Vectors

Reading: Strang 1.1

Learning objective: Understand the 3 equivalent ways of thinking about a vector.

The Real Numbers

In this module, all the numbers we use will be real numbers.

Examples of real numbers are

$$0, 1, 2, 3, \ldots$$

$$-1, -2, -3, \ldots$$

$$\frac{3}{2}, \frac{5}{4}, \frac{87}{92}$$

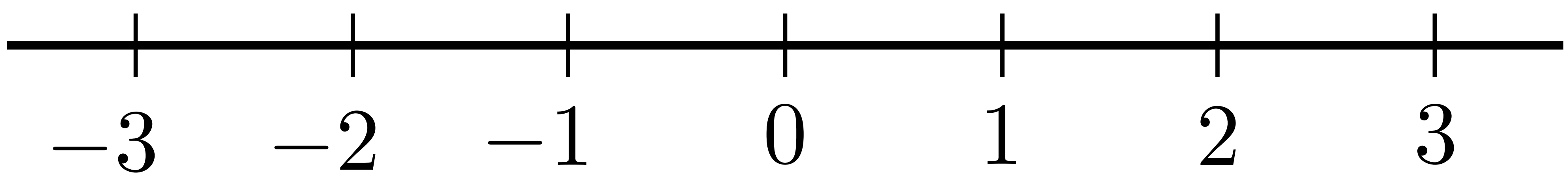
$$\sqrt{2}, \pi \approx 3.1415926\ldots$$

What's an example of a number that is not real?

The Real Numbers

The set of all real numbers is denoted by $\mathbb{R}$.

We can picture real numbers on a number line.

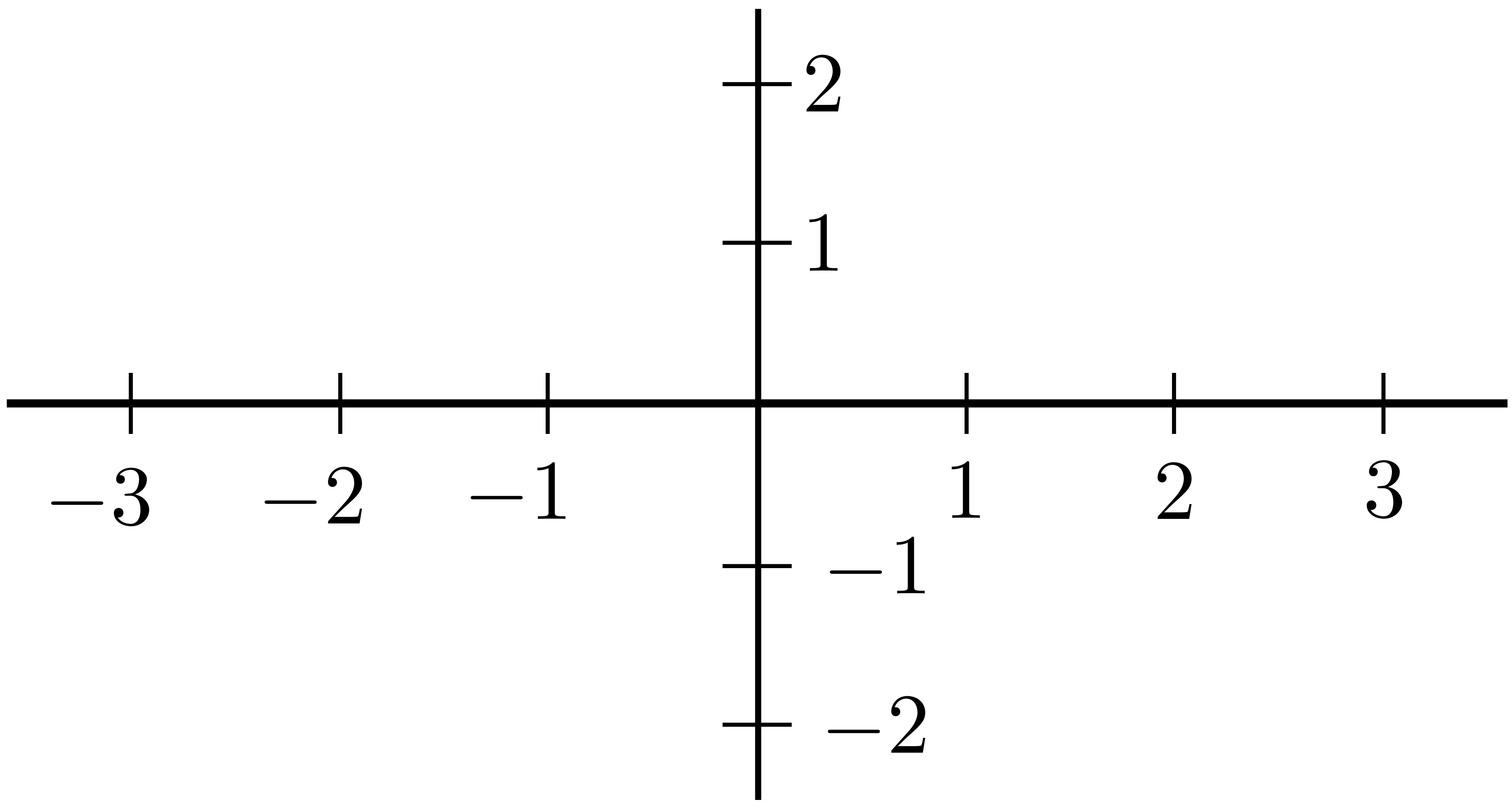


In this module, we will call real numbers scalars, to distinguish from what comes next…

Pairs of Real Numbers

Now we move up one level of complexity and consider pairs of real numbers.

Examples: $(0, 0), (1, 2), (-2.37, -1.5), (-2, 2)$



Pairs of Real Numbers

The set of all pairs of real numbers is denoted as

$\mathbb{R} \times \mathbb{R}$ or more simply as $\mathbb{R}^2$

We call a pair of real numbers a vector.

We write a vector like this $(-3, 2)$, or like this

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$ (column vector)

These two ways of writing are interchangeable.

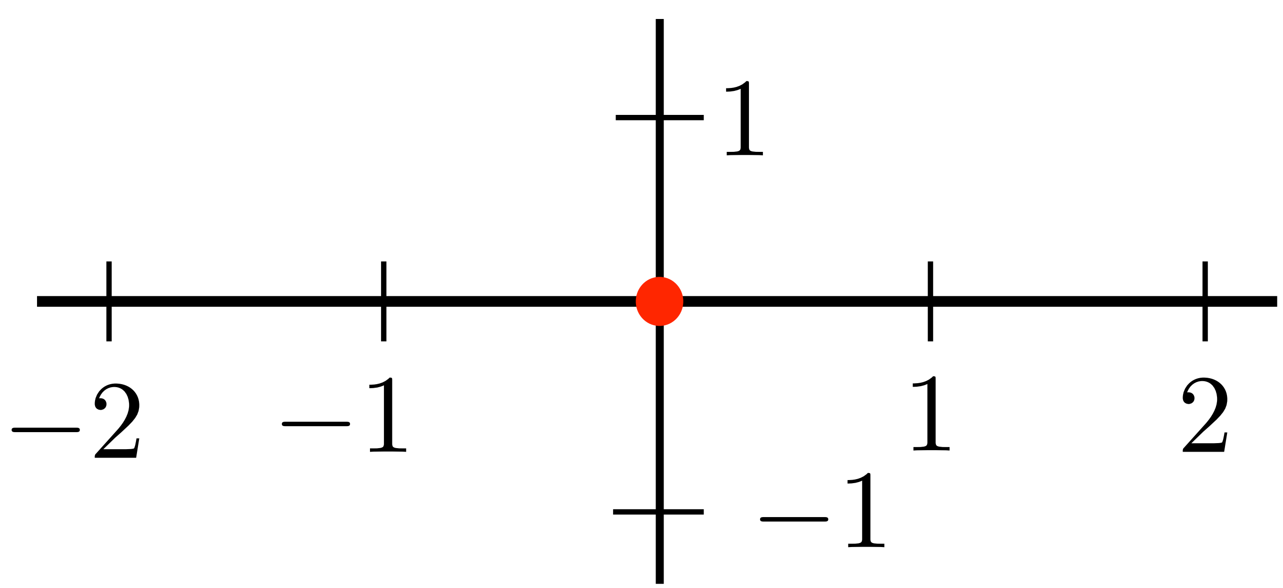
The Zero Vector

One vector is especially important: the zero vector.

This is the vector $(0, 0)$.

We have a special notation for it: $\mathbf{0} = (0, 0)$.

If writing by hand you can use $\vec{0}$ instead.



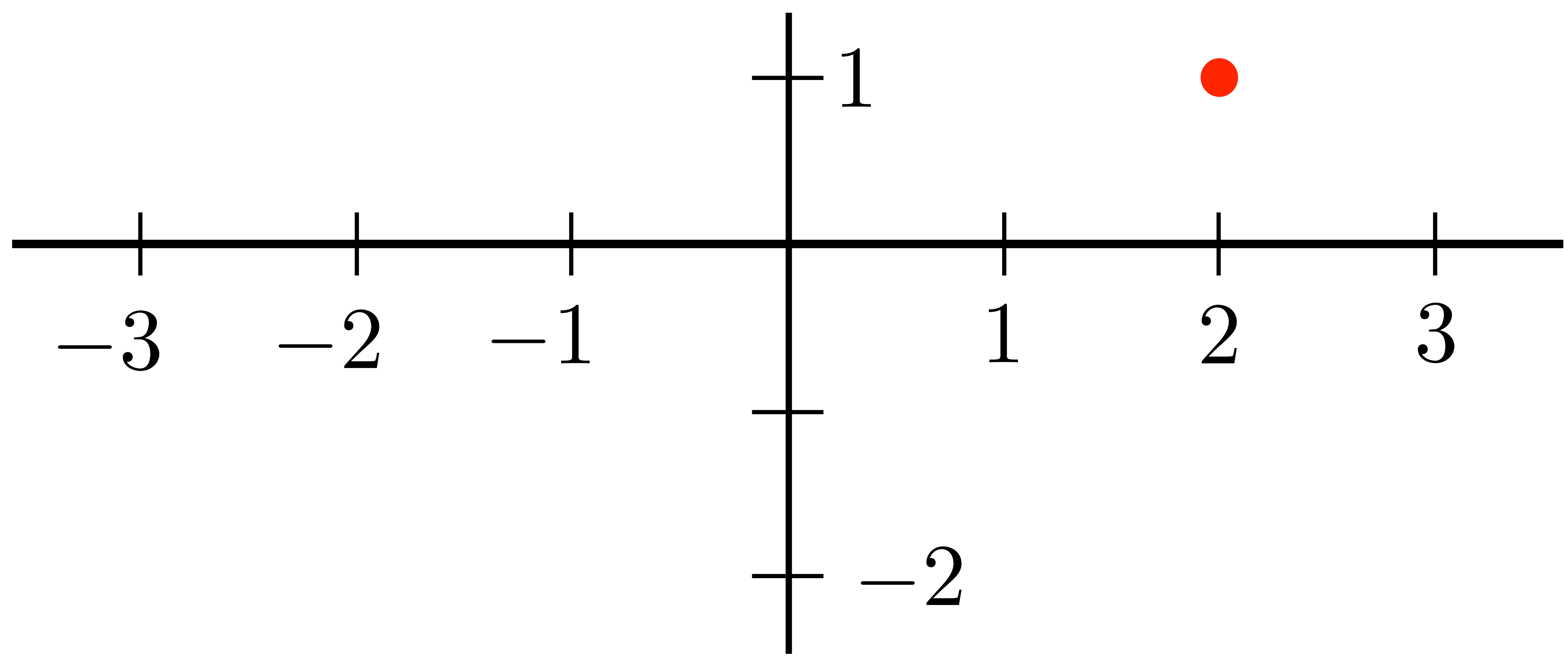
The zero vector sits at the origin of the axes.

