1) a)

Gaussian Distribution:

$$N(x \mid \mu, \delta^2) = \frac{1}{(2\pi\delta^2)^{1/2}} \exp\left\{ -\frac{1}{2\delta^2}(x-\mu)^2 \right\}$$

To the above formula is normalized:

$$\int_{-\infty}^{\infty} N(x \mid \mu, \delta^2)\, dx = 1.$$

Let $I = \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} (x-\mu)^2 \right) dx$

$$I = \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} (x)^2 \right) dx$$

$$\Leftrightarrow I^2 = \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} x^2 \right) dx \cdot \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} x^2 \right) dx$$

$$\Leftrightarrow I^2 = \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} x^2 \right) dx \cdot \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} y^2 \right) dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2} (x^2+y^2) \right) dx\, dy \qquad (1)$$

Let $x = r.\cos\theta$

$y = r.\sin\theta$

$\Rightarrow x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2.$

Using the Jacobian determinant:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

$$(1) \Leftrightarrow \int_0^{2\pi}\int_0^{\infty} \exp\left(\frac{-1}{2\sigma^2}.r^2\right) r \, d_r \, d_\theta$$

$$= 2\pi\int_0^{\infty} \exp\left(\frac{-r^2}{2\sigma^2}\right) r \, d_r = 2\pi . A. \qquad (2)$$

Let : $u = r^2 \Rightarrow d_u = 2r d_r$

$$A = \int_0^{\infty} \exp\left(\frac{-u}{2\sigma^2}\right)\frac{d_u}{2} = \frac{1}{2}\int_0^{\infty} \exp\left(\frac{-u}{2\sigma^2}\right) d_u$$

$(e^u)' = u' . e^u$ $\int e^{ax+b} dx = \frac{1}{a} . e^{ax+b}$

Thứ
Ngày . . No.

$$(2) \Leftrightarrow 2\pi . \frac{1}{2} \int_0^\infty \exp\left(\frac{-u}{2\delta^2}\right) du$$

$$= \pi \left[\exp\left(\frac{-u}{2\delta^2}\right) . (-2\delta^2) \right] \Big|_0^\infty$$

$$= +2\pi\delta^2$$

$$\Rightarrow I^2 = +2\pi\delta^2 \Leftrightarrow I = (2\pi\delta^2)^{\frac{1}{2}}$$

So,

$$\int_{-\infty}^{\infty} N(x|\mu, \delta^2) dx = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}} . \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\delta^2}(x-\mu)^2\right) dx$$

$$\Leftrightarrow \int_{-\infty}^{\infty} N(x|\mu, \delta^2) dx = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}} . I = \frac{1}{(2\pi\delta^2)^{1/2}} . (2\pi\delta^2)^{\frac{1}{2}}$$

$$\Leftrightarrow \int_{-\infty}^{\infty} N(x|\mu, \delta^2) dx = 1 \quad (\text{proved})$$

Thứ
Ngày . . No.

b)
We have:

$$E(x) = \int_{-\infty}^{\infty} N(x \mid \mu, \delta^2) . x . dx$$

$$\Leftrightarrow E(x) = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}} . \underbrace{\int_{-\infty}^{\infty} x . \exp\left(-\frac{1}{2\delta^2}(x-\mu)^2\right) dx}_{A.}$$

Let $\chi = \frac{x-\mu}{\sqrt{2}\delta} \Rightarrow d_\chi = \frac{\sqrt{2}}{2\delta} d_x$

$$A = \int_{-\infty}^{\infty} (\sqrt{2}\delta . \chi + \mu) . \exp(-\chi^2) . \sqrt{2}\delta^2 d_\chi$$

$$A = \sqrt{2}.\delta \int_{-\infty}^{\infty} (\sqrt{2}\delta \chi + \mu) . \exp(-\chi^2) d_\chi$$

$$= \sqrt{2}.\delta . \left[\int_{-\infty}^{\infty} \sqrt{2}\delta \chi . \exp(-\chi^2) d_\chi + \mu \int_{-\infty}^{\infty} \exp(-\chi^2) d_\chi \right]$$

$$= \sqrt{2}.\delta \left[\sqrt{2}\delta \underbrace{\int_{-\infty}^{\infty} \chi . \exp(-\chi^2) d\chi}_{0} + \mu \int_{-\infty}^{\infty} \exp(-\chi^2) d_\chi \right]$$

$$_0\int^{\infty} \exp(-t^2)\,d_t = \frac{\sqrt{\pi}}{2}$$

Thứ

Ngày . .

