Rules for Matrix Operations

Reading: Strang 2.4

Learning objective: Be able to apply linearity of matrix multiplication. Understand why matrix multiplication is associative.

Matrix Multiplication

We have now seen three views of matrix multiplication:

Column View: The i^{th} column of AB equals A times the i^{th} column of B.

$$(AB)[:, i] = AB[:, i]$$

Row View: The i^{th} row of AB equals the i^{th} row of A times B.

$$(AB)[i, :] = A[i, :]B$$

Row-Column View: The (i, j) entry of AB equals the i^{th} row of A times the j^{th} column of B.

$$(AB)[i, j] = \langle A[i, :], B[:, j] \rangle$$

Matrix Multiplication is Linear

Matrix-vector multiplication is a linear function.

- § $A(c \cdot \vec{v}) = c \cdot (A\vec{v})$ for any $c \in \mathbb{R}, v \in \mathbb{R}^n$
- § $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

It is easy to multiply a linear combination of vectors by a matrix:

$$A(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdots + c_k\vec{v}_k) = c_1 \cdot A\vec{v}_1 + c_2 \cdot A\vec{v}_2 + \cdots + c_k \cdot A\vec{v}_k$$

Distributive Law

We can use the linearity of matrix-vector multiplication to show that matrix multiplication from the left is distributive:

$$C(A+B) = CA + CB$$

Proof: We use the column picture of matrix multiplication.

What is the i^{th} column of the result on the left hand side?

$$C(A[:,i] + B[:,i]) = CA[:,i] + CB[:,i]$$

This equality uses linearity of matrix-vector multiplication.

Now we have the i^{th} column of the right hand side.

Distributive Law

There is also a distributive law for matrix multiplication from the right:

$$(A + B)C = AC + BC$$

You could prove this similarly using the row picture of matrix multiplication.

Associativity

Now we come to the most important property, matrix multiplication is associative.

$$A(BC) = (AB)C$$

The sequence in which we do the multiplications does not matter.

We can write products of matrices without parentheses.

Associativity

Our definition of matrix multiplication was motivated by having the following property:

$$A(B\vec{v}) = (AB)\vec{v}$$

This property is why matrix multiplication is associative.

$$A(BC) = (AB)C$$

We prove this using the column picture of matrix multiplication.

The i^{th} column of BC is $BC[:, i]$.

$$A(B\vec{v}) = (AB)\vec{v} \qquad \bigstar$$

This property is why matrix multiplication is associative.

$$A(BC) = (AB)C$$

Proof: What is the i^{th} column of the left hand side (LHS)?

$$A(BC)[:,i]$$

What is the i^{th} column of BC?

$$BC[:,i]$$

Substituting this in, the i^{th} column of the LHS is

$$A(BC[:,i]) = (AB)C[:,i] \qquad \text{using } \bigstar$$

which is the i^{th} column of the right hand side (RHS).

Inverses

Reading: Strang 2.5

Learning objective: Understand the formal definition of inverses and also their intuitive action. Be able to compute the inverses of elementary matrices.

Definition: Invertible

Definition: A square matrix A is invertible if and only if there is a matrix B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

Here I is the identity matrix of the same size as A.

A square matrix that is not invertible is called singular.

Example: AMORTA

Let's look at the elementary matrix E implementing the row operation $R_2' = R_2 - 3R_1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2' = R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Say we want to return the second row to its original state.

How do we undo this operation?

Example

To undo the row operation $R_2' = R_2 - 3R_1$ we can add three times row one to row two.

$$R_2' = R_2 - 3R_1 \implies R_2 = R_2' + 3R_1$$

The elementary matrix that adds 3 times row 1 to row 2 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Call this matrix F.

I claim that F undoes the action of E.

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{F}\left(\underbrace{\begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{E}\underbrace{\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}}_{\vec{b}}\right) = \begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}b_1\\ b_2 - 3b_1\\ b_3\end{bmatrix}$$

$$= \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

Conclusion: $F(E\vec{b}) = \vec{b}$ for any vector $\vec{b}$.

$$F(E\vec{b}) = (FE)\vec{b}$$

What does this mean about FE?

Check

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F}$$

Conclusion

The matrix E is invertible.

There is a matrix, namely F, such that

$$EF = I \qquad \text{and} \qquad FE = I$$

The matrix F is also invertible.

In fact, every elementary matrix adding a multiple of one row to another is invertible.

This is witnessed by the elementary matrix that undoes the operation.

Example: Row Swaps

Let's look at the other kind of elementary matrix we have encountered: one that exchanges two rows.

When multiplied on the left, this matrix swaps rows 1 and 3.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

How can we undo the operation of swapping rows?

Example: Row Swaps

We swap the rows again!

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For any elementary matrix R doing a row swap,

$$RR = I\,.$$

Any row swap matrix is invertible.

Punchline

A big question we are interested in is

When is a square matrix A invertible?

