  
  REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Actions

- add `TIMES2` to the list of theorems used by rewrite tactic
- use hol-mode to expand the extended rewrite tactic
- goal is solved, so let's add closing parenthesis and semicolon

- we have a finished tactic proving our goal
- notice that `GEN_TAC` is not needed
- let's polish the proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  “!l. LENGTH (APPEND l l) = 2 * LENGTH l”,  
  GEN_TAC >>  
  REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Polished Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  “!l. LENGTH (APPEND l l) = 2 * LENGTH l”,  
  REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

- let's prove something slightly more complicated
- drop old goal by pressing M-h d
(menu-entry HOL - Goalstack - Drop goal)
- set up goal on `goalStack` (M-h g)
- at same time start writing proof script

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘!x1 x2 x3 11 12 13.  
  (MEM x1 11 /\ MEM x2 12 /\ MEM x3 13) /\  
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>  
  ~(ALL_DISTINCT (11 ++ 12 ++ 13))‘‘,
```

Current Goal

```
!x1 x2 x3 11 12 13.  
  (MEM x1 11 /\ MEM x2 12 /\ MEM x3 13) /\  
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>  
  ~ALL_DISTINCT (11 ++ 12 ++ 13)
```

- let's strip the goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘!x1 x2 x3 11 12 13.  
  (MEM x1 11 /\ MEM x2 12 /\ MEM x3 13) /\  
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>  
  ~(ALL_DISTINCT (11 ++ 12 ++ 13))‘‘,  
  REPEAT STRIP_TAC
```

Current Goal

```
!x1 x2 x3 11 12 13.  
  (MEM x1 11 /\ MEM x2 12 /\ MEM x3 13) /\  
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>  
  ~ALL_DISTINCT (11 ++ 12 ++ 13)
```

- let's strip the goal

Proof Script

```
val LENGTH_APPEND_SAME = prove (  
  ‘‘!1. LENGTH (APPEND 1 1) = 2 * LENGTH 1‘‘,  
  REPEAT STRIP_TAC
```

Actions

- add `REPEAT STRIP_TAC` to proof script
- expand this tactic using hol-mode

Current Goal

F

```
-----  
0.  MEM x1 11          4.  x2 <= x3  
1.  MEM x2 12          5.  x3 <= SUC x1  
2.  MEM x3 13          6.  ALL_DISTINCT (11 ++ 12 ++ 13)  
3.  x1 <= x2
```

- oops, we did too much, we would like to keep ALL_DISTINCT in goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (''...',  
REPEAT GEN_TAC >> STRIP_TAC
```

Actions

- undo REPEAT STRIP_TAC (M-h b)
- expand more fine-tuned strip tactic

Current Goal

$\sim \text{ALL_DISTINCT } (11 \ ++ \ 12 \ ++ \ 13)$

0. MEM x1 11	3. x1 <= x2
1. MEM x2 12	4. x2 <= x3
2. MEM x3 13	5. x3 <= SUC x1

- now let's simplify `ALL_DISTINCT`
- search suitable theorems with `DB.match`
- use them with rewrite tactic

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘...’’,  
REPEAT GEN_TAC >> STRIP_TAC >>  
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND]
```

Current Goal

$$\sim((\text{ALL_DISTINCT } 11 \wedge \text{ALL_DISTINCT } 12 \wedge !e. \text{MEM } e \ 11 \implies \sim \text{MEM } e \ 12) \wedge \text{ALL_DISTINCT } 13 \wedge !e. \text{MEM } e \ 11 \wedge \text{MEM } e \ 12 \implies \sim \text{MEM } e \ 13)$$

0. MEM x1 11	3. x1 <= x2
1. MEM x2 12	4. x2 <= x3
2. MEM x3 13	5. x3 <= SUC x1

- from assumptions 3, 4 and 5 we know $x2 = x1 \wedge x2 = x3$
- let's deduce this fact by `DECIDE_TAC`

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove ('...',  
  REPEAT GEN_TAC >> STRIP_TAC >>  
  REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>  
  '(x2 = x1) \wedge (x2 = x3)' by DECIDE_TAC
```

Current Goals — 2 subgoals, one for each disjunct

$\sim((\text{ALL_DISTINCT } 11 \wedge \text{ALL_DISTINCT } 12 \wedge !e. \text{MEM } e \ 11 \implies \sim \text{MEM } e \ 12) \wedge$
 $\text{ALL_DISTINCT } 13 \wedge !e. \text{MEM } e \ 11 \wedge \text{MEM } e \ 12 \implies \sim \text{MEM } e \ 13)$

0. MEM x1 11	4. x2 <= x3
1. MEM x2 12	5. x3 <= SUC x1
2. MEM x3 13	6a. x2 = x1
3. x1 <= x2	6b. x2 = x3

- both goals are easily solved by first-order reasoning
- let's use `METIS_TAC[]` for both subgoals

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (''...',  
  REPEAT GEN_TAC >> STRIP_TAC >>  
  REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>  
  '(x2 = x1) \/\ (x2 = x3)' by DECIDE_TAC >> (  
    METIS_TAC[]  
  ));
```

Finished Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (  
  ‘‘!x1 x2 x3 l1 l2 l3.  
    (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\  
    ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>  
    ~(ALL_DISTINCT (l1 ++ l2 ++ l3))‘‘,  
  REPEAT GEN_TAC >> STRIP_TAC >>  
  REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>  
  ‘(x2 = x1) \/ (x2 = x3)’ by DECIDE_TAC >> (  
    METIS_TAC[]  
  ));
```

- notice that proof structure is explicit
- parentheses and indentation used to mark new subgoals

Part IX

Induction Proofs



- mathematical (a. k. a. natural) induction principle:
If a property P holds for 0 and $P(n)$ implies $P(n + 1)$ for all n ,
then $P(n)$ holds for all n .
- HOL is expressive enough to encode this principle as a theorem.

$$\vdash !P. P\ 0 \ /\ \ (!n. P\ n \ ==> P\ (SUC\ n)) \ ==> !n. P\ n$$

- Performing mathematical induction in HOL means applying this theorem (e. g. via `HO_MATCH_MP_TAC`)
- there are many similarish induction theorems in HOL
- Example: complete induction principle

$$\vdash !P. (!n. (!m. m < n ==> P\ m) ==> P\ n) ==> !n. P\ n$$

- **structural induction** theorems are an important special form of induction theorems
- they describe performing induction on the structure of a datatype
- Example: $\vdash !P. P [] \wedge (!t. P t \implies !h. P (h::t)) \implies !l. P l$
- structural induction is used very frequently in HOL
- for each algebraic datatype, there is an induction theorem

- there are many induction theorems in HOL
 - ▶ datatype definitions lead to induction theorems
 - ▶ recursive function definitions produce corresponding induction theorems
 - ▶ recursive relation definitions give rise to induction theorems
 - ▶ many are manually defined
- Examples

$\vdash !P. P [] \wedge (!l. P l \implies !x. P (SNOC\ x\ l)) \implies !l. P l$

$\vdash !P. P\ FEMPTY \wedge$
 $(!f. P\ f \implies !x\ y. x\ NOTIN\ FDOM\ f \implies P\ (f\ |+ (x,y))) \implies !f. P\ f$

$\vdash !P. P\ \{\}\ \wedge$
 $(!s. FINITE\ s \wedge P\ s \implies !e. e\ NOTIN\ s \implies P\ (e\ INSERT\ s)) \implies$
 $!s. FINITE\ s \implies P\ s$

$\vdash !R\ P. (!x\ y. R\ x\ y \implies P\ x\ y) \wedge (!x\ y\ z. P\ x\ y \wedge P\ y\ z \implies P\ x\ z) \implies$
 $!u\ v. R^+\ u\ v \implies P\ u\ v$

- the tactic `Induct` (or `Induct_on`) is usually used to start induction proofs
- it looks at the type of the quantifier (or its argument) and applies the default induction theorem for this type
- this is usually what one needs
- other (non default) induction theorems can be applied via `INDUCT_THEN` or `HO_MATCH_MP_TAC`
- similarish `Cases_on` picks and applies default case-split theorems

- let's prove via induction
$$!11\ 12. \text{ REVERSE } (11 ++ 12) = \text{ REVERSE } 12 ++ \text{ REVERSE } 11$$
- we set up the goal and start an induction proof on `11`

Proof Script

```
val REVERSE_APPEND = prove (  
  ‘‘!11 12. REVERSE (11 ++ 12) = REVERSE 12 ++ REVERSE 11‘‘,  
  Induct
```

- the induction tactic produced two cases
- base case:

`!12. REVERSE ([] ++ 12) = REVERSE 12 ++ REVERSE []`

- induction step:

`!h 12. REVERSE (h::11 ++ 12) = REVERSE 12 ++ REVERSE (h::11)`

`!12. REVERSE (11 ++ 12) = REVERSE 12 ++ REVERSE 11`

- both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_APPEND = prove (''  
!11 12. REVERSE (11 ++ 12) = REVERSE 12 ++ REVERSE 11'',  
Induct >| [  
  REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_NIL],  
  ASM_REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_ASSOC]  
]);
```

- let's prove via induction
$$!1. \text{ REVERSE } (\text{REVERSE } 1) = 1$$
- we set up the goal and start an induction proof on 1

Proof Script

```
val REVERSE_REVERSE = prove (  
  ‘‘!1. REVERSE (REVERSE 1) = 1‘‘,  
  Induct
```

- the induction tactic produced two cases
- base case:

`REVERSE (REVERSE []) = []`

- induction step:

`!h. REVERSE (REVERSE (h::l1)) = h::l1`

`REVERSE (REVERSE 1) = 1`

- again both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_REVERSE = prove (  
  ‘‘!l. REVERSE (REVERSE l) = l’’,  
  Induct >| [  
    REWRITE_TAC[REVERSE_DEF],  
    ASM_REWRITE_TAC[REVERSE_DEF, REVERSE_APPEND, APPEND]  
  ]);
```

Part X

Basic Definitions



- there are **conservative definition principles** for types and constants
- conservative means that all theorems that can be proved in extended theory can also be proved in original one
- however, such extensions make the theory more comfortable
- definitions introduce **no new inconsistencies**
- the HOL community has a very strong tradition of a purely definitional approach

- **axioms** are a different approach
- they allow postulating arbitrary properties, i. e. extending the logic with arbitrary theorems
- this approach might introduce new inconsistencies
- in HOL axioms are very rarely needed
- using definitions is often considered more elegant
- it is hard to keep track of axioms
- use axioms only if you really know what you are doing

- **oracles** are families of axioms
- however, they are used differently than axioms
- they are used to enable usage of external tools and knowledge
- you might want to use an external automated prover
- this external tool acts as an oracle
 - ▶ it provides answers
 - ▶ it does not explain or justify these answers
- you don't know, whether this external tool might be buggy
- all theorems proved via it are tagged with a special oracle-tag
- tags are propagated
- this allows keeping track of everything depending on the correctness of this tool

- Common oracle-tags
 - ▶ `DISK_THM` — theorem was written to disk and read again
 - ▶ `HolSatLib` — proved by MiniSat
 - ▶ `HolSmtLib` — proved by external SMT solver
 - ▶ `fast_proof` — proof was skipped to compile a theory rapidly
 - ▶ `cheat` — we cheated :-)
- `cheating` via e. g. the `cheat` tactic means skipping proofs
- it can be helpful during proof development
 - ▶ test whether some lemmata allow you finishing the proof
 - ▶ skip lengthy but boring cases and focus on critical parts first
 - ▶ experiment with exact form of invariants
 - ▶ ...
- cheats should be removed reasonable quickly
- HOL warns about cheats and skipped proofs

- definitions can't introduce new inconsistencies
- they force you to state all assumed properties at one location
- however, you still need to be careful
- Is your definition really expressing what you had in mind ?
- Does your formalisation correspond to the real world artefact ?
- How can you convince others that this is the case ?
- we will discuss methods to deal with this later in this course
 - ▶ formal sanity
 - ▶ conformance testing
 - ▶ code review
 - ▶ comments, good names, clear coding style
 - ▶ ...
- this is highly complex and needs a lot of effort in general

- HOL allows to introduce new constants with certain properties, provided the existence of such constants has been shown

Specification of EVEN and ODD

```
> EVEN_ODD_EXISTS
val it = |- ?even odd. even 0 /\ ~odd 0 /\ (!n. even (SUC n) <=> odd n) /\
          (!n. odd (SUC n) <=> even n)

> val EO_SPEC = new_specification ("EO_SPEC", ["EVEN", "ODD"], EVEN_ODD_EXISTS);
val EO_SPEC = |- EVEN 0 /\ ~ODD 0 /\ (!n. EVEN (SUC n) <=> ODD n) /\
              (!n. ODD (SUC n) <=> EVEN n)
```

- `new_specification` is a convenience wrapper
 - ▶ it uses existential quantification instead of Hilbert's choice
 - ▶ deals with pair syntax
 - ▶ stores resulting definitions in theory
- `new_specification` captures the underlying principle nicely

- special case: new constant defined by equality

Specification with Equality

```
> double_EXISTS
val it =
|- ?double. (!n. double n = (n + n))

> val double_def = new_specification ("double_def", ["double"], double_EXISTS);
val double_def =
|- !n. double n = n + n
```

- there is a specialised methods for such simple definitions

Non Recursive Definitions

```
> val DOUBLE_DEF = new_definition ("DOUBLE_DEF", 'DOUBLE n = n + n')
val DOUBLE_DEF =
|- !n. DOUBLE n = n + n
```

- all variables occurring on right-hand-side (rhs) need to be arguments
 - ▶ e.g. `new_definition (... , 'F n = n + m')` fails
 - ▶ `m` is free on rhs
- all type variables occurring on rhs need to occur on lhs
 - ▶ e.g. `new_definition ("IS_FIN_TY",
 'IS_FIN_TY = FINITE (UNIV : 'a set)')` fails
 - ▶ `IS_FIN_TY` would lead to inconsistency
 - ▶ `|- FINITE (UNIV : bool set)`
 - ▶ `|- ~FINITE (UNIV : num set)`
 - ▶ `T <=> FINITE (UNIV:bool set) <=>
 IS_FIN_TY <=>
 FINITE (UNIV:num set) <=> F`
 - ▶ therefore, such definitions can't be allowed

- function specification do not need to define the function precisely
- multiple different functions satisfying one spec are possible
- functions resulting from such specs are called **underspecified**
- underspecified functions are still total, one just lacks knowledge
- one common application: modelling **partial functions**
 - ▶ functions like e. g. **HD** and **TL** are total
 - ▶ they are defined for empty lists
 - ▶ however, it is not specified, which value they have for empty lists
 - ▶ only known: **HD [] = HD []** and **TL [] = TL []**

