Dimension of the Left Nullspace

Reading: Strang 3.6

Learning objective: Be able to find a basis for the left nullspace of a matrix.

Four Subspaces

Let A be an m-by-n matrix with $EA = R = \mathrm{rref}(A)$.

Say that R has r many pivots.

nullspace $N(A) \subseteq \mathbb{R}^n$

dimension: $n - r$

basis: special solutions

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

dimension: $m - r$

row space $C(A^T) \subseteq \mathbb{R}^n$

dimension: r

basis: nonzero rows of R.

column space $C(A) \subseteq \mathbb{R}^m$

dimension: r

basis: first r cols of E^{-1}

= pivot cols of A.

Left Nullspace

Let A be an m-by-n matrix with r many pivots.

The left nullspace of A is the set

$$\begin{aligned}\{\vec{u} : \vec{u}^T A = \vec{0}_n^T\} &= \{\vec{u} : A^T\vec{u} = \vec{0}_n\} \\ &= N(A^T)\end{aligned}$$

Because $r = \dim(C(A)) = \dim(C(A^T))$ the n-by-m matrix A^T also has r pivots.

Thus $\dim(N(A^T)) = m - r$.

Basis for the Left Nullspace

We could find a basis for the left nullspace by taking the special solutions to $A^T\vec{x} = \vec{0}_n$.

Let's see another way.

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

In the case of our running example:

$$\underbrace{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$$

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In general, the last $m - r$ rows of R will be all-zero.

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In general, the last $m - r$ rows of R will be all-zero.

The i^{th} row of R is equal to $E(i,:)A$.

This means the last $m - r$ rows of E are in the left nullspace of A.

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

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Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Basis for the Left Nullspace

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Proof: We have seen that the last $m - r$ rows of E are in the left nullspace of A.

Moreover, they are linearly independent.

Because E is invertible, its rows are linearly independent.

We also know the dimension of the left nullspace is $m - r$.

Basis for the Left Nullspace

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Proof: We have $m - r$ linearly independent vectors in a subspace of dimension $m - r$.

These vectors must span the left nullspace of A.

If a vector $\vec{v} \in N(A^T)$ was not in their span, then we would have a sequence of $m - r + 1$ linearly independent vectors in $N(A^T)$, a contradiction.

Summary

Let A be an m-by-n matrix with r pivots.

$$EA = R = \mathrm{rref}(A)$$

nullspace $N(A) \subseteq \mathbb{R}^n$

dimension: $n - r$

basis: special solutions

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

dimension: $m - r$

basis: last $m - r$ rows of E.

row space $C(A^T) \subseteq \mathbb{R}^n$

dimension: r

basis: nonzero rows of R.

column space $C(A) \subseteq \mathbb{R}^m$

dimension: r

basis: first r cols of E^{-1}

= pivot cols of A.

Finding a Basis: Example

Reading: Strang 3.6

Learning objective: Apply the row space or column space method to find a basis to the span of a set of vectors.

Example Application

Let

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v_5} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Find a basis for $\operatorname{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\})$.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v_5} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Find a basis for $\operatorname{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\})$.

We can approach this in two ways:

1) Make a matrix A whose columns are $\vec{v}_1, \ldots, \vec{v}_5$.

Then find a basis for the column space of A .

2) Make a matrix B whose rows are $\vec{v}_1, \ldots, \vec{v}_5$.

Then find a basis for the row space of B.

Approach 1: Columns

Make a matrix A whose columns are $\vec{v}_1, \ldots, \vec{v}_5$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 2 & -1 & 1 \end{bmatrix}$$

Find a basis for the column space of A.

We can proceed by doing Gaussian elimination to identify the pivot columns.

