Basis

Reading: Strang 3.5

Learning objective: Be able to explain the motivation behind the definition of a basis.

Basis: Motivation

A basis is a way to give a "name" to every vector in a vector space.

What makes a naming system good?

- § Every vector should have a name.
- § No vector has two different names.

In other words, every vector should have a unique name.

Basis: Example

The vectors $(1,0),(0,1)$ are a basis for $\mathbb{R}^2$.

This basis gives our usual names for vectors in $\mathbb{R}^2$.

The name (a_1,a_2) corresponds to the vector

$$a_1 \cdot (1,0) + a_2 \cdot (0,1)$$

This is a good naming system. Every vector in $\mathbb{R}^2$ can be written as a unique linear combination of $(1,0),(0,1)$.

Another Example

Another example of a basis for $\mathbb{R}^2$ are the vectors $(1,1),(1,-1)$.

Every vector in $\mathbb{R}^2$ can be written as a unique linear combination of these two vectors.

Basis: Motivation

Let V be a vector space. We want to "name" the vectors in V by writing them as a linear combination of some set of vectors $v_1, \ldots, v_n$.

When $v_1, \ldots, v_n$ are a basis for V, this provides a good naming system.

§ Every vector should have a name.

$$\text{span}(\{v_1, \ldots, v_n\}) = V$$

§ No vector has two different names.

$v_1, \ldots, v_n$ are linearly independent.

Linear Independence

Let's see why linear independence ensures no vector has two different names.

Let $v_1, \ldots, v_n$ be linearly independent and let $w \in V$.

Suppose that w has two different "names", that is

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$
$$w = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$

Then

$$w - w = \mathbf{0} = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n$$

Linear Independence

Let $v_1, \ldots, v_n$ be linearly independent and let $w \in \mathrm{span}(\{v_1, \ldots, v_n\})$.

Suppose that w has two different "names", that is

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$
$$w = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$

Then

$$w - w = \mathbf{0} = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n$$

As $v_1, \ldots, v_n$ are linearly independent this means

$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$$

Thus w cannot have two different names.

Basis: Definition

Let V be a vector space. We say that the sequence of vectors $v_1, \ldots, v_n$ is a basis for V if and only if

- § $\operatorname{span}(\{v_1, \ldots, v_n\}) = V$
- § the sequence $v_1, \ldots, v_n$ is linearly independent.

Basis: Intuition

Let V be a vector space. A basis for V is a sequence of vectors $v_1, \ldots, v_n$ such that

1) $\mathrm{span}(\{v_1, \ldots, v_n\}) = V$

2) the sequence $v_1, \ldots, v_n$ is linearly independent.

The first condition is easier to satisfy with more vectors.

The second condition is easier to satisfy with fewer vectors.

The two conditions together give just the right number of vectors.

Basis: Intuition

Let V be a vector space. A basis for V is a sequence of vectors $v_1, \ldots, v_n$ such that

1) $\mathrm{span}(\{v_1, \ldots, v_n\}) = V$

2) the sequence $v_1, \ldots, v_n$ is linearly independent.

A basis is a smallest sequence of vectors whose span is V.

→ No vector is redundant.

A basis is a largest set of linearly independent vectors in V.

→ No independent vector can be added.

Example 1

The vectors $(1,0,0),(0,1,0),(0,0,1)$ are a basis for $\mathbb{R}^3$.

In general if I is an n-by-n identity matrix, the columns of I form a basis for $\mathbb{R}^n$.

This is called the standard basis of $\mathbb{R}^n$.

Example 2

Let A be an n-by-n invertible matrix.

The columns of A form a basis for $\mathbb{R}^n$.

As A is invertible, $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.

$\longrightarrow$ The span of the columns is $\mathbb{R}^n$.

As A is invertible, $N(A) = \{\vec{0}_n\}$.

$\longrightarrow$ The columns of A are linearly independent.

Example 3

Let $M_{3,3}$ be the vector space of 3-by-3 matrices, and T the subspace of upper-triangular matrices.

The sequence of matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

form a basis for T.

We have already seen that these matrices span T and are linearly independent.

Example 4

Let F be the vector space of real valued functions and P_2 be the subspace of polynomials of degree ≤ 2.

The functions $1, x, x^2$ form a basis for P_2.

In general, for the subspace P_n of polynomials of degree $\leq n$, the functions

$$1, x, x^2, \ldots, x^n$$

form a basis for P_n.

