MH1200 Problem Set 11 Solutions

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Problem 1. Let A be a 4-by-4 matrix. If you exchange the first two rows of A, does the column space of A stay the same? How about the row space? How about the nullspace?

Solution: If we exchange the first two rows of A, the row space and the nullspace will stay the same. The column space can change.

Let E be any invertible matrix. Note that

$$A\vec{u} = \vec{0} \iff EA\vec{u} = E\vec{0} \iff EA\vec{u} = \vec{0} \ ,$$

thus the nullspace of A and EA is the same.

The rowspace of EA is contained in the rowspace of A, as every row of EA is a linear combination of rows of A. Similarly, the rowspace of $E^{-1}EA = A$ is contained in the row space of EA as every row of $E^{-1}EA$ is a linear combination of rows of EA. Thus A and EA have the same row space.

The column space of A can change, as seen by the following example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ .$$

After we swap rows 1 and 2, the row space contains the vector $(2, 1, 0, 0)$, which is not in the columnspace of A.

Problem 2. Let A be an m-by-n matrix of rank r. Suppose there are $b \in \mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ has *no solution*. What inequalities must hold among m, n and r ? Can it be the case that $N(A^T) = \{\vec{0}_m\}$?

Solution: This means that the column space of A is not equal to $\mathbb{R}^m$. Thus the dimension of the column space, r, must be less than m. We also have $r \leq n$ as r is also the dimension of the row space, which is at most n. There can be any relationship between m and n.

The dimension of the left nullspace is $m - r$, thus as $r < m$ the dimension of the left nullspace is at least 1. The left nullspace cannot be $\{\vec{0}_m\}$.

Problem 3. Let A be the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 5 & 2 \\ 2 & 0 & 0 & 2 & -4 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & 4 & 0 & 3 & 6 \end{bmatrix}.$$

1. Find $R = \text{rref}(A)$, the reduced row echelon form of A.

2. Let $\hat{R}$ be the matrix R with any zero rows removed. Find a matrix $\hat{B}$ such that $A = \hat{B}\hat{R}$. Hint: The columns of $\hat{B}$ are determined by the columns of A corresponding to pivot columns of R.

3. Find a basis for the row space, column space, nullspace and left nullspace of A.

Solution

1. Doing some Gaussian elimination gives

$$\begin{bmatrix} 1 & 4 & 0 & 5 & 2 \\ 2 & 0 & 0 & 2 & -4 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & 4 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 5 & 2 \\ 0 & -8 & 0 & -8 & -8 \\ 0 & 4 & 1 & 4 & 3 \\ 0 & 8 & 0 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 5 & 2 \\ 0 & -8 & 0 & -8 & -8 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 4 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

   We have found the reduced row echelon form of A.

2. The pivot columns of R are columns $1, 2, 3$. Thus the columns of $\hat{B}$ must be columns $1, 2, 3$ of A.

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

3. A basis for the row space of A is given by the rows of $\hat{R}$. That is a basis for the row space of A is $(1, 0, 0, 1, -2), (0, 1, 0, 1, 1), (0, 0, 1, 0, -1)$.

   A basis for the column space of A is given by the columns of $\hat{B}$. A basis for the column space of A is $(1, 2, -1, -1), (4, 0, 0, 4), (0, 0, 1, 0)$.

   From R we can determine the special solutions to the equation $R\vec{x} = \vec{0}$. The special solution corresponding to x_4 is $(-1, -1, 0, 1, 0)$. The special solution corresponding to x_5 is $(2, -1, 1, 0, 1)$. A basis for the nullspace of A is $(-1, -1, 0, 1, 0), (2, -1, 1, 0, 1)$.

As there are three pivots, the dimension of the left nullspace will be 1. Let E be the invertible matrix such that $EA = R$. Since the last row of R is all-zero, the last row of E will be in the left nullspace. Further as the dimension of the left nullspace is 1, the last row of E will be a basis for the left nullspace of A.

I will work slightly differently, essentially constructing the last row of E without having to figure out all of E. How did we create an all-zero row in the Gaussian elimination above? In the first round of Gaussian elimination, we added the first row of A to the last row of A.

$$\begin{bmatrix} 0 & 8 & 0 & 8 & 8 \end{bmatrix} = A(1,:) + A(4,:)$$

In the second round of Gaussian elimination, we then added the second row of A', the matrix created after the first round of Gaussian elimination, to the last row. This already created an all-zero row. As the second row of A' is $A(2,:) - 2 \cdot A(1,:)$ we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A(1,:) + A(4,:) + A(2,:) - 2 \cdot A(1,:) = -A(1,:) + A(2,:) + A(4,:) \ .$$

We have found a linear combination of the rows equal to the all-zero row vector. A basis for the left nullspace of A is $(-1, 1, 0, 1)$.

