Chapter 6

Linear Recurrences

"Everything goes, everything comes back; eternally rolls the wheel of being." (Friedrich Nietzsche)

This chapter is dedicated to linear recurrences, a special type of equations that defines a sequence, that is a series of terms of the form

$$a_0, a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$$

recursively, that is such that each term a_n is defined as a function of the preceding terms. A recursive linear recurrence must be accompanied by initial conditions, that is information about some of the first terms such as a_0 or a_0, a_1.

Example 57. The Fibonacci sequence is defined by

$$f_n = f_{n-1} + f_{n-2},\ f_0 = 0,\ f_1 = 1.$$

To compute f_2, we have

$$f_2 = f_1 + f_0 = 1.$$

To compute f_3, we have

$$f_3 = f_2 + f_1 = 1 + 1 = 2.$$

We will see in this chapter two methods to solve linear recurrences involving one or two preceding terms.

Recurrence Relation

A *recurrence relation* is an equation that *recursively defines a sequence,* i.e., each term of the sequence is defined as a function of the preceding terms

A recursive formula must be accompanied by *initial conditions* (information about the beginning of the sequence).

Fibonacci Sequence

$$\boxed{f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0,\ f_1 = 1}$$

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Leonardo Pisano Bigolio (c. 1170-c. 1250)

Image from wikipedia

The first method is called *backtracking*, and consists of taking a linear recurrence defining a_n, and replace the terms $a_{n-1}, a_{n-2}, \dots$ with the relation that defines a_n, but where n is replaced by $n-1$, $n-2$, etc. This is best illustrated on an example.

Example 58. Consider the linear recurrence

$$a_n = a_{n-1} + 3,\ a_1 = 2.$$

Then

$$\begin{aligned} a_{n-1} &= a_{n-2} + 3 \\ a_{n-2} &= a_{n-3} + 3 \\ a_{n-3} &= a_{n-4} + 3 \end{aligned}$$

and so on and so forth. Therefore

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3 \\ &= (a_{n-3} + 3) + 6 = a_{n-3} + 3 \cdot 3 \\ &= \dots \\ &= a_1 + 3(n-1). \end{aligned}$$

The last equality follows because a generic term is of the form $a_{n-i} + 3i$, therefore when $n - i = 1$, $i = n - 1$. By plugging the initial condition, we conclude

$$a_n = 2 + 3(n-1).$$

Once the solution has been found, you may wonder how to check whether this is the right answer. One way to do it is by proving it by induction!

Example 59. Let us provide a proof by induction for the above example. Define $P(n) =$ "$a_n = 2 + 3(n-1)''$. Then $P(1) =$ "$a_1 = 2$", which is the initial condition, is true. Suppose $P(k) =$ "$a_k = 2 + 3(k-1)''$ is true. We want to prove $P(k+1)$.

$$\begin{aligned} a_{k+1} &= a_k + 3 \\ &= 2 + 3(k-1) + 3 \\ &= 2 + 3k \end{aligned}$$

as desired.

Solving Recurrence Relation

- *Backtracking* is a technique for finding explicit formula for recurrence relation
- E.g., say $a_n = a_{n-1}+3$ and $a_1=2$
- $a_n = a_{n-1}+3 = (a_{n-2}+3)+3 = a_{n-2}+2*3$
  $= (a_{n-3}+3)+2*3 = a_{n-3}+3*3$
  $= (a_{n-2}+3)+3 = a_{n-2}+2*3$
  …
  $= a_1+(n-1)*3$
  $a_n = 2+(n-1)*3$

Homogeneous Relation Of Degree *d*

A *linear homogeneous relation* of degree *d* is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

Examples

- The Fibonacci sequence
- The relation: $a_n=2a_{n-1}$ (degree 1)
- But **not** the relation: $a_n=2a_{n-1}+1$

The *characteristic equation* of the above relation is

$$x^d = c_1 x^{d-1} + c_2 a x^{d-2} + \ldots + c_d$$

Definition 24. A *linear homogeneous relation* of degree d is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}.$$

Its *characteristic equation* is

$$x^d = c_1 x_{d-1} + c_2 x_{d-2} + \ldots + c_d.$$

The characteristic equation is obtained from $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$ by replacing a_i by x^i:

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} - \ldots - c_d x^{n-d} = 0,$$

then factor out x^{n-d} to get

$$x^{n-d}(x^d - c_1 x^{d-1} - c_2 x^{d-2} - \ldots - c_d) = 0.$$

The meaning of replacing a_i by x^i is that we are looking for a solution a_n of the linear recurrence which takes the form $a_n = x^n$.