```
val MY_HD_EXISTS = prove (“‘?hd. !x xs. (hd (x::xs) = x)‘”, ...);  
val MY_HD_SPEC =  
  new_specification ("MY_HD_SPEC", ["MY_HD"], MY_HD_EXISTS)
```


- HOL allows introducing non-empty subtypes of existing types
- a predicate $P : \text{ty} \rightarrow \text{bool}$ describes a subset of an existing type ty
- ty may contain type variables
- only **non-empty** types are allowed
- therefore a non-emptiness proof ex-thm of form $?e. P\ e$ is needed
- `new_type_definition (op-name, ex-thm)` then introduces a new type `op-name` specified by P

- lets try to define a type `dlist` of lists containing no duplicates
- predicate `ALL_DISTINCT : 'a list -> bool` is used to define it
- easy to prove theorem `dlist_exists: |- ?1. ALL_DISTINCT 1`
- `val dlist_TY_DEF = new_type_definitions("dlist",
dlist_exists)` defines a new type `'a dlist` and returns a theorem

```
|- ?(rep : 'a dlist -> 'a list).  
    TYPE_DEFINITION ALL_DISTINCT rep
```

- `rep` is a function taking a `'a dlist` to the list representing it
 - ▶ `rep` is injective
 - ▶ a list satisfies `ALL_DISTINCT` iff there is a corresponding `dlist`

- `define_new_type_bijections` can be used to define bijections between old and new type

```
> define_new_type_bijections {name="dlist_tybij", ABS="abs_dlist",  
    REP="rep_dlist", tyax=dlist_TY_DEF}
```

```
val it =  
  |- (!a. abs_dlist (rep_dlist a) = a) /\  
    (!r. ALL_DISTINCT r <=> (rep_dlist (abs_dlist r) = r))
```

- other useful theorems can be automatically proved by
 - ▶ `prove_abs_fn_one_one`
 - ▶ `prove_abs_fn_onto`
 - ▶ `prove_rep_fn_one_one`
 - ▶ `prove_rep_fn_onto`

- primitive definition principles are easily explained
- they lead to conservative extensions
- however, they are cumbersome to use
- LCF approach allows implementing more convenient definition tools
 - ▶ **Datatype** package
 - ▶ **TFL** (Total Functional Language) package
 - ▶ **IndDef** (Inductive Definition) package
 - ▶ **quotientLib** Quotient Types Library
 - ▶ ...

- the **Datatype** package allows to define datatypes conveniently
- the **TFL** package allows to define (mutually recursive) functions
- the **EVAL** conversion allows evaluating those definitions
- this gives many HOL developments the feeling of a functional program
- there is really a close connection between functional programming and definitions in HOL
 - ▶ functional programming design principles apply
 - ▶ **EVAL** is a great way to test quickly, whether your definitions are working as intended

Functional Programming Example



```
> Datatype 'mylist = E | L 'a mylist'
val it = (): unit

> Define '(mylen E = 0) /\ (mylen (L x xs) = SUC (mylen xs))'
Definition has been stored under "mylen_def"
val it =
  |- (mylen E = 0) /\ !x xs. mylen (L x xs) = SUC (mylen xs):
  thm

> EVAL ''mylen (L 2 (L 3 (L 1 E)))''
val it =
  |- mylen (L 2 (L 3 (L 1 E))) = 3:
  thm
```

- the **Datatype** package allows to define SML style datatypes easily
- there is support for
 - ▶ algebraic datatypes
 - ▶ record types
 - ▶ mutually recursive types
 - ▶ ...
- many constants are automatically introduced
 - ▶ constructors
 - ▶ case-split constant
 - ▶ size function
 - ▶ field-update and accessor functions for records
 - ▶ ...
- many theorems are derived and stored in current theory
 - ▶ injectivity and distinctness of constructors
 - ▶ nchotomy and structural induction theorems
 - ▶ rewrites for case-split, size and record update functions
 - ▶ ...

Tree Datatype in SML

```
datatype ('a,'b) btree =   Leaf of 'a
                          | Node of ('a,'b) btree * 'b * ('a,'b) btree
```

Tree Datatype in HOL

```
Datatype 'btree =   Leaf 'a
                   | Node btree 'b btree'
```

Tree Datatype in HOL — Deprecated Syntax

```
Hol_datatype 'btree =   Leaf of 'a
                       | Node of btree => 'b => btree'
```


btree_distinct

```
|- !a2 a1 a0 a. Leaf a <> Node a0 a1 a2
```

btree_11

```
|- (!a a'. (Leaf a = Leaf a') <=> (a = a')) /\  
  (!a0 a1 a2 a0' a1' a2'.  
    (Node a0 a1 a2 = Node a0' a1' a2') <=>  
    (a0 = a0') /\ (a1 = a1') /\ (a2 = a2'))
```

btree_nchotomy

```
|- !bb. (?a. bb = Leaf a) \/ (?b b1 b0. bb = Node b b1 b0)
```

btree_induction

```
|- !P. (!a. P (Leaf a)) /\  
  (!b b0. P b /\ P b0 ==> !b1. P (Node b b1 b0)) ==>  
  !b. P b
```

btreesize_def

```
|- (!f f1 a. btree_size f f1 (Leaf a) = 1 + f a) /\
  (!f f1 a0 a1 a2.
    btree_size f f1 (Node a0 a1 a2) =
      1 + (btree_size f f1 a0 + (f1 a1 + btree_size f f1 a2)))
```

btreecase_def

```
|- (!a f f1. btree_CASE (Leaf a) f f1 = f a) /\
  (!a0 a1 a2 f f1. btree_CASE (Node a0 a1 a2) f f1 = f1 a0 a1 a2)
```

btreecase_cong

```
|- !M M' f f1.
  (M = M') /\ (!a. (M' = Leaf a) ==> (f a = f' a)) /\
  (!a0 a1 a2.
    (M' = Node a0 a1 a2) ==> (f1 a0 a1 a2 = f1' a0 a1 a2)) ==>
  (btree_CASE M f f1 = btree_CASE M' f' f1')
```

Enumeration type in SML

```
datatype my_enum = E1 | E2 | E3
```

Enumeration type in HOL

```
Datatype 'my_enum = E1 | E2 | E3'
```

`my_enum_nchotomy`

```
|- !P. P E1 /\ P E2 /\ P E3 ==> !a. P a
```

`my_enum_distinct`

```
|- E1 <> E2 /\ E1 <> E3 /\ E2 <> E3
```

`my_enum2num_thm`

```
|- (my_enum2num E1 = 0) /\ (my_enum2num E2 = 1) /\ (my_enum2num E3 = 2)
```

`my_enum2num_num2my_enum`

```
|- !r. r < 3 <=> (my_enum2num (num2my_enum r) = r)
```

Record type in SML

```
type rgb = { r : int, g : int, b : int }
```

Record type in HOL

```
Datatype 'rgb = <| r : num; g : num; b : num |>'
```

rgb_component_equality

```
|- !r1 r2. (r1 = r2) <=>  
          (r1.r = r2.r) /\ (r1.g = r2.g) /\ (r1.b = r2.b)
```

rgb_nchotomy

```
|- !rr. ?n n0 n1. rr = rgb n n0 n1
```

rgb_r_fupd

```
|- !f n n0 n1. rgb n n0 n1 with r updated_by f = rgb (f n) n0 n1
```

rgb_updates_eq_literal

```
|- !r n1 n0 n.  
  r with <|r := n1; g := n0; b := n|> = <|r := n1; g := n0; b := n|>
```

Datatype Package - Example IV

- nested record types are not allowed
- however, mutual recursive types can mitigate this restriction

Filesystem Datatype in SML

```
datatype file = Text of string
              | Dir of {owner : string ,
                       files : (string * file) list}
```

Not Supported Nested Record Type Example in HOL

```
Datatype 'file = Text string
              | Dir <| owner : string ;
                   files : (string # file) list |>'
```

Filesystem Datatype - Mutual Recursion in HOL

```
Datatype 'file = Text string
              | Dir directory
;
directory = <| owner : string ;
             files : (string # file) list |>'
```

- there is no support for co-algebraic types
- the Datatype package could be extended to do so
- other systems like Isabelle/HOL provide high-level methods for defining such types

Co-algebraic Type Example in SML — Lazy Lists

```
datatype 'a lazylist = Nil
                  | Cons of ('a * (unit -> 'a lazylist))
```


- Datatype package allows to define many useful datatypes
- however, there are many limitations
 - ▶ some types cannot be defined in HOL, e. g. empty types
 - ▶ some types are not supported, e. g. co-algebraic types
 - ▶ there are bugs (currently e. g. some trouble with certain mutually recursive definitions)
- biggest restrictions in practice (in my opinion and my line of work)
 - ▶ no support for co-algebraic datatypes
 - ▶ no nested record datatypes
- depending on datatype, different sets of useful lemmata are derived
- most important ones are added to **TypeBase**
 - ▶ tools like **Induct_on**, **Cases_on** use them
 - ▶ there is support for pattern matching

Total Functional Language (TFL) package



- TFL package implements support for terminating functional definitions
- Define defines functions from high-level descriptions
- there is support for pattern matching
- look and feel is like function definitions in SML
- based on **well-founded recursion** principle
- Define is the most common way for definitions in HOL

- a relation $R : 'a \rightarrow 'a \rightarrow \text{bool}$ is called **well-founded**, iff there are no infinite descending chains

$\text{wellfounded } R = \sim ?f. !n. R (f (\text{SUC } n)) (f n)$

- Example: $\$< : \text{num} \rightarrow \text{num} \rightarrow \text{bool}$ is well-founded
- if arguments of recursive calls are smaller according to well-founded relation, the recursion terminates
- this is the essence of termination proofs

- a well-founded relation R can be used to define recursive functions
- this recursion principle is called **WFREC** in HOL
- idea of **WFREC**
 - ▶ if arguments get smaller according to R , perform recursive call
 - ▶ otherwise abort and return **ARB**
- **WFREC** always defines a function
- if all recursive calls indeed decrease according to R , the original recursive equations can be derived from the **WFREC** representation
- TFL uses this internally
- however, this is well-hidden from the user

Simple Definitions

```
> val DOUBLE_def = Define 'DOUBLE n = n + n'
val DOUBLE_def =
  |- !n. DOUBLE n = n + n:
  thm

> val MY_LENGTH_def = Define '(MY_LENGTH [] = 0) /\
                               (MY_LENGTH (x::xs) = SUC (MY_LENGTH xs))'
val MY_LENGTH_def =
  |- (MY_LENGTH [] = 0) /\ !x xs. MY_LENGTH (x::xs) = SUC (MY_LENGTH xs):
  thm

> val MY_APPEND_def = Define '(MY_APPEND [] ys = ys) /\
                               (MY_APPEND (x::xs) ys = x :: (MY_APPEND xs ys))'
val MY_APPEND_def =
  |- (!ys. MY_APPEND [] ys = ys) /\
    (!x xs ys. MY_APPEND (x::xs) ys = x::MY_APPEND xs ys):
  thm
```

- **Define** feels like a function definition in HOL
- it can be used to define "terminating" recursive functions
- **Define** is implemented by a large, non-trivial piece of SML code
- it uses many heuristics
- outcome of **Define** sometimes hard to predict
- the input descriptions are only hints
 - ▶ the produced function and the definitional theorem might be different
 - ▶ in simple examples, quantifiers added
 - ▶ pattern compilation takes place
 - ▶ earlier "conjuncts" have precedence

Define - More Examples

```
> val MY_HD_def = Define 'MY_HD (x :: xs) = x'
val MY_HD_def = |- !x xs. MY_HD (x::xs) = x : thm

> val IS_SORTED_def = Define '
  (IS_SORTED (x1 :: x2 :: xs) = ((x1 < x2) /\ (IS_SORTED (x2::xs)))) /\
  (IS_SORTED _ = T)'
val IS_SORTED_def =
  |- (!xs x2 x1. IS_SORTED (x1::x2::xs) <=> x1 < x2 /\ IS_SORTED (x2::xs)) /\
    (IS_SORTED [] <=> T) /\ (!v. IS_SORTED [v] <=> T)

> val EVEN_def = Define '(EVEN 0 = T) /\ (ODD 0 = F) /\
  (EVEN (SUC n) = ODD n) /\ (ODD (SUC n) = EVEN n)'
val EVEN_def =
  |- (EVEN 0 <=> T) /\ (ODD 0 <=> F) /\ (!n. EVEN (SUC n) <=> ODD n) /\
    (!n. ODD (SUC n) <=> EVEN n) : thm

> val ZIP_def = Define '(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\
  (ZIP _ _ = [])'
val ZIP_def =
  |- (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys) /\
    (!v1. ZIP [] v1 = []) /\ (!v4 v3. ZIP (v3::v4) [] = []) : thm
```

- **Define** introduces (if needed) the function using **WFREC**
- intended definition derived as a theorem
- the theorems are stored in current theory
- usually, one never needs to look at it

Examples

```
val IS_SORTED_primitive_def =  
|- IS_SORTED =  
  WFREC (@R. WF R /\ !x1 xs x2. R (x2::xs) (x1::x2::xs))  
    (\IS_SORTED a.  
      case a of  
        [] => I T  
      | [x1] => I T  
      | x1::x2::xs => I (x1 < x2 /\ IS_SORTED (x2::xs)))  
  
|- !R M. WF R ==> !x. WFREC R M x = M (RESTRICT (WFREC R M) R x) x  
|- !f R x. RESTRICT f R x = (\y. if R y x then f y else ARB)
```


- `Define` automatically defines induction theorems
- these theorems are stored in current theory with suffix `ind`
- use `DB.fetch "-" "something_ind"` to retrieve them
- these induction theorems are useful to reason about corresponding recursive functions

Example

```
val IS_SORTED_ind = |- !P.  
  ((!x1 x2 xs. P (x2::xs) ==> P (x1::x2::xs)) /\  
   P [] /\  
   (!v. P [v])) ==>  
  !v. P v
```

- **Define** might fail for various reasons to define a function
 - ▶ such a function cannot be defined in HOL
 - ▶ such a function can be defined, but not via the methods used by TFL
 - ▶ TFL can define such a function, but its heuristics are too weak and user guidance is required
 - ▶ there is a bug :-)
- **termination** is an important concept for **Define**
- it is easy to misunderstand termination in the context of HOL
- we need to understand what is meant by termination

- in SML it is natural to talk about termination of functions
- in the HOL logic there is no concept of execution
- thus, there is no concept of termination in HOL

3 characterisations of a function $f : \text{num} \rightarrow \text{num}$

- ▶ $\vdash !n. f\ n = 0$
- ▶ $\vdash (f\ 0 = 0) \wedge !n. (f\ (\text{SUC}\ n) = f\ n)$
- ▶ $\vdash (f\ 0 = 0) \wedge !n. (f\ n = f\ (\text{SUC}\ n))$

Is f terminating? All 3 theorems are equivalent.

- it is useful to think in terms of termination
- the TFL package implements heuristics to define functions that would terminate in SML
- the TFL package uses well-founded recursion
- the required well-founded relation corresponds to a termination proof
- therefore, it is very natural to think of **Define** searching a termination proof
- important: this is the idea behind this function definition package, not a property of HOL

HOL is not limited to "terminating" functions

- one can define "non-terminating" functions in HOL
- however, one cannot do so (easily) with **Define**

Definition of WHILE in HOL