Problem 4. Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and the nullspace?

Solution: We can use the idea of the rank revealing factorization to achieve this. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix} \ .$$

By construction every column of A is linear combination of $(1, 0, 1)$ and $(1, 2, 0)$ (the columns of the left factor) and also every row of A is a linear combination of $(1, 0, 1)$ and $(1, 2, 0)$ (the rows of the right factor). This means that $C(A) \subseteq \text{span}(\{(1, 0, 1), (1, 2, 0)\})$ and $C(A^T) \subseteq \text{span}(\{(1, 0, 1), (1, 2, 0)\})$. Looking at the columns of A we see that they are not all multiples of each other, thus $\dim(C(A)) \geq 2$. This means that $C(A) = \text{span}(\{(1, 0, 1), (1, 2, 0)\})$. We can use the same argument to see $C(A^T) = \text{span}(\{(1, 0, 1), (1, 2, 0)\})$.

We know that vectors in the nullspace are orthogonal to vectors in the row space. The vector $(1, 0, 1)$ cannot be in both the nullspace and the row space as it is not orthogonal to itself.

Problem 5. Let

$$\vec{v}_1 = (1, -2, 0, 0, 3), \ \vec{v}_2 = (2, -5, -3, -2, 6), \ \vec{v}_3 = (0, 5, 15, 10, 0), \ \vec{v}_4 = (2, 1, 15, 8, 6)$$

Let $S = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\})$. Determine the dimension of S and find a basis for S *among* the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

Solution: Let's make a matrix A whose columns are the vectors $\vec{v_1}, \dots, \vec{v_4}$. Then S will be the column space of A. If we do Gaussian elimination on A and bring it into row echelon form, we know that the columns of A corresponding to the pivot columns will be a basis of the column space of A. Furthermore, the elements of this basis will be columns of A, as needed in this question.

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 1 \\ 0 & -3 & 15 & 15 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 5 \\ 0 & -3 & 15 & 15 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 pivots, thus the dimension of the column space of A is 3, and so is the dimension of S. The pivot columns are columns $1, 2, 4$, thus $\vec{v}_1, \vec{v}_2, \vec{v}_4$ will be a basis for S.

Problem 6. Let P_3 be the vector space of polynomials of degree ≤ 3. Let S be the subspace $S = \{p \in P_3 : p(1) = 0\}$. Determine a basis for S.

Solution: Let us write out the set S in more detail.

$$S = \{a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 : a_0 + a_1 + a_2 + a_3 = 0, a_0, a_1, a_2, a_3 \in \mathbb{R}\} \ .$$

The coefficients of polynomials in S must satisfy the single linear equation

$$a_0 + a_1 + a_2 + a_3 = 0 \ . \tag{1}$$

This is a nullspace problem, and we know how to find the general solution to this equation, it is $\{(-s-t-r, s, t, r) : s, t, r \in \mathbb{R}\}$. A basis for the set of solutions to Equation 1 is given by the three special solutions $(-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)$. We claim that the 3 polynomials with these coefficient vectors are a basis for S.

Claim: A basis for S is given by the polynomials

$$-1 + x, -1 + x^2, -1 + x^3 \ .$$

Proof. First note that each of $-1 + x, -1 + x^2, -1 + x^3$ are elements of S. As $\{(-s-t-r, s, t, r) : s, t, r \in \mathbb{R}\}$ is the general solution to Equation 1, any polynomial whose coefficients satisfy Equation 1 can be written as $s \cdot (-1 + x) + t \cdot (-1 + x^2) + r \cdot (-1 + x^3)$. This implies that $S = \text{span}(\{-1 + x, -1 + x^2, -1 + x^3\})$.

To show these functions are linearly independent, consider a linear combination of them equal to the constant zero function:

$$a_1 \cdot (-1 + x) + a_2 \cdot (-1 + x^2) + a_3 \cdot (-1 + x^3) = -a_1 - a_2 - a_3 + a_1 x + a_2 x^2 + a_3 x^3 = \mathbf{0} \ .$$

As $1, x, x^2, x^3$ are linearly independent, this means that $a_3 = 0, a_2 = 0, a_1 = 0$ and thus the functions are linearly independent as claimed. □

Problem 7. 1. Let $M_{2,2}$ be the vector space of 2-by-2 matrices. Find a basis and compute the dimension of $M_{2,2}$.

2. We have seen that the set S of 3-by-3 *symmetric* matrices is a subspace of $M_{3,3}$, the set of all 3-by-3 matrices. Find a basis for S and determine its dimension.