Representations of Vectors

We have already seen two ways of thinking about a vector:

As a pair of real numbers: $(2, 1)$ (algebraic view)

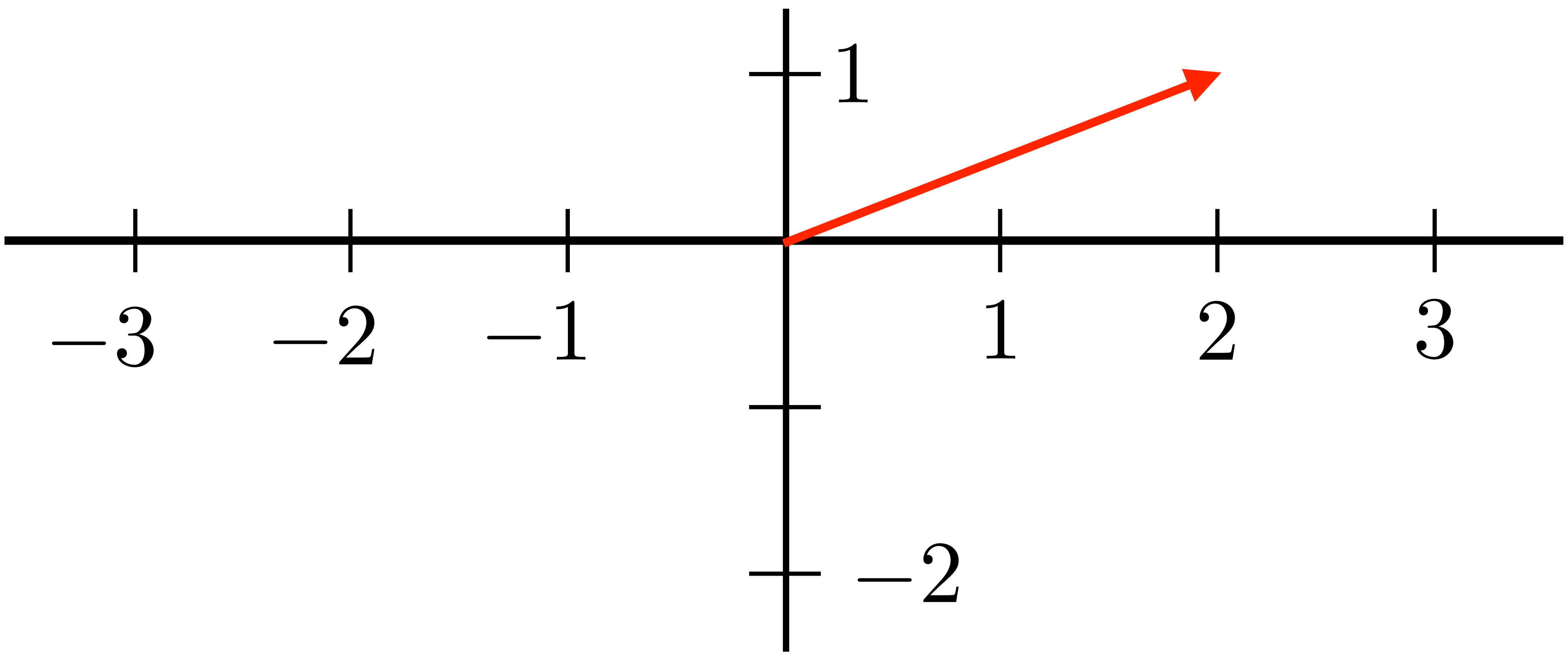
As a point in the plane: (geometric view)



Third Representation

There is a third way of thinking about a vector, as an arrow.

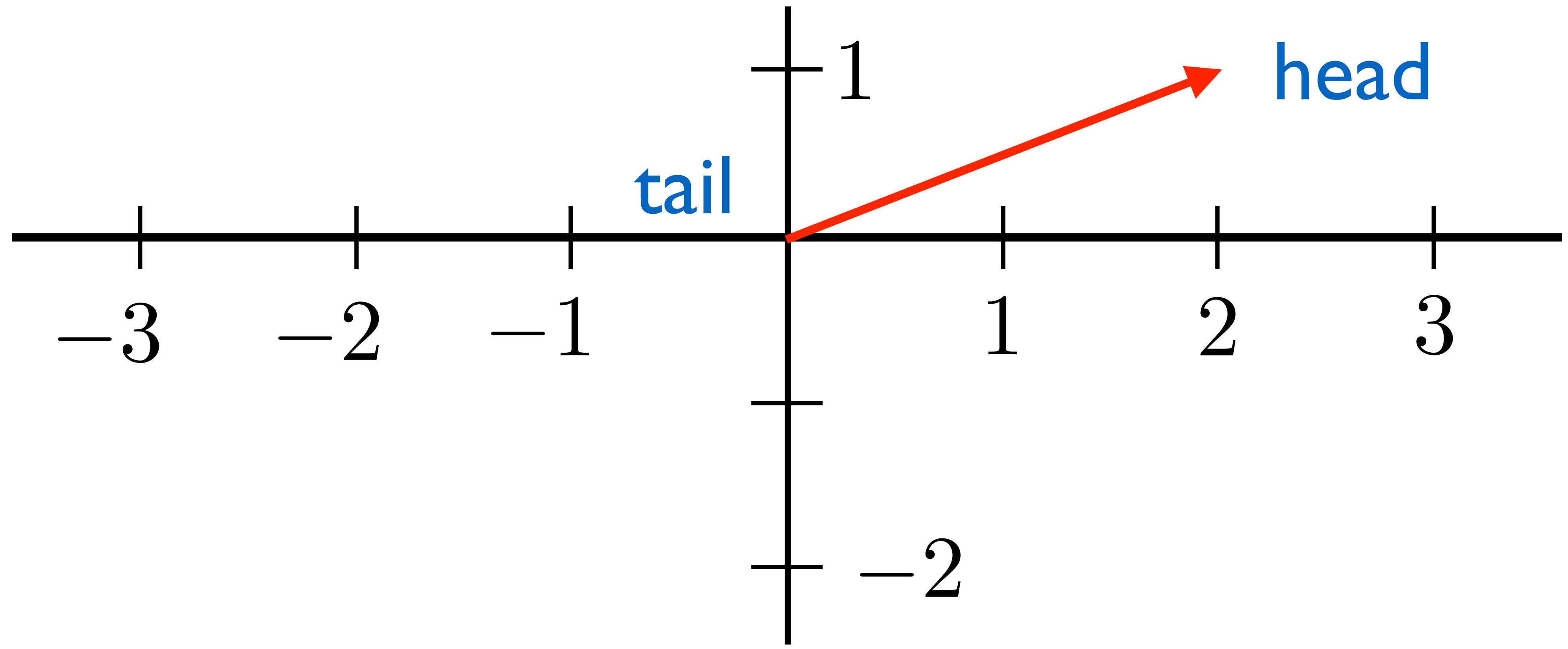
this is also the vector $(2,1)$



We can picture the vector $(2,1)$ as an arrow from the origin $(0,0)$ to the point $(2,1)$.

Anatomy of Arrows

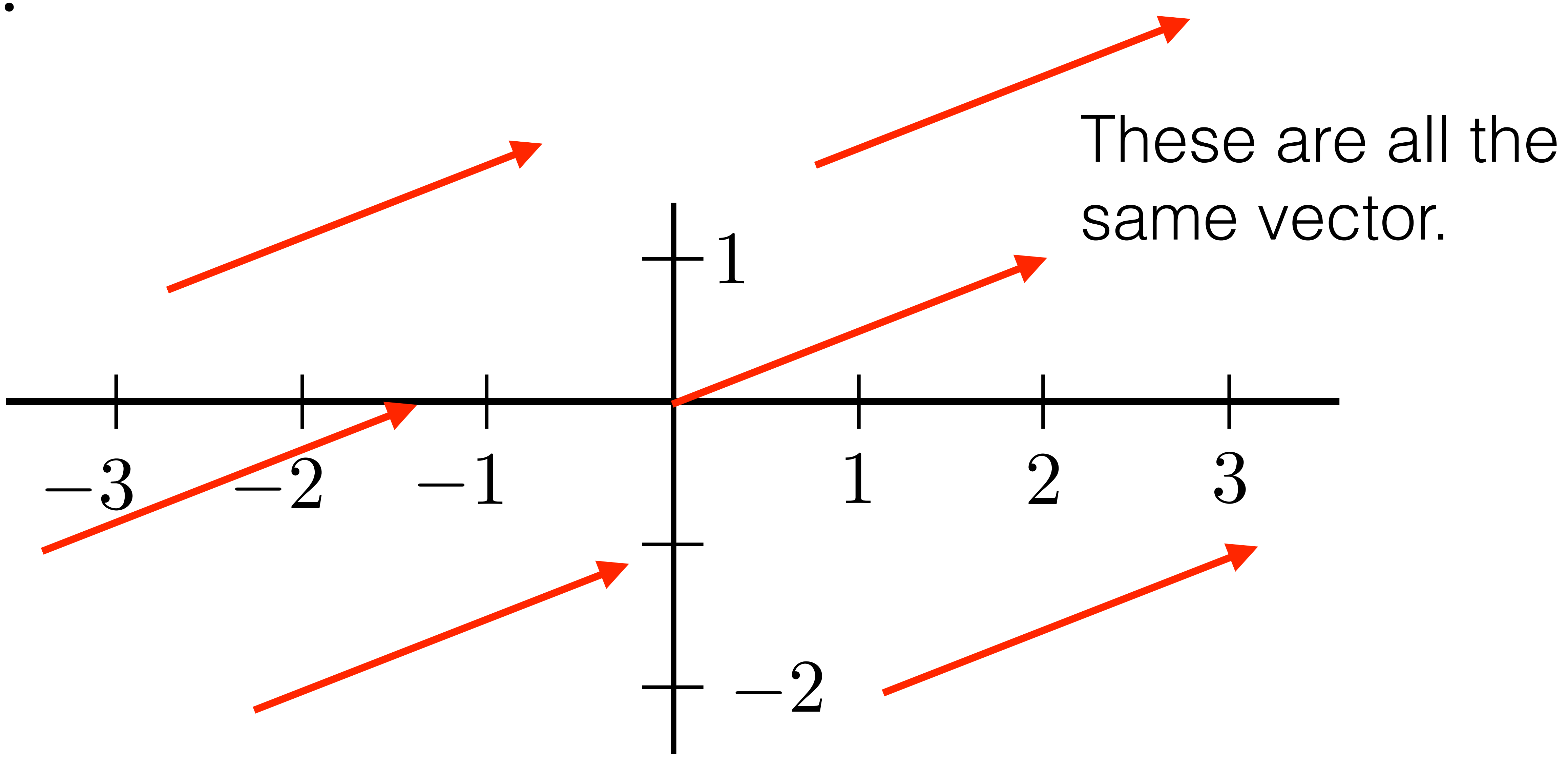
An arrow has a head and a tail.



Arrows can be moved

An arrow is determined by its length and direction.

As an arrow is moved around, it still represents the same vector.



Equivalence between points and arrows

Note that a point in the plane uniquely defines an arrow: the arrow that goes from $\mathbf{0}$ to the point.

Similarly, an arrow uniquely defines a point: translate the arrow so the tail is at $\mathbf{0}$ and take the point where the head is.

Equivalence between points and pairs

The pair of numbers (a, b) defines the point that is a units over on the horizontal axis and b units up on the vertical axis.

Likewise, a point in the plane defines a pair of numbers by its distance from 0 along the horizontal and vertical axes, respectively.

Algebra and Geometry

Part of the usefulness of linear algebra is the equivalence of these algebraic and geometric points of view.

Often it is easier to manipulate vectors using the algebraic point of view.

Thinking about the geometric meaning of these manipulations can provide insight into what is happening.

Triples of Real Numbers

Now we move up one level of complexity and consider triples of real numbers.

Examples: $(0,0,0), (1,-1,1), (1.5,0,-1)$

We can also write them as column vectors like this.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

These two ways of writing them mean the same thing.

Triples of Real Numbers

The set of all triples of real numbers is denoted as

$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or more simply as $\mathbb{R}^3$

$$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$$

Triples of real numbers are also examples of vectors!

We call such a vector three-dimensional.