No.

We have $\exp(-x^2)$ is a even function.

$$\Rightarrow \int_{-\infty}^{\infty} \exp(-x^2)\,d_x = 2\int_0^{\infty} \exp(-x^2)\,dx.$$

$$= 2.\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

$$\Rightarrow A = \sqrt{2}.\delta\,(.\mu\sqrt{\pi})$$

So,

$$E(x) = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}}.\sqrt{2}\,\delta\,.\mu.\sqrt{\pi}$$

$$\Leftrightarrow E(x) = \mu \quad (\text{Proved})$$

c)

$$Var(x) = E[x^2] - E[x]^2$$

$$+\ E[x^2] = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}}.\underbrace{\int_{-\infty}^{\infty} x^2.\exp\left(-\frac{1}{2\delta^2}(x^2-\mu)^2\right)d_x}_{B}$$

Let $t = \frac{x-\mu}{\sqrt{2}.\delta} \Rightarrow d_t = \frac{\sqrt{2}}{2.\delta}\,dx$

$$\int_a^b f(t)\,G(t)\,d_t = \left[F(t)G(t)\right]\Big|_a^b - \int_a^b F(t)\,g(t)\,d_t$$

Thứ

Ngày . . No.

$$b = \int_{-\infty}^{\infty} (\sqrt{2}\,\delta t + \mu)^2 \cdot \exp(-t^2) \cdot \sqrt{2}\,\delta\, d_t$$

$$= \sqrt{2}\cdot\delta \int_{-\infty}^{\infty} \left[\left(2\delta^2 t^2 + 2\sqrt{2}\,\delta t.\mu + \mu^2\right)\right] \exp(-t^2)\, d_t$$

$$= \sqrt{2}\cdot\delta \left[\left(\int_{-\infty}^{\infty} 2\delta^2 t^2 \cdot \exp(-t^2)\, d_t\right) + \left(\int_{-\infty}^{\infty} 2\sqrt{2}\,\delta\,\mu . t \cdot \exp(-t^2)\, d_t\right)\right.$$

$$\left. + \left(\int_{-\infty}^{\infty} \mu^2 \cdot \exp(-t^2)\, d_t\right)\right]$$

$$= \sqrt{2}\cdot\delta \cdot \left[\left(2\delta^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\, d_t\right) + \left(2\sqrt{2}\,\delta\mu \underbrace{\int_{-\infty}^{\infty} t \cdot \exp(-t^2)\, d_t}_{0}\right)\right.$$

$$\left. + \left(\mu^2 \underbrace{\int_{-\infty}^{\infty} \exp(-t^2)\, d_t}_{\sqrt{\pi}}\right)\right]$$

$$= \sqrt{2}\,\delta \cdot \left[2\delta^2 \left(\left[-\frac{t}{2}\exp(-t^2)\right]\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2}\exp(-t^2)\, d_t\right) + \right.$$

$$\left. \mu^2 \cdot \sqrt{\pi}\right]$$

Thứ
Ngày
No.

$$= \sqrt{2}\,\delta \cdot \left[2\delta^2 \left(0 + \frac{1}{2} \cdot \sqrt{\pi} \right) + \mu^2 \cdot \sqrt{\pi} \right]$$

$$= \sqrt{2} \cdot \delta \left[\delta^2 \cdot \sqrt{\pi} + \mu^2 \cdot \sqrt{\pi} \right]$$

$$= \sqrt{2} \cdot \delta \cdot \sqrt{\pi} \left(\delta^2 + \mu^2 \right)$$

$$\Rightarrow E[X^2] = \frac{1}{(2\pi\delta^2)^{\frac{1}{2}}} \cdot \sqrt{2} \cdot \delta \cdot \sqrt{\pi} \left(\delta^2 + \mu^2 \right)$$

$$= \delta^2 + \mu^2$$

$$\Rightarrow Var(x) = E[X^2] - E[X]^2$$

$$= \delta^2 + \mu^2 - \mu^2 \quad (\text{from } b)$$

$$= \delta^2 \quad (\text{proved})$$

d)

Thứ
Ngày . . No.