The main punchline is:

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

We will (slowly) prove this over the next lectures.

Along the way we will find other conditions equivalent to invertibility (there are many).

Fun with Permutation Matrices

Learning objective: To see that matrices can represent interesting actions!

Permutation Matrices

We have already encountered permutation matrices.

It is a matrix that can be formed by rearranging the rows of the identity matrix.

§ A permutation matrix can be written as a product of row swap matrices.

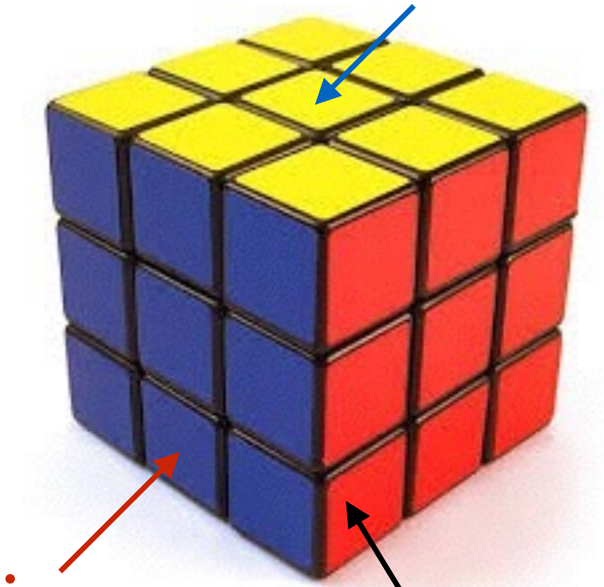
§ A permutation matrix has exactly one one in every row and column.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fun with Perm. Matrices

Permutation matrices can describe the moves in permutation puzzles.

Let's look at some permutation matrices for moves on the rubik's cube.



Basic terminology:

Edge cubies: have two sides. 12 of them.

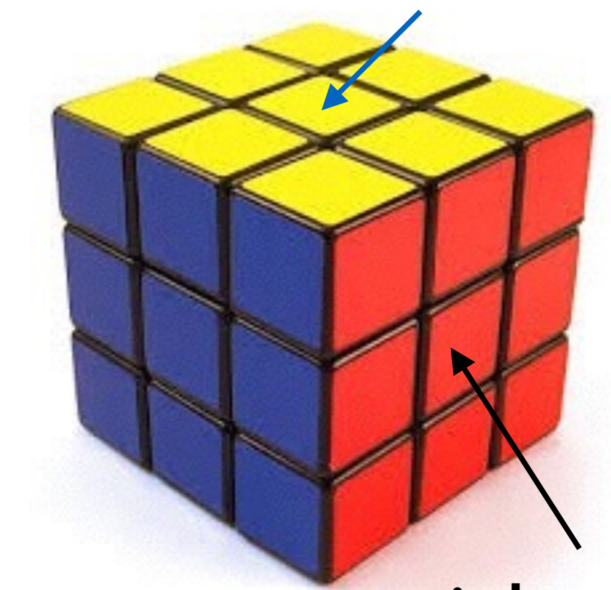
Corner cubie: have three sides. 8 of them.

Center cubie: doesn't move. 6 of them.

Edge cubies: have two sides. 12 of them.

Corner cubie: have three sides. 8 of them.

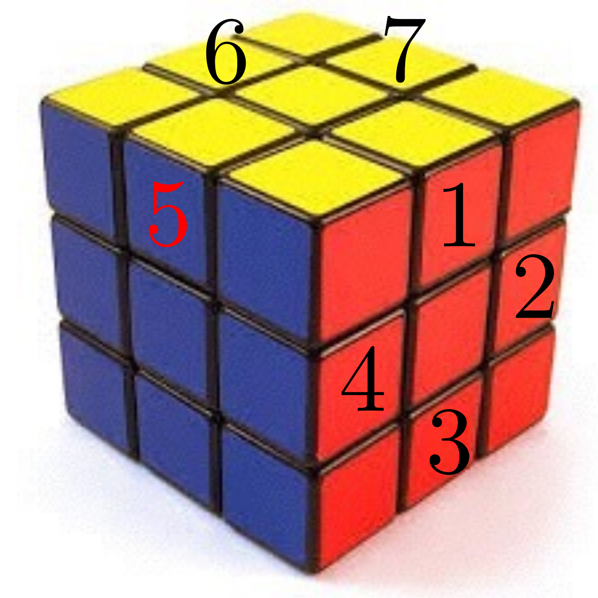
Center cubies: don't move. 6 of them.



Under turning the faces, edges stay edges, corners stay corners.

So the matrices are not too big, we are just going to look at edge cubies on the right and up faces.

Label the edge positions on right and up faces by $1, 2, \ldots, 7$



What does turning the right face clockwise a quarter turn do?