Example 60. The Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ is a homogeneous relation. Let us compute its characteristic equation:

$$x^n - x^{n-1} - x^{n-2} = 0 \iff x^{n-2}(x^2 - x - 1) = 0$$

therefore $x^2 - x - 1 = 0$ is the characteristic equation.

Let us focus on quadratic characteristic equations, that is of the form

$$x^2 - c_1 x - c_2 = 0$$

which corresponds to linear recurrences of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}.$$

This means that we are looking for a solution a_n of the linear recurrence which is of the form $a_n = x^n$. If such an x exists, then it must satisfy the characteristic equation $x^2 - c_1 x - c_2 = 0$, that is, we are looking for roots (or zeroes) of the characteristic equation. Suppose that $x^2 - c_1 x - c_2 = 0$ has two distinct real roots s_1, s_2, then

$$s_1^2 - c_1 s_1 - c_2 = 0,\ s_2^2 - c_1 s_2 - c_2 = 0.$$

Theorem

If the *characteristic equation* $x^2-c_1x-c_2=0$ (of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$) has

- two distinct roots s_1,s_2, then the explicit formula for the sequence a_n is

$$\boxed{u.s_1^n+v.s_2^n}$$

- a single root s, then the explicit formula for a_n is

$$\boxed{u.s^n+v.n.s^n}$$

where u & v are determined by initial conditions.

Example

Determine the number of bit strings (i.e., comprising 0/1s) of length n that contains *no adjacent 0s.*

- C_n = this number of bit strings
- A binary string with no adjacent 0s is constructed by
  - Adding "1" to any string w of length n-1 satisfying the condition, or
  - Adding "10" to any string v of length n-2 satisfying the condition

- Thus $C_n = C_{n-1} + C_{n-2}$ where $C_1=2$ (0,1), $C_2=3$ (01, 10, 11)

Therefore

$$s_1^n - c_1 s_1^{n-1} - c_2 s^{n-2} = 0,\ s_2^n - c_1 s_2^{n-1} - c_2 s^{n-2} = 0$$

and we have that if s is a solution of $x^2 - c_1 x - c_2 = 0$ then s^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. This tells us that solutions of a_n are composed of s_1^n, s_2^n. Note the term "composed" is used, because if a sequence a'_n also satisfies the recurrence of a_n, then $a_n + a'_n$ satisfies the recurrence of a_n as well, as does multiples of a_n (see Exercise 49).

This means that the final solution is really a composition of s_1^n, s_2^n, namely a solution for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is given by

$$a_n = u s_1^n + v s_2^n,$$

where u, v depend on the initial conditions (that is on a_0, a_1).

Suppose now that $x^2 - c_1 x - c_2 = 0$ has one double real root s, that is $x^2 - c_1 x - c_2 = (x - s)^2$. Then

$$s^2 - c_1 s - c_2 = 0,$$

and s^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ as for the case of two distinct roots. We obtained the characteristic equation from

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} = x^{n-2}(x^2 - c_1 x - c_2) = 0.$$

If s is a root of this equation, then s is a root of its derivative:

$$n x^{n-1} - c_1 (n-1) x^{n-2} - c_2 (n-2) x^{n-3} = 0.$$

Therefore s satisfies both

$$s^n = c_1 s^{n-1} + c_2 s^{n-2},\ n s^n = c_1 (n-1) s^{n-1} + c_2 (n-2) s^{n-2}. \tag{6.1}$$

If we combine s^n and ns^n, as we did for s_1^n and s_2^n, we get

$$a_n = u s^n + v n s^n,$$

and $a_{n+1} = u s^{n+1} + v(n+1) s^{n+1}$. We are left to check

$$\begin{aligned} c_1 a_{n+1} + c_2 a_{n+2} &= c_1 (u s^{n+1} + v(n+1) s^{n+1}) + c_2 (u s^{n+2} + v(n+2) s^{n+2}) \\ &= u[c_1 s^{n+1} + c_2 s^{n+2}] + v[c_1 (n+1) s^{n+1} + c_2 (n+2) s^{n+2}] \\ &= u s^{n+2} + v(n+2) s^{n+2} \\ &= a_{n+2} \end{aligned}$$

using (6.1), which is consistent with our recurrence relation.