Triples of Real Numbers

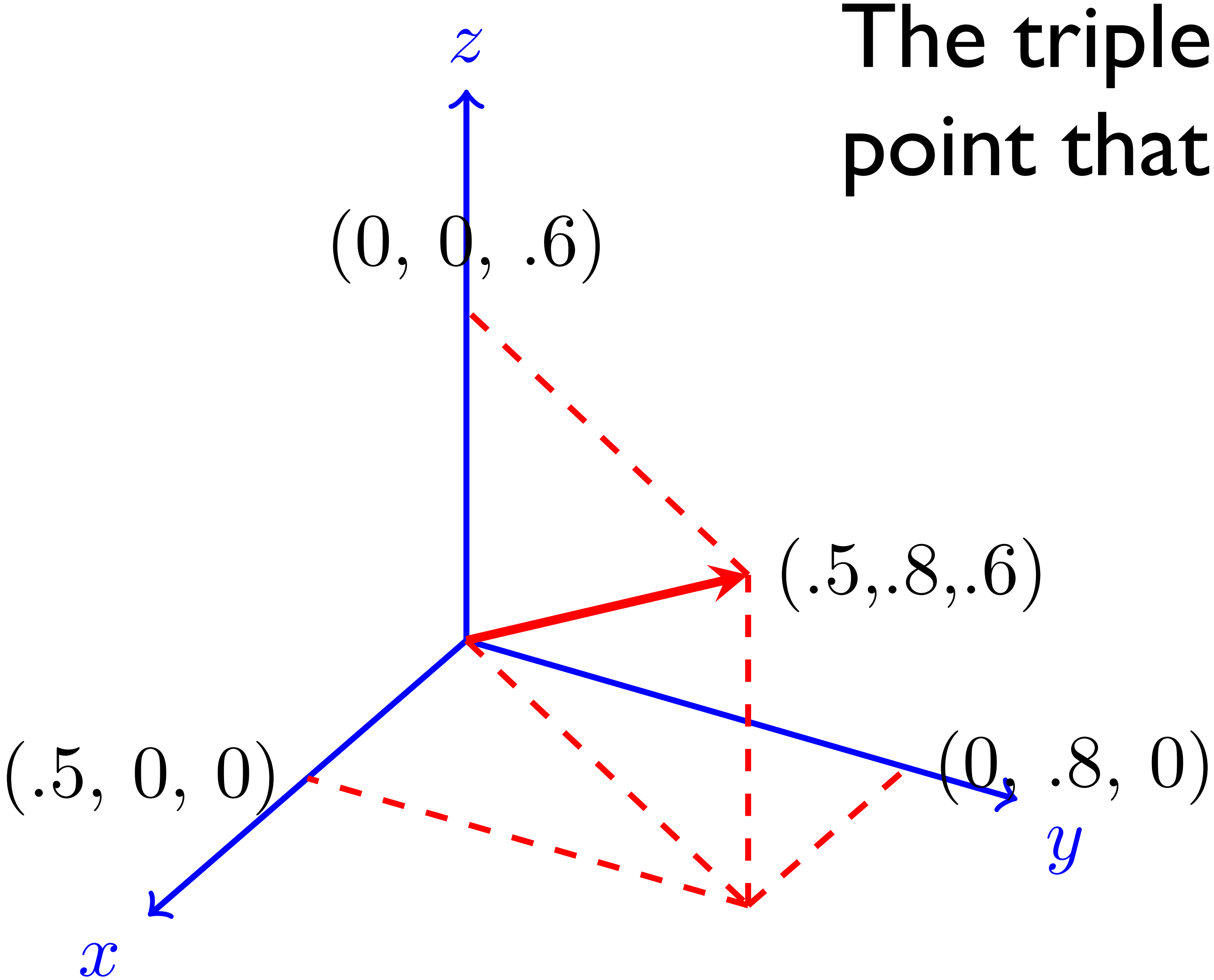
Everything we've said so far applies similarly to triples of real numbers.

There is a three-dimensional zero vector:

$$\mathbf{0} = (0, 0, 0).$$

Points in 3D Space

We have the same three representations of three-dimensional vectors.



The triple $(.5, .8, .6)$ defines the point that is:

$.5$ units over on the x-axis.

$.8$ units over on the y-axis.

$.6$ units over on the z-axis.

One Hour Later…

88-tuples of Real Numbers

Now we move up one level of complexity and consider tuples of 88 real numbers.

The set of all 88-tuples of real numbers is denoted $\mathbb{R}^{88}$.

The zero vector is

$$\mathbf{0} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

Wait!

Instead of considering each case separately, let's develop a general theory that applies to vectors of any dimension.

n-dimensional vectors

Let n be some positive integer.

We call an n-tuple of real numbers an n-dimensional vector.

In general, it looks like this: $\vec{a} = (a_1, \ldots, a_n)$

these are called the coordinates or components of the vector.

We write vectors with arrows above them to distinguish them from scalars.

n-dimensional vectors

The set of all n-dimensional vectors is denoted

$$\mathbb{R}^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in \mathbb{R}\}$$

There is an n-dimensional zero vector that we denote

$$\mathbf{0} = (0, \ldots, 0)$$

insert the right number
of zeros here.

If n is not clear from context, we write $\mathbf{0}_n$.

n-dimensional vectors

All we have said carries over to n-dimensional vectors.

We can think about them in three ways:

§ As a tuple of n real numbers.

§ As a point in n-dimensional space.

§ As an arrow in n-dimensional space.

Questions

How can you visualize n-dimensional space for $n > 3$?

Well, you can't really. This is what makes the first representation—which we can manipulate—so useful.

Questions

What is the use of higher dimensional vectors?

Think back to the example of housing prices.

We can represent a flat by a vector.

$$\text{flat1} = (\underset{\text{price}}{850000}, \overset{\text{sq. feet}}{1614}, \underset{\text{\#beds}}{3}, \overset{\text{\#bath}}{2}, \underset{\text{district}}{21}, \overset{\text{year built}}{1999}, \underset{\text{lease type}}{99})$$

This naturally creates a 7-dimensional vector!

Operations with vectors

Reading: Strang 1.1

Learning objective: Visualize linear combinations of vectors.

Operations with vectors

Now that we have defined vectors, let's see what we can do with them.

There are two basic operations:

- § We can add vectors together.
- § We can multiply a vector by a scalar.

Recall that a scalar is just a single real number.

Vector Addition: Algebraic View

Vector addition is done coordinate-wise.

$$(3,5,2)+(4,-1,0)=(7,4,2)$$

If you think of coordinates as representing different goods, this makes sense.

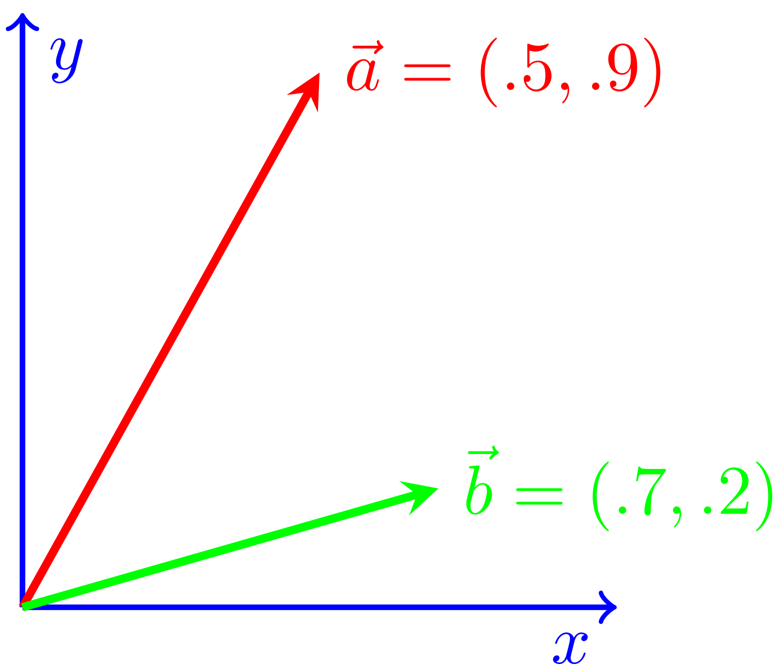
(apples, bananas, oranges)

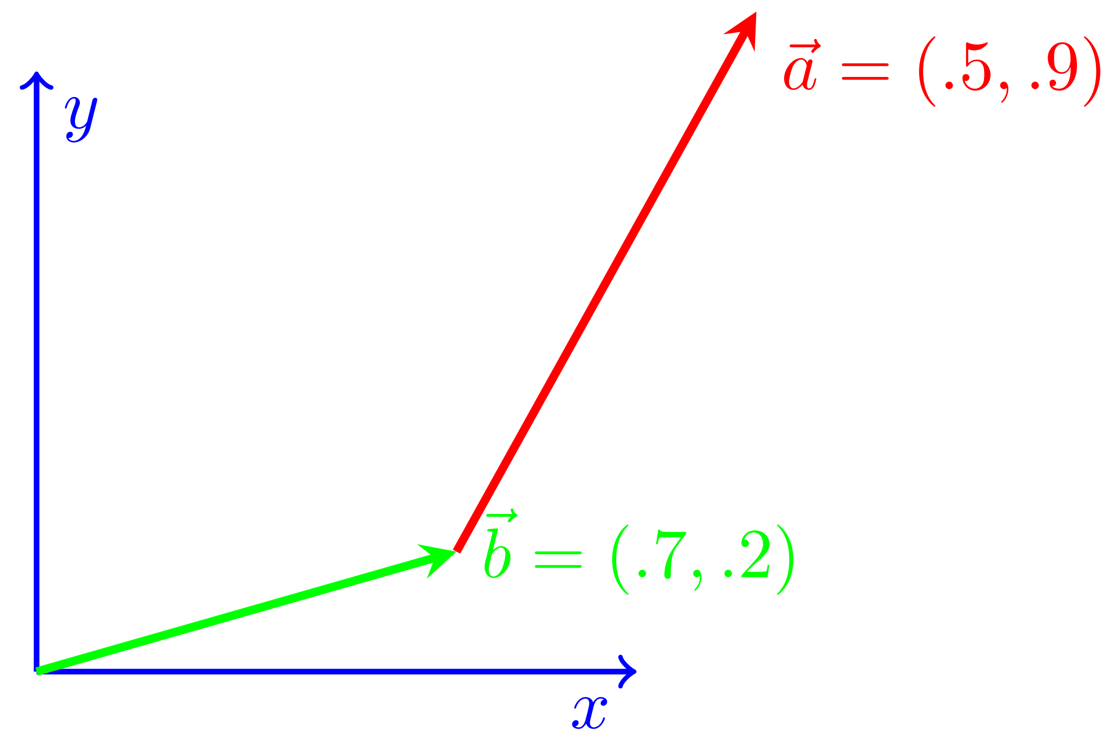
In general,

$$(a_1,\ldots,a_n)+(b_1,\ldots,b_n)=(a_1+b_1,\ldots,a_n+b_n)$$

Vector Addition: Geometric View

Vector addition can be visualized with arrows.