The corresponding columns of A will be a basis for the column space of A.

$$
\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 2 & -1 & 1 \end{bmatrix}
\xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \\ R_4' = R_4 - R_1}]{}
\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & -2 & 2 & -2 & 0 \end{bmatrix}
$$

$$
\xrightarrow{R_4' = R_4 - R_2}
\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}
\xrightarrow{R_4' = R_4 - R_3}
\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}
$$

Now we have reached a row echelon form.

Columns 1,2,4,5 contain pivots.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 2 & -1 & 1 \end{bmatrix}$$

A

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

row echelon form

Columns 1,2,4,5 contain pivots.

Thus columns 1,2,4,5 of A form a basis for the column space of A.

A basis for $\mathrm{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\})$ is given by $\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5$.

Notice that in this case our basis is a subset of the original set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$.

Column Space

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 2 & -1 & 1 \end{bmatrix}$$

$$A$$

Thus columns 1,2,4,5 of A form a basis for the column space of A.

The column space of A is $\mathbb{R}^4$.

If there was a $\vec{w} \in \mathbb{R}^4$ with $\vec{w} \notin \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\})$ then $\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5, \vec{w}$ would be a lin. independent sequence.

This is a contradiction as $\dim(\mathbb{R}^4) = 4$.

Note

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Note that in general the column space of A is not the same as the column space of a (reduced) row echelon form of A .

The column space can change under elementary row operations.

$$C(A) = \{t \cdot (1,1) : t \in \mathbb{R}\} \qquad C(R) = \{t \cdot (1,0) : t \in \mathbb{R}\}$$

However the dimension of the column space does not change under elementary row operations.

Approach 2: Rows

Make a matrix B whose rows are $\vec{v}_1, \ldots, \vec{v}_5$.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

Find a basis for the row space of B .

We can proceed by Gaussian elimination.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_5' = R_5 - R_1]{\substack{R_2' = R_2 - R_1 \\ R_4' = R_4 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow[R_4 \leftrightarrow R_5]{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4' = R_4 - R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\underset{B}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}} \qquad \underset{R}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

We have arrived at R in row echelon form

The row space of B and R is the same.

$$EB = R \implies C(R^T) \subseteq C(B^T)$$

$$B = E^{-1}R \implies C(B^T) \subseteq C(R^T)$$

$$\underset{R}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

The nonzero rows of R are a basis for the row space of R .

It is clear the the nonzero rows of R span the row space of R.

They are also linearly independent.

To find the row space, it suffices to find a row echelon form, you don't have to go all the way to a rref.

Linear Independence

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R$$

The nonzero rows do not pass the easy test, but we can use the "staircase" structure to show lin. independence.

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking at the first component we see that $a_1 = 0$.

Then, looking at the second component shows $a_2 = 0$.

Continuing in this way we see that all coefficients are zero.

These vectors are linearly independent.

This will hold for the nonzero rows in any row echelon form.

Approach 2: Rows

Make a matrix B whose rows are $\vec{v}_1, \ldots, \vec{v}_5$.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

The nonzero rows of a row echelon form for B form a basis for the row space of B .

A basis for $\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\})$ is given by

$$(1, 1, 1, 1), (0, -2, 0, -2), (0, 0, -2, -2), (0, 0, 0, 2)$$

Orthogonality of the four subspaces

Reading: Strang 4.1

Learning objective: Apply the definition of orthogonal subspaces to the row and nullspace.

Row Space and Nullspace

Let A be an m-by-n matrix.

Both the row space and nullspace of A are subspaces of $\mathbb{R}^n$.

As they live in the same vector space, we can look at the relationship between the row space and the null space.

What vectors are in both the row space of A and the nullspace of A ?

Orthogonal Vectors

Two vectors $\vec{v}, \vec{w}$ are orthogonal if and only if $\langle \vec{v}, \vec{w} \rangle = 0$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -1 & 0 \end{bmatrix}$$

The vector $(-1, 2, -1)$ is in the nullspace of A.

This vector is orthogonal to each row of A.