Example

- Now solve $C_n = C_{n-1} + C_{n-2}$ where $C_1=2$, $C_2=3$
- Characteristic equation: $x^2-x-1=0$
- Its roots are $(1+\sqrt{5})/2$

  $(1-\sqrt{5})/2$

Recall roots of quadratic eqn.

$$a.x^2 + b.x + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

- Thus

$$\boxed{C_n = u.(\tfrac{1+\sqrt{5}}{2})^n + v.(\tfrac{1-\sqrt{5}}{2})^n}$$

Example

Initial conditions give us:

$$C_1 = u.(\tfrac{1+\sqrt{5}}{2}) + v.(\tfrac{1-\sqrt{5}}{2}) = 2$$

$$\text{i.e., } \tfrac{u+v}{2} + \tfrac{(u-v)\sqrt{5}}{2} = 2$$

$$C_2 = u.(\tfrac{1+\sqrt{5}}{2})^2 + v.(\tfrac{1-\sqrt{5}}{2})^2 = 3$$

$$\text{i.e., } \tfrac{3(u+v)}{2} + \tfrac{(u-v)\sqrt{5}}{2} = 3$$

Solving, we get

$$u = \frac{\sqrt{5}+3}{2\sqrt{5}}$$

$$v = \frac{\sqrt{5}-3}{2\sqrt{5}}$$

Example 61. Suppose we want to determine the number of bit strings of length n that contains no adjacent zeroes. We denote this number by C_n. We first observe that there are two ways of obtaining such sequences from a smaller sequence. One may take any string:

- of length $n-1$ satisfying the condition and add a 1 (one cannot add a 0, since the sequence may finish by 0),
- of length $n-2$ satisfying the condition and add 10 (one cannot add 00, or 01 since the string could finish by 0, and 11 is included above).

Therefore the linear recurrence involved is

$$C_n = C_{n-1} + C_{n-2}.$$

The characteristic equation is obtained from

$$x^n - x^{n-1} - x^{n-2} = x^{n-2}(x^2 - x - 1) = 0,$$

it is $x^2 - x - 1 = 0$. To find its roots, we compute

$$\frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

The solution is then

$$C_n = u(\frac{1+\sqrt{5}}{2})^n + v(\frac{1-\sqrt{5}}{2})^n.$$

We are left to find u, v based on the initial conditions. They are $C_1 = 2$ (the strings are 0 and 1), while $C_2 = 3$ (the strings are 11,01,10). This gives us two equations for two unknowns:

$$\begin{aligned} u\left(\frac{1+\sqrt{5}}{2}\right) + v\left(\frac{1-\sqrt{5}}{2}\right) &= 2 \\ u\left(\frac{1+\sqrt{5}}{2}\right)^2 + v\left(\frac{1-\sqrt{5}}{2}\right)^2 &= 3. \end{aligned}$$

which can be simplified to

$$\begin{aligned} \frac{u+v}{2} + \sqrt{5}\frac{u-v}{2} &= 2 \\ 3\frac{u+v}{2} + \sqrt{5}\frac{u-v}{2} &= 3. \end{aligned}$$

Set $a = (u+v)/2$, $b = \sqrt{5}(u-v)/2$. We need to solve

$$a + b = 2,\ 3a + b = 3 \Rightarrow b = \frac{3}{2},\ a = \frac{1}{2}.$$

Thus

$$v = \frac{\sqrt{5}-3}{2\sqrt{5}},\ u = 1 - \frac{\sqrt{5}-3}{2\sqrt{5}} = \frac{\sqrt{5}+3}{2\sqrt{5}}.$$

6.1 Supplementary notes

Here is another explanation regarding characteristic equations which uses linear algebra. Suppose we want to solve the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. Write it in matrix form as

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

The second equation may seem redundant, but it is clearly correct! Since $a_{n-1} = c_1 a_{n-2} + c_2 a_{n-3}$, we further have