```
|- !P g x. WHILE P g x = if P x then WHILE P g (g x) else x
```

Execution Order

There is no "execution order". One can easily define a complicated constant function:

```
(myk : num -> num) (n:num) = (let x = myk (n+1) in 0)
```

Unsound Definitions

A function $f : \text{num} \rightarrow \text{num}$ with the following property cannot be defined in HOL unless HOL has an inconsistency:

```
!n. f n = ((f n) + 1)
```

Such a function would allow to prove $0 = 1$.

- TFL uses various heuristics to find a well-founded relation
- however, these heuristics may not be strong enough
- in such cases the user can provide a well-founded relation manually
- the most common well-founded relations are **measures**
- measures map values to natural numbers and use the less relation
`|- !(f:'a -> num) x y. measure f x y <=> (f x < f y)`
- all measures are well-founded: `|- !f. WF (measure f)`
- moreover, existing well-founded relations can be combined
 - ▶ lexicographic order **LEX**
 - ▶ list lexicographic order **LLEX**
 - ▶ ...

- if `Define` fails to find a termination proof, `Hol_defn` can be used
- `Hol_defn` defers termination proofs
- it derives termination conditions and sets up the function definitions
- all results are packaged as a value of type `defn`
- after calling `Hol_defn` the defined function(s) can be used
- however, the intended definition theorem has not been derived yet
- to derive it, one needs to
 - ▶ provide a well-founded relation
 - ▶ show that termination conditions respect that relation
- `Defn.tprove` and `Defn.tgoal` are intended for this
- proofs usually start by providing relation via tactic `WF_REL_TAC`

Manual Termination Proof Example 1



```
> val qsort_defn = Hol_defn "qsort" '  
  (qsort ord [] = []) /\  
  (qsort ord (x::rst) =  
    (qsort ord (FILTER ($~ o ord x) rst)) ++  
    [x] ++  
    (qsort ord (FILTER (ord x) rst)))'
```

```
val qsort_defn = HOL function definition (recursive)
```

Equation(s) :

```
[...] |- qsort ord [] = []  
[...] |- qsort ord (x::rst) =  
  qsort ord (FILTER ($~ o ord x) rst) ++ [x] ++  
  qsort ord (FILTER (ord x) rst)
```

Induction : ...

Termination conditions :

0. !rst x ord. R (ord, FILTER (ord x) rst) (ord, x::rst)
1. !rst x ord. R (ord, FILTER (\$~ o ord x) rst) (ord, x::rst)
2. WF R

Manual Termination Proof Example 2



```
> Defn.tgoal qsort_defn
```

Initial goal:

?R.

```
WF R /\
(!rst x ord. R (ord,FILTER (ord x)      rst) (ord,x::rst)) /\
(!rst x ord. R (ord,FILTER ($~ o ord x) rst) (ord,x::rst))
```

Manual Termination Proof Example 2



```
> Defn.tgoal qsort_defn
```

Initial goal:

```
?R.
```

```
WF R /\
(!rst x ord. R (ord,FILTER (ord x)      rst) (ord,x::rst)) /\
(!rst x ord. R (ord,FILTER ($~ o ord x) rst) (ord,x::rst))
```

```
> e (WF_REL_TAC 'measure (\(_, l). LENGTH l)')
```

```
1 subgoal :
```

```
(!rst x ord. LENGTH (FILTER (ord x) rst) < LENGTH (x::rst)) /\
(!rst x ord. LENGTH (FILTER (\x'. ~ord x x') rst) < LENGTH (x::rst))
```

```
> ...
```

Manual Termination Proof Example 3



```
> val (qsort_def, qsort_ind) =  
  Defn.tprove (qsort_defn,  
    WF_REL_TAC 'measure (\(., 1). LENGTH 1)' >> ...)
```

```
val qsort_def =  
|- (qsort ord [] = []) /\  
  (qsort ord (x::rst) =  
    qsort ord (FILTER ($~ o ord x) rst) ++ [x] ++  
    qsort ord (FILTER (ord x) rst))
```

```
val qsort_ind =  
|- !P. (!ord. P ord []) /\  
  (!ord x rst.  
    P ord (FILTER (ord x) rst) /\  
    P ord (FILTER ($~ o ord x) rst) ==>  
    P ord (x::rst)) ==>  
  !v v1. P v v1
```

Part XI

Good Definitions



- using *good* definitions is very important
 - ▶ good definitions are vital for **clarity**
 - ▶ **proofs** depend a lot on the form of definitions
- unluckily, it is hard to state what a good definition is
- even harder to come up with good definitions
- let's look at it a bit closer anyhow

- HOL guarantees that theorems do indeed hold
- However, does the theorem mean what you think it does?
- you can separate your development in
 - ▶ main theorems you care for
 - ▶ auxiliary stuff used to derive your main theorems
- it is essential to understand your main theorems

Guarded by HOL

- proofs checked
- internal, technical definitions
- technical lemmata
- proof tools

Manual review needed for

- meaning of main theorems
- meaning of definitions used by main theorems
- meaning of types used by main theorems

- it is essential to understand your main theorems
 - ▶ you need to understand all the definitions directly used
 - ▶ you need to understand the indirectly used ones as well
 - ▶ you need to convince others that you express the intended statement
 - ▶ therefore, it is vital to **use very simple, clear definitions**
- defining concepts is often the main development task
- checking resulting model against real artefact is vital
 - ▶ testing via e. g. **EVAL**
 - ▶ formal sanity
 - ▶ conformance testing
- wrong models are main source of error when using HOL
- proofs, auxiliary lemmata and auxiliary definitions
 - ▶ can be as technical and complicated as you like
 - ▶ correctness is guaranteed by HOL
 - ▶ reviewers don't need to care

- good definitions can shorten proofs significantly
- they improve maintainability
- they can improve automation drastically
- unluckily for proofs definitions often need to be technical
- this contradicts clarity aims

How to come up with good definitions



- unluckily, it is hard to state what a good definition is
- it is even harder to come up with them
 - ▶ there are often many competing interests
 - ▶ a lot of experience and detailed tool knowledge is needed
 - ▶ much depends on personal style and taste
- general advice: use more than one definition
 - ▶ in HOL you can derive equivalent definitions as theorems
 - ▶ define a concept as clearly and easily as possible
 - ▶ derive equivalent definitions for various purposes
 - ★ one very close to your favourite textbook
 - ★ one nice for certain types of proofs
 - ★ another one good for evaluation
 - ★ ...
- lessons from functional programming apply

Objectives

- clarity (readability, maintainability)
- performance (runtime speed, memory usage, ...)

General Advice

- use the powerful type-system
- use many small function definitions
- encode invariants in types and function signatures

Good Definitions – no number encodings



- many programmers familiar with C encode everything as a number
- enumeration types are very cheap in SML and HOL
- use them instead

Example Enumeration Types

In C the result of an order comparison is an integer with 3 equivalence classes: 0, negative and positive integers. In SML and HOL, it is better to use a variant type.

```
val _ = Datatype 'ordering = LESS | EQUAL | GREATER';
```

```
val compare_def = Define '  
  (compare LESS    lt eq gt = lt)  
  /\ (compare EQUAL lt eq gt = eq)  
  /\ (compare GREATER lt eq gt = gt) ';
```

```
val list_compare_def = Define '  
  (list_compare cmp [] [] = EQUAL) /\ (list_compare cmp [] l2 = LESS)  
  /\ (list_compare cmp l1 [] = GREATER)  
  /\ (list_compare cmp (x::l1) (y::l2) = compare (cmp (x:'a) y)  
    (* x<y *) LESS  
    (* x=y *) (list_compare cmp l1 l2)  
    (* x>y *) GREATER) ';
```

- the type-checker is your friend
 - ▶ it helps you find errors
 - ▶ code becomes more robust
 - ▶ using good types is a great way of writing self-documenting code
- therefore, use many types
- even use types isomorphic to existing ones

Virtual and Physical Memory Addresses

Virtual and physical addresses might in a development both be numbers. It is still nice to use separate types to avoid mixing them up.

```
val _ = Datatype 'vaddr = VAddr num';  
val _ = Datatype 'paddr = PAddr num';
```

```
val virt_to_phys_addr_def = Define '  
  virt_to_phys_addr (VAddr a) = PAddr( translation of a )';
```

- often people use tuples where records would be more appropriate
- using large tuples quickly becomes awkward
 - ▶ it is easy to mix up order of tuple entries
 - ★ often types coincide, so type-checker does not help
 - ▶ no good error messages for tuples
 - ★ hard to decipher type mismatch messages for long product types
 - ★ hard to figure out which entry is missing at which position
 - ★ non-local error messages
 - ★ variable in last entry can hide missing entries
- records sometimes require slightly more proof effort
- however, records have many benefits

- using records
 - ▶ introduces field names
 - ▶ provides automatically defined accessor and update functions
 - ▶ leads to better type-checking error messages
- records improve readability
 - ▶ accessors and update functions lead to shorter code
 - ▶ field names act as documentation
- records improve maintainability
 - ▶ improved error messages
 - ▶ much easier to add extra fields

- try to encode as many invariants as possible in the types
- this allows the type-checker to ensure them for you
- you don't have to check them manually any more
- your code becomes more robust and clearer

Network Connections (Example by Yaron Minsky from Jane Street)

Consider the following datatype for network connections. It has many implicit invariants.

```
datatype connection_state = Connected | Disconnected | Connecting;
```

```
type connection_info = {  
  state           : connection_state,  
  server          : inet_address,  
  last_ping_time  : time option,  
  last_ping_id    : int option,  
  session_id      : string option,  
  when_initiated  : time option,  
  when_disconnected : time option  
}
```


Network Connections (Example by Yaron Minsky from Jane Street) II

The following definition of `connection_info` makes the invariants explicit:

```
type connected      = { last_ping          : (time * int) option,  
                        session_id         : string };  
type disconnected    = { when_disconnected : time };  
type connecting     = { when_initiated     : time };  
  
datatype connection_state =  
  Connected of connected  
| Disconnected of disconnected  
| Connecting of connecting;  
  
type connection_info = {  
  state : connection_state,  
  server : inet_address  
}
```

Objectives

- clarity (readability)
- good for proofs
- performance (good for automation, easily evaluable, ...)

General Advice

- same advice as for functional programming applies
- use even smaller definitions
 - ▶ introduce auxiliary definitions for important function parts
 - ▶ use extra definitions for important constants
 - ▶ ...
- tiny definitions
 - ▶ allow keeping proof state small by unfolding only needed ones
 - ▶ allow many small lemmata
 - ▶ improve maintainability

Technical Issues

- write definitions such that they work well with HOL's tools
- this requires you to know HOL well
- a lot of experience is required
- general advice
 - ▶ avoid explicit case-expressions
 - ▶ prefer curried functions

Example

```
val ZIP_GOOD_def = Define '(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\  
                           (ZIP _ _ = [])'
```

```
val ZIP_BAD1_def = Define 'ZIP xs ys = case (xs, ys) of  
                               (x::xs, y::ys) => (x,y)::(ZIP xs ys)  
                               | (_, _) => []'
```

```
val ZIP_BAD2_def = Define '(ZIP (x::xs, y::ys) = (x,y)::(ZIP (xs, ys))) /\  
                           (ZIP _ = [])'
```

Multiple Equivalent Definitions

- satisfy competing requirements by having multiple equivalent definitions
- derive them as theorems
- initial definition should be as clear as possible
 - ▶ clarity allows simpler reviews
 - ▶ simplicity reduces the likelihood of errors

Example - ALL_DISTINCT

```
|- (ALL_DISTINCT [] <=> T) /\
    (!h t. ALL_DISTINCT (h::t) <=> ~MEM h t /\ ALL_DISTINCT t)

|- !l. ALL_DISTINCT l <=>
    (!x. MEM x l ==> (FILTER ($= x) l = [x]))

|- !ls. ALL_DISTINCT ls <=> (CARD (set ls) = LENGTH ls):
```

Formal Sanity

- to ensure correctness test your definitions via e. g. **EVAL**
- in HOL testing means symbolic evaluation, i. e. proving lemmata
- **formally proving sanity check lemmata** is very beneficial
 - ▶ they should express core properties of your definition
 - ▶ thereby they check your intuition against your actual definitions
 - ▶ these lemmata are often useful for following proofs
 - ▶ using them improves robustness and maintainability of your development
- I highly recommend using formal sanity checks

```
> val ALL_DISTINCT = Define '  
  (ALL_DISTINCT [] = T) /\  
  (ALL_DISTINCT (h::t) = ~MEM h t /\ ALL_DISTINCT t)';
```

Example Sanity Check Lemmata

```
|- ALL_DISTINCT []  
|- !x xs. ALL_DISTINCT (x::xs) <=> ~MEM x xs /\ ALL_DISTINCT xs  
|- !x. ALL_DISTINCT [x]  
|- !x xs. ~(ALL_DISTINCT (x::x::xs))  
|- !l. ALL_DISTINCT (REVERSE l) <=> ALL_DISTINCT l  
|- !x l. ALL_DISTINCT (SNOC x l) <=> ~MEM x l /\ ALL_DISTINCT l  
|- !l1 l2. ALL_DISTINCT (l1 ++ l2) <=>  
  ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2
```

Formal Sanity Example II 1



```
> val ZIP_def = Define ‘  
  (ZIP [] ys = []) /\ (ZIP xs [] = []) /\  
  (ZIP (x::xs) (y::ys) = (x, y)::(ZIP xs ys))‘
```

```
val ZIP_def =  
  |- (!ys. ZIP [] ys = []) /\ (!v3 v2. ZIP (v2::v3) [] = []) /\  
    (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)
```

- above definition of **ZIP** looks straightforward
- small changes cause heuristics to produce different theorems
- use formal sanity lemmata to compensate

```
> val ZIP_def = Define ‘  
  (ZIP xs [] = []) /\ (ZIP [] ys = []) /\  
  (ZIP (x::xs) (y::ys) = (x, y)::(ZIP xs ys))‘
```

```
val ZIP_def =  
  |- (!xs. ZIP xs [] = []) /\ (!v3 v2. ZIP [] (v2::v3) = []) /\  
    (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys0)
```

```
val ZIP_def =  
  |- (!ys. ZIP [] ys = []) /\ (!v3 v2. ZIP (v2::v3) [] = []) /\  
    (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)
```

Example Formal Sanity Lemmata