Many Bases

The plural of basis is bases.

A vector space can have many different bases.

The columns of any n-by-n invertible matrix provide a basis for $\mathbb{R}^n$.

The key fact is that any two bases for a vector space have the same number of elements.

This is how we define the dimension of a vector space V: the number of elements in any basis for V.

Dimension

Reading: Strang 3.5

Learning objective: Be able to derive the dimension lemma for subspaces of $\mathbb{R}^k$.

Dimension Lemma

Lemma: Let $S \subseteq \mathbb{R}^k$ be a subspace and suppose that $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_m\}) = S$. If $n > m$ then any sequence of vectors $\vec{w}_1, \ldots, \vec{w}_n \in S$ is linearly dependent.

Proof:

As $\vec{v}_1, \ldots, \vec{v}_m$ span S, we can write each $\vec{w}_i$ in terms of $\vec{v}_1, \ldots, \vec{v}_m$.

$$
\begin{aligned}
\vec{w}_1 &= a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{m1}\vec{v}_m \\
\vec{w}_2 &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{m2}\vec{v}_m \\
&\vdots \\
\vec{w}_n &= a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \cdots + a_{mn}\vec{v}_m
\end{aligned}
$$

Dimension Lemma

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\end{aligned}$$

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

Dimension Lemma

$$\underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{w_1} & \vec{w_2} & \cdots & \vec{w_n} \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}_{A}$$

Recall that we are given $n > m$ and want to show that $w_1, \ldots, w_n$ are linearly dependent.

The matrix A has more columns than rows. It must have a free column after Gaussian elimination.

There exists a nonzero vector $\vec{u}$ such that $A\vec{u} = \vec{0}_m$.

Dimension Lemma

$$\underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}_{A}$$

There exists a nonzero vector $\vec{u}$ such that $A\vec{u} = \vec{0}_m$.

Then $W\vec{u} = VA\vec{u} = V\vec{0}_m = \vec{0}_k$.

Thus the columns of the matrix W are linearly dependent.

The sequence of vectors $\vec{w}_1, \ldots, \vec{w}_n$ is linearly dependent.

Dimension Theorem

Lemma: Let $S \subseteq \mathbb{R}^k$ be a subspace and suppose that $\mathrm{span}(\{\vec{v}_1, \ldots, \vec{v}_m\}) = S$. If $n > m$ then any sequence of vectors $\vec{w}_1, \ldots, \vec{w}_n \in S$ is linearly dependent.

Theorem: Let $S \subseteq \mathbb{R}^k$ be a subspace and suppose that $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are two bases for S. Then $m = n$.

In other words:

Any two bases for S have the same number of elements.

Dimension Theorem

Lemma: Let $S \subseteq \mathbb{R}^k$ be a subspace and suppose that $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_m\}) = S$. If $n > m$ then any sequence of vectors $\vec{w}_1, \ldots, \vec{w}_n$ is linearly dependent.

Theorem: Let $S \subseteq \mathbb{R}^k$ be a subspace and suppose that $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are two bases for S. Then $m = n$.

Proof: Suppose that $n > m$. Then as $\vec{v}_1, \ldots, \vec{v}_m$ span S, by the dimension lemma this means $\vec{w}_1, \ldots, \vec{w}_n$ are linearly dependent, a contradiction.

We also reach a contradiction when $m > n$, thus $m = n$.

Finite-dimensional spaces

A vector space V is called finite-dimensional if there is a finite set $\{v_1, \ldots, v_n\}$ with $\text{span}(\{v_1, \ldots, v_n\}) = V$.

We will only discuss bases of finite-dimensional vector spaces.

Of the vector spaces we have discussed, only the space of real-valued functions is not finite-dimensional.

Dimension Theorem

We just proved the dimension theorem for subspaces of $\mathbb{R}^k$.

The same theorem holds for any finite-dimensional vector space:

Let V be a finite-dimensional vector space and $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be two bases for V. Then $m = n$.

It can be proved in the same way, and we will omit the proof.

Dimension

Definition: Let V be a finite-dimensional vector space. The dimension of V is the number of elements in any basis for V.

This definition makes sense because of the dimension theorem.

Examples

The dimension of $\mathbb{R}^n$ is n.

The dimension of the space of 3-by-3 upper triangular matrices is 6.

We have seen a basis is given by

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

The dimension of P_n, the space of polynomials of degree at most n, is $n+1$.