Example Class 1

Elementary Number Theory

Outline

- Modulo n
- Binary arithmetic
- Applications of binary arithmetic

Modulo n

- Recall

$$a \equiv b \pmod{n}$$

- If $a \equiv b \pmod{n}$, then $a-b = qn$ and $a=qn+b$.
- We represent integers mod n as {0,1,…,n-1} (thanks to the Euclidean division)
- We have addition and multiplication modulo n.

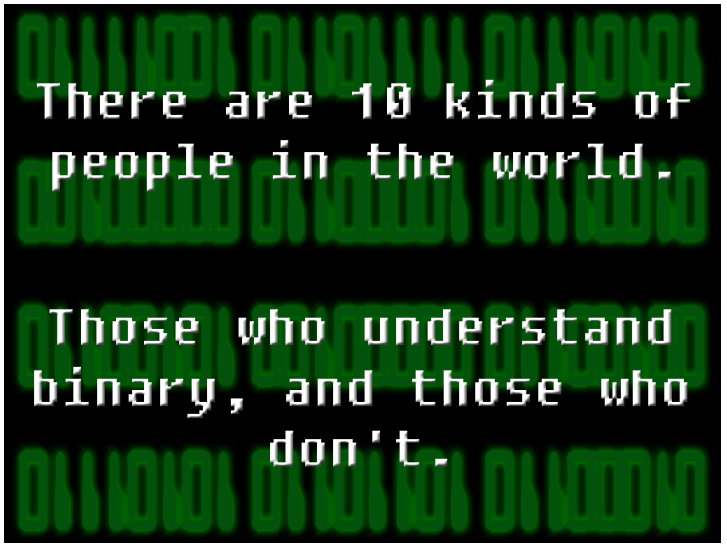
Integers mod 2

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1

Counting in Binary

$$10010_2 = 0*2^0 + 1*2^1 + 0*2^2 + 0*2^3 + 1*2^4$$
$$= 18_{10}$$
$$= 8.10^0 + 1.10^1$$



Binary Vectors

- A vector is a row (or column) array containing numbers.
- (1,0,0,1) is a binary row vector (of length 4)
- One can add two binary vectors component wise (vector addition).

Binary Vectors: Example

- For example (1,0,0,1)+(0,0,1,1)=(1,0,1,0)
- S=set of binary vectors of length n, Δ=vector addition. Is S closed under Δ?
- Different from counting in binary!

Binary in the Real World

- Storage of data across multiple hard disks
- Data is in binary format.
- Data needs to be stored so as to tolerate disk failures.

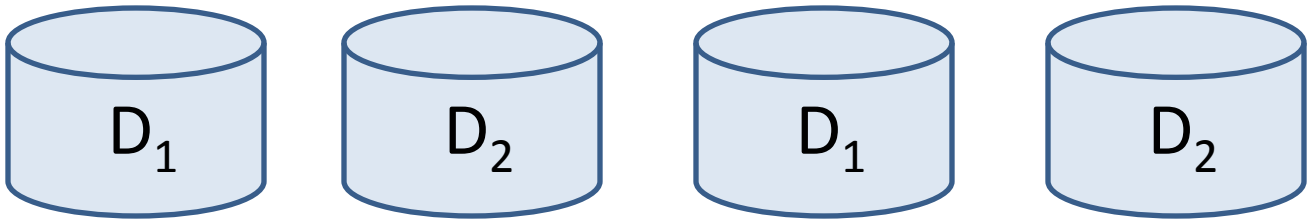
Image belongs to Microsoft

Example (I)

- Suppose you want to store 200GB of (binary) data

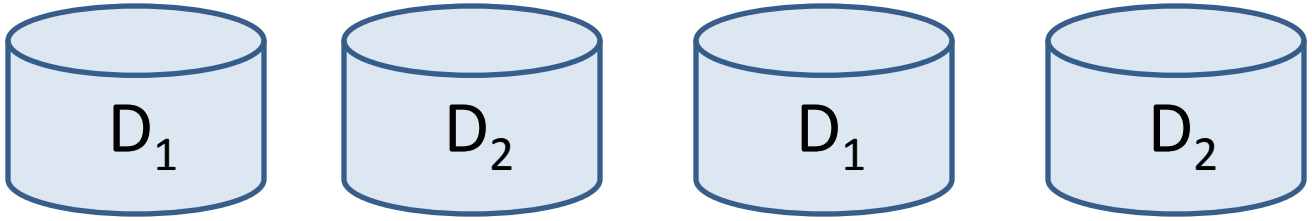
- Option 1: buy 4 disks of 100 GB each, store 2 copies of your data.

$D = (D_1, D_2)$



Example (II)

$D = (D_1, D_2)$



- Good thing: if one hard disk fails, your data is safe.
- Bad thing: you paid for 4 hard disks instead of 2.

- Can we think of a better solution?