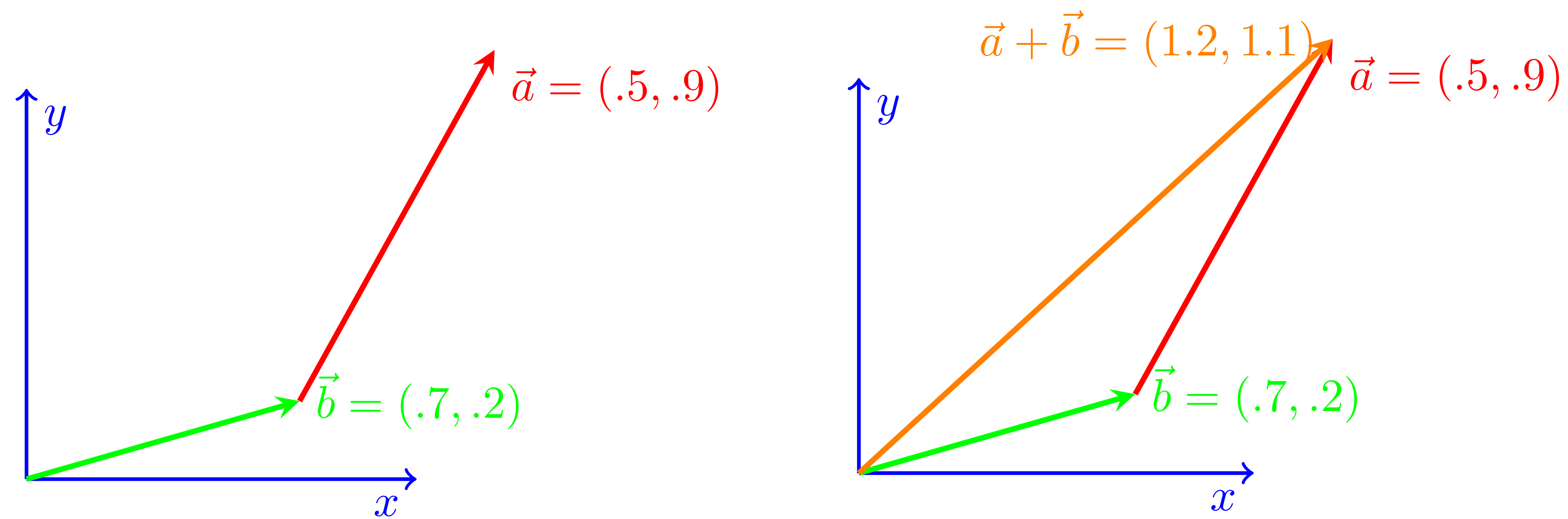


To compute $\vec{a} + \vec{b}$ we translate $\vec{a}$ to put its tail at the head of $\vec{b}$.

Remember translation does not change an arrow.

Vector Addition: Geometric View

Vector addition can be visualized with arrows.



The arrow from $(0, 0)$ to the head of $\vec{a}$ is $\vec{a} + \vec{b}$.

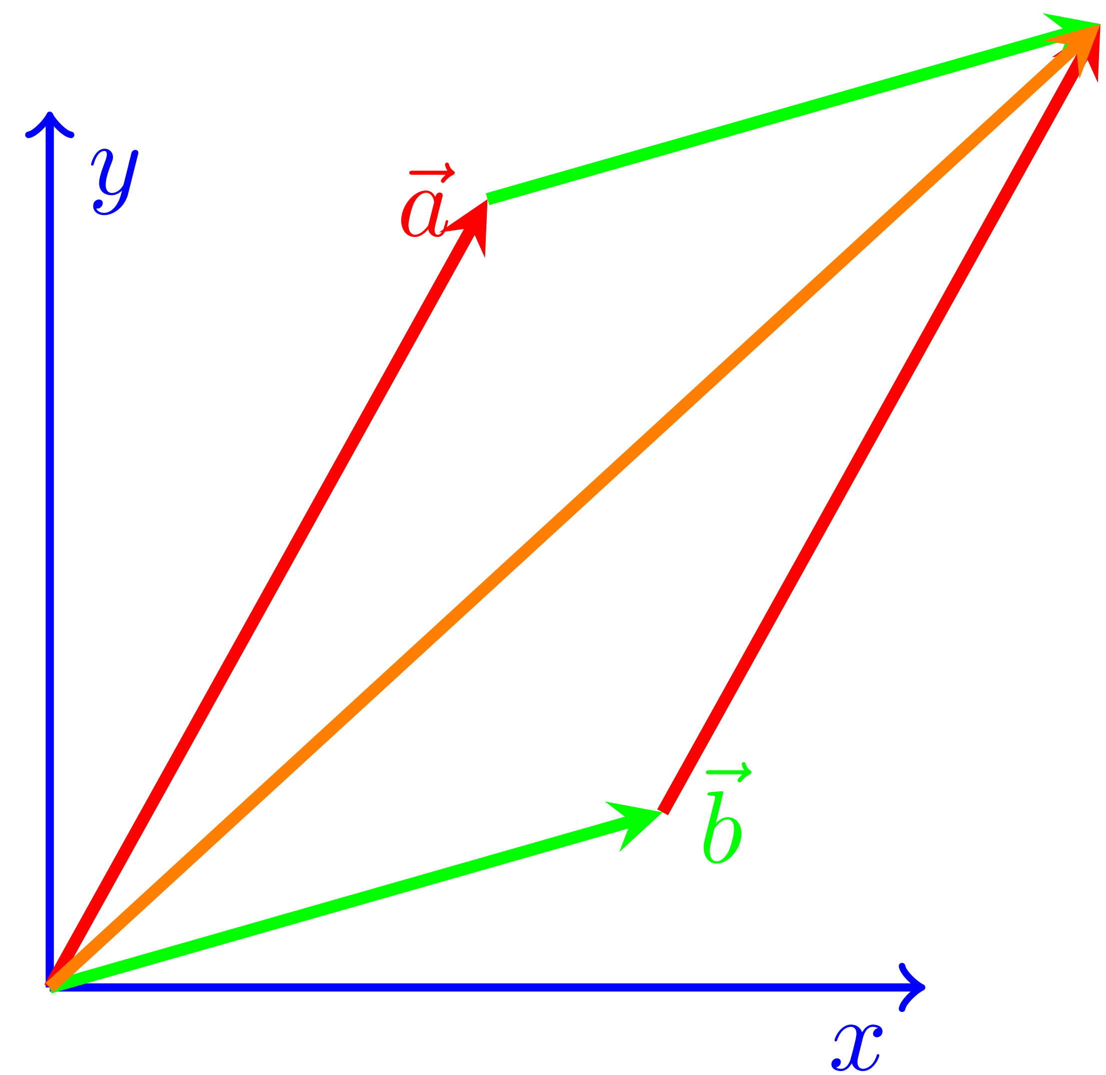
Commutativity

Note that it does not matter if we translate $\vec{a}$ to the head of $\vec{b}$ or translate $\vec{b}$ to the head of $\vec{a}$.

We end up at the same place.

Vector addition is commutative

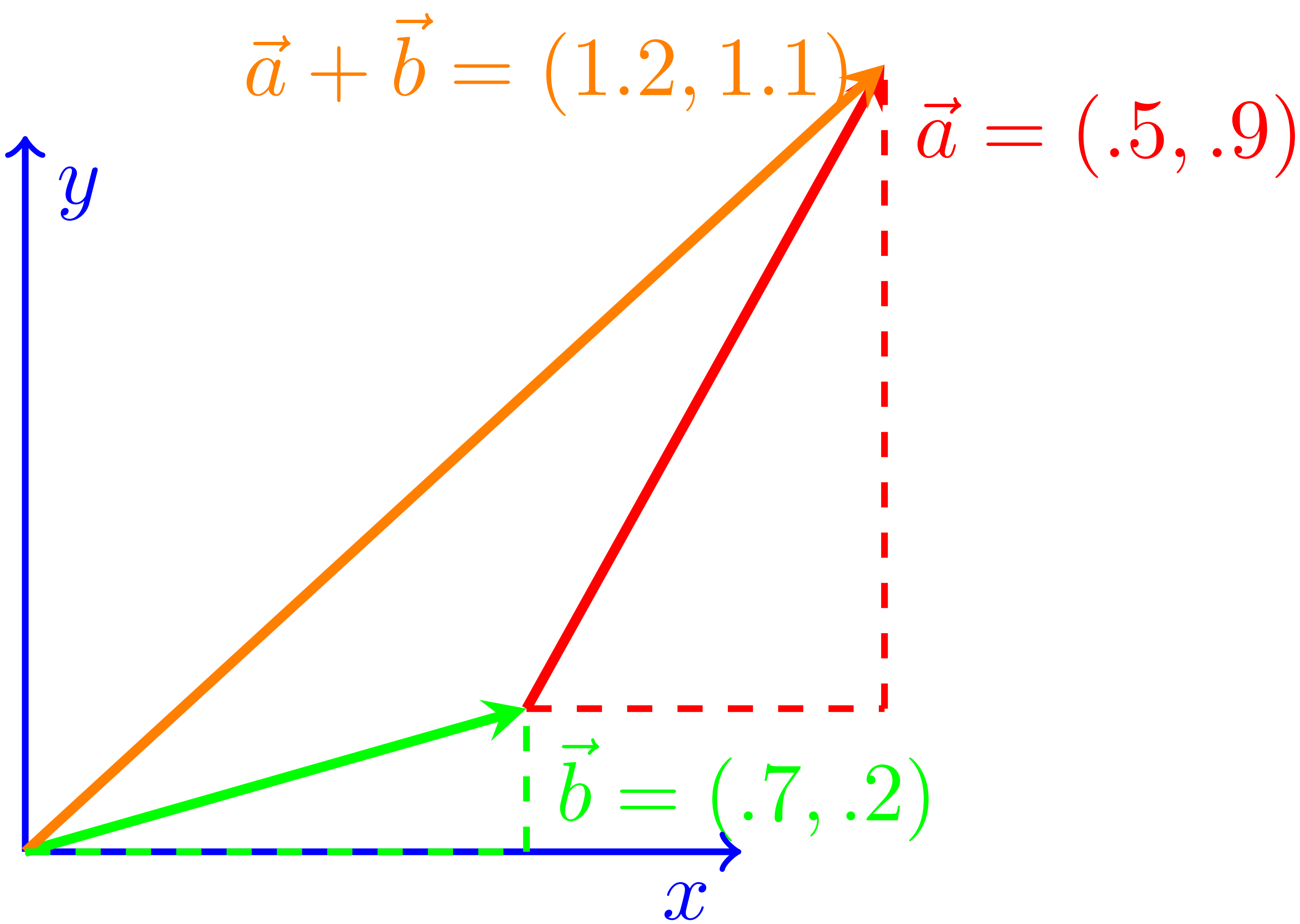
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$



What is the name of this shape?

Algebraic vs. Geometric

We can see the equivalence of the algebraic and geometric pictures of vector addition.



The x-coordinate of $\vec{a}+\vec{b}$ is that of $\vec{a}$ plus that of $\vec{b}$.

Scalar Multiplication

The second vector operation is scalar multiplication.

Algebraic view: scalar multiplication also acts coordinate-wise. Each coordinate is multiplied by the scalar.