Orthogonal Vectors

In general, for an m-by-n matrix A, any vector $\vec{v} \in N(A)$ is orthogonal to each row of A.

If $\vec{v} \in N(A)$ then $A\vec{v} = \vec{0}_m$.

$$A\vec{v} = \vec{0}_m \implies A(i,:)\vec{v} = 0 \quad \text{for each row } i.$$

Thus $\vec{v}$ is orthogonal to each row of A.

Orthogonal Subspaces

We can say more: for every $\vec{v} \in N(A)$ and $\vec{w} \in C(A^T)$

$$\langle \vec{w}, \vec{v} \rangle = 0$$

Every vector in the nullspace is orthogonal to every vector in the row space.

If $\vec{v} \in N(A)$ then $A\vec{v} = \vec{0}_m$.

If $\vec{w} \in C(A^T)$ then $\vec{w} = A^T\vec{u}$ for some $\vec{u} \in \mathbb{R}^m$.

$$\begin{aligned} \langle \vec{w}, \vec{v} \rangle = \vec{w}^T\vec{v} = (A^T\vec{u})^T\vec{v} &= \vec{u}^T A\vec{v} \\ &= \vec{u}^T\vec{0}_m = 0 \end{aligned}$$

Orthogonal Subspaces

Every vector in the nullspace is orthogonal to every vector in the row space.

We say that the row space and the nullspace are orthogonal subspaces of $\mathbb{R}^n$.

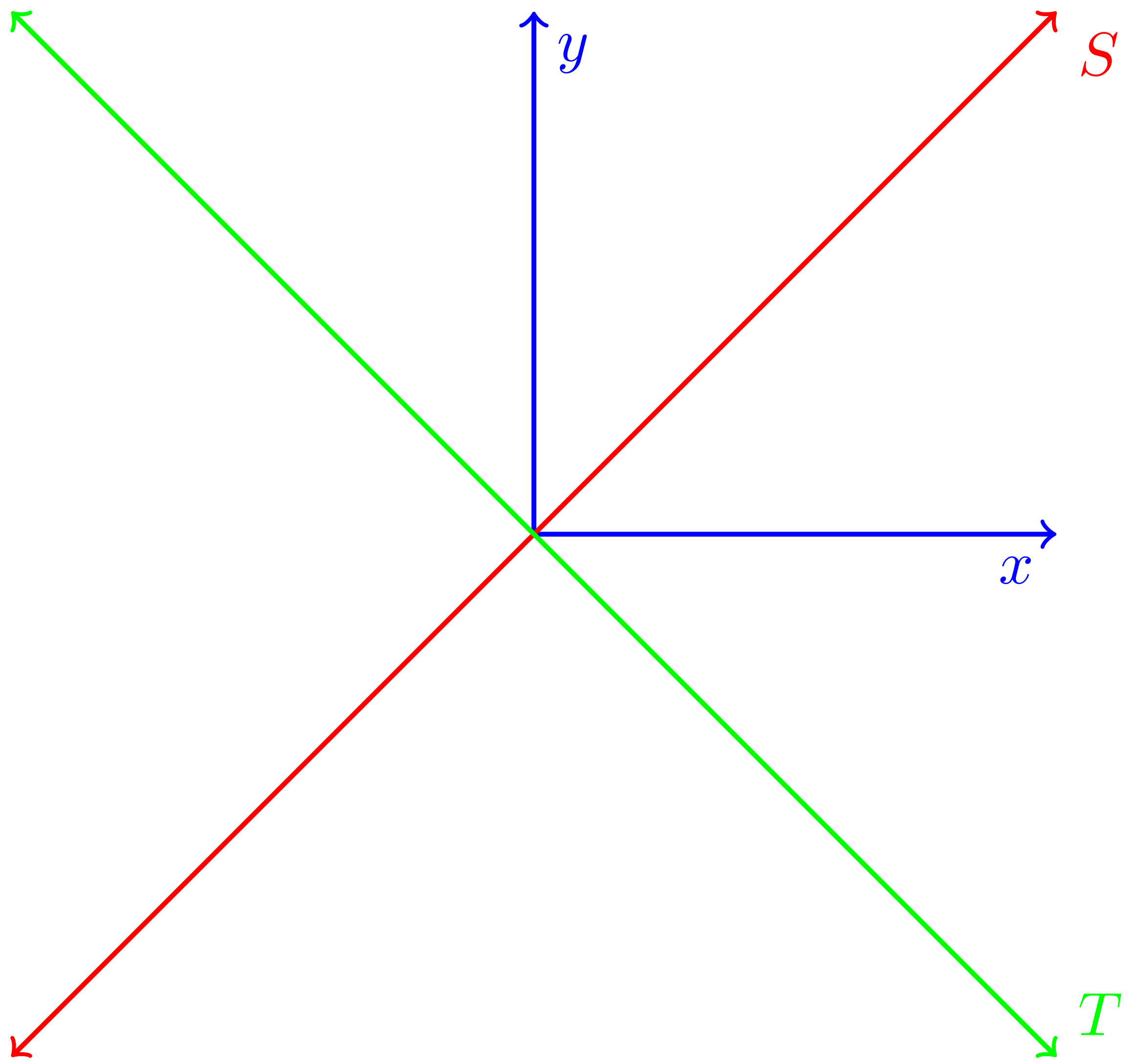
Definition: Two subspaces $S, T \subseteq \mathbb{R}^n$ are said to be orthogonal iff $\langle \vec{w}, \vec{v} \rangle = 0$ for all $\vec{w} \in S$ and $\vec{v} \in T$.