What was in position 1 goes to position 2

What was in position 2 goes to position 3

What was in position 3 goes to position 4

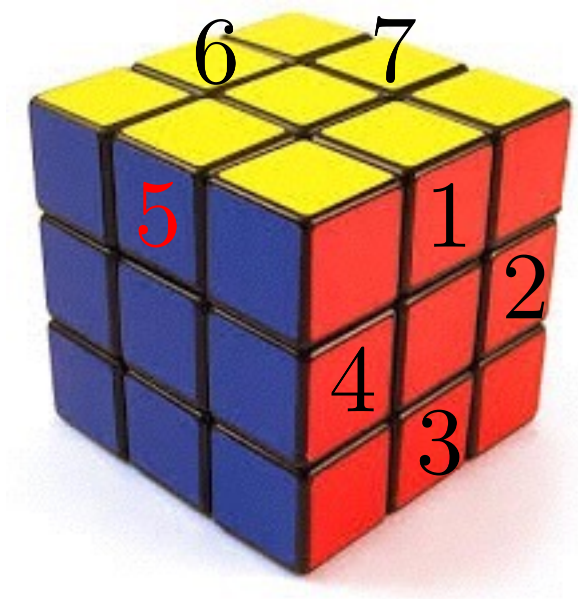
What was in position 4 goes to position 1

What was in position 5 goes to position 5

What was in position 6 goes to position 6

What was in position 7 goes to position 7

Identify position i with 7 dimensional column vector e_i with 1 in i^{th} entry.



$$\underset{e_1}{\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\end{bmatrix}}, \cdots, \underset{e_7}{\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\end{bmatrix}}$$

Now we can make a matrix representing a quarter turn of right face.

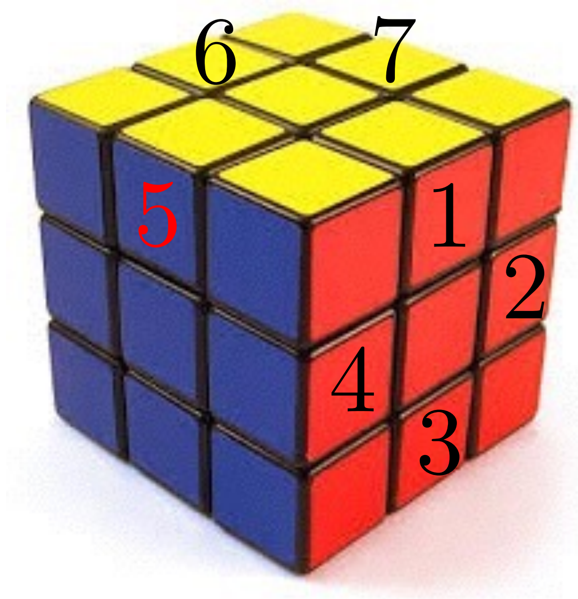
This matrix sends:

$$e_1 \to e_2 \quad e_2 \to e_3 \quad e_3 \to e_4 \quad e_4 \to e_1$$

$$e_5 \to e_5 \quad e_6 \to e_6 \quad e_7 \to e_7$$

What matrix does this?

Right quarter turn sends:



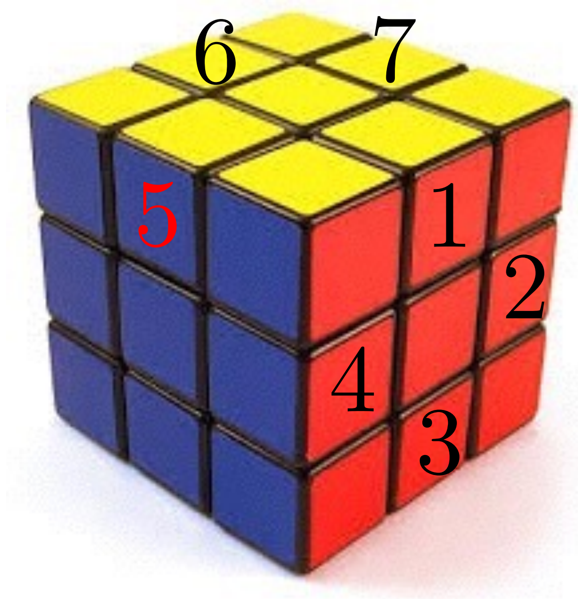
$$e_1 \to e_2 \quad e_2 \to e_3 \quad e_3 \to e_4 \quad e_4 \to e_1$$

$$e_5 \to e_5 \quad e_6 \to e_6 \quad e_7 \to e_7$$

What matrix does this?

$$R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can do the same thing with a clockwise quarter turn of the Up face:



$$e_1 \to e_5 \qquad e_2 \to e_2 \qquad e_3 \to e_3 \qquad e_4 \to e_4$$

$$e_5 \to e_6 \qquad e_6 \to e_7 \qquad e_7 \to e_1$$

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we can multiply these matrices to see the state of the cube after a sequence of moves.