```
|- (!xs. ZIP xs [] = []) /\ (!ys. ZIP [] ys = []) /\  
  (!y ys x xs. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)  
|- !xs ys. LENGTH (ZIP xs ys) = MIN (LENGTH xs) (LENGTH ys)  
|- !x y xs ys. MEM (x, y) (ZIP xs ys) ==> (MEM x xs /\ MEM y ys)  
|- !xs1 xs2 ys1 ys2. LENGTH xs1 = LENGTH ys1 ==>  
  (ZIP (xs1++xs2) (ys1++ys2) = (ZIP xs1 ys1 ++ ZIP xs2 ys2))  
...
```

- in your proofs use sanity lemmata, not original definition
- this makes your development robust against
 - ▶ small changes to the definition required later
 - ▶ changes to [Define](#) and its heuristics
 - ▶ bugs in function definition package

Part XII

Deep and Shallow Embeddings



- often one models some kind of formal language
- important design decision: use **deep** or **shallow** embedding
- in a nutshell:
 - ▶ shallow embeddings just model semantics
 - ▶ deep embeddings model syntax as well
- a shallow embedding directly uses the HOL logic
- a deep embedding
 - ▶ defines a datatype for the syntax of the language
 - ▶ provides a function to map this syntax to a semantic

Example: Embedding of Propositional Logic I



- propositional logic is a subset of HOL
- a shallow embedding is therefore trivial

```
val sh_true_def      = Define 'sh_true = T';
val sh_var_def       = Define 'sh_var (v:bool) = v';
val sh_not_def       = Define 'sh_not b = ~b';
val sh_and_def       = Define 'sh_and b1 b2 = (b1 /\ b2)';
val sh_or_def        = Define 'sh_or b1 b2 = (b1 \/ b2)';
val sh_implies_def   = Define 'sh_implies b1 b2 = (b1 ==> b2)';
```

Example: Embedding of Propositional Logic II



- we can also define a datatype for propositional logic
- this leads to a deep embedding

```
val _ = Datatype 'bvar = BVar num'  
val _ = Datatype 'prop = d_true | d_var bvar | d_not prop  
                | d_and prop prop | d_or prop prop  
                | d_implies prop prop';
```

```
val _ = Datatype 'var_assignment = BAssign (bvar -> bool)'  
val VAR_VALUE_def = Define 'VAR_VALUE (BAssign a) v = (a v)'
```

```
val PROP_SEM_def = Define '  
  (PROP_SEM a d_true = T) /\  
  (PROP_SEM a (d_var v) = VAR_VALUE a v) /\  
  (PROP_SEM a (d_not p) = ~(PROP_SEM a p)) /\  
  (PROP_SEM a (d_and p1 p2) = (PROP_SEM a p1 /\ PROP_SEM a p2)) /\  
  (PROP_SEM a (d_or p1 p2) = (PROP_SEM a p1 \/ PROP_SEM a p2)) /\  
  (PROP_SEM a (d_implies p1 p2) = (PROP_SEM a p1 ==> PROP_SEM a p2))'
```

Shallow

- quick and easy to build
- extensions are simple

Deep

- can reason about syntax
- allows verified implementations
- sometimes tricky to define
 - ▶ e. g. bound variables

Important Questions for Deciding

- Do I need to reason about syntax?
- Do I have hard to define syntax like bound variables?
- How much time do I have?

Example: Embedding of Propositional Logic III



- with deep embedding one can easily formalise syntactic properties like
 - ▶ Which variables does a propositional formula contain?
 - ▶ Is a formula in negation-normal-form (NNF)?
- with shallow embeddings
 - ▶ syntactic concepts can't be defined in HOL
 - ▶ however, they can be defined in SML
 - ▶ no proofs about them possible

```
val _ = Define '  
  (IS_NNF (d_not d_true) = T) /\ (IS_NNF (d_not (d_var v)) = T) /\  
  (IS_NNF (d_not _) = F) /\  
  
  (IS_NNF d_true = T) /\ (IS_NNF (d_var v) = T) /\  
  (IS_NNF (d_and p1 p2) = (IS_NNF p1 /\ IS_NNF p2)) /\  
  (IS_NNF (d_or p1 p2) = (IS_NNF p1 /\ IS_NNF p2)) /\  
  (IS_NNF (d_implies p1 p2) = (IS_NNF p1 /\ IS_NNF p2))'
```

Verified Programs

- are formalised in HOL
- their properties have been proven once and for all
- all runs have proven properties
- are usually less sophisticated, since they need verification
- is what one wants ideally
- often require deep embedding

Verifying Programs

- are written in meta-language
- they produce a separate proof for each run
- only certain that current run has properties
- allow more flexibility, e. g. fancy heuristics
- good pragmatic solution
- shallow embedding fine

Summary Deep vs. Shallow Embeddings



- deep embeddings require more work
- they however allow reasoning about syntax
 - ▶ induction and case-splits possible
 - ▶ a semantic subset can be carved out syntactically
- syntax sometimes hard to define for deep embeddings
- combinations of deep and shallow embeddings common
 - ▶ certain parts are deeply embedded
 - ▶ others are embedded shallowly

Part XIII

Rewriting



- simplification via rewriting was already a strength of Edinburgh LCF
- it was further improved for Cambridge LCF
- HOL inherited this powerful rewriter
- equational reasoning is still the main workhorse
- there are many different equational reasoning tools in HOL
 - ▶ `Rewrite` library
inherited from Cambridge LCF
you have seen it in the form of `REWRITE_TAC`
 - ▶ `computeLib` — fast evaluation
built for speed, optimised for ground terms
seen in the form of `EVAL`
 - ▶ `simpLib` — Simplification
sophisticated rewrite engine, HOL's main workhorse
not discussed in this lecture, yet
 - ▶ ...

- we have seen primitive inference rules for equality before

$$\frac{\begin{array}{c} \Gamma \vdash s = t \\ \Delta \vdash u = v \\ \text{types fit} \end{array}}{\Gamma \cup \Delta \vdash s(u) = t(v)} \text{ COMB}$$

$$\frac{\begin{array}{c} \Gamma \vdash s = t \\ x \text{ not free in } \Gamma \end{array}}{\Gamma \vdash \lambda x. s = \lambda x. t} \text{ ABS}$$

$$\frac{\begin{array}{c} \Gamma \vdash s = t \\ \Delta \vdash t = u \end{array}}{\Gamma \cup \Delta \vdash s = u} \text{ TRANS}$$

$$\frac{}{\vdash t = t} \text{ REFL}$$

- these rules allow us to replace any subterm with an equal one
- this is the core of rewriting

- in HOL, equality reasoning is implemented by **conversions**
- a conversion is a SML function of type `term -> thm`
- given a term `t`, a conversion
 - ▶ produces a theorem of the form `|- t = t'`
 - ▶ raises an **UNCHANGED** exception or
 - ▶ fails, i. e. raises an **HOL_ERR** exception

Example

```
> BETA_CONV ``(\x. SUC x) y``  
val it = |- (\x. SUC x) y = SUC y
```

```
> BETA_CONV ``SUC y``  
Exception-HOL_ERR ... raised
```

```
> REPEATC BETA_CONV ``SUC y``  
Exception- UNCHANGED raised
```

- similar to tactics and tacticals there are **conversionals** for conversions
- conversionals allow building conversions from simpler ones
- there are many of them
 - ▶ THENC
 - ▶ ORELSEC
 - ▶ REPEATC
 - ▶ TRY_CONV
 - ▶ RAND_CONV
 - ▶ RATOR_CONV
 - ▶ ABS_CONV
 - ▶ ...

- for rewriting depth-conversionals are important
- a depth-conversional applies a conversion to all subterms
- there are many different ones
 - ▶ `ONCE_DEPTH_CONV c` — top down, applies `c` once at highest possible positions in distinct subterms
 - ▶ `TOP_SWEEP_CONV c` — top down, like `ONCE_DEPTH_CONV`, but continues processing rewritten terms
 - ▶ `TOP_DEPTH_CONV c` — top down, like `TOP_SWEEP_CONV`, but try top-level again after change
 - ▶ `DEPTH_CONV c` — bottom up, recurse over subterms, then apply `c` repeatedly at top-level
 - ▶ `REDEPTH_CONV c` — bottom up, like `DEPTH_CONV`, but revisits subterms

- it remains to rewrite terms at top-level
- this is achieved by REWR_CONV
- given a term t and a theorem $\vdash t_1 = t_2$, REWR_CONV t thm
 - ▶ searches an instantiation of term and type variables such that t_1 becomes α -equivalent to t
 - ▶ fails, if no instantiation is found
 - ▶ otherwise, instantiate the theorem and get $\vdash t_1' = t_2'$
 - ▶ return theorem $\vdash t = t_2'$

Example

term LENGTH [1;2;3], theorem $\vdash \text{LENGTH } ((x:'a)::xs) = \text{SUC } (\text{LENGTH } xs)$
 found type instantiation: $[('a' \rightarrow 'num')]$
 found term instantiation: $[('x:num' \rightarrow '1'; 'xs' \rightarrow '[2;3]')]$
 returned theorem: $\vdash \text{LENGTH } [1;2;3] = \text{SUC } (\text{LENGTH } [2;3])$

- the tricky part is finding the instantiation
- this problem is called the (term) **matching** problem

- given term `t_org` and a term `t_goal` try to find
 - ▶ type substitution `ty_s`
 - ▶ term substitution `tm_s`
- such that $\text{subst } tm_s (\text{inst } ty_s \ t_org) \stackrel{\alpha}{=} t_goal$
- this can be easily implemented by a recursive search

<code>t_org</code>	<code>t_goal</code>	action
<code>t1_org t2_org</code>	<code>t1_goal t2_goal</code>	recurse
<code>t1_org t2_org</code>	otherwise	fail
<code>\x. t_org x</code>	<code>\y. t_goal y</code>	match types of x, y and recurse
<code>\x. t_org x</code>	otherwise	fail
<code>const</code>	<code>same const</code>	match types
<code>const</code>	otherwise	fail
<code>var</code>	anything	try to bind var, take care of existing bindings

t_org

```
LENGTH ((x:'a)::xs)
[]:'a list
0
b /\ T
b /\ b
b /\ b
!x:num. P x /\ Q x
!x:num. P x /\ Q x
!x:num. P x /\ Q x
```

t_goal

```
LENGTH [1;2;3]
[]:'b list
0
(P (x:'a) ==> Q) /\ T
P x /\ P x
P x /\ P y
!y:num. P' y /\ Q' y
!y. (2 = y) /\ Q' y
!y. (y = 2) /\ Q' y
```

subst

```
'a → num, x → 1, xs → [2;3]
'a → 'b
empty substitution
b → P x ==> Q
b → P x
fail
P → P', Q → Q'
P → ($= 2), Q → Q'
fail
```

- it is often very annoying that the last match in the list above fails
- it prevents us for example rewriting $!y. (2 = y) /\ Q y$ to $(!y. (2=y)) /\ (!y. Q y)$
- Can we do better? Yes, with higher order (term) matching.

- term matching searches for substitutions such that t_org becomes α -equivalent to t_goal
- **higher order term matching** searches for substitutions such that t_org becomes t_subst such that the $\beta\eta$ -normalform of t_subst is α -equivalent equivalent to $\beta\eta$ -normalform of t_goal , i. e.

higher order term matching is aware of the semantics of λ

β -reduction $(\lambda x. f) y = f[y/x]$

η -conversion $(\lambda x. f x) = f$ where x is not free in f

- the HOL implementation expects t_org to be a **higher-order pattern**
 - ▶ t_org is β -reduced
 - ▶ if X is a variable that should be instantiated, then all arguments should be distinct variables
- for other forms of t_org , HOL's implementation might fail
- higher order matching is used by [HO_REWR_CONV](#)

Examples Higher Order Term Matching



t_org

$!x:\text{num}. P\ x \wedge Q\ x$
 $!x. P\ x \wedge Q\ x$
 $!x. P\ x \wedge Q$
 $!x. P\ (x, x)$
 $!x. P\ (x, x)$

t_goal

$!y. (y = 2) \wedge Q'\ y$
 $!x. P\ x \wedge Q\ x \wedge Z\ x$
 $!x. P\ x \wedge Q\ x$
 $!x. Q\ x$
 $!x. \text{FST}\ (x, x) = \text{SND}\ (x, x)$

substs

$P \rightarrow (\lambda y. y = 2), Q \rightarrow Q'$
 $Q \rightarrow \lambda x. Q\ x \wedge Z\ x$
fails
fails
 $P \rightarrow \lambda xx. \text{FST}\ xx = \text{SND}\ xx$

**Don't worry, it might look complicated, but
in practice it is easy to get a feeling for higher order matching.**

- the rewrite library combines `REWR_CONV` with depth conversions
- there are many different conversions, rules and tactics
- at their core, they all work very similarly
 - ▶ given a list of theorems, a set of rewrite theorems is derived
 - ★ split conjunctions
 - ★ remove outermost universal quantification
 - ★ introduce equations by adding `= T` (or `= F`) if needed
 - ▶ `REWR_CONV` is applied to all the resulting rewrite theorems
 - ▶ a depth-conversion is used with resulting conversion
- for performance reasons an efficient indexing structure is used
- by default implicit rewrites are added

- REWRITE_CONV
- REWRITE_RULE
- REWRITE_TAC
- ASM_REWRITE_TAC
- ONCE_REWRITE_TAC
- PURE_REWRITE_TAC
- PURE_ONCE_REWRITE_TAC
- . . .

- similar to `Rewrite` lib, but uses higher order matching
- internally uses `HO_REWR_CONV`
- similar conversions, rules and tactics as `Rewrite` lib
 - ▶ `Ho_Rewrite.REWRITE_CONV`
 - ▶ `Ho_Rewrite.REWRITE_RULE`
 - ▶ `Ho_Rewrite.REWRITE_TAC`
 - ▶ `Ho_Rewrite.ASM_REWRITE_TAC`
 - ▶ `Ho_Rewrite.ONCE_REWRITE_TAC`
 - ▶ `Ho_Rewrite.PURE_REWRITE_TAC`
 - ▶ `Ho_Rewrite.PURE_ONCE_REWRITE_TAC`
 - ▶ ...