$$2 \cdot (.7, .2) = (1.4, .4)$$

$$-3 \cdot (1, -2, 4) = (-3, 6, -12)$$

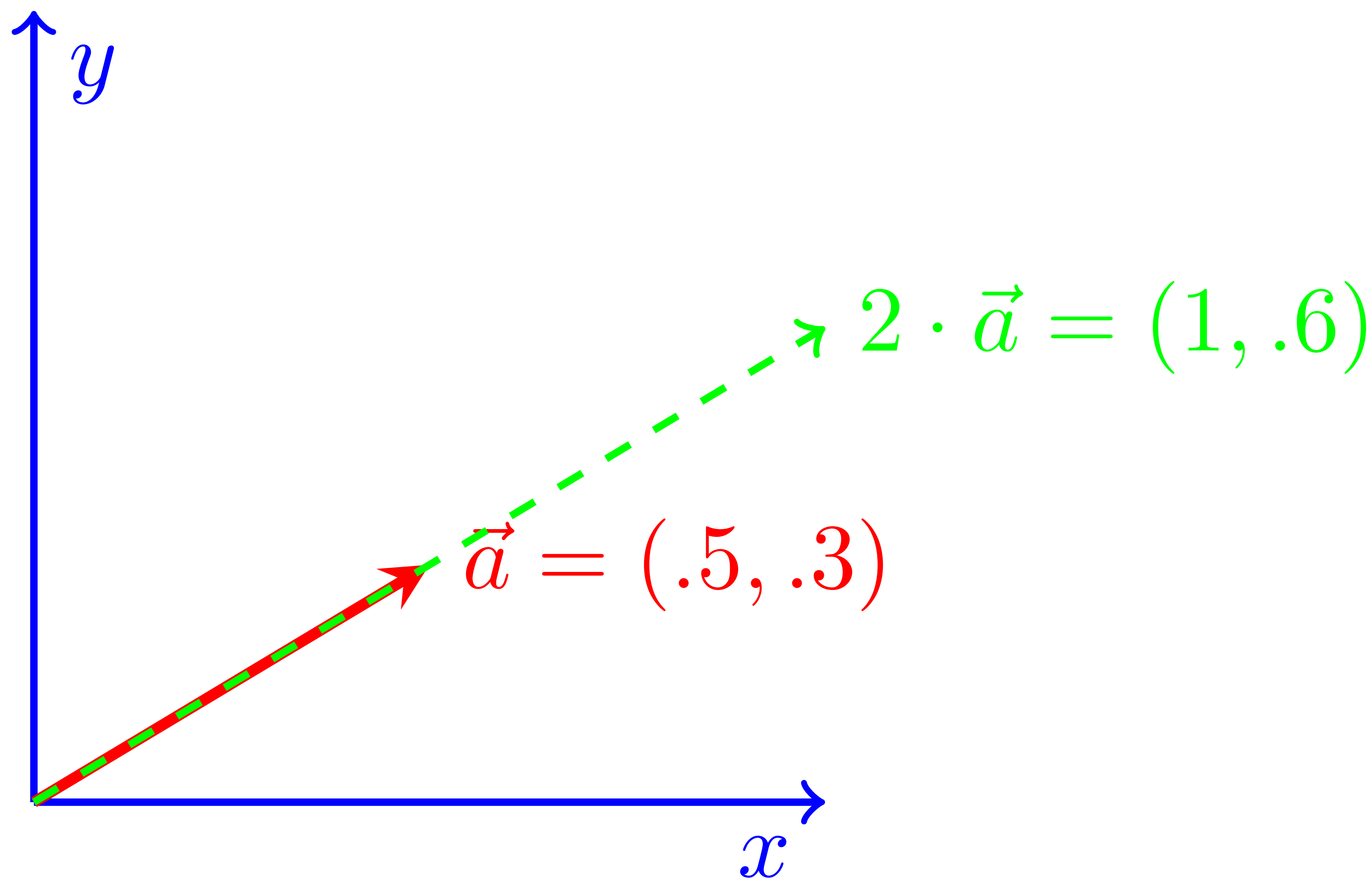
$$-1 \cdot (1, -1) = -(1, -1) = (-1, 1)$$

In general,

$$c \cdot (a_1, \ldots, a_n) = (c \cdot a_1, \ldots, c \cdot a_n)$$

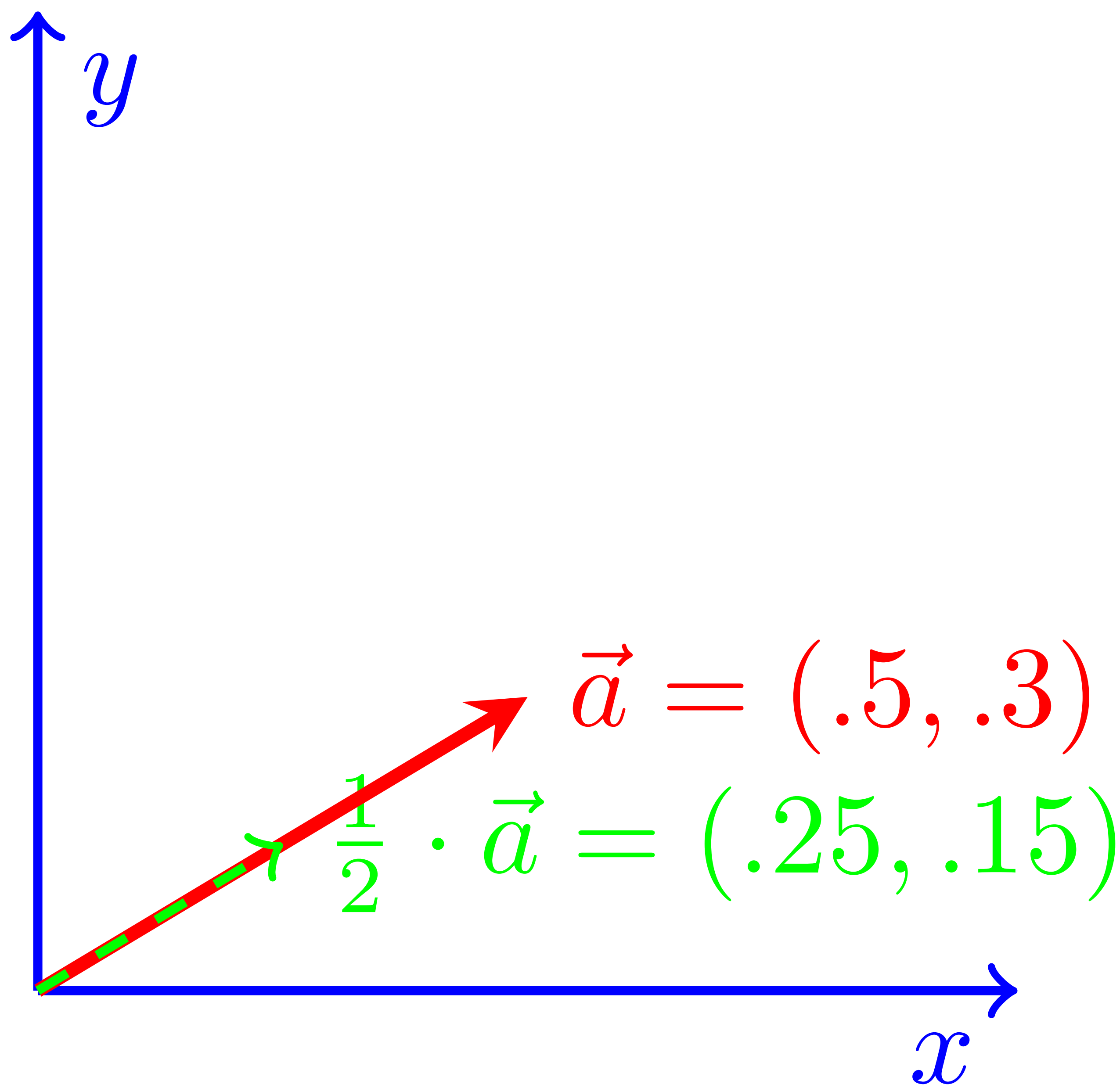
Scalar Multiplication: Geometric View

From the geometric view, multiplication by a positive scalar changes the length of an arrow, but not its direction.



Scalar Multiplication: Geometric View

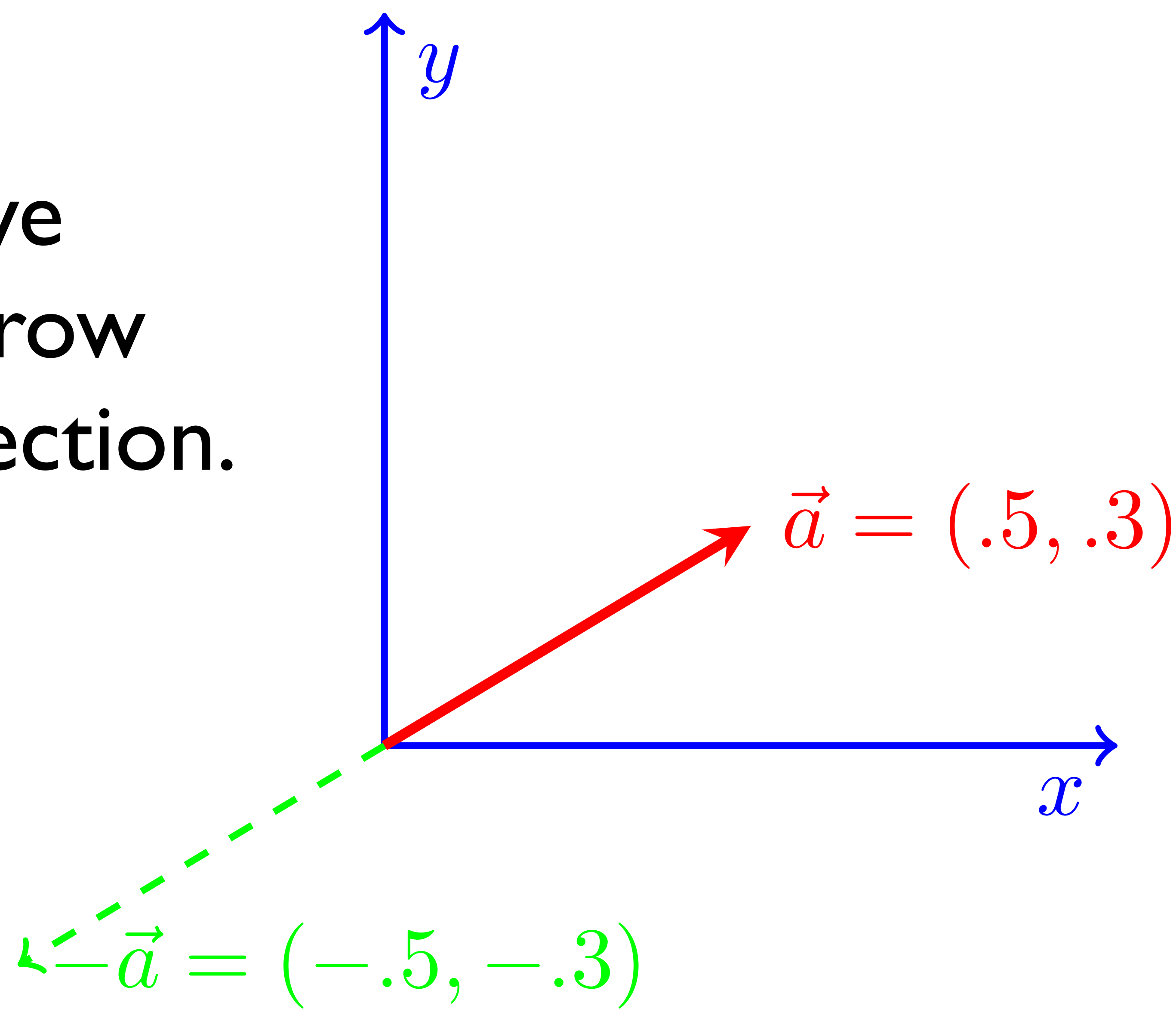
From the geometric view, multiplication by a positive scalar changes the length of an arrow, but not its direction.



