Row Swaps

Reading: Strang 2.3

Learning objective: Be able to identify a row swap matrix and understand their role in constructing permutation matrices.

Elementary Matrix

A matrix that implements a row operation of Gaussian elimination is called an elementary matrix.

We have seen the form of elementary matrices corresponding to adding a multiple of one row to another.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

This matrix does the operation $R_3' = R_3 - 3R_2$.

Row Swap

We have gone over another row operation in Gaussian elimination: exchanging two rows.

What is the elementary matrix corresponding to this operation?

Say we want to swap rows 1 and 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

start with the identity matrix

perform the row swap on the identity matrix

Row Swap

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix}$$

Composing row swaps

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ swaps rows 1 and 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ swaps rows 2 and 3.

What is the following product?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Composing row swaps

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What if we do the row swaps in the opposite order?

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Not commutative

We have discovered that matrix multiplication is in general not commutative:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

In general, $AB \neq BA$.

This is not a defect of matrix multiplication, but a property of actions in the world.

It really matters in what order you swap rows!

Terminology

As the order in which we multiply matters, saying "A times B" can be ambiguous.

For the product AB we can say "A multiplies B on the left".

For the product BA we can say "A multiplies B on the right".

Permutation Matrix

A permutation matrix is a matrix formed by permuting the rows of the identity matrix.

Every row and column of a permutation matrix has exactly one one (because the identity matrix does).

Every permutation matrix is a product of row swap matrices.

Example

Here is an example of a permutation matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

How can we write this as a product of row swap matrices?

A library can be completely rearranged just by exchanging two books at a time.

Views of Matrix Multiplication

Reading: Strang 2.4

Learning objective: Become comfortable multiplying matrices using the column picture, row picture, and dot product picture.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A(B[:,1]) & A(B[:,2]) & \cdots & A(B[:,n]) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

We arrived at the definition of matrix multiplication by thinking about the columns of the product.

From a theoretical point of view, this is probably the most important picture of matrix multiplication.

For actually computing products, however, there is another view in terms of dot products that is generally more convenient.

Matrix-vector multiplication

Let's go back to matrix-vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Dot product view

Let's go back to matrix-vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix}$$

The first component of the result is the dot product of the first row of A with $\vec{b}$.

Likewise for the second and third components.

Dot product view

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix}$$

$$= \begin{bmatrix} \langle A[1,:], \vec{b} \rangle \\ \langle A[2,:], \vec{b} \rangle \\ \langle A[3,:], \vec{b} \rangle \end{bmatrix}$$

The i^{th} component of $A\vec{b}$ is the dot product of the i^{th} row of A with $\vec{b}$.

First Column

We can apply the dot product view of matrix-vector multiplication to matrix-matrix multiplication.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A(B[:,1]) & A(B[:,2]) & \cdots & A(B[:,n]) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

The first column of the product is $A(B[:,1])$.

$$A(B[:,1]) = \begin{bmatrix} \langle A[1,:], B[:,1] \rangle \\ \langle A[2,:], B[:,1] \rangle \\ \vdots \\ \langle A[k,:], B[:,1] \rangle \end{bmatrix}$$

Second Column

We can apply the dot product view of matrix-vector multiplication to matrix-matrix multiplication.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A(B[:,1]) & A(B[:,2]) & \cdots & A(B[:,n]) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

The second column of the product is $A(B[:,2])$.

$$A(B[:,2]) = \begin{bmatrix} \langle A[1,:], B[:,2] \rangle \\ \langle A[2,:], B[:,2] \rangle \\ \vdots \\ \langle A[k,:], B[:,2] \rangle \end{bmatrix}$$

Dot product view

In general, we have the following picture.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \langle A[1,:], B[:,1] \rangle & \langle A[1,:], B[:,2] \rangle & \cdots & \langle A[1,:], B[:,n] \rangle \\ \vdots & & \ddots & \vdots \\ \langle A[k,:], B[:,1] \rangle & \langle A[k,:], B[:,2] \rangle & \cdots & \langle A[k,:], B[:,n] \rangle \end{bmatrix}$$

$$\boxed{\text{The } (i,j) \text{ entry of } AB \text{ is the dot product of } A[i,:] \text{ and } B[:,j].}$$

Example

Compute the product.

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}$$

Example 2

Compute the product.

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}$$

Row picture

This leads us to the row picture of matrix multiplication.

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} = u_1 \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} + u_2 \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \\ + u_3 \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

The result is a linear combination of the rows of A.