```
> REWRITE_CONV [LENGTH] ‘‘LENGTH [1;2]’’  
val it = |- LENGTH [1; 2] = SUC (SUC 0)
```

```
> ONCE_REWRITE_CONV [LENGTH] ‘‘LENGTH [1;2]’’  
val it = |- LENGTH [1; 2] = SUC (LENGTH [2])
```

```
> REWRITE_CONV [] ‘‘A /\ A /\ ~A’’  
Exception- UNCHANGED raised
```

```
> PURE_REWRITE_CONV [NOT_AND] ‘‘A /\ A /\ ~A’’  
val it = |- A /\ A /\ ~A <=> A /\ F
```

```
> REWRITE_CONV [NOT_AND] ‘‘A /\ A /\ ~A’’  
val it = |- A /\ A /\ ~A <=> F
```

```
> REWRITE_CONV [FORALL_AND_THM] ‘‘!x. P x /\ Q x /\ R x’’  
Exception- UNCHANGED raised
```

```
> Ho_Rewrite.REWRITE_CONV [FORALL_AND_THM] ‘‘!x. P x /\ Q x /\ R x’’  
val it = |- !x. P x /\ Q x /\ R x <=> (!x. P x) /\ (!x. Q x) /\ (!x. R x)
```

- the `Rewrite` and `Ho_Rewrite` library provide powerful infrastructure for term rewriting
- thanks to clever implementations they are reasonably efficient
- basics are easily explained
- however, efficient usage needs some experience

- to use rewriting efficiently, one needs to understand about term rewriting systems
- this is a large topic
- unluckily, it cannot be covered here in detail for time constraints
- however, in practise you quickly get a feeling
- important points in practise
 - ▶ ensure termination of your rewrites
 - ▶ make sure they work nicely together

Theory

- choose well-founded order \prec
- for each rewrite theorem $\vdash t_1 = t_2$ ensure $t_2 \prec t_1$

Practice

- informally define for yourself what **simpler** means
- ensure each rewrite makes terms simpler
- good heuristics
 - ▶ subterms are simpler than whole term
 - ▶ use an order on functions

- a proper subterm is always simpler
 - ▶ !l. APPEND [] l = l
 - ▶ !n. n + 0 = n
 - ▶ !l. REVERSE (REVERSE l) = l
 - ▶ !t1 t2. if T then t1 else t2 <=> t1
 - ▶ !n. n * 0 = 0
- the right hand side should not use extra vars, throwing parts away is usually simpler
 - ▶ !x xs. (SNOC x xs = []) = F
 - ▶ !x xs. LENGTH (x::xs) = SUC (LENGTH xs)
 - ▶ !n x xs. DROP (SUC n) (x::xs) = DROP n xs

- it is useful to consider some functions simple and other complicated
- replace complicated ones with simple ones
- never do it in the opposite direction
- clear examples
 - ▶ `| - !m n. MEM m (COUNT_LIST n) <=> (m < n)`
 - ▶ `| - !ls n. (DROP n ls = []) <=> (n >= LENGTH ls)`
- unclear example
 - ▶ `| - !L. REVERSE L = REV L []`

- some equations can be used in both directions
- one should decide on one direction
- this implicitly defines a **normalform** one wants terms to be in
- examples
 - ▶ $\vdash !f\ l.\ \text{MAP}\ f\ (\text{REVERSE}\ l) = \text{REVERSE}\ (\text{MAP}\ f\ l)$
 - ▶ $\vdash !l1\ l2\ l3.\ l1\ ++\ (l2\ ++\ l3) = l1\ ++\ l2\ ++\ l3$

- some equations immediately lead to non-termination, e. g.
 - ▶ $\vdash !m\ n. m + n = n + m$
 - ▶ $\vdash !m. m = m + 0$
- slightly more subtle are rules like
 - ▶ $\vdash !n. \text{fact } n = \text{if } (n = 0) \text{ then } 1 \text{ else } n * \text{fact}(n-1)$
- often combination of multiple rules leads to non-termination
this is especially problematic when adding to predefined sets of rewrites
 - ▶ $\vdash !m\ n\ p. m + (n + p) = (m + n) + p$ and
 $\vdash !m\ n\ p. (m + n) + p = m + (n + p)$

Rewrites working together

- rewrite rules should not compete with each other
- if a term ta can be rewritten to $ta1$ and $ta2$ applying different rewrite rules, then $ta1$ and $ta2$ it should be possible to further rewrite them both to a common tb
- this can often be achieved by adding extra rewrite rules

Example

Assume we have the rewrite rules $\mid-$ $\text{DOUBLE } n = n + n$ and $\mid-$ $\text{EVEN } (\text{DOUBLE } n) = T$.

With these the term $\text{EVEN } (\text{DOUBLE } 2)$ can be rewritten to

- T or
- $\text{EVEN } (2 + 2)$.

To avoid a hard to predict result, $\text{EVEN } (2+2)$ should be rewritten to T . Adding an extra rewrite rule $\mid-$ $\text{EVEN } (n + n) = T$ achieves this.

Rewrites working together II



- to design rewrite systems that work well, normalforms are vital
- a term is in **normalform**, if it cannot be rewritten any further
- one should have a clear idea what the normalform of common terms looks like
- all rules should work together to establish this normalform
- the right-hand-side of each rule should be in normalform
- the left-hand-side should not be simplifiable by any other rule
- the order in which rules are applied should not influence the final result

- `computeLib` is the library behind `EVAL`
- it is a rewriting library designed for evaluating ground terms (i. e. terms without variables) efficiently
- it uses a call-by-value strategy similar to SML's
- it uses first order term matching
- it performs β reduction in addition to rewrites

- `computeLib` uses `compsets` to store its rewrites
- a `compset` stores
 - ▶ rewrite rules
 - ▶ extra conversions
- the extra conversions are guarded by a term pattern for efficiency
- users can define their own compsets
- however, `computeLib` maintains one special compset called `the_compset`
- `the_compset` is used by `EVAL`

- EVAL uses `the_compset`
- tools like the `Datatype` or `TFL` libraries automatically extend `the_compset`
- this way, EVAL knows about (nearly) all types and functions
- one can extended `the_compset` manually as well
- rewrites exported by `Define` are good for ground terms but may lead to non-termination for non-ground terms
- `zDefine` prevents `TFL` from automatically extending `the_compset`

- **simplib** is a sophisticated rewrite engine
- it is HOL's main workhorse
- it provides
 - ▶ higher order rewriting
 - ▶ usage of context information
 - ▶ conditional rewriting
 - ▶ arbitrary conversions
 - ▶ support for decision procedures
 - ▶ simple heuristics to avoid non-termination
 - ▶ fancier preprocessing of rewrite theorems
 - ▶ ...
- it is very powerful, but compared to **Rewrite** lib sometimes slow

- `simpLib` uses **`simpsets`**
- `simpsets` are special datatypes storing
 - ▶ rewrite rules
 - ▶ conversions
 - ▶ decision procedures
 - ▶ congruence rules
 - ▶ ...
- in addition there are `simpset-fragments`
- `simpset-fragments` contain similar information as `simpsets`
- fragments can be added to and removed from `simpsets`
- common usage: basic `simpset` combined with one or more `simpset-fragments`, e. g.
 - ▶ `list_ss ++ pairSimps.gen_beta_ss`
 - ▶ `std_ss ++ QI_ss`
 - ▶ ...

- a call to the simplifier takes as arguments
 - ▶ a simpset
 - ▶ a list of rewrite theorems
- common high-level entry points are
 - ▶ `SIMP_CONV ss thmL` — conversion
 - ▶ `SIMP_RULE ss thmL` — rule
 - ▶ `SIMP_TAC ss thmL` — tactic without considering assumptions
 - ▶ `ASM_SIMP_TAC ss thmL` — tactic using assumptions to simplify goal
 - ▶ `FULL_SIMP_TAC ss thmL` — tactic simplifying assumptions with each other and goal with assumptions
 - ▶ `REV_FULL_SIMP_TAC ss thmL` — similar to `FULL_SIMP_TAC` but with reversed order of assumptions
- there are many derived tools not discussed here

```
> SIMP_CONV bool_ss [LENGTH] ‘‘LENGTH [1;2]‘‘  
val it = |- LENGTH [1; 2] = SUC (SUC 0)
```

```
> SIMP_CONV std_ss [LENGTH] ‘‘LENGTH [1;2]‘‘  
val it = |- LENGTH [1; 2] = 2
```

```
> SIMP_CONV list_ss [] ‘‘LENGTH [1;2]‘‘  
val it = |- LENGTH [1; 2] = 2
```

Current GoalStack

P (SUC (SUC x0)) (SUC (SUC y0))

0. SUC y1 = y2
1. x1 = SUC x0
2. y1 = SUC y0
3. SUC x1 = x2

Action

FULL_SIMP_TAC std_ss []

Resulting GoalStack

P (SUC (SUC x0)) y2

0. SUC (SUC y0) = y2
1. x1 = SUC x0
2. y1 = SUC y0
3. SUC x1 = x2

Current GoalStack

P (SUC (SUC x0)) y2

0. SUC (SUC y0) = y2
1. x1 = SUC x0
2. y1 = SUC y0
3. SUC x1 = x2

Action

REV_FULL_SIMP_TAC std_ss []

Resulting GoalStack

P x2 y2

0. SUC (SUC y0) = y2
1. x1 = SUC x0
2. y1 = SUC y0
3. SUC (SUC x0) = x2

- `pure_ss` — empty simpset
- `bool_ss` — basic simpset
- `std_ss` — standard simpset
- `arith_ss` — arithmetic simpset
- `list_ss` — list simpset
- `real_ss` — real simpset

- many theories and libraries provide their own simpset-fragments
- `PRED_SET_ss` — simplify sets
- `STRING_ss` — simplify strings
- `QI_ss` — extra quantifier instantiations
- `gen_beta_ss` — β reduction for pairs
- `ETA_ss` — η conversion
- `EQUIV_EXTRACT_ss` — extract common part of equivalence
- `CONJ_ss` — use conjunctions for context
- `LIFT_COND_ss` — lifting if-then-else
- ...

- in contrast to `Rewrite` lib the simplifier can run arbitrary conversions
- most common and useful conversion is probably β -reduction
- `std_ss` has support for basic arithmetic and numerals
- it also has simple, syntactic conversions for instantiating quantifiers
 - ▶ $\lambda x. \dots \wedge (x = c) \wedge \dots \implies \dots$
 - ▶ $\lambda x. \dots \vee \sim(x = c) \vee \dots$
 - ▶ $\lambda x. \dots \wedge (x = c) \wedge \dots$
- besides very useful conversions, there are decision procedures as well
- the most frequently used one is probably the arithmetic decision procedure you already know from `DECIDE`

Examples I



```
> SIMP_CONV std_ss [] ``(\x. x + 2) 5``
```

```
val it = |- (\x. x + 2) 5 = 7
```

```
> SIMP_CONV std_ss [] ``!x. Q x /\ (x = 7) ==> P x``
```

```
val it = |- (!x. Q x /\ (x = 7) ==> P x) <=> (Q 7 ==> P 7)``
```

```
> SIMP_CONV std_ss [] ``?x. Q x /\ (x = 7) /\ P x``
```

```
val it = |- (?x. Q x /\ (x = 7) /\ P x) <=> (Q 7 /\ P 7)``
```

```
> SIMP_CONV std_ss [] ``x > 7 ==> x > 5``
```

```
Exception- UNCHANGED raised
```

```
> SIMP_CONV arith_ss [] ``x > 7 ==> x > 5``
```

```
val it = |- (x > 7 ==> x > 5) <=> T
```

- the simplifier supports higher order rewriting
- this is often very handy
- for example it allows moving quantifiers around easily

Examples

```
> SIMP_CONV std_ss [FORALL_AND_THM] ‘‘!x. P x /\ Q /\ R x’’
```

```
val it = |- (!x. P x /\ Q /\ R x) <=>  
          (!x. P x) /\ Q /\ (!x. R x)
```

```
> SIMP_CONV std_ss [GSYM RIGHT_EXISTS_AND_THM, GSYM LEFT_FORALL_IMP_THM]
```

```
‘‘!y. (P y /\ (?x. y = SUC x)) ==> Q y’’  
val it = |- (!y. P y /\ (?x. y = SUC x) ==> Q y) <=>  
          !x. P (SUC x) ==> Q (SUC x)
```

- a great feature of the simplifier is that it can use context information
- by default simple context information is used like
 - ▶ the precondition of an implication
 - ▶ the condition of `if-then-else`
- one can configure which context to use via congruence rules
 - ▶ e. g. by using `CONJ_ss` one can easily use context of conjunctions
 - ▶ warning: using `CONJ_ss` can be slow
- using context often simplifies proofs drastically
 - ▶ using `Rewrite` lib, often a goal needs to be split and a precondition moved to the assumptions
 - ▶ then `ASM_REWRITE_TAC` can be used
 - ▶ with `SIMP_TAC` there is no need to split the goal

```
> SIMP_CONV std_ss [] ``((l = []) ==> P l) /\ Q l``
```

```
val it = |- ((l = []) ==> P l) /\ Q l <=>  
            ((l = []) ==> P []) /\ Q l
```

```
> SIMP_CONV arith_ss [] ``if (c /\ x < 5) then (P c /\ x < 6) else Q c``
```

```
val it = |- (if c /\ x < 5 then P c /\ x < 6 else Q c) <=>  
            if c /\ x < 5 then P T else Q c:
```

```
> SIMP_CONV std_ss [] ``P x /\ (Q x /\ P x ==> Z x)``
```

Exception- UNCHANGED raised

```
> SIMP_CONV (std_ss++boolSimps.CONJ_ss) [] ``P x /\ (Q x /\ P x ==> Z x)``
```

```
val it = |- P x /\ (Q x /\ P x ==> Z x) <=> P x /\ (Q x ==> Z x)
```


- perhaps the most powerful feature of the simplifier is that it supports conditional rewriting
- this means it allows **conditional** rewrite theorems of the form
 $\mid - \text{cond} \implies (t1 = t2)$
- if the simplifier finds a term $t1'$ it can rewrite via $t1 = t2$ to $t2'$, it tries to discharge the assumption cond'
- for this, it calls itself recursively on cond'
 - ▶ all the decision procedures and all context information is used
 - ▶ conditional rewriting can be used
 - ▶ to prevent divergence, there is a limit on recursion depth
- if $\text{cond}' = \text{True}$ can be shown, $t1'$ is rewritten to $t2'$
- otherwise $t1'$ is not modified

Conditional Rewriting Example



- consider the conditional rewrite theorem

`!1 n. LENGTH l <= n ==> (DROP n l = [])`

- let's assume we want to prove

`(DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7]`

- we can without conditional rewriting

- ▶ show `|- LENGTH [1;2;3;4] <= 7`
- ▶ use this to discharge the precondition of the rewrite theorem
- ▶ use the resulting theorem to rewrite the goal

- with conditional rewriting, this is all automated

`> SIMP_CONV list_ss [DROP_LENGTH_TOO_LONG]`

`“(DROP 7 [1;2;3;4]) ++ [5;6;7]”`

`val it = |- DROP 7 [1; 2; 3; 4] ++ [5; 6; 7] = [5; 6; 7]`

- conditional rewriting often shortens proofs considerably

Proof with Rewrite