2)

a) Suppose x : a D-dimensional vector.

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix} \Rightarrow \mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

Covariance matrix Σ:

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix} \Rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$$

Σ is symmetric $\Rightarrow \Sigma_{aa}$; Σ_{bb} are symmetrix.

$\Sigma_{ab} = \Sigma_{ba}^T$.

We have:

$$-\frac{1}{2}(x-\mu)^T \cdot \Sigma^{-1}(x-\mu) = -\frac{1}{2}(x-\mu)^T A(x-\mu)$$

$$= -\frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{ab}(x_b-\mu_b)$$

$$-\frac{1}{2}(x_b-\mu_b)^T A_{ba}(x_a-\mu_a) - \frac{1}{2}(x_b-\mu_b)^T A_{bb}(x_b-\mu_b)$$

$$= -\frac{1}{2} x_a^T A_{aa}^{-1} x_a + x_a^T (A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) + C. \quad (1)$$

KOKUYO

Thứ
Ngày
No.

Compare with Gauss distribution:

$$\Delta^2 = \frac{-1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + C$$

(1) $\Rightarrow \Sigma_{a|b} = A_{aa}^{-1}$

$$\mu_{a|b} = \Sigma_{a|b} \left(A_{aa} \mu_a - A_{ab} (x_b - \mu_b) \right)$$

$$= \mu_a - A_{aa}^{-1} A_{ab} (x_b - \mu_b)$$

Using Schur complement.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CMD^{-1} & D^{-1}CMBD^{-1} \end{pmatrix}, \quad M = (A - BD^{-1}C)^{-1}$$

$$\Rightarrow A_{aa} = \left(\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba} \right)^{-1}$$

$$A_{ab} = -\left(\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba} \right)^{-1} \cdot \Sigma_{ab} \Sigma_{bb}^{-1}$$

So,

$$\mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (x_b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

Thứ

Ngày . . No.

$$\Rightarrow p(x_a | x_b) = \mathcal{N}(x_{a|b} | \mu_{a|b}; \Sigma_{a|b})$$

b) The Marginal Distribution:

$$p(x_a) = \int p(x_a, x_b)\, d_{x_b}$$

We have quadratic form related to x_b.

$$-\frac{1}{2} x_b^T A_{bb} x_b + x_b^T m = -\frac{1}{2}(x_b - A_{bb}^{-1} m)^T A_{bb}$$

$$(x_b - A_{bb}^{-1} m) + \frac{1}{2} m^T A_{bb}^{-1} m$$

with $m = A_{bb}\mu_b - A_{ba}(x_a - \mu_a)$

~~Integrate~~ If we intergate over unnormalized Gauss.

$$\int \exp\left(-\frac{1}{2}(x_b - A_{bb}^{-1} m)^T A_{bb}(x_b - A_{bb}^{-1} m)\right) d_{x_b}$$

$\Rightarrow$ The remaining:

$$-\frac{1}{2} x_a^T (A_{aa} - A_{ab} A_{bb}^{-1} A_{ba}) x_a + x_a^T (A_{aa} - A_{ab} A_{bb}^{-1} A_{ba})^{-1} \mu_a$$

$+ \; C.$

Similarly,

$$E[x_a] = \mu_a$$

$$\mathrm{Cov}(x_a) = \Sigma_{aa}$$

$$\Rightarrow p(x_a) = N(x_a \mid \mu_a, \Sigma_{aa})$$