Row picture

$$\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\right)\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11} & a_{12} & a_{13}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$$

This is $A[1,:]B$. The first row of A times B .

Row picture

Changing the order of multiplications:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \right)$$

This is equal to $(AB)[1,:]$, the first row of AB.

The first row of AB is $A[1,:]B$.

Row Picture

If A is a k-by-m matrix and B is a m-by-n matrix, then the product AB is defined and is a k-by-n matrix.

The i^{th} row of AB is $A[i,:]B$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} \cdots & A[1,:]B & \cdots \\ \cdots & A[2,:]B & \cdots \\ & \vdots & \\ \cdots & A[k,:]B & \cdots \end{bmatrix}$$

Row Picture

If A is a k-by-m matrix and B is a m-by-n matrix, then the product AB is defined and is a k-by-n matrix.

The i^{th} row of AB is $A[i,:]B$.

Every row of AB is a linear combination of the rows of B .

Elementary matrices

The row picture is a good way to think about the action of elementary matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Example

This example shows the benefit of choosing the most appropriate view of multiplication for the situation.

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

For what values of a, b, c does $AB = BA$?

Rules for Matrix Operations

Reading: Strang 2.4

Learning objective: Be able to determine if a function is linear. Understand why matrix multiplication is associative.

Overview

So far we have talked a lot about matrix multiplication.

Matrix multiplication is definitely the most interesting matrix operation!

Your brain can relax for the next few minutes as we talk about easier matrix operations like addition and multiplication by scalars.

We will come back to matrix multiplication for the grand finale: showing matrix multiplication is associative!

Matrix Equality

What does the "equals" sign mean when applied to matrices?

Two matrices are equal if and only if they are the same size and agree in every entry.

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 8 & 16 \\ 1 & 32 & 64 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & c \\ 1 & 8 & 16 \\ 1 & 32 & 64 \end{bmatrix}$$

For what value of c are these two matrices equal?

Multiplication by a scalar

For a matrix A and scalar c the matrix cA is formed by multiplying each entry of A by c.

Examples:

$$2\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 12 \end{bmatrix}$$

$$1.5\begin{bmatrix} 6 & 10 & 2 \\ 4 & 20 & -4 \\ 14 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 15 & 3 \\ 6 & 30 & -6 \\ 21 & 3 & 0 \end{bmatrix}$$

$$-1\begin{bmatrix} 1 & 5 & -2 \\ 6 & -4 & 8 \\ 7 & 3 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 5 & -2 \\ 6 & -4 & 8 \\ 7 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -5 & 2 \\ -6 & 4 & -8 \\ -7 & -3 & 1 \end{bmatrix}$$

Addition

We can add two matrices A and B of the same size.

The (i, j) entry of $A + B$ is equal to $A[i, j] + B[i, j]$.

Examples:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & 7 & 9 \\ 7 & 19 & 37 & 61 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$$

Properties

For any matrices A, B, C of the same size:

- $A + B = B + A$ (commutativity)
- $A + (B + C) = (A + B) + C$ (associativity)
- $c(A + B) = cA + cB$ (distributivity)

Follow from the analogous properties of real numbers.

Note that $A + B + C$ is well defined.

Linearity

OK, brain back on! Let's talk about matrix-vector multiplication.

We are going to show a very important property of matrix-vector multiplication: it is a linear function.

Let A be a m-by-n matrix. We can define a function by left multiplication by A:

$$\vec{v} \in \mathbb{R}^n \longrightarrow \boxed{A} \longrightarrow A\vec{v} \in \mathbb{R}^m$$

Linearity

We are going to show a very important property of matrix-vector multiplication: it is a linear function.

$$\vec{v} \in \mathbb{R}^n \longrightarrow \boxed{A} \longrightarrow A\vec{v} \in \mathbb{R}^m$$

This means it satisfies the following two properties:

- § $A(c \cdot \vec{v}) = c \cdot (A\vec{v})$ for any $c \in \mathbb{R}, v \in \mathbb{R}^n$
- § $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

Linearity

§ $A(c \cdot \vec{v}) = c \cdot (A\vec{v})$ for any $c \in \mathbb{R}, v \in \mathbb{R}^n$

Use distributivity of scalar mult. over vector add.:

$$c \cdot (\vec{u} + \vec{w}) = c \cdot \vec{u} + c \cdot \vec{w}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ c \cdot v_3 \end{bmatrix}$$

Linearity

§ $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

Use distributivity of scalar addition:

$$(c + d) \cdot \vec{w} = c \cdot \vec{w} + d \cdot \vec{w}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Linearity

In general any function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called linear if it satisfies these two properties:

- $f(c \cdot \vec{v}) = c \cdot f(\vec{v})$ for any $c \in \mathbb{R}, v \in \mathbb{R}^n$
- $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

Question: If f is linear, what is $f(\vec{0}_n)$?