```
prove ((' (DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7] ',  
'DROP 7 [1;2;3;4] = []' by (  
  MATCH_MP_TAC DROP_LENGTH_TOO_LONG >>  
  REWRITE_TAC[LENGTH] >>  
  DECIDE_TAC  
) >>  
ASM_REWRITE_TAC[APPEND])
```

Proof with Simplifier

```
prove ((' (DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7] ',  
SIMP_TAC list_ss [])
```

Notice that `DROP_LENGTH_TOO_LONG` is part of `list_ss`.

- conditional rewriting is a very powerful technique
- decision procedures and sophisticated rewrites can be used to discharge preconditions without cluttering proof state
- it provides a powerful search for theorems that apply
- however, if used naively, it can be slow
- moreover, to work well, rewrite theorems need to be of a special form

- if the pattern is too general, the simplifier becomes very slow
- consider the following, trivial but hopefully educational example

Looping example

```
> val my_thm = prove ('~P ==> (P = F)', PROVE_TAC[])
> time (SIMP_CONV std_ss [my_thm]) 'P1 /\ P2 /\ P3 /\ ... /\ P10'
runtime: 0.84000s,    gctime: 0.02400s,    systime: 0.02400s.
Exception- UNCHANGED raised

> time (SIMP_CONV std_ss []) 'P1 /\ P2 /\ P3 /\ ... /\ P10'
runtime: 0.00000s,    gctime: 0.00000s,    systime: 0.00000s.
Exception- UNCHANGED raised
```

- ▶ notice that the rewrite is applied at plenty of places (quadratic in number of conjuncts)
- ▶ notice that each backchaining triggers many more backchainings
- ▶ each has to be aborted to prevent diverging
- ▶ as a result, the simplifier becomes very slow
- ▶ incidentally, the conditional rewrite is useless

- good conditional rewrites $\mid- c \implies (l = r)$ should mention only variables in c that appear in l
- if c contains extra variables $x_1 \dots x_n$, the conditional rewrite engine has to search instantiations for them
- this mean that conditional rewriting is trying discharge the precondition $?x_1 \dots x_n. c$
- the simplifier is usually not able to find such instances

Transitivity

```
> val P_def = Define 'P x y = x < y';  
> val my_thm = prove ('!x y z. P x y ==> P y z ==> P x z', ...)  
> SIMP_CONV arith_ss [my_thm] 'P 2 3 /\ P 3 4 ==> P 2 4'  
Exception- UNCHANGED raised
```

```
(* However transitivity of < build in via decision procedure *)  
> SIMP_CONV arith_ss [P_def] 'P 2 3 /\ P 3 4 ==> P 2 4'  
val it =  $\mid- P\ 2\ 3\ /\ P\ 3\ 4 \implies P\ 2\ 4 \iff T$ :
```

Conditional Rewriting Pitfalls III



- let's look in detail why `SIMP_CONV` did not make progress above

```
> set_trace "simplifier" 2;
> SIMP_CONV arith_ss [my_thm] 'P 2 3 /\ P 3 4 ==> P 2 4'
[468000]: more context: |- !x y z. P x y ==> P y z ==> P x z
[468000]: New rewrite: |- (?y. P x y /\ P y z) ==> (P x z <=> T)
...
[584000]: more context: [.] |- P 2 3 /\ P 3 4
[584000]: New rewrite: [.] |- P 2 3 <=> T
[584000]: New rewrite: [.] |- P 3 4 <=> T
[588000]: rewriting P 2 4 with |- (?y. P x y /\ P y z) ==> (P x z <=> T)
[588000]: trying to solve: ?y. P 2 y /\ P y 4
[588000]: rewriting P 2 y with |- (?y. P x y /\ P y z) ==> (P x z <=> T)
[592000]: trying to solve: ?y'. P 2 y' /\ P y' y
...
[596000]: looping - cut
...
[608000]: looping - stack limit reached
...
[640000]: couldn't solve: ?y. P 2 y /\ P y 4
Exception- UNCHANGED raised
```

Conditional vs. Unconditional Rewrite Rules



- conditional rewrite rules are often much more powerful
- however, `Rewrite` lib does not support them
- for this reason there are often two versions of rewrite theorems

drop example

- `DROP_LENGTH_NIL` is a useful rewrite rule:
`| - !l. DROP (LENGTH l) l = []`
- in proofs, one needs to be careful though to preserve exactly this form
 - ▶ one should not (partly) evaluate `LENGTH l` or modify `l` somehow
- with the conditional rewrite rule `DROP_LENGTH_TOO_LONG` one does not need to be as careful
`| - !l n. LENGTH l <= n ==> (DROP n l = [])`
 - ▶ the simplifier can use simplify the precondition using information about `LENGTH` and even arithmetic decision procedures

- some theorems given in the list of rewrites to the simplifier are used for special purposes
- there are marking functions that mark these theorems
 - ▶ `Once : thm -> thm` use given theorem at most once
 - ▶ `Ntimes : thm -> int -> thm` use given theorem at most the given number of times
 - ▶ `AC : thm -> thm -> thm` use given associativity and commutativity theorems for AC rewriting
 - ▶ `Cong : thm -> thm` use given theorem as a congruence rule
- these special forms are easy ways to add this information to a simpset
- it can be directly set in a simpset as well

Example Once



```
> SIMP_CONV pure_ss [Once ADD_COMM] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (b + a = c + d)

> SIMP_CONV pure_ss [Ntimes ADD_COMM 2] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (a + b = c + d)

> SIMP_CONV pure_ss [ADD_COMM] ``a + b = c + d``
Exception- UNCHANGED raised

> ONCE_REWRITE_CONV [ADD_COMM] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (b + a = d + c)

> REWRITE_CONV [ADD_COMM] ``a + b = c + d``
... diverges ...
```

- the simpset `srw_ss()` is maintained by the system
 - ▶ it is automatically extended by new type-definitions
 - ▶ theories can extend it via `export_rewrites`
 - ▶ libs can augment it via `augment_srw_ss`
- the stateful simpset contains many rewrites
- it is very powerful and easy to use

Example

```
> SIMP_CONV (srw_ss()) [] ‘‘case [] of [] => (2 + 4)‘‘  
val it = |- (case [] of [] => 2 + 4 | v::v1 => ARB) = 6
```

- the stateful simpset is very powerful and easy to use
- however, results are hard to predict
- proofs using it unwisely are hard to maintain
- the stateful simpset can expand too much
 - ▶ bigger, harder to read proof states
 - ▶ high level arguments become hard to see
- whether to use the stateful simpset depends on personal proof style
- I advise to not use `srw_ss` at the beginning
- once you got a good intuition on how the simplifier works, make your own choice

- it is complicated to add arbitrary decision procedures to a simpset
- however, adding simple conversions is straightforward
- a conversion is described by a `stdconvdata` record

```
type stdconvdata = {  
  name: string,      (* name for debugging *)  
  pats: term list,   (* list of patterns, when to try conv *)  
  conv: conv         (* the conversion *)  
}
```

- use `std_conv_ss` to create simpset-fragment

Example

```
val WORD_ADD_ss =  
  simpLib.std_conv_ss  
    {conv = CHANGED_CONV WORD_ADD_CANON_CONV,  
     name = "WORD_ADD_CANON_CONV",  
     pats = [''words$word_add (w:'a word) y[']]}
```

- the simplifier is HOL's main workhorse for automation
- it is very powerful
- conditional rewriting very powerful
 - ▶ here only simple examples were presented
 - ▶ experiment with it to get a feeling
- many advanced features not discussed here at all
 - ▶ using congruence rules
 - ▶ writing own decision procedures
 - ▶ rewriting with respect to arbitrary congruence relations

Warning

The simplifier is very powerful. Make sure you understand it and are in control when using it. Otherwise your proofs easily become lengthy, convoluted and hard to maintain.

Part XIV

Advanced Definition Principles



- a relation is a function from some arguments to `bool`
- the following example types are all types of relations:
 - ▶ `: 'a -> 'a -> bool`
 - ▶ `: 'a -> 'b -> bool`
 - ▶ `: 'a -> 'b -> 'c -> 'd -> bool`
 - ▶ `: ('a # 'b # 'c) -> bool`
 - ▶ `: bool`
 - ▶ `: 'a -> bool`
- relations are closely related to sets
 - ▶ $R\ a\ b\ c \iff (a, b, c) \in \{(a, b, c) \mid R\ a\ b\ c\}$
 - ▶ $(a, b, c) \in S \iff (\backslash a\ b\ c. (a, b, c) \in S)\ a\ b\ c$

- relations are often defined by a set of **rules**

Definition of Reflexive-Transitive Closure

The transitive reflexive closure of a relation $R : 'a \rightarrow 'a \rightarrow \text{bool}$ can be defined as the least relation $\text{RTC } R$ that satisfies the following rules:

$$\frac{R \ x \ y}{\text{RTC } R \ x \ y} \quad \frac{}{\text{RTC } R \ x \ x} \quad \frac{\text{RTC } R \ x \ y \quad \text{RTC } R \ y \ z}{\text{RTC } R \ x \ z}$$

- if the rules are monoton, a least and a greatest fix point exists (Knaster-Tarski theorem)
- least fixpoints give rise to **inductive relations**
- greatest fixpoints give rise to **coinductive relations**

- `(Co)IndDefLib` provides infrastructure for defining (co)inductive relations
- given a set of rules `Hol_(co)reln` defines (co)inductive relations
- 3 theorems are returned and stored in current theory
 - ▶ a rules theorem — it states that the defined constant satisfies the rules
 - ▶ a cases theorem — this is an equational form of the rules showing that the defined relation is indeed a fixpoint
 - ▶ a (co)induction theorem
- additionally a strong (co)induction theorem is stored in current theory

Example: Transitive Reflexive Closure



```
> val (RTC_REL_rules, RTC_REL_ind, RTC_REL_cases) = Hol_reln '  
    (!x y.    R x y                                ==> RTC_REL R x y) /\  
    (!x.      RTC_REL R x x) /\  
    (!x y z.  RTC_REL R x y /\ RTC_REL R y z ==> RTC_REL R x z)'  
  
val RTC_REL_rules = |- !R.  
    (!x y. R x y ==> RTC_REL R x y) /\ (!x. RTC_REL R x x) /\  
    (!x y z. RTC_REL R x y /\ RTC_REL R y z ==> RTC_REL R x z)  
  
val RTC_REL_cases = |- !R a0 a1.  
    RTC_REL R a0 a1 <=>  
    (R a0 a1 \/ (a1 = a0) \/ ?y. RTC_REL R a0 y /\ RTC_REL R y a1)
```

Example: Transitive Reflexive Closure II



```
val RTC_REL_ind = |- !R RTC_REL'.  
  ((!x y. R x y ==> RTC_REL' x y) /\ (!x. RTC_REL' x x) /\  
    (!x y z. RTC_REL' x y /\ RTC_REL' y z ==> RTC_REL' x z)) ==>  
  (!a0 a1. RTC_REL R a0 a1 ==> RTC_REL' a0 a1)
```

```
> val RTC_REL_strongind = DB.fetch "-" "RTC_REL_strongind"
```

```
val RTC_REL_strongind = |- !R RTC_REL'.  
  (!x y. R x y ==> RTC_REL' x y) /\ (!x. RTC_REL' x x) /\  
  (!x y z.  
    RTC_REL R x y /\ RTC_REL' x y /\ RTC_REL R y z /\  
    RTC_REL' y z ==>  
    RTC_REL' x z) ==>  
  (!a0 a1. RTC_REL R a0 a1 ==> RTC_REL' a0 a1)
```

Example: EVEN



```
> val (EVEN_REL_rules, EVEN_REL_ind, EVEN_REL_cases) = Hol_reln
  '(EVEN_REL 0 /\ (!n. EVEN_REL n ==> (EVEN_REL (n + 2))))';

val EVEN_REL_cases =
  |- !a0. EVEN_REL a0 <=> (a0 = 0) \/\ ?n. (a0 = n + 2) /\ EVEN_REL n

val EVEN_REL_rules =
  |- EVEN_REL 0 /\ !n. EVEN_REL n ==> EVEN_REL (n + 2)

val EVEN_REL_ind = |- !EVEN_REL'.
  (EVEN_REL' 0 /\ (!n. EVEN_REL' n ==> EVEN_REL' (n + 2))) ==>
  (!a0. EVEN_REL a0 ==> EVEN_REL' a0)
```

- notice that in this example there is exactly one fixpoint
- therefore, for these rules the inductive and coinductive relation coincide

Example: Dummy Relations



```
> val (DF_rules, DF_ind, DF_cases) = Hol_reln
    '(!n. DF (n+1) ==> (DF n))'

> val (DT_rules, DT_coind, DT_cases) = Hol_coreln
    '(!n. DT (n+1) ==> (DT n))'

val DT_coind =
  |- !DT'. (!a0. DT' a0 ==> DT' (a0 + 1)) ==> !a0. DT' a0 ==> DT a0

val DF_ind =
  |- !DF'. (!n. DF' (n + 1) ==> DF' n) ==> !a0. DF a0 ==> DF' a0

val DT_cases = |- !a0. DT a0 <=> DT (a0 + 1):
val DF_cases = |- !a0. DF a0 <=> DF (a0 + 1):
```

- notice that the definitions of **DT** and **DF** look like a non-terminating recursive definition
- **DT** is always true, i. e. $\vdash !n. DT\ n$
- **DF** is always false, i. e. $\vdash !n. \sim(DF\ n)$

- `quotientLib` allows to define types as quotients of existing types with respect to **partial equivalence relation**
- each equivalence class becomes a value of the new type
- partiality allows ignoring certain values of original type
- `quotientLib` allows to lift definitions and lemmata as well
- details are technical and won't be presented here

- let's assume we have an implementation of finite sets of numbers as binary trees with
 - ▶ type `binset`
 - ▶ binary tree invariant `WF_BINSET : binset -> bool`
 - ▶ constant `empty_binset`
 - ▶ add and member functions `add : num -> binset -> binset`,
`mem : binset -> num -> bool`
- we can define a partial equivalence relation by

```
binset_equiv b1 b2 := (  
  WF_BINSET b1 /\ WF_BINSET b2 /\  
  (!n. mem b1 n <=> mem b2 n))
```
- this allows defining a quotient type of sets of numbers
- functions `empty_binset`, `add` and `mem` as well as lemmata about them can be lifted automatically

- quotient types are sometimes very useful
 - ▶ e. g. rational numbers are defined as a quotient type
- there is powerful infrastructure for them
- many tasks are automated
- however, the details are technical and won't be discussed here

- pattern matching ubiquitous in functional programming
- pattern matching is a powerful technique
- it helps to write concise, readable definitions
- very handy and frequently used for interactive theorem proving
- however, it is **not directly supported** by HOL's logic
- representations in HOL
 - ▶ sets of equations as produced by **Define**
 - ▶ decision trees (printed as case-expressions)

- we have already used top-level pattern matches with the TFL package
- **Define** is able to handle them
 - ▶ all the semantic complexity is taken care of
 - ▶ no special syntax or functions remain
 - ▶ no special rewrite rules, reasoning tools needed afterwards
- **Define** produces a set of equations
- this is the recommended way of using pattern matching in HOL

Example