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-2} \\ a_{n-3} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

and by repeating the argument, we obtain

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

Next compute the characteristic polynomial of

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$$

which by definition is

$$\det \begin{bmatrix} c_1 - x & c_2 \\ 1 & -x \end{bmatrix} = -x(c_1 - x) - c_2 = x^2 - xc_1 - c_2.$$

Suppose that this polynomial has two distinct roots s_1 and s_2, which are distinct (this method does not apply otherwise). Then we can write

$$x^2 - xc_1 - c_2 = (x - s_1)(x - s_2).$$

Let $m_1 = (m_{11}, m_{12})$ be the vector (called eigenvector) that satisfies

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} = s_1 \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix}$$

and let $m_2 = (m_{21}, m_{22})$ be the vector (the other eigenvector) that satisfies

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_{21} \\ m_{22} \end{bmatrix} = s_2 \begin{bmatrix} m_{21} \\ m_{22} \end{bmatrix}.$$

Putting both together, we get that

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} = \begin{bmatrix} s_1 m_{11} & s_2 m_{21} \\ s_1 m_{12} & s_2 m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

or in other words

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}^{-1}.$$

Note that the matrix is invertible because $s_1 \neq s_2$.

We go back to our recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, which is in matrix

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} \iff \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}^{-1} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

Now we expression the linear recurrence using the initial conditions:

$$\begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} \iff \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} s_1^{n-1} & 0 \\ 0 & s_2^{n-1} \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

Now without knowing the value of m_{ij}, one can already tell that a_{n-1} is of the form

$$a_{n-1} = u s_1^{n-1} + v s_2^{n-1}$$

for some constant u, v to be determined.

This tells us how to write a generic solution a_{n-1} as a function of s_1, s_2 which are the roots of $x^2 - x c_1 - c_2 = 0$.

Exercises for Chapter 6

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Exercise 49. Let $a_n = c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Exercise 50. Solve the following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

Exercise 51. Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - b_{n-2},\ b_0 = 2,\ b_1 = 4.$$

Examples for Chapter 6

Linear Recurrence relations are often useful to analyze algorithms, such as divide-and-conquer algorithms. We will illustrate this using the game of Hanoi tower.

In this game, the goal is to move n disks ranked from the largest at the bottom to the smallest on top from one post to another. The only permitted action is to remove the top disk from a post and drop it onto another post. The rule is that a larger disk can never lie above a smaller disk on any post. When $n = 3$ disks, the Hanoi tower game can be solved in 7 steps. But say one would like to know how many steps it would take to solve it for $n = 50$ disks, how could this be figured out?

The method is to derive a linear recurrence relation, and then to solve it. To find a linear recurrence relation, notice that to solve the Hanoi tower game for $n = 3$ disks, the following steps are done:

1. Solve a Hanoi tower game for $n = 2$ disks,
2. Move the largest disk,
3. Solve another Hanoi tower game for $n = 2$ disks.

.

In fact, this is true in general, which yields the linear recurrence

$$T_n = 2T_{n-1} + 1,$$

where T_n denotes the number of steps for n disks.

We solve this linear recurrence using backtracking. Note that

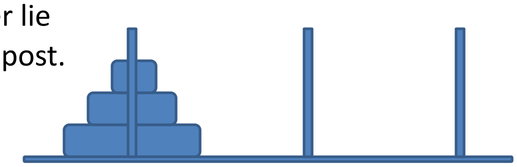
$$T_{n-1} = 2T_{n-2} + 1,\ T_{n-2} = 2T_{n-3} + 1,\ T_{n-3} = 2T_{n-4} + 1, \ldots$$

