Exercises for Chapter 7

Exercise 52.

1. Show that
$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$
for $1 \leq k \leq l$, where by definition
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \; n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$
You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Exercise 53. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

Exercise 54. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:
$$P(A \cap B), \; P(A), \; P(A \cup B), \; P(A \times B).$$

Exercise 55. Prove by contradiction that for two sets A and B
$$(A - B) \cap (B - A) = \varnothing.$$

Exercise 56. Let $P(C)$ denote the power set of C. Prove that for two sets A and B
$$P(A) = P(B) \iff A = B.$$

Exercise 57. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 58. Show that the empty set is a subset of all non-null sets.

Exercise 59. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Exercise 60. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 61. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 62. How many subsets of $\{1, \dots, n\}$ are there with an even number of elements? Justify your answer.

Exercise 63. Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Exercise 64. Let A, B, C be sets. Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Exercise 65. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 66. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 67. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Exercise 68. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Exercise 69. Let A, B, C be sets. Prove or disprove the following set equality:

$$A \times (B - C) = (A \times B) - (A \times C).$$