Determinants

Reading: Strang 5.1

Learning objective: Be able to compute the determinant by Gaussian elimination.

Review

Last week we defined the determinant function by the the three properties we want it to satisfy.

These properties are chosen to make it to easy to see how the determinant of a matrix changes under elementary row operations.

We can then use Gaussian elimination to show our main result about the determinant: $\det(A) \neq 0$ if and only if A is invertible.

Three Properties

The determinant is a function from square matrices to the real numbers that satisfies the following properties:

1) The determinant of the n-by-n identity matrix is one.

2) The determinant changes sign when two rows are exchanged.

3) The determinant is a linear function of each row separately.

Third Property

3) The determinant is a linear function of each row separately.

This property is saying

§ $\det(A(1,:),\ldots,t\cdot A(i,:),\ldots,A(n,:)) = t\cdot\det(A(1,:),\ldots,A(i,:),\ldots,A(n,:))$

§ $$\begin{aligned}\det(A(1,:),\ldots,A(i,:)+\vec{b}^T,\ldots,A(n,:)) = &\det(A(1,:),\ldots,A(i,:),\ldots,A(n,:)) \\ &+\det(A(1,:),\ldots,\vec{b}^T,\ldots,A(n,:))\end{aligned}$$

Consequences

Now let's look at some consequences of the properties we have given.

Consequence 1: If two rows of A are equal then $\det(A) = 0$.

This is a consequence of the determinant changing sign when two rows are exchanged.

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \Longrightarrow \begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

Zero Row

Consequence 2: If A has an all-zero row then $\det(A) = 0$.

This is a consequence of the linearity of the determinant in each row.

$$\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 \cdot 0 & 0 \cdot 0 & 0 \cdot 0 \\ g & h & i \end{vmatrix} = 0$$

If we multiply the second row by zero, this multiplies the determinant by zero.

Determinant and AMORTA

Consequence 3: The determinant is unchanged by adding a multiple of one row to another.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{vmatrix}$$

Reason: By linearity in the second row:

$$\begin{vmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

Reason: By linearity in the second row:

$$\begin{vmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

For the last equality we used that the determinant of a matrix with two equal rows is zero.

Elementary row operations

Now we know how the determinant is affected by all elementary row operations.

§ Adding a multiple of one row to another: determinant is unchanged.

§ Exchanging two rows: determinant is multiplied by -1.

§ Multiplying a row by c: determinant is multiplied by c.

This allows us to compute determinants by Gaussian elimination.

Diagonal Matrix

Theorem: The determinant of a diagonal matrix is the product of its diagonal entries.

Reason: We express each row as a scalar times the corresponding row of the identity matrix and apply linearity:

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 5 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 5 \cdot 4 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 5 \cdot 4 \cdot 3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 5 \cdot 4 \cdot 3$$

Upper Triangular Matrix

Theorem: The determinant of an upper triangular matrix is the product of its diagonal entries.

Reason: First say all diagonal entries are nonzero.

Then we can, starting from the bottom, add a multiple of one row to another to create a diagonal matrix.

This does not change the determinant.

$$\begin{vmatrix} 5 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

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The determinant of a diagonal matrix is the product of its diagonal entries.

$$\begin{vmatrix} 5 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 5 \cdot 4 \cdot 3$$

Theorem: The determinant of an upper triangular matrix is the product of its diagonal entries.

Reason: Now say some diagonal entry is zero.

Then by the action of adding a multiple of one row to another we can create an all-zero row.

The determinant will be zero.

Lower Triangular Matrix

We can argue in a similar fashion for a lower triangular matrix.

Theorem: The determinant of a lower triangular matrix is the product of its diagonal entries.

Example

Let's compute the determinant of this matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

We proceed by Gaussian elimination. Adding a multiple of one row to another does not change the determinant.

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{vmatrix}$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{4}{3}$$

Determinant and Invertibility

Theorem: The absolute value of $\det(A)$ equals the absolute value of the product of the pivots of A.

Reason: By adding a multiple of one row to another and doing row exchanges, we can transform A into an upper triangular matrix U.

Neither of these operations changes the absolute value of the determinant.