3. A matrix with $A = -A^T$ is called *skew-symmetric*. Let $W = \{A \in M_{3,3} : A = -A^T\}$ be the subspace of 3-by-3 skew-symmetric matrices. Find a basis for W and determine its dimension.

Solution: 1. An arbitrary element of $M_{2,2}$ is of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

for real numbers a, b, c, d. By separation of parameters we see that this is equal to

$$a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .$$

Thus every matrix in $M_{2,2}$ can be written as a linear combination of the four matrices $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Now let us show that these four matrices are linearly independent. The zero element of $M_{2,2}$ is the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. To check linear independence we look at solutions to the equation

$$a_1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

This means $a_1 = a_2 = a_3 = a_4 = 0$, thus these four matrices are linearly independent. (This is an instance where we can use the 'easy test' for linear independence.)

We have shown that the four matrices span $M_{2,2}$ and are linearly independent, therefore they are a basis for $M_{2,2}$. This is called the *standard basis* for $M_{2,2}$. The dimension of $M_{2,2}$ is 4.

Similarly for the vector space $M_{m,n}$ of m-by-n matrices a basis is given by the sequence of $m \cdot n$ many matrices E_{ij} where $1 \leq i \leq m, 1 \leq j \leq n$ and

$$E_{ij}(k, \ell) = \begin{cases} 1 & \text{if } k = i, \ell = j \\ 0 & \text{otherwise} \end{cases} .$$

The dimension of $M_{m,n}$ is $m \cdot n$.

2. A general 3-by-3 symmetric matrix is of the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} .$$

By separation of parameters this is equal to

$$a\cdot\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+b\cdot\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+c\cdot\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+d\cdot\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+e\cdot\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+f\cdot\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\quad.$$

Thus the span of these 6 matrices is equal to the set of symmetric 3-by-3 matrices.

These 6 matrices are also linearly independent by the *easy test*. Each matrix is nonzero in an entry where all the other matrices are zero. Thus these 6 matrices form a basis for S. The dimension of S is 6.

3. Let's consider a general 3-by-3 matrix A

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$$

If $A=-A^T$, this means $a_{11}=a_{22}=a_{33}=0$ and $a_{21}=-a_{12}, a_{31}=-a_{13}, a_{32}=-a_{23}$. Thus a general skew-symmetric matrix is of the form

$$\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{bmatrix}\quad.$$

Again by separation of parameters we can express this as a linear combination of 3 matrices

$$\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{bmatrix}=a_{12}\cdot\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}+a_{13}\cdot\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+a_{23}\cdot\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}$$

Thus we see that the span of these 3 matrices is the set of all 3-by-3 skew-symmetric matrices. We can also see that the 3 matrices on the right are linearly independent by the easy test. Thus they form a basis for W. The dimension of W is 3.

Problem 8. Let

$$A=\begin{bmatrix}3&2&1\\6&4&2\\9&6&3\end{bmatrix}$$

1. Determine the rank of A
2. Find vectors $\vec{u},\vec{v}\in\mathbb{R}^3$ such that $A=\vec{u}\vec{v}^T$.
3. Show that any matrix of the form $\vec{u}\vec{v}^T$ for $\vec{u}\in\mathbb{R}^m, \vec{v}\in\mathbb{R}^n$ has rank at most 1.
4. Show that any m-by-n matrix B that has rank 1 can be written as $B=\vec{u}\vec{v}^T$ for some vectors $\vec{u},\vec{v}$.

Solution:

1. We find the rank by Gaussian elimination, which in this case terminates in a hurry! The second row is 2 times the first and the third row is three times the first. Thus in row echelon form there is only one nonzero row and the rank is 1.

2. We can write
$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

3. For a matrix $A = \vec{u}\vec{v}^T$, every column of A is a multiple of $\vec{u}$. Thus $C(A) \subseteq \{t \cdot \vec{u} : t \in \mathbb{R}\}$ is a subspace of dimension at most 1. As the rank is defined to be the dimension of the column space, this implies the rank of A is at most 1.

4. Let R be the reduced row echelon form of B and E an invertible matrix such that $B = ER$. Now as B has rank 1, only the first row of R can be nonzero. The $\hat{R}$ be the matrix R with all zero rows removed. Thus $\hat{R}$ will be a row vector. As the zero rows of R do not affect the product, we can let $\hat{E}$ be the first column of E and have $ER = \hat{E}\hat{R}$. This is the desired expression of B as a column vector times a row vector.

Problem 9. We have seen two "easy" ways to prove $S \subseteq \mathbb{R}^n$ is a subspace. Namely, to write S as the span of a set of vectors, or to give a matrix A whose nullspace is equal to S.