```
> val ZIP_def = Define '(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\  
                        (ZIP [] [] = [])'  
val ZIP_def = |- (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys) /\  
                (ZIP [] [] = [])
```

- sometimes one does not want to use this compilation by TFL
 - ▶ one wants to use pattern-matches somewhere nested in a term
 - ▶ one might not want to introduce a new constant
 - ▶ one might want to avoid using TFL for technical reasons
- in such situations, case-expressions can be used
- their syntax is similar to the syntax used by SML

Example

```
> val ZIP_def = Define 'ZIP xs ys = case (xs, ys) of
                                (x::xs, y::ys) => (x,y)::(ZIP xs ys)
                                | ([], []) => []'

val ZIP_def = |- !ys xs. ZIP xs ys =
  case (xs,ys) of
    ([],[]) => []
  | ([],v4::v5) => ARB
  | (x::xs',[]) => ARB
  | (x::xs',y::ys') => (x,y)::ZIP xs' ys'
```

- the datatype package defines case-constants for each datatype
- the parser contains a pattern compilation algorithm
- case-expressions are by the parser compiled to decision trees using case-constants
- pretty printer prints these decision trees as case-expressions again

Example

```
val ZIP_def = |- !ys xs. ZIP xs ys =  
  pair_CASE (xs,ys)  
    (\v v1.  
      list_CASE v (list_CASE v1 [] (\v4 v5. ARB))  
        (\x xs'. list_CASE v1 ARB (\y ys'. (x,y)::ZIP xs' ys'))):
```

- using case expressions feels very natural to functional programmers
- case-expressions allow concise, well-readable definitions
- however, there are also many drawbacks
- there is large, complicated code in the parser and pretty printer
 - ▶ this is outside the kernel
 - ▶ parsing a pretty-printed term can result in a non α -equivalent one
 - ▶ there are bugs in this code (see e.g. Issue #416 reported 8 May 2017)
- the results are hard to predict
 - ▶ heuristics involved in creating decision tree
 - ▶ however, it is beneficial that proofs follow this internal, volatile structure

- technical issues
 - ▶ it is tricky to reason about decision trees
 - ▶ rewrite rules about case-constants needs to be fetched from `TypeBase`
 - ★ alternative `srw_ss` often does more than wanted
 - ▶ partially evaluated decision-trees are not pretty printed nicely any more
- underspecified functions
 - ▶ decision trees are exhaustive
 - ▶ they list underspecified cases explicitly with value `ARB`
 - ▶ this can be lengthy
 - ▶ `Define` in contrast hides underspecified cases

Partial Proof Script

```
val _ = prove (“!l1 l2.  
  (LENGTH l1 = LENGTH l2) ==>  
    ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))”),  
  
ONCE_REWRITE_TAC [ZIP_def]
```

Current Goal

```
!l1 l2.  
  (LENGTH l1 = LENGTH l2) ==>  
  (((case (l1,l2) of  
    ([],[]) => []  
    | ([],v4::v5) => ARB  
    | (x::xs',[]) => ARB  
    | (x::xs',y::ys') => (x,y)::ZIP xs' ys') =  
    []) <=> (l1 = []) /\ (l2 = []))
```


Partial Proof Script

```
val _ = prove (“!11 12.  
  (LENGTH 11 = LENGTH 12) ==>  
  ((ZIP 11 12 = []) <=> ((11 = []) /\ (12 = [])))“,  
  
ONCE_REWRITE_TAC [ZIP_def] >>  
REWRITE_TAC[pairTheory.pair_case_def] >> BETA_TAC
```

Current Goal

```
!11 12.  
  (LENGTH 11 = LENGTH 12) ==>  
  (((case 11 of  
    [] => (case 12 of [] => [] | v4::v5 => ARB)  
    | x::xs' => case 12 of [] => ARB | y::ys' => (x,y)::ZIP xs' ys') =  
    []) <=> (11 = []) /\ (12 = []))
```

Partial Proof Script

```
val _ = prove (“!l1 l2.  
  (LENGTH l1 = LENGTH l2) ==>  
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))”),  
  
ONCE_REWRITE_TAC [ZIP_def] >>  
Cases_on ‘l1’ >| [  
  REWRITE_TAC[listTheory.list_case_def]
```

Current Goal

```
!l2.  
  (LENGTH [] = LENGTH l2) ==>  
  (((case ([],l2) of  
    ([],[]) => []  
    | ([],v4::v5) => ARB  
    | (x::xs',[]) => ARB  
    | (x::xs',y::ys') => (x,y)::ZIP xs' ys') =  
    []) <=> (l2 = []))
```

- case expressions are natural to functional programmers
- they allow concise, readable definitions
- however, fancy parser and pretty-printer needed
 - ▶ trustworthiness issues
 - ▶ sanity check lemmata advisable
- reasoning about case expressions can be tricky and lengthy
- proofs about case expression often hard to maintain
- therefore, use top-level pattern matching via **Define** if easily possible

Part XV

Maintainable Proofs



- proofs are hopefully still used in a few weeks, months or even years
- often the environment changes slightly during the lifetime of a proof
 - ▶ your definitions change slightly
 - ▶ your own lemmata change (e. g. become more general)
 - ▶ used libraries change
 - ▶ HOL changes
 - ★ automation becomes more powerful
 - ★ rewrite rules in certain simpsets change
 - ★ definition packages produce slightly different theorems
 - ★ autogenerated variable-names change
 - ★ ...
- even if HOL and used libraries are stable, proofs often go through several iterations
- often they are adapted by someone else than the original author
- therefore it is important that proofs are easily maintainable

- maintainability is closely linked to other desirable properties of proofs
- proofs should be
 - ▶ easily understandable
 - ▶ well-structured
 - ▶ robust
 - ★ they should be able to cope with minor changes to environment
 - ★ if they fail they should do so at sensible points
 - ▶ reusable
- How can one write proofs with such properties?
- as usual, there are no easy answers but plenty of good advice
- I recommend following the advice of **ProofStyle** manual
- parts of this advice as well as a few extra points are discussed in the following

- format your proof such that it easily understandable
- make the structure of the proof very clear
- **show clearly where subgoals start and stop**
- use indentation to mark proofs of subgoals
- use empty lines to separate large proofs of subgoals
- use comments where appropriate

Bad Example Term Formatting

```
prove (“!11 12. 11 <> [] ==> LENGTH 12 <
LENGTH (11 ++ 12)“,
...)
```

Good Example Term Formatting

```
prove (“!11 12. 11 <> [] ==>
      (LENGTH 12 < LENGTH (11 ++ 12))“,
...)
```


Bad Example Subgoals

```
prove (“!11 12. 11 <> [] ==> (LENGTH 12 < LENGTH (11 ++ 12))“,
Cases >>
  REWRITE_TAC[] >>
  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>
  REPEAT STRIP_TAC >>
  DECIDE_TAC)
```

Improved Example Subgoals

At least show when a subgoal starts and ends

```
prove (“!11 12. 11 <> [] ==> (LENGTH 12 < LENGTH (11 ++ 12))“,
Cases >> (
  REWRITE_TAC[]
) >>
  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>
  REPEAT STRIP_TAC >>
  DECIDE_TAC)
```

Good Example Subgoals

Make sure `REWRITE_TAC` is only applied to first subgoal and proof fails, if it does not solve this subgoal.

```
prove (''!11 12. 11 <> [] ==> (LENGTH 12 < LENGTH (11 ++ 12))'',
Cases >- (
  REWRITE_TAC[]
) >>
REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>
REPEAT STRIP_TAC >>
DECIDE_TAC)
```

Alternative Good Example Subgoals

Alternative good formatting using THENL

```
prove (‘‘!11 12. 11 <> [] ==> (LENGTH 12 < LENGTH (11 ++ 12))‘‘,  
Cases >| [  
  REWRITE_TAC[],  
  
  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>  
  REPEAT STRIP_TAC >>  
  DECIDE_TAC  
)
```

Another Bad Example Subgoals

Bad formatting using THENL

```
prove (‘‘!11 12. 11 <> [] ==> (LENGTH 12 < LENGTH (11 ++ 12))‘‘,  
Cases >| [REWRITE_TAC[],  
  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>  
  REPEAT STRIP_TAC >> DECIDE_TAC])
```

- use semicoli after each declaration
 - ▶ if exception is raised during interactive processing (e. g. by a failing proof), previous successful declarations are kept
 - ▶ it sometimes leads to better error messages in case of parsing errors
- use plenty of parentheses to make structure very clear
- don't ignore parser warnings
 - ▶ especially warnings about multiple possible parse trees are likely to lead to unstable proofs
 - ▶ understand why such warnings occur and make sure there is no problem
- format your development well
 - ▶ use indentation
 - ▶ use linebreaks at sensible points
 - ▶ don't use overlong lines
 - ▶ ...
- don't use `open` in middle of files
- personal opinion: avoid unicode in source files

- follow standard design principles
 - ▶ **KISS** principle
 - ▶ **“premature optimization is the root of all evil”** (Donald Knuth)
- don't try to be overly clever
- simple proofs are preferable
- proof-checking-speed mostly unimportant
- conciseness not a value in itself but desirable if it helps
 - ▶ readability
 - ▶ maintainability
- abstraction is often desirable, but also has a price
 - ▶ don't use too complex, artificial definitions and lemmata

Too much abstraction Example

```
val TOO_ABSTRACT_LEMMA = prove (‘‘
!(size :’a -> num) (P : ’a -> bool) (combine : ’a -> ’a -> ’a).
  (!x. P x ==> (0 < size x)) /\
  (!x1 x2. size x1 + size x2 <= size (combine x1 x2)) ==>

  (!x1 x2. P x1 ==> (size x2 < size (combine x1 x2)))‘‘,
... )

prove (‘‘!l1 l2. l1 <> [] ==> (LENGTH l2 < LENGTH (l1 ++ l2))‘‘,
  some proof using ABSTRACT_LEMMA
)
```

- a common mistake is to use too clever tactics
 - ▶ intended to work on many (sub)goals
 - ▶ using **TRY** and other fancy trial and error mechanisms
 - ▶ intended to replace multiple simple, clear tactics
- typical case: a tactic containing **TRY** applied to many subgoals
- it is often hard to see why such tactics work
- if something goes wrong, they are hard to debug
- general advice: don't factor with tactics, instead use definitions and lemmata

Too Clever Tactics Example I



Bad Example Subgoals

```
prove ('!l1 l2. l1 <> [] ==> (LENGTH l2 < LENGTH (l1 ++ l2))',
Cases >> (
  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>
  REPEAT STRIP_TAC >>
  DECIDE_TAC
))
```

Alternative Good Example Subgoals II

```
prove ('!l1 l2. l1 <> [] ==> (LENGTH l2 < LENGTH (l1 ++ l2))',
Cases >> SIMP_TAC list_ss [])
```

```
prove ('!l1 l2. l1 <> [] ==> (LENGTH l2 < LENGTH (l1 ++ l2))',
Cases >| [
  REWRITE_TAC[],

  REWRITE_TAC[listTheory.LENGTH, listTheory.LENGTH_APPEND] >>
  REPEAT STRIP_TAC >>
  DECIDE_TAC
])
```


Bad Example

```
val oadd_def = Define '(oadd (SOME n1) (SOME n2) = (SOME (n1 + n2))) /\
                        (oadd _ _ = NONE)';
val osub_def = Define '(osub (SOME n1) (SOME n2) = (SOME (n1 - n2))) /\
                        (osub _ _ = NONE)';
val omul_def = Define '(omul (SOME n1) (SOME n2) = (SOME (n1 * n2))) /\
                        (omul _ _ = NONE)';

val obin_NONE_TAC =
  Cases_on 'o1' >> Cases_on 'o2' >>
  SIMP_TAC std_ss [oadd_def, osub_def, omul_def];

val oadd_NONE = prove (
  '!'o1 o2. (oadd o1 o2 = NONE) <=> (o1 = NONE) \/ (o2 = NONE)',
  obin_NONE_TAC);
val osub_NONE = prove (
  '!'o1 o2. (osub o1 o2 = NONE) <=> (o1 = NONE) \/ (o2 = NONE)',
  obin_NONE_TAC);
val omul_NONE = prove (
  '!'o1 o2. (omul o1 o2 = NONE) <=> (o1 = NONE) \/ (o2 = NONE)',
  obin_NONE_TAC);
```

Good Example

```
val obin_def = Define '(obin op (SOME n1) (SOME n2) = (SOME (op n1 n2))) /\
                        (obin _ _ _ = NONE)';

val oadd_def = Define 'oadd = obin $+';
val osub_def = Define 'osub = obin $-';
val omul_def = Define 'omul = obin $*';

val obin_NONE = prove (
  '!'op o1 o2. (obin op o1 o2 = NONE) <=> (o1 = NONE) \/\ (o2 = NONE)',
  Cases_on 'o1' >> Cases_on 'o2' >> SIMP_TAC std_ss [obin_def]);

val oadd_NONE = prove (
  '!'o1 o2. (oadd o1 o2 = NONE) <=> (o1 = NONE) \/\ (o2 = NONE)',
  REWRITE_TAC[oadd_def, obin_NONE]);

val osub_NONE = prove (
  '!'o1 o2. (osub o1 o2 = NONE) <=> (o1 = NONE) \/\ (o2 = NONE)',
  REWRITE_TAC[osub_def, obin_NONE]);

val omul_NONE = prove (
  '!'o1 o2. (omul o1 o2 = NONE) <=> (o1 = NONE) \/\ (o2 = NONE)',
  REWRITE_TAC[omul_def, obin_NONE]);
```

Use many subgoals and lemmata



- often it is beneficial to use subgoals
 - ▶ they structure long proofs well
 - ▶ they help keeping the proof state clean
 - ▶ they mark clearly what one tries to proof
 - ▶ they provide points where proofs can break sensibly
- general enough subgoals should become lemmata
 - ▶ this improves reusability
 - ▶ proof script for main lemma becomes shorter
 - ▶ proofs are disentangled

- the following example is taken from exercise 5
- we try to prove `!1. IS_WEAK_SUBLIST_FILTER 1 1`
- given are following definitions and lemmata