Example

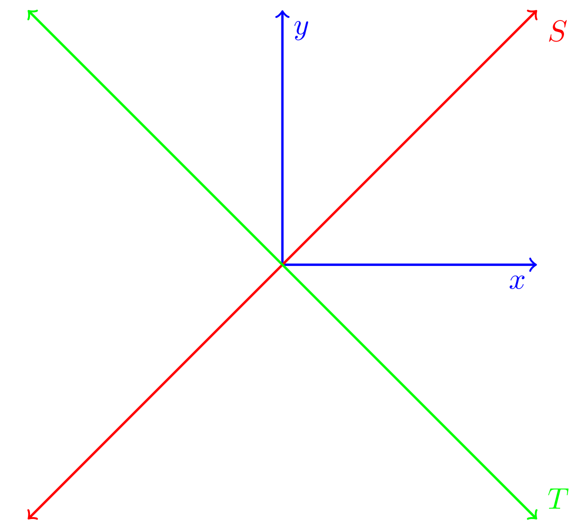
Let $S = \{t \cdot (1,1) : t \in \mathbb{R}\}$
and $T = \{t \cdot (1,-1) : t \in \mathbb{R}\}$.

These are orthogonal subspaces of $\mathbb{R}^2$.

$$\begin{bmatrix} a & a \end{bmatrix} \begin{bmatrix} b \\ -b \end{bmatrix} = ab - ab = 0$$



Intersection



Let $S, T \subseteq \mathbb{R}^n$ be orthogonal subspaces.

What vectors can be in both S and T ?

If $\vec{v} \in S \cap T$ then $\vec{v}$ has to be orthogonal to itself:

$$\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2 = 0$$

The only vector with this property is $\vec{v} = \vec{0}_n$.

Theorem: If $S, T \subseteq \mathbb{R}^n$ are orthogonal subspaces then

$$S \cap T = \{\vec{0}_n\}$$

Intersection of row and nullspace

Let A be an m-by-n matrix. As the row space and nullspace of A are orthogonal subspaces:

$$\boxed{C(A^T) \cap N(A) = \{\vec{0}_n\}}$$

The only vector in both the row space and the nullspace of A is $\vec{0}_n$.