Image of a line

Say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function.

Let $\vec{u} \neq \vec{v} \in \mathbb{R}^n$, and consider the line going through them:

$$c \cdot \vec{u} + (1-c) \cdot \vec{v} \qquad\qquad c \in \mathbb{R}$$

What is the image of this line under f ?

That is, where does f map this line?

$$f(c \cdot \vec{u} + (1-c) \cdot \vec{v}) = f(c \cdot \vec{u}) + f((1-c) \cdot \vec{v})$$

$$= c \cdot f(\vec{u}) + (1-c) \cdot f(\vec{v})$$

Image of a line

Say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function.

$$\begin{aligned} f(c \cdot \vec{u} + (1-c) \cdot \vec{v}) &= f(c \cdot \vec{u}) + f((1-c) \cdot \vec{v}) \\ &= c \cdot f(\vec{u}) + (1-c) \cdot f(\vec{v}) \end{aligned}$$

If $f(\vec{u}) \neq f(\vec{v})$ then this will be on the line going through $f(\vec{u}), f(\vec{v})$.

If $f(\vec{u}) = f(\vec{v})$ then for any value of c this will be $f(\vec{u})$.

Hint: This may help with question 6 on problem set 3.

Distributive Law

We can use the linearity of matrix-vector multiplication to show that matrix multiplication from the left is distributive:

$$C(A+B) = CA + CB$$

Proof: We use the column picture of matrix multiplication.

What is the i^{th} column of the result on the left hand side?

$$C(A[:,i] + B[:,i]) = CA[:,i] + CB[:,i]$$

The equality uses linearity of matrix-vector multiplication.

Now we have the i^{th} column of the right hand side.

Distributive Law

There is also a distributive law for matrix multiplication from the right:

$$(A + B)C = AC + BC$$

You could prove this similarly using the row picture of matrix multiplication.

Associativity

Finally, we come to the most important property, matrix multiplication is associative.

$$A(BC) = (AB)C$$

The sequence in which we do the multiplications does not matter.

We can write products of matrices without parentheses.

Associativity

Our definition of matrix multiplication was motivated by having the following property:

$$A(B\vec{v}) = (AB)\vec{v}$$

This property is why matrix multiplication is associative.

$$A(BC) = (AB)C$$

We prove this using the column picture of matrix multiplication.

The i^{th} column of BC is $BC[:, i]$.

$$A(B\vec{v}) = (AB)\vec{v} \qquad \bigstar$$

This property is why matrix multiplication is associative.

$$A(BC) = (AB)C$$

Proof: What is the i^{th} column of the left hand side (LHS)?

$$A(BC)[:, i]$$

What is the i^{th} column of BC?

$$BC[:, i]$$

Substituting this in, the i^{th} column of the LHS is

$$A(BC[:, i]) = (AB)C[:, i] \qquad \text{using } \bigstar$$

which is the i^{th} column of the right hand side (RHS).

Outer Products

Reading: Strang 2.4

Learning objective: Understand the outer product picture of matrix multiplication.

Outer Product

An outer product is the product of a column vector and a row vector.

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

This terminology contrasts with the inner product, another name for the dot product.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Missing Blocks

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Outer product view

This gives us another view of matrix multiplication!

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \left(\underset{A_1}{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}} + \underset{A_2}{\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}} \right) \left(\underset{B_1}{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}} + \underset{B_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} \right)$$

As $A_1 B_2 = A_2 B_1 = \mathbf{0}_{2\times 2}$ this is

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Outer product view

If A is a k-by-m matrix and B is a m-by-n matrix, then the product AB is defined and is a k-by-n matrix.

AB is the sum of m outer products:

$$AB = A[:,1]B[1,:] + A[:,2]B[2,:] + \cdots + A[:,m]B[m,:]$$

Summary: views of matrix multiplication

We have now seen 4 pictures of matrix multiplication.

- § Column picture.
- § Row picture.
- § Row-column picture: dot (inner) products.
- § Column-row picture: outer products.

All these pictures give the same answer!