Inverses

Reading: Strang 2.5

Learning objective: Understand when a product of matrices is invertible. Be familiar with the "big list" of conditions equivalent to invertibility.

Review

Definition: A square matrix A is invertible if and only if there is a matrix B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

Here I is the identity matrix of the same size as A.

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

1) The "easy" direction: if A is invertible then Gaussian elimination produces n pivots.

because $A\vec{x} = \vec{0}$ has a unique solution.

2) If elimination produces n pivots, then A has a right inverse.

because $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$.

3) If elimination produces n pivots, then A has a left inverse.

because Gauss-Jordan produces the identity matrix and is done by left multiplication by elem. matrices.

Computing the Inverse

We can compute the inverse of A by solving $A\vec{x} = \vec{e}_i$, where $\vec{e}_i$ is the i^{th} column of the identity matrix, for every i.

These solutions form the columns of A^{-1}.

We can organize this computation by forming the super-augmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 3 & 9 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Right vs Left

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 3 & 9 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

explains gaussian jordan elimination

This computation is solving for a right inverse

$$AX = I$$

But we can also see why it gives a left inverse.

The steps of Gauss-Jordan elimination do

$$AX = I \rightarrow E_1 AX = E_1 \rightarrow E_k \cdots E_2 E_1 AX = E_k \cdots E_2 E_1$$

If $E_k \cdots E_2 E_1 A = I$ then $X = E_k \cdots E_2 E_1$.

Homogeneous Equations

A homogeneous system of equations is one where the right hand side vector is the zero vector $\vec{0}$.

In this case, the right hand side remains $\vec{0}$ throughout Gaussian elimination.

Theorem: A square matrix A is invertible if and only if

$$A\vec{x} = \vec{0}$$

has a unique solution.

Reason: $A\vec{x} = \vec{0}$ has a unique solution if and only if A has a pivot in every column.

Question

Suppose that A and B are invertible.

Is AB invertible?

Question

Suppose that B is singular.

Is AB invertible for some A?

Left Inverse Suffices

Normally to show the square matrix A is invertible, we have to find B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

Theorem: If $BA = I$ for square matrices A, B then A is invertible.

Proof: We show the contrapositive. If A is singular then there is a vector $\vec{u} \neq \vec{0}$ with $A\vec{u} = \vec{0}$.

Then $BA\vec{u} = B\vec{0} = \vec{0}$, thus BA is singular and cannot be the identity matrix.

Right Inverse Suffices

Suppose that A has a right inverse. There is a matrix B such that $AB = I$.

This means B has a left inverse, and so B is invertible.

As B is invertible, it also has a right inverse, which must equal its left inverse A (left inverse = right inverse).

This means $BA = I$ and A is invertible.

The Big List

Let A be a square matrix. The following are equivalent:

- A is invertible.
- Gaussian elimination produces a full set of pivots.
- $A\vec{x} = \vec{0}$ has a unique solution.
- A has a left inverse.
- A has a right inverse.
- The reduced row echelon form of A is the identity matrix.

LU Decomposition

Reading: Strang 2.6

Learning objective: Be able to compute the LU decomposition of a matrix and understand its application to solving systems of linear equations.

Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We will find this beautiful LU decomposition.

We start with Gaussian elimination and keep track of the elementary matrices implementing the row operations.

Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \\ R_4' = R_4 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix}$$

Multiplying the corresponding elementary matrices together we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix}$$

Next round of G.E.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \xrightarrow[\substack{R_3' = R_3 - 2R_2 \\ R_4' = R_4 - 3R_2}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$$

Multiplying the corresponding elementary matrices together we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$$

Question

What is its inverse?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

Last round of G.E.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \xrightarrow[R_4' = R_4 - 3R_3]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have reached an upper triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

Putting everything together:

$$
\underset{\substack{\uparrow \\ \text{third round} \\ \text{of G.E.}}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}}
\underset{\substack{\uparrow \\ \text{second round} \\ \text{of G.E.}}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}}
\underset{\substack{\uparrow \\ \text{first round} \\ \text{of G.E.}}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}
\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}}
=
\underset{U}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
$$

We have seen that the matrices implementing each round of Gaussian elimination are invertible.

To obtain the LU decomposition, we undo these operations to obtain an expression for A.

Undo

We undo each operation in turn, starting from the most recent.

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}}_{E_3^{-1}}\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

Undo

We undo each operation in turn, starting from the most recent.

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}}_{E_3^{-1}}\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&3&0&1\end{bmatrix}}_{E_2^{-1}}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}}_{E_3^{-1}}\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

Undo

We undo each operation in turn, starting from the most recent.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}}_{E_2^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}}_{E_3^{-1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{E_1^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}}_{E_2^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}}_{E_3^{-1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Final Result

$$\underset{A}{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}} = \underset{E_1^{-1}}{\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix}} \underset{E_2^{-1}}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&3&0&1\end{bmatrix}} \underset{E_3^{-1}}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}} \underset{U}{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}$$

From the problem set, the product of lower triangular matrices is lower triangular.

Set $L = E_1^{-1}E_2^{-1}E_3^{-1}$.

$$\underset{A}{\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}} = \underset{L}{\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}} \underset{U}{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}$$

Big Picture

Let's review what happened.

In this Gaussian elimination we only used the operation of adding a multiple of one row to a row below it.

The corresponding elementary matrices are lower triangular.

In matrix form Gaussian elimination becomes

$$E_k \cdots E_2 E_1 A = U$$

where U is upper triangular and each E_i is lower triangular.

Big Picture

In matrix form Gaussian elimination becomes

$$E_k \cdots E_2 E_1 A = U$$

where U is upper triangular and each E_i is lower triangular.

The inverse of each E_i is also lower triangular.

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

The product of lower triangular matrices is lower triangular:

$$A = LU$$

where $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

Summary

Theorem: If Gaussian elimination on the square matrix A proceeds without row swaps, then there is a factorization

$$A = LU$$

where L is lower triangular and U is upper triangular.

Row Swaps

The trouble with row swaps is simply that they are not lower triangular.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

We will see next time how to deal with matrices where Gaussian elimination requires row swaps.

Application

This is how square linear systems are actually solved in practice! Say that $A = LU$

$$\begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$

First solve

$$\begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \begin{bmatrix} \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$

Then

$$\begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{y} \end{bmatrix}$$

Application

Together we have

$$L\vec{y} = \vec{b}$$
$$U\vec{x} = \vec{y}$$
$$\Rightarrow \quad L(U\vec{x}) = \vec{b}$$

Recall that solving $U\vec{x} = \vec{y}$ for upper triangular U can be done by back substitution with about n^2 operations.

Similarly $L\vec{y} = \vec{b}$ can be solved with about n^2 operations by forward substitution.

Gaussian elimination, on the other hand, takes about $\frac{2}{3}n^3$ arithmetic operations.

Application

Together we have

$$L\vec{y} = \vec{b}$$
$$U\vec{x} = \vec{y}$$

$$L(U\vec{x}) = \vec{b}$$

Once we have found an LU decomposition $A = LU$, we can quickly solve $A\vec{x} = \vec{b}$ for any $\vec{b}$.

This is how an inverse is computed by computer (not with Gauss-Jordan elimination).

"inv performs an LU decomposition of the input matrix. It then uses the results to form a linear system whose solution is the matrix inverse inv(X)."

Python Experiment

Let's compare the performance in python of two ways of solving a system of linear equations.

$$>>>\ x = \mathrm{np.matmul}(\mathrm{np.linalg.inv}(A), b)$$

$$>>>\ x = \mathrm{np.linalg.solve}(A, b)$$