Thus $|\det(A)| = |\det(U)|$, and $|\det(U)|$ is the absolute value of the product of the pivots.

Determinant and Invertibility

Corollary: A is invertible iff $\det(A) \neq 0$.

Reason: A is invertible iff it has a full set of pivots.

From the last theorem $\det(A) \neq 0$ iff A has a full set of pivots.

Determinant of a Product

Next we will derive a very useful property of the determinant.

$$\boxed{\det(AB) = \det(A)\det(B)}$$

For example, this tells us the determinant of the inverse of an invertible matrix.

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\implies \det(A^{-1}) = \frac{1}{\det(A)}$$

Determinant of a Product

$$\boxed{\det(AB) = \det(A)\det(B)}$$

First, we will prove a special case:

$$\det(EB) = \det(E)\det(B)$$

where E is an elementary matrix.

We show this for each of the three types of elementary matrices in turn.

AMORTA

$$\det(EB) = \det(E)\det(B)$$

Say that E is an elementary matrix corresponding to adding a multiple of one row to another.

Then $\det(EB) = \det(B)$. Adding a multiple of one row to another does not change the determinant.

Also $\det(E) = 1$. E is a triangular matrix with ones on the diagonal.

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Row Swap

$$\det(EB) = \det(E)\det(B)$$

Say that E is an elementary matrix corresponding to exchanging two rows.

Then $\det(EB) = -\det(B)$. Exchanging two rows multiplies the determinant by -1.

Also $\det(E) = -1$. E is formed by swapping two rows of the identity matrix, which has determinant 1.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

Multiplying a row by a scalar

$$\det(EB) = \det(E)\det(B)$$

Say that E is an elementary matrix corresponding to multiplying a row by c.

Then $\det(EB) = c\det(B)$. By multilinearity, multiplying a row by c multiplies the determinant by c.

Also $\det(E) = c$. E is formed by multiplying a row of the identity matrix by c.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{vmatrix} = c$$

Product of Elementary Matrices

Now we have finished the special case

$$\det(EB) = \det(E)\det(B)$$

where E is an elementary matrix.

Let's look at a consequence of this:

If $E_1, \ldots, E_k$ are all elementary matrices, then

$$\det(E_1 E_2 \cdots E_k) = \det(E_1)\det(E_2)\cdots\det(E_k)$$

Determinant of a Product

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Proof: First suppose that A is singular.

Then AB is singular as well.

Thus $\det(A) = 0$ and $\det(AB) = 0$.

The equality holds.

Determinant of a Product

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Proof: Now suppose A is invertible.

Then from the problem set, A can be written as a product of elementary matrices:

$$A = E_1 E_2 \cdots E_k$$

Note that

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_k)$$

Determinant of a Product

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Proof:

$$A = E_1 E_2 \cdots E_k$$

$$\begin{aligned}
\det(AB) &= \det(E_1 E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2)\det(E_3 \cdots E_k B) \\
&= \det(E_1)\cdots\det(E_k)\det(B) \\
&= \det(A)\det(B)
\end{aligned}$$

Permutation Matrix

We can apply this theorem to permutation matrices.

Every permutation matrix is a product of row swap matrices.

$$P = R_k \cdots R_2 R_1$$

$$\begin{aligned}\det(P) &= \det(R_k) \cdots \det(R_2)\det(R_1) \\ &= (-1)^k\end{aligned}$$

as each row swap matrix has determinant -1.

Example

The permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a product of two row swap matrices.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore its determinant is 1.

Permutation Matrices

We have seen that that for a permutation matrix P

$$P^{-1} = P^T$$

By the product theorem for determinant,

$$\det(P)\det(P^T) = 1$$

As $\det(P) \in \{-1, 1\}$ this means

$$\det(P) = \det(P^T)$$

Determinant and Transpose

For an upper triangular matrix U as well

$$\det(U) = \det(U^T)$$

The determinant of U is the product of its diagonal entries.

The determinant of U^T is also the product of its diagonal entries as it is lower triangular.

And U and U^T have the same entries on the diagonal.

Determinant and Transpose

For any square matrix A

$$\boxed{\det(A) = \det(A^T)}$$

Tomorrow we will see why this is true.