Scalar Multiplication: Geometric View

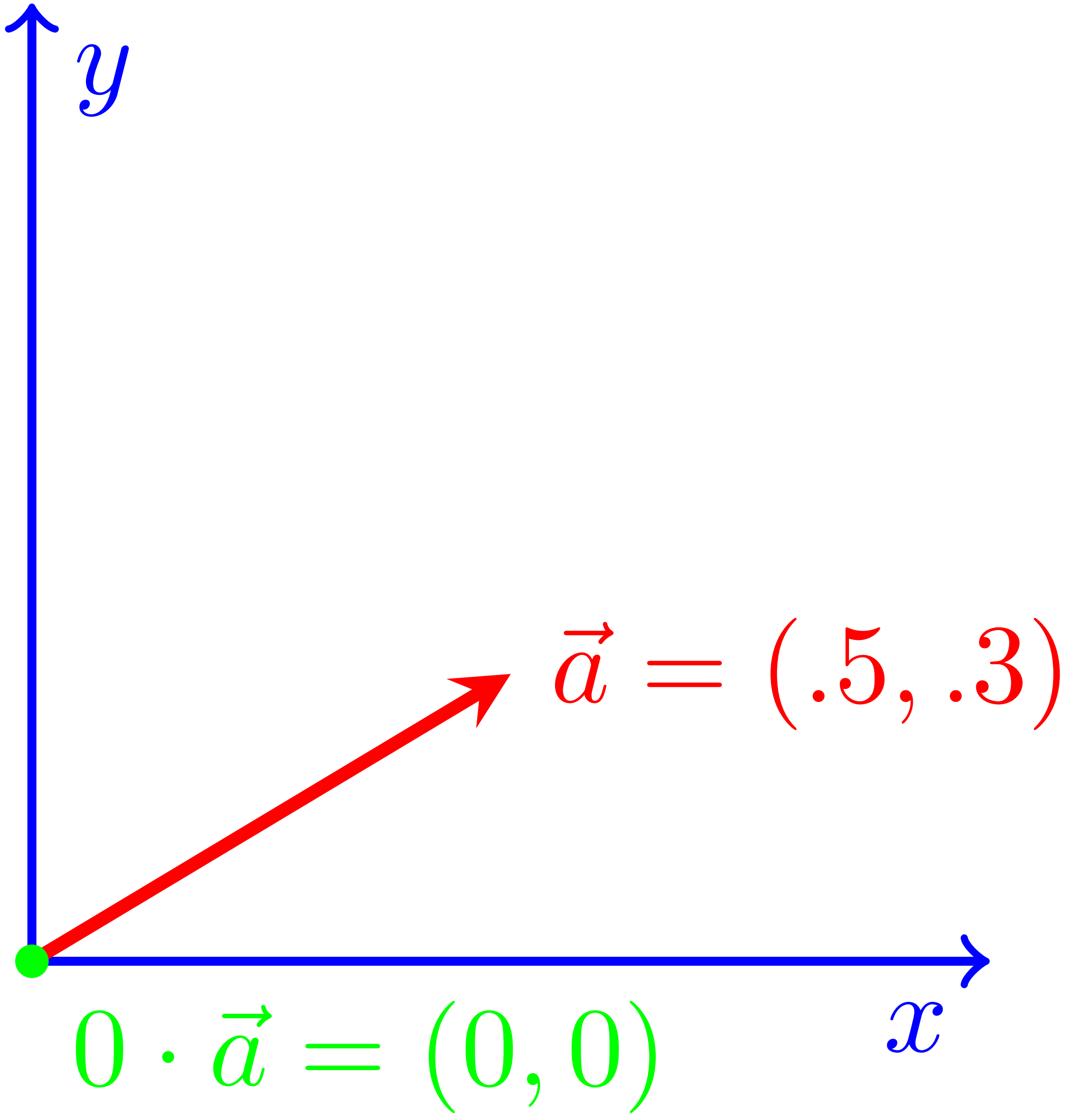
From the geometric view, multiplication by a positive scalar changes the length of an arrow, but not its direction.

Multiplying by a negative scalar will make the arrow point the opposite direction.



Zero Vector

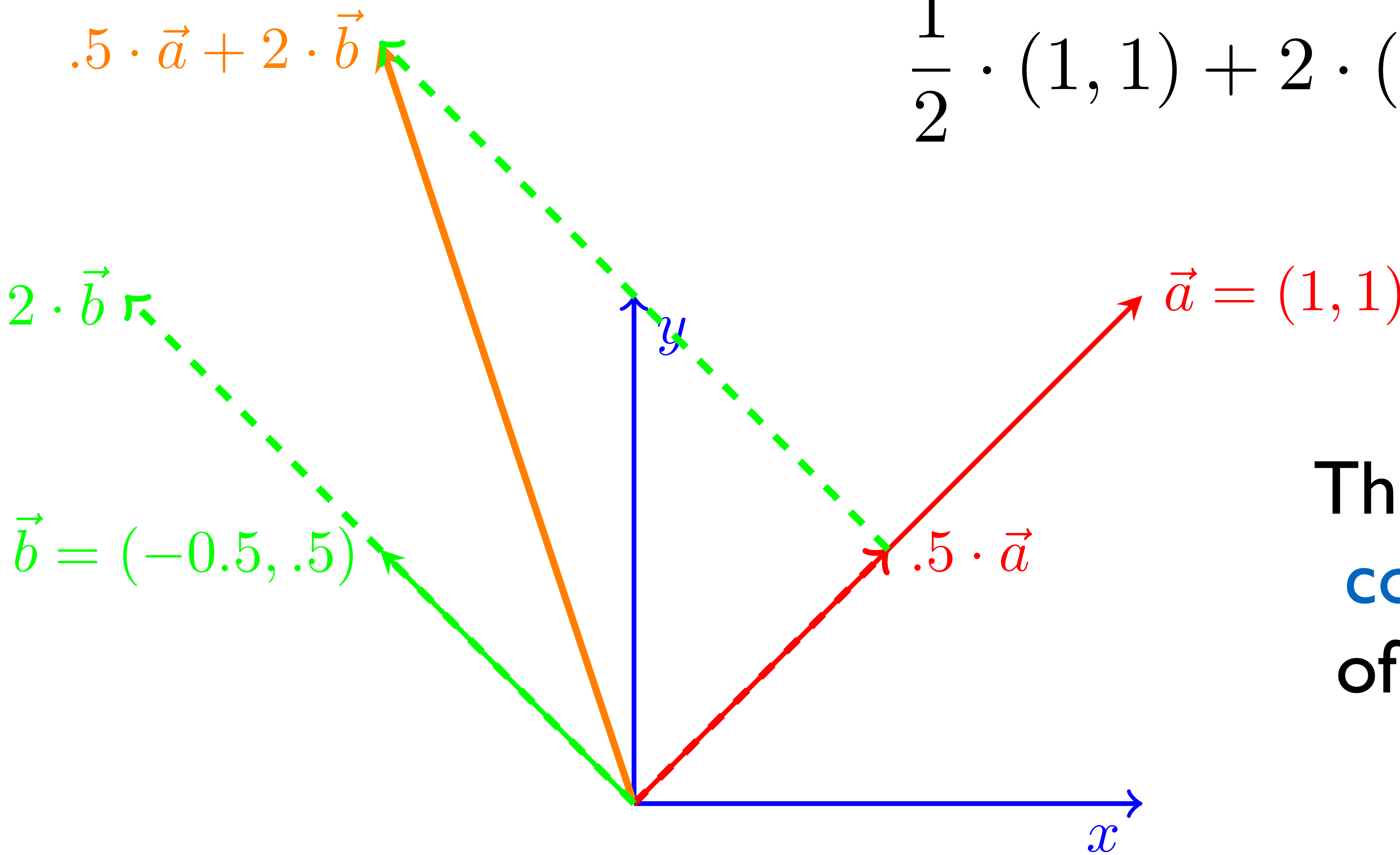
If we multiply any vector by 0, we get the zero vector.



Linear Combinations

We can do both of these operations at the same time.

$$\frac{1}{2} \cdot (1,1) + 2 \cdot (-.5, .5)$$



This is a linear combination of $\vec{a}$ and $\vec{b}$.

Linear Combinations

One could argue that linear algebra is the study of linear combinations of vectors.

This is a fundamental operation that we will endlessly talk about.

It is important to become very familiar with taking linear combinations.

Example 1

Let $\vec{a} = (1, 0)$ and $\vec{b} = (0, 1)$ be two vectors.

What is the set of all linear combinations of $\vec{a}$ and $\vec{b}$?

$$\{c_1 \cdot \vec{a} + c_2 \cdot \vec{b} : c_1, c_2 \in \mathbb{R}\}$$

Any vector $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

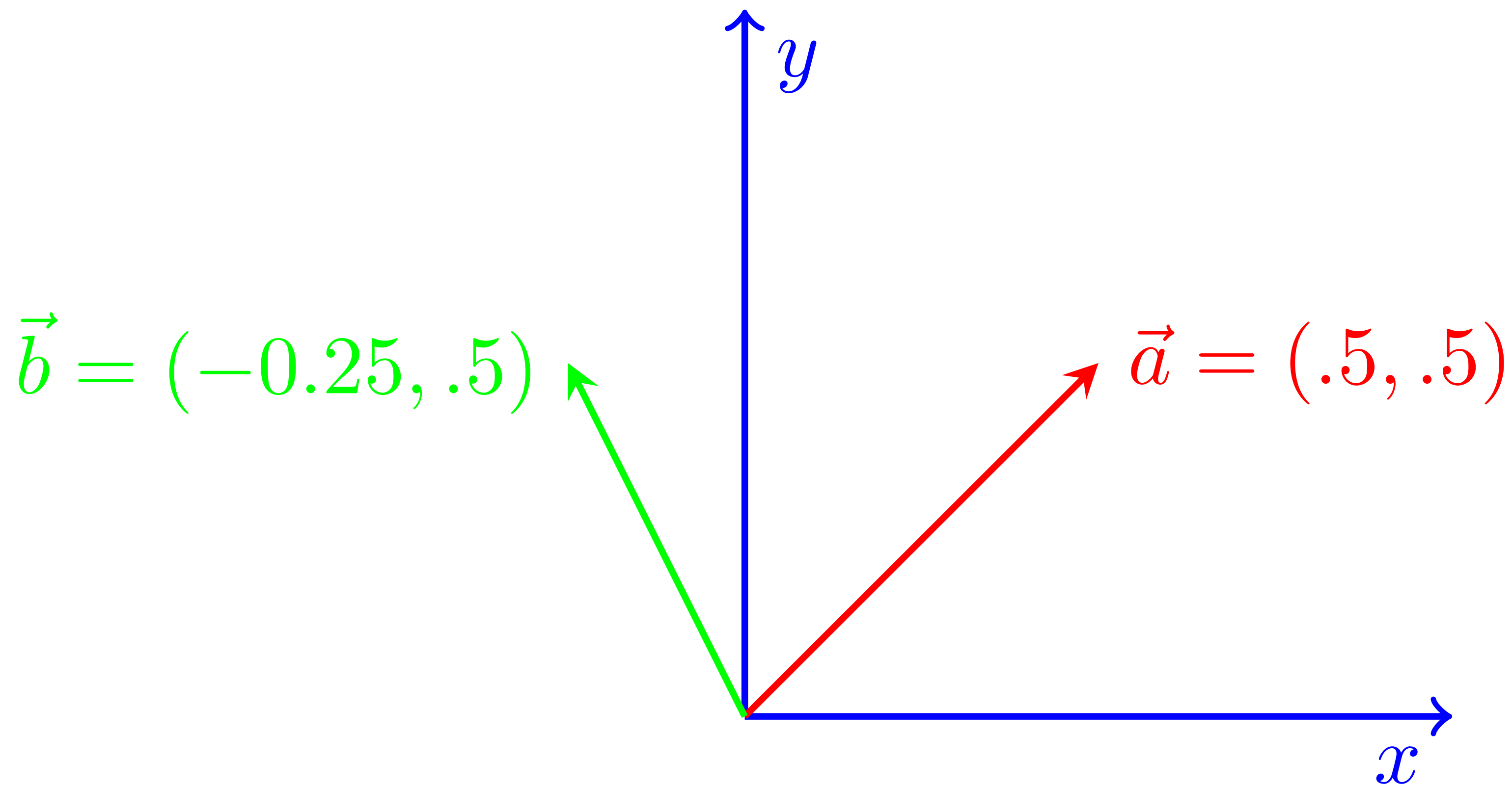
$$(u_1, u_2) = u_1 \cdot (1, 0) + u_2 \cdot (0, 1)$$

Example 2

Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{1}{2})$ be two vectors.

What is the set of all linear combinations of $\vec{a}$ and $\vec{b}$?