Then

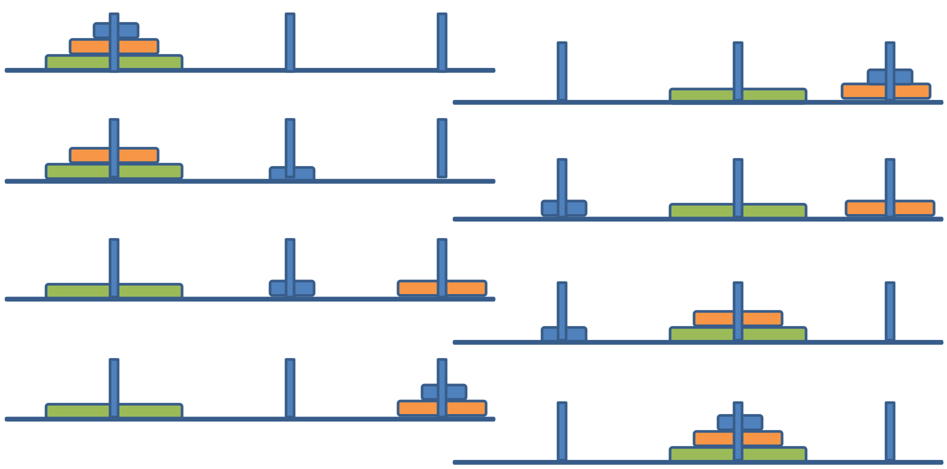
$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2T_{n-2} + 1) = 4T_{n-2} + 3 \\ &= 4(2T_{n-3} + 1) + 3 = 8T_{n-3} + 7 \\ &= 8(2T_{n-4} + 1) + 7 = 16T_{n-4} + 15 \\ &= \ldots \end{aligned}$$

Hanoi Tower

- **Goal:** move all n disks in the same order, but on a different post.
- **Only permitted action:** remove the top disk from a post and drop it onto another post.
- **Rule:** a larger disk can never lie above a smaller disk on any post.



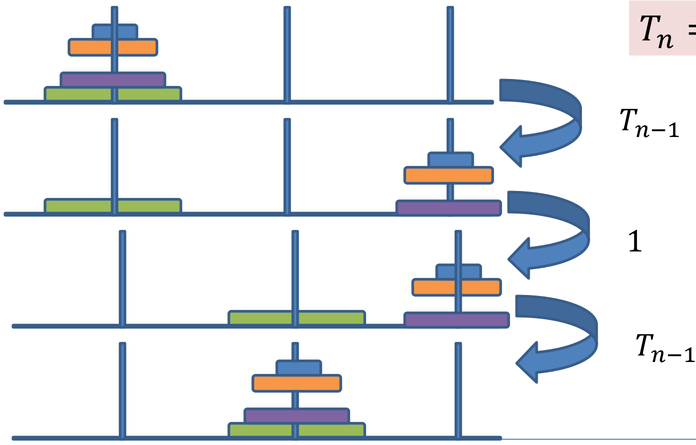
Hanoi Tower (n=3)



Find a Recurrence

- T_n = minimum number of steps needed to move an n-disk tower from one post to another

$T_n = 2T_{n-1}+1$



Backtracking

$T_1 = 1$ $T_n = 2T_{n-1}+1$

Backtracking

$T_1 = 1$ $\quad T_n = 2T_{n-1}+1$

T_n
$= 2T_{n-1} + 1 = 2(2T_{n-2} + 1) + 1 \Leftarrow 3$
$= 4T_{n-2} + 3 = 4(2T_{n-3} + 1) + 3 \Leftarrow 7$
$= 8T_{n-3} + 7 \Leftarrow 15$

$T_n = 2^n - 1$

Induction

- P(n)=``$T_n = 2^n - 1$''
- Basis step: P(1)=``$T_1 = 1$''
- Inductive step: suppose P(n) is true.
- To show, P(n+1).
- $T_{n+1} = 2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$

We notice that a general term is of the form

$$2^i T_{n-i} + (2^i - 1),$$

therefore when $n - i = 1$, $i = n - 1$, and we get

$$T_n = 2^{n-1} T_1 + (2^{n-1} - 1)$$

with $T_1 = 1$. Thus finally

$$T_n = 2^n - 1.$$

Once a solution has been found by backtracking, it is advised to confirm that the solution is sound, by performing a proof by induction. Here $P(n) =$ "$T_n = 2^n - 1$. The basis step is $P(1) =$ "$T_1 = 2^1 - 1 = 1$" which is true. The induction step is to assume that $P(k) =$ "$T_k = 2^k - 1$" is true. We need to prove $P(k+1)$. But

$$T_{k+1} = 2T_k + 1 = 2 \cdot 2^k + 1 = 2^{k+1}.$$

as desired!