We have further seen that every subspace of $\mathbb{R}^n$ can be written as the span of a set of vectors, as we in fact proved that every subspace $S \subseteq \mathbb{R}^n$ has a *basis*.

For every subspace $S \subseteq \mathbb{R}^n$, can we also find a matrix A such that the nullspace of A is equal to S? We explore this question here by means of an example.

Let $S = \text{span}(\{(1, 2, 1, 1), (1, 1, 2, -3)\})$. Our goal is to find a matrix A whose nullspace is equal to S.

1. Find a basis for the nullspace of the matrix
$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & -3 \end{bmatrix} .$$

2. Make a matrix A whose rows are the basis vectors you found from part (1). Show that the nullspace of A is equal to S.

Solution:

1. To find a basis for the null space of B we bring it into reduced row echelon form
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

From here we can read off the special solutions

$$\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \\ 0 \\ 1 \end{bmatrix},$$

which form a basis for the null space.

2. As directed, we form a matrix A whose rows are the special solutions we just found.

$$A = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 7 & -4 & 0 & 1 \end{bmatrix}$$

We can directly verify that both $(1, 2, 1, 1)$ and $(1, 1, 2, -3)$ are in the nullspace of A. As the nullspace of A is a subspace this also means that $S = \text{span}(\{(1, 2, 1, 1), (1, 1, 2, -3)\})$ is contained in the nullspace of A. Further the rows of A are linearly independent (by the easy test) meaning that A has 2 pivots. Thus the dimension of the nullspace of A is $4 - 2 = 2$. This means that S is equal to the nullspace of A.

This problem is an example of the general fact we saw in lecture that $S = (S^\perp)^\perp$ for a subspace S. We initially create the matrix B such that $S = C(B^T)$. The nullspace of B will then be $S^\perp$. If we then create a matrix whose rows are a basis for $S^\perp$, *its* nullspace will be $(S^\perp)^\perp = S$. This shows how to express S as the nullspace of a matrix.

Problem 10. Let V be a finite-dimensional vector space. For subspaces $U, W \subseteq V$ define their *sum* as $U + W = \{u + w : u \in U, w \in W\}$.

1. Show that $U + W$ is a subspace.

2. Suppose that $U \cap W = \{\mathbf{0}\}$. Show that then

$$\dim(U + W) = \dim(U) + \dim(W) \ .$$

Solution

1. Let $\mathbf{0}$ be the zero element of V. Then $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$ as $\mathbf{0}$ is in both U and W. Now we show that $U + W$ is closed under taking linear combinations. Let $v_1, v_2 \in U + W$. Thus $v_1 = u_1 + w_1, v_2 = u_2 + w_2$ for some $u_1, u_2 \in U$ and $w_1, w_2 \in W$. As U, W are subspaces, also $a_1u_1 + a_2u_2 \in U$ and $a_1w_1 + a_2w_2 \in W$ for any scalars $a_1, a_2 \in \mathbb{R}$. Therefore

$$\begin{aligned} (a_1u_1 + a_2u_2) + (a_1w_1 + a_2w_2) &= a_1(u_1 + w_1) + a_2(u_2 + w_2) \\ &= a_1v_1 + a_2v_2 \end{aligned}$$

is in $U + W$.

2. Let $u_1, \ldots, u_s$ be a basis for U and $w_1, \ldots, w_r$ be a basis for W (thus $\dim(U) = s$ and $\dim(W) = r$). We want to show that $\dim(U + W) = s + r$. We can see that $U + W \subseteq$ span($\{u_1, \ldots, u_s, w_1, \ldots, w_r\}$). It thus suffices to show that $u_1, \ldots, u_s, w_1, \ldots, w_r$ are linearly independent. Suppose that we have a linear combination

$$a_1 u_1 + \cdots + a_s u_s + b_1 w_1 + \cdots + b_r w_r = \mathbf{0} \ ,$$

we want to show that $a_1 = \cdots = a_s = b_1 = \cdots = b_r = 0$. Rearranging this equation, we see that

$$a_1 u_1 + \cdots + a_s u_s = -(b_1 w_1 + \cdots + b_r w_r) \ .$$

Thus the vector $v = a_1 u_1 + \cdots + a_s u_s = -(b_1 w_1 + \cdots + b_r w_r)$ is in $U \cap W$ and therefore $v = \mathbf{0}$. This in turn means that $a_1 = \cdots = a_s = 0$ as $u_1, \ldots, u_s$ are linearly independent, and similarly $b_1 = \cdots = b_r = 0$ as $w_1, \ldots, w_r$ are linearly independent.

More generally, it holds that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) \ .$$