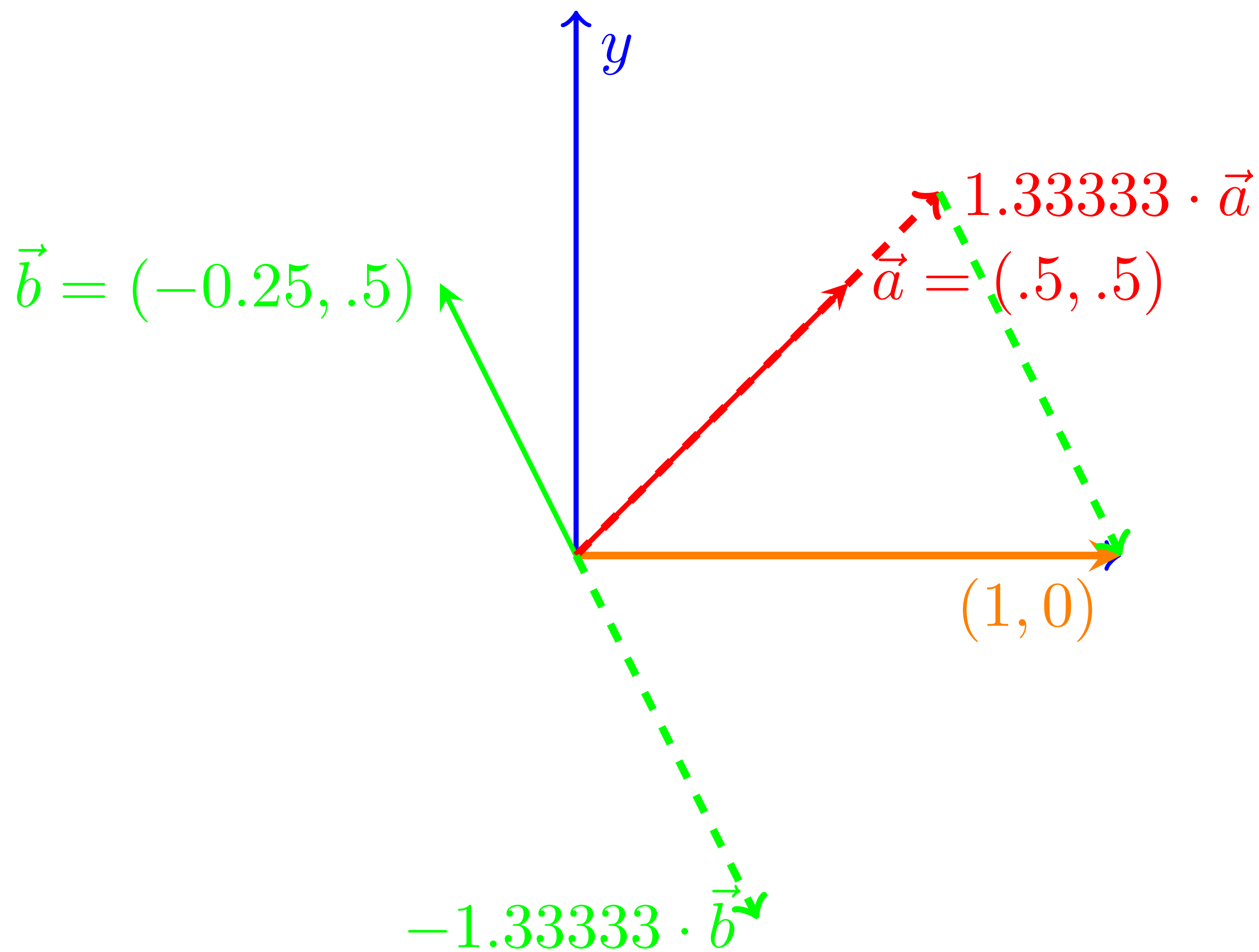
$$\{c_1 \cdot \vec{a} + c_2 \cdot \vec{b} : c_1, c_2 \in \mathbb{R}\}$$



Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{1}{2})$ be two vectors.

What is the set of all linear combinations of $\vec{a}$ and $\vec{b}$?

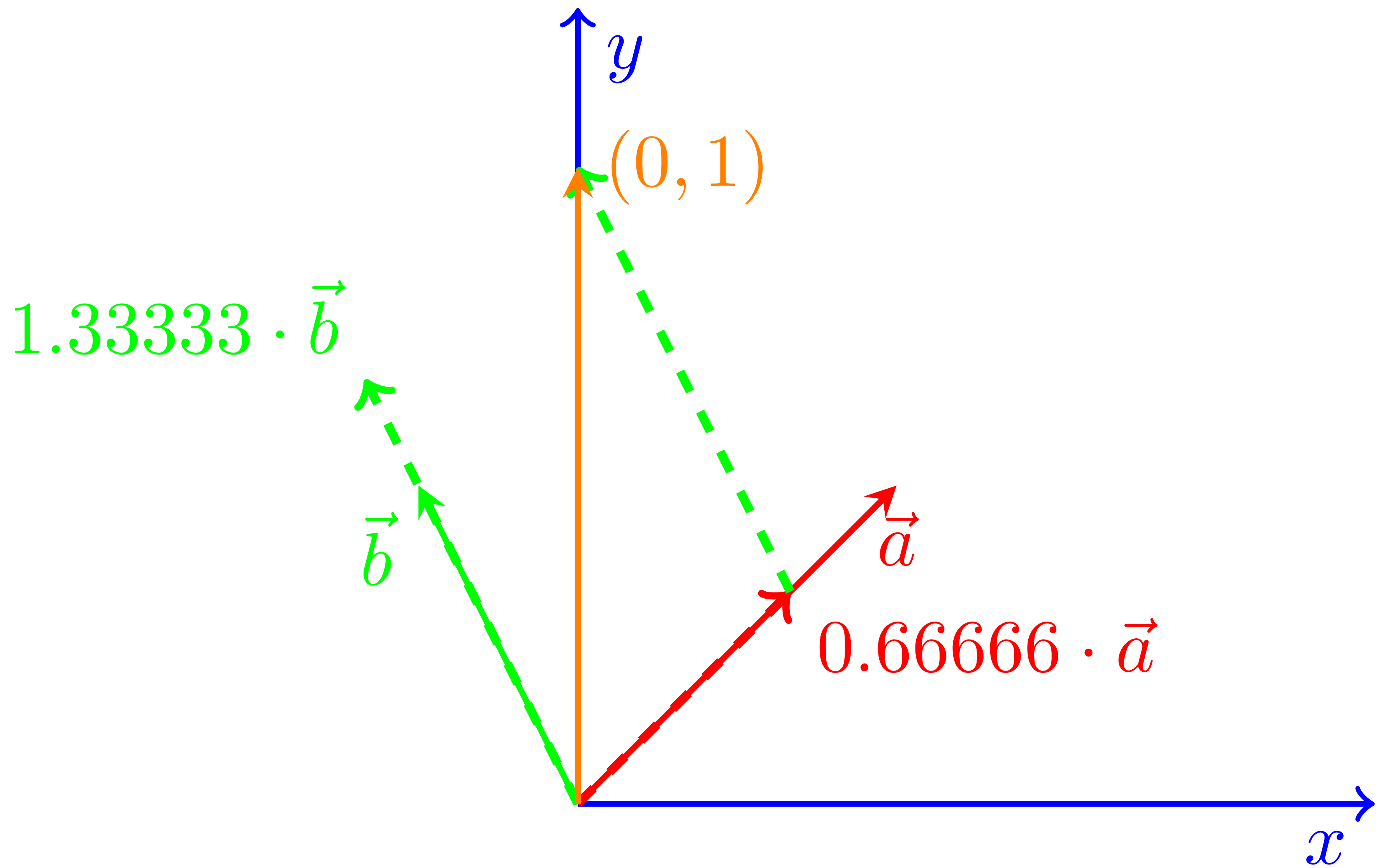
Note that $(1, 0) = \frac{4}{3} \cdot (\frac{1}{2}, \frac{1}{2}) - \frac{4}{3} \cdot (-\frac{1}{4}, \frac{1}{2})$



Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{1}{2})$ be two vectors.

What is the set of all linear combinations of $\vec{a}$ and $\vec{b}$?

Also $(0,1) = \frac{2}{3} \cdot (\frac{1}{2}, \frac{1}{2}) + \frac{4}{3} \cdot (-\frac{1}{4}, \frac{1}{2})$



Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{1}{2})$ be two vectors.

What is the set of all linear combinations of $\vec{a}$ and $\vec{b}$?

We can write $(1, 0)$ and $(0, 1)$ linear combinations of $\vec{a}$ and $\vec{b}$.

This means all two-dimensional vectors can be written as linear combinations of $\vec{a}$ and $\vec{b}$.

$$\{c_1 \cdot \vec{a} + c_2 \cdot \vec{b} : c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2$$

Do you see why?

Example 2 shows the typical situation.

Usually the set of all linear combinations of two two-dimensional vectors will be $\mathbb{R}^2$.

But this is not always the case.

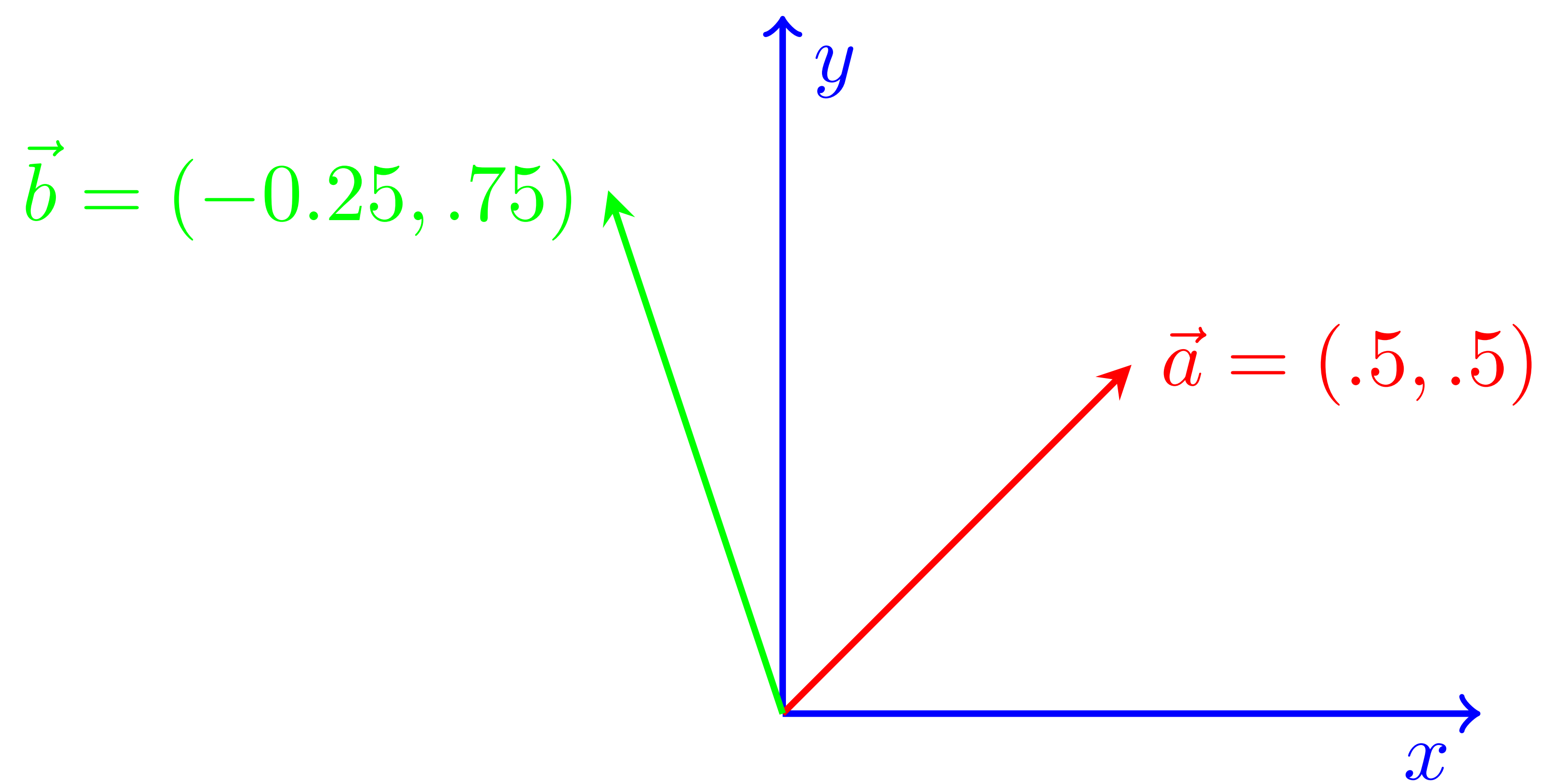
Can you think of a situation where it is not?

Example 3

Now for something trickier.

Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{3}{4})$ be two vectors.

What does the set $\{c \cdot \vec{a} + (1-c) \cdot \vec{b} : c \in \mathbb{R}\}$ look like?