Example (III)

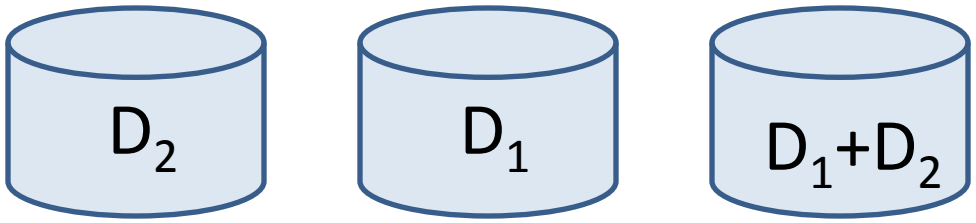
a	b	a+b
0	0	0
0	1	1
1	0	1
1	1	0

- Suppose a and b are bits, and take

  a, b, a+b

- Do the same thing with disks

$D = (D_1, D_2)$



Example (IV): Parity (RAID)

- Binary vector addition:

Drive 1: **01101101**
Drive 2: **11010100**

01101101
XOR 11010100

10111001 ← Store in Drive 3

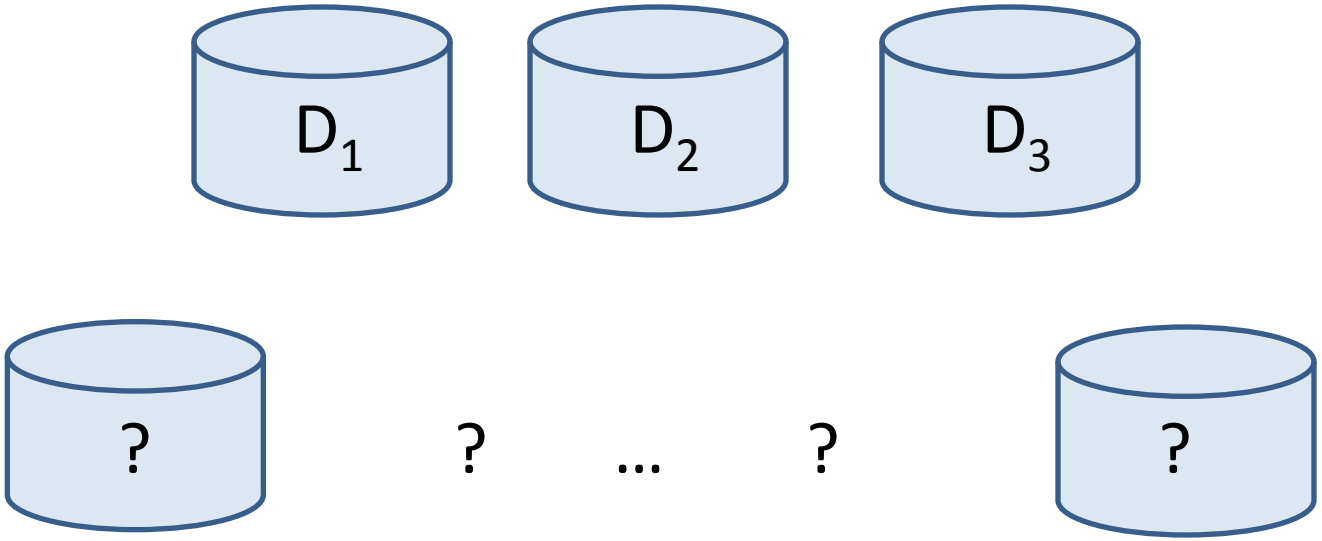
You can still loose one disk, but paid for only 3.

Another Example (I)

- You want to store 150 GB data.
- Now you are buying storage devices, each of 50GB capacity.
- This time, even if any arbitrary two devices fail, you still want to recover all your data!

Another Example (II)

$D = (D_1, D_2, D_3)$

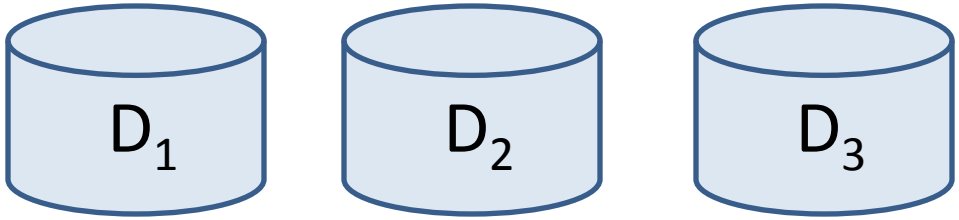


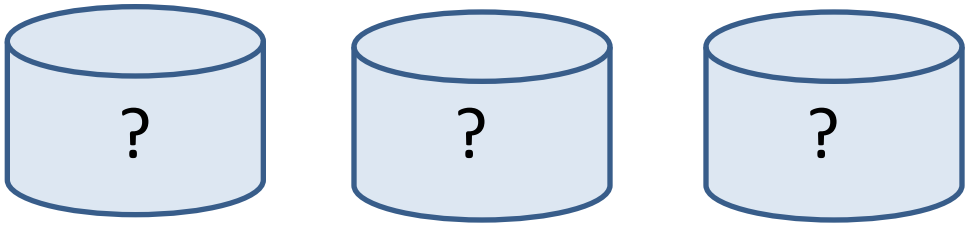
We can achieve what we want with a total of 9 disks. Can we do better?

Another Example (III)

$D = (D_1, D_2, D_3)$

D_1+D_2
D_1+D_3
D_2+D_3
$D_1+D_2+D_3$





To tolerate two failures, we need each D_i to be present at least 3 times.

Summary

- Modulo n
- Binary arithmetic
  - Binary vectors
  - Counting in binary
- Applications of binary arithmetic
  - storage