```
val FILTER_BY_BOOLS_def = Define '  
  FILTER_BY_BOOLS b1 l = MAP SND (FILTER FST (ZIP (b1, l)))';  
  
val IS_WEAK_SUBLIST_FILTER_def = Define 'IS_WEAK_SUBLIST_FILTER l1 l2 =  
  ?(b1 : bool list). (LENGTH b1 = LENGTH l1) /\ (l2 = FILTER_BY_BOOLS b1 l1)';  
  
val FILTER_BY_BOOLS_REWRITES = store_thm ("FILTER_BY_BOOLS_REWRITES",  
  '(FILTER_BY_BOOLS [] [] = []) /\  
    (!b b1 x xs. (FILTER_BY_BOOLS (b::b1) (x::xs) =  
      if b then x::(FILTER_BY_BOOLS b1 xs) else FILTER_BY_BOOLS b1 xs))',  
  REWRITE_TAC [FILTER_BY_BOOLS_def, ZIP, MAP, FILTER] >>  
  Cases_on 'b' >> REWRITE_TAC [MAP]);
```

First Version

```
val IS_WEAK_SUBLIST_FILTER_REFL = store_thm ("IS_WEAK_SUBLIST_FILTER_REFL",
  ``!l. IS_WEAK_SUBLIST_FILTER l l``,
  REWRITE_TAC[IS_WEAK_SUBLIST_FILTER_def] >>
  Induct_on 'l' >- (
    Q.EXISTS_TAC '[]' >>
    SIMP_TAC list_ss [FILTER_BY_BOOLS_REWRITES]
  ) >>
  FULL_SIMP_TAC std_ss [] >>
  GEN_TAC >>
  Q.EXISTS_TAC 'T::bl' >>
  ASM_SIMP_TAC list_ss [FILTER_BY_BOOLS_REWRITES])
```

- the proof mixes properties of `IS_WEAK_SUBLIST_FILTER` and properties of `FILTER_BY_BOOLS`
- it is hard to see what the main idea is

Subgoal Example III



- the following proof separates the property of `FILTER_BY_BOOLS` as a subgoal
- the main idea becomes clearer

Subgoal Version

```
val IS_WEAK_SUBLIST_FILTER_REFL = store_thm ("IS_WEAK_SUBLIST_FILTER_REFL",
  ``!l. IS_WEAK_SUBLIST_FILTER l l``,
GEN_TAC >>
REWRITE_TAC[IS_WEAK_SUBLIST_FILTER_def] >>
'FILTER_BY_BOOLS (REPLICATE (LENGTH l) T) l = l' suffices_by (
  METIS_TAC[LENGTH_REPLICATE]
) >>
Induct_on 'l' >> (
  ASM_SIMP_TAC list_ss [FILTER_BY_BOOLS_REWRITES, REPLICATE]
))
```

- the subgoal is general enough to justify a lemma
- the structure becomes even cleaner
- this improves reusability

Lemma Version

```
val FILTER_BY_BOOLS_REPL_T = store_thm ("FILTER_BY_BOOLS_REPL_T",  
  ‘!l. FILTER_BY_BOOLS (REPLICATE (LENGTH l) T) l = l’  
  Induct >> ASM_REWRITE_TAC [REPLICATE, FILTER_BY_BOOLS_REWRITES, LENGTH]);  
  
val IS_WEAK_SUBLIST_FILTER_REFL = store_thm ("IS_WEAK_SUBLIST_FILTER_REFL",  
  ‘!l. IS_WEAK_SUBLIST_FILTER l l’  
  GEN_TAC >>  
  REWRITE_TAC[IS_WEAK_SUBLIST_FILTER_def] >>  
  Q.EXISTS_TAC ‘REPLICATE (LENGTH l) T’ >>  
  SIMP_TAC list_ss [FILTER_BY_BOOLS_REPL_T, LENGTH_REPLICATE])
```

Avoid Autogenerated Names



- many HOL-tactics introduce new variable names
 - ▶ Induct
 - ▶ Cases
 - ▶ ...
- the new names are often very artificial
- even worse, generated names might change in future
- proof scripts using autogenerated names are therefore
 - ▶ hard to read
 - ▶ potentially fragile
- therefore rename variables after they have been introduced
- HOL has multiple tactics supporting renaming
- most useful is `rename1` 'pat', it searches for pattern and renames vars accordingly

Bad Example

```
prove ('!l. 1 < LENGTH l ==> (?x1 x2 l'. l = x1::x2::l'))',  
GEN_TAC >>  
Cases_on 'l' >> SIMP_TAC list_ss [] >>  
Cases_on 't' >> SIMP_TAC list_ss [])
```

Good Example

```
prove ('!l. 1 < LENGTH l ==> (?x1 x2 l'. l = x1::x2::l'))',  
GEN_TAC >>  
Cases_on 'l' >> SIMP_TAC list_ss [] >>  
rename1 'LENGTH l2' >>  
Cases_on 'l2' >> SIMP_TAC list_ss [])
```

Proof State before rename1

```
1 < SUC (LENGTH t) ==> ?x2 l'. t = x2::l'
```

Proof State after rename1

```
1 < SUC (LENGTH l2) ==> ?x2 l'. l2 = x2::l'
```

Part XVI

Overview of HOL 4



- in this course we discussed the basics of HOL 4
- you were encouraged to learn more on your own in exercises
- there is a lot more to learn even after the end of the course
 - ▶ many more libraries
 - ▶ proof tools
 - ▶ existing formalisations
 - ▶ ...
- to really use HOL well, you should continue learning
- to help getting started, a short overview is presented here

HOL Bare Source Directories



The following source directories are the very basis of HOL. They are required to build `hol.bare`.

- `src/portableML` – common stuff for PolyML and MoscowML
- `src/prekernel`
- `src/0` – Standard Kernel
- `src/logging-kernel` – Logging Kernel
- `src/experimental-kernel` – Experimental Kernel
- `src/postkernel`
- `src/opentheory`
- `src/parse`
- `src/bool`
- `src/1`
- `src/proofman`

On top of `hol.bare`, there are many basic theories and tools. These are all required for building the main `hol` executable.

- `src/compute` – fast ground term rewriting
- `src/HolSat` – SAT solver interfaces
- `src/taut` – propositional proofs using `HolSat`
- `src/marker` – marking terms
- `src/q` – parsing support
- `src/combin` – combinators
- `src/lite` – some simple lib with various stuff
- `src/refute` – refutation prover, normal forms
- `src/metis` – first order resolution prover
- `src/meson` – first order model elimination prover

- `src/simp` – simplifier
- `src/holyhammer` – tool for finding Metis proofs
- `src/tactictoe` – machine learning tool for finding proofs
- `src/IndDef` – (co)inductive relation definitions
- `src/basicProof` – library containing proof tools
- `src/relation` – relations and order theory
- `src/one` – unit type theory
- `src/pair` – tuples
- `src/sum` – sum types
- `src/tfl` – defining terminating functions
- `src/option` – option types

- `src/num` – numbers and arithmetic
- `src/pred_set` – predicate sets
- `src/datatype` – Datatype package
- `src/list` – list theories
- `src/monad` – monads
- `src/quantHeuristics` – instantiating quantifiers
- `src/unwind` – lib for unwinding structural hardware definitions
- `src/pattern_matches` – pattern matches alternative
- `src/bossLib` – main HOL lib loaded at start

`bossLib` is one central library. It loads all basic theories and libraries and provides convenient wrappers for the most common tools.

Besides the basic libraries and theories that are required and loaded by `hol`, there are many more developments in HOL's source directory.

- `src/sort` – sorting lists
- `src/string` – strings
- `src/TeX` – exporting LaTeX code
- `src/res_quan` – restricted quantifiers
- `src/quotient` – quotient type package
- `src/finite_map` – finite map theory
- `src/bag` – bags a. k. a. multisets
- `src/n-bit` – machine words

- `src/ring` – reasoning about rings
- `src/integer` – integers
- `src/llists` – lazy lists
- `src/path` – finite and infinite paths through a transition system
- `src/patricia` – efficient finite map implementations using trees
- `src/emit` – emitting SML and OCaml code
- `src/search` – traversal of graphs that may contain cycles

- `src/rational` – rational numbers
- `src/real` – real numbers
- `src/complex` – complex numbers
- `src/HolQbf` – quantified boolean formulas
- `src/HolSmt` – support for external SMT solvers
- `src/float` – IEEE floating point numbers
- `src/floating-point` – new version of IEEE floating point numbers
- `src/probability` – some probability theory
- `src/temporal` – shallow embedding of temporal logic
- ...

The directory examples hosts many theories and libraries as well. There is not always a clear distinction between an example and a development in `src`. However, in general examples are more specialised and often larger. They are not required to follow HOL's coding style as much as developments in `src`.

- `examples/balanced_bst` – finite maps via balanced trees
- `examples/unification` – (nominal) unification
- `examples/Crypto` – various block ciphers
- `examples/elliptic` – elliptic curve cryptography
- `examples/formal-languages` – regular and context free formal languages
- `examples/computability` – basic computability theory

- [examples/set-theory](#) – axiomatic formalisation of set theory
- [examples/lambda](#) – lambda calculus
- [examples/acl2](#) – connection to ACL2 prover
- [examples/theorem-prover](#) – soundness proof of Milawa prover
- [examples/PSL](#) – formalisation of PSL
- [examples/HolBdd](#) – Binary Decision Diagrams
- [examples/HolCheck](#) – basic model checker
- [examples/temporal_deep](#) – deep embedding of temporal logics and automata

- [examples/pgcl](#) formalisation of pGCL (the Probabilistic Guarded Command Language)
- [examples/dev](#) – some hardware compilation
- [examples/STE](#) – symbolic trajectory evaluation
- [examples/separationLogic](#) – formalisation of separation logic
- [examples/ARM](#) – formalisation of ARM architecture
- [examples/l3-machine-code](#) – l3 language
- [examples/machine-code](#) – compilers and decompilers to machine-code
- ...

- some useful tools are a bit hidden in the HOL sources
- moreover there are developments outside the main HOL 4 sources
 - ▶ CakeML <https://cakeml.org>
- keep in touch with community to continue learning about HOL 4
 - ▶ mailing-list hol-info
 - ▶ GitHub <https://github.com/HOL-Theorem-Prover/HOL>
 - ▶ <https://hol-theorem-prover.org>
- if you continue using HOL, please consider sharing your work with the community

Part XVII

Other Interactive Theorem Provers



- at the beginning we very briefly discussed other theorem provers
- now, with more knowledge about HOL 4 we can discuss other provers and their differences to HOL 4 in more detail
- HOL 4 is a good system
- it is very well suited for the tasks required by the PROSPER project
- however, as always **choose the right tool for your task**
- you might find a different prover more suitable for your needs
- hopefully this course has enabled you to learn to use other provers on your own without much trouble

- based on classical higher order logic
- logic is sweet spot between expressivity and automation
- + very trustworthy thanks to LCF approach
- + simple enough to understand easily
- + very easy to write custom proof tools, i. e. own automation
- + reasonably fast and efficient
- decent automation
- no user-interface
- no special proof language
- no IDE, very little editor support

- mainly developed by Peter Homeier
<http://www.trustworthytools.com/>
- extension of HOL 4
 - + logic extended by kinds
 - + allows type operator variables
 - + allows quantification over type variables
- + sometimes handy to e. g. model category theory
- not very actively developed
- HOL 4 usually sufficient and better supported

- mainly developed by John Harrison
 - <https://github.com/jrh13/hol-light>
 - cleanup and reimplementing of HOL in OCaml
 - little legacy code
 - however, still very similar to HOL 4
- + much better automation for real analysis
- + cleaner
- OCaml introduces some minor issues with trustworthiness
 - some other libs and tools of HOL 4 are missing
 - HOL 4 has bigger community

- Isabelle is also a descendant of LCF
- originally developed by Larry Paulson in Cambridge
<https://www.cl.cam.ac.uk/research/hvg/Isabelle/>
- meanwhile also developed at TU Munich by Tobias Nipkow
<http://www21.in.tum.de>
- huge contributions by Markarius Wenzel
<http://sketis.net>
- Isabelle is a generic theorem prover
- most used instantiation is Isabelle/HOL
- other important one is Isabelle/ZF

- logic of Isabelle / HOL very similar to HOL's logic
 - ▶ meta logic leads to meta level quantification and object level quantification
 - + type classes
 - + powerful module system
 - + existential variables
 - ▶ ...
- Isabelle is implemented using the LCF approach
- it uses SML (Poly/ML)
- many original tools (e. g. simplifier) similar to HOL
- focused as HOL on equational reasoning
- many tools are exchanged between HOL 4 and Isabelle / HOL
 - ▶ Metis
 - ▶ Sledgehammer
 - ▶ ...

- + a lot of engineering went into Isabelle/HOL
- + it has a very nice GUI
 - ▶ IDE based on JEdit
 - ▶ special language for proofs (Isar)
 - ▶ good error messages
 - ▶ ...
- + very good automation
- + efficient implementations
- + many libraries (Archive of Formal Proof)
- + excellent code extraction
- + good documentation
- + easy for new users

- special proof language Isar used
- this allows to write **declarative proofs**
 - ▶ very high level
 - ▶ easy to read by humans
 - ▶ very robust
 - ▶ very good tool support
 - ▶ ...
- however, tactical proofs are not easily accessible any more
 - ▶ many intermediate goals need to be stated (declared) explicitly
 - ▶ this can be very tedious
 - ▶ tools like verification condition generators are hard to use

- + Isabelle/HOL provides excellent out of the box automation
- + it provides a very nice user interface
- + it is very nice for new users
- however, this comes at a price
 - ▶ multiple layers added between kernel and user
 - ▶ hard to understand all these layers
 - ▶ a lot of knowledge is needed to write your own automation
- hard to write own automation
- Isabelle/HOL due to focus on declarative proofs not well suited for e. g. PROSPER

- Coq is a proof assistant using the Calculus of Inductive Constructions
 - inspired by HOL 88
 - backward proofs as in HOL 4 used
 - however, very big differences
 - ▶ much more powerful logic
 - ▶ dependent types
 - ▶ constructive logic
 - ▶ not exactly following LCF approach
- + good user interface
- + very good community support

- + Coq's logic is very powerful
- + it is very natural for mathematicians
- + very natural for language theory
- + allows reasoning about proofs
 - allows to add axioms as needed
 - as a result, Coq is used often to
 - ▶ formalise mathematics
 - ▶ formalise programming language semantics
 - ▶ reason about proof theory

- Coq's power comes at a price
 - there is not much automation
 - proofs tend to be very long
 - ▶ they are very simple though
 - + comparably easy to maintain
 - Coq's proof checking can be very slow
 - when verifying programs or hardware you notice that HOL was designed for this purpose
 - ▶ need for **obvious** termination is tedious
 - ▶ missing automation
 - ▶ very slow

- there are many good theorem provers out there
- **pick the right tool for your purpose**
- the HOL theorem prover is a good system for many purposes
- for PROSPER it is a good choice
- I encourage you to continue learning about HOL and interactive theorem proving in general
- if you have any questions feel free to contact me (Thomas Tuerk, email **thomas@tuerk-brechen.de**)