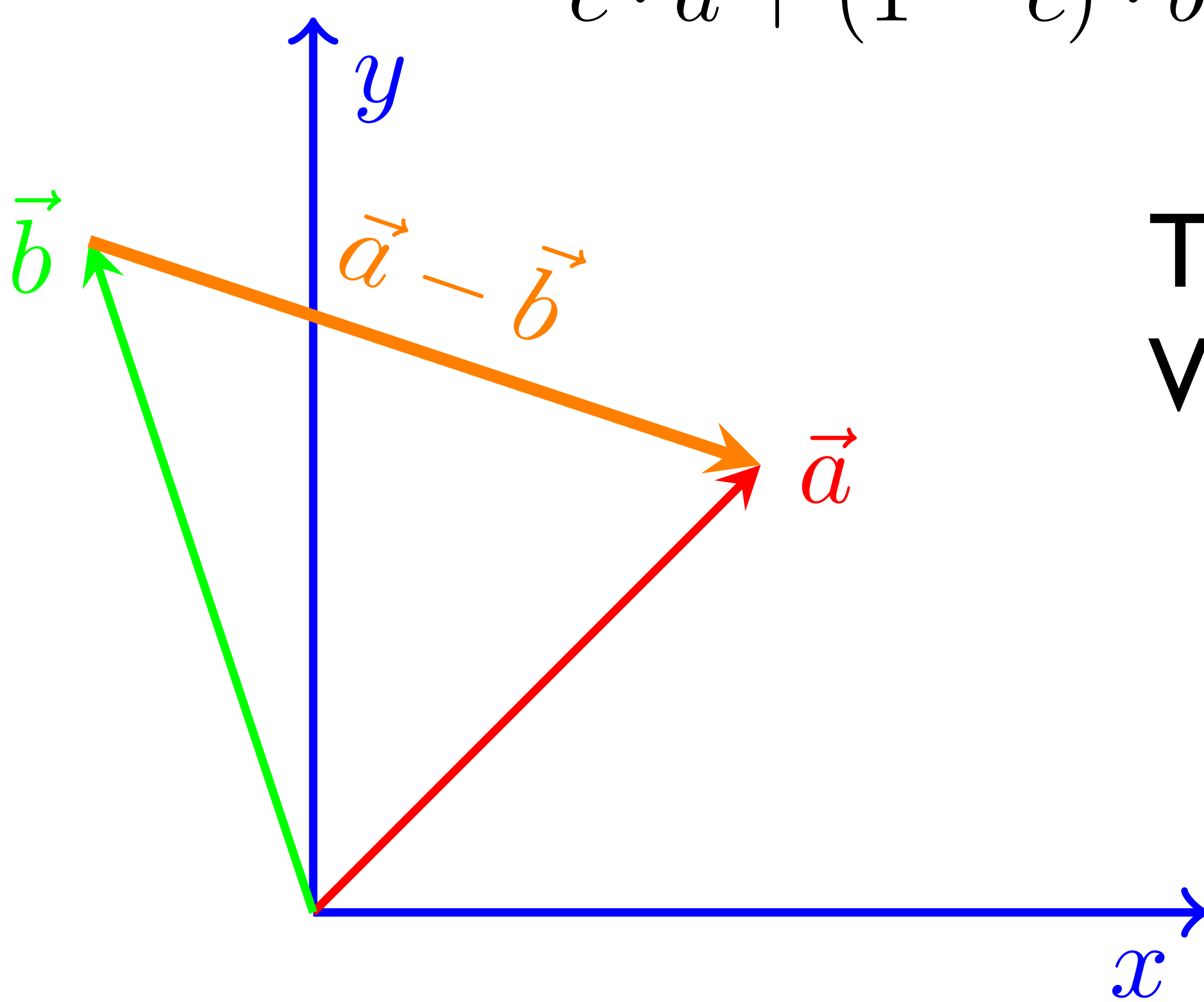


Let $\vec{a} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{b} = (-\frac{1}{4}, \frac{3}{4})$ be two vectors.

What does the set $\{c \cdot \vec{a} + (1-c) \cdot \vec{b} : c \in \mathbb{R}\}$ look like?

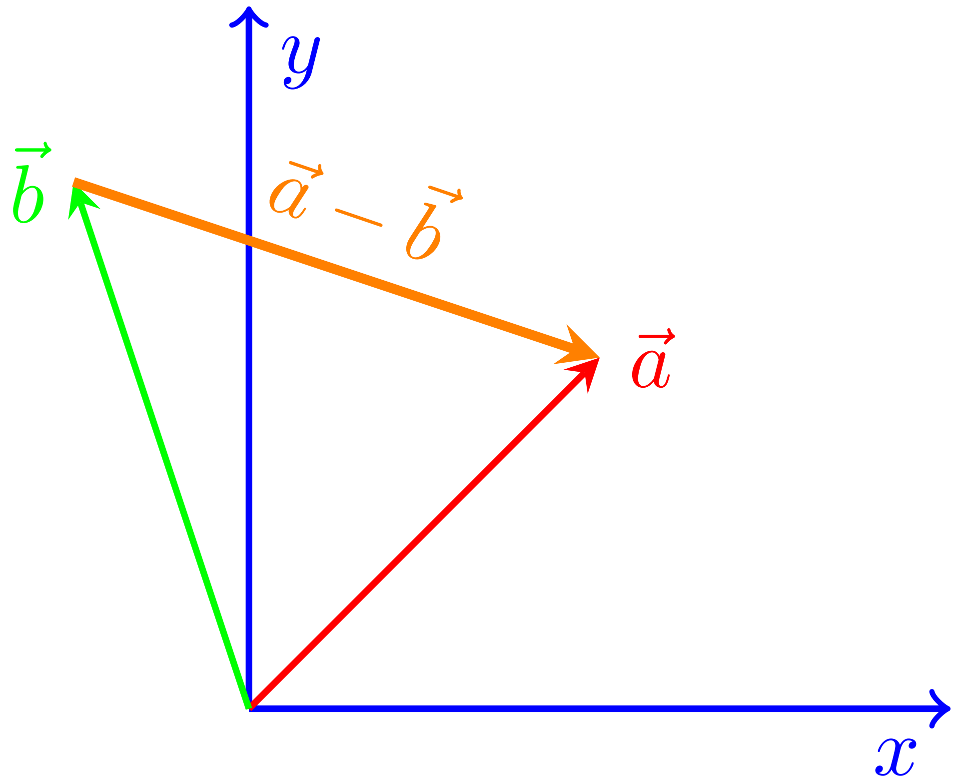
OK, let's break it down. Expand out the sum:

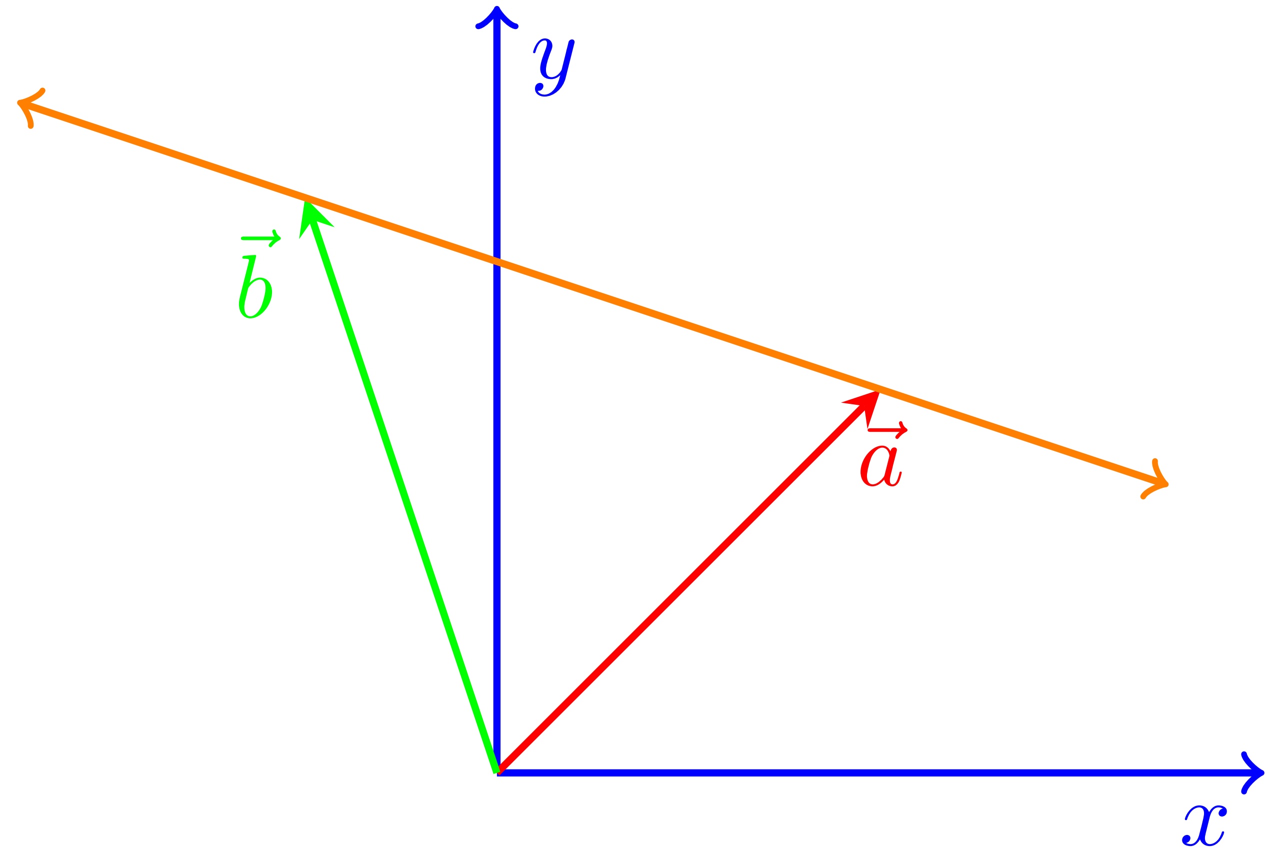
$$c \cdot \vec{a} + (1-c) \cdot \vec{b} = \vec{b} + c \cdot (\vec{a} - \vec{b})$$



This looks more manageable.
What is $\vec{a} - \vec{b}$?

$$c \cdot \vec{a} + (1 - c) \cdot \vec{b} = \vec{b} + c \cdot (\vec{a} - \vec{b})$$





The set $\{c \cdot \vec{a} + (1 - c) \cdot \vec{b} : c \in \mathbb{R}\}$ is the line through the points $\vec{a}$ and $\vec{b}$.

Properties of Vector Operations

Reading: Strang 1.1

Learning objective: Manipulate linear combinations using the properties of vector addition and scalar multiplication.

Properties of vector addition

- Commutativity

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

- Associativity

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

This means it doesn't matter in what order we add the vectors up.

Distributivity Properties

§ Scalar multiplication distributes over vector addition

$$c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$$

§ Scalar multiplication distributes over scalar addition

$$(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$$

These properties follow from the definition of vector addition and scalar multiplication and the fact that

$$c \cdot (a + b) = c \cdot a + c \cdot b$$

Application

Let's look at an application of these properties:

Claim: Let $\vec{a}, \vec{b} \in \mathbb{R}^2$. If

$$(1,0) = c_1 \cdot \vec{a} + c_2 \cdot \vec{b}$$

and

$$(0,1) = d_1 \cdot \vec{a} + d_2 \cdot \vec{b}$$

then for any vector $\vec{u} \in \mathbb{R}^2$ there exist scalars $e_1, e_2 \in \mathbb{R}$ such that

$$\vec{u} = e_1 \cdot \vec{a} + e_2 \cdot \vec{b}$$

Application

The key idea here is this:

If $\vec{v}$ is a linear combination of $\vec{a}, \vec{b}$ and $\vec{w}$ is a linear combination of $\vec{a}, \vec{b}$, then

$$c \cdot \vec{v} + d \cdot \vec{w}$$

is also a linear combination of $\vec{a}, \vec{b}$.

$$(1,0) = c_1 \cdot \vec{a} + c_2 \cdot \vec{b}$$

and

$$(0,1) = d_1 \cdot \vec{a} + d_2 \cdot \vec{b}$$

We want to show that any $\vec{u} = (u_1, u_2)$ is a linear combination of $\vec{a}, \vec{b}$.

$$\begin{aligned}(u_1, u_2) &= u_1 \cdot (1,0) + u_2 \cdot (0,1) \\ &= u_1 \cdot (c_1 \cdot \vec{a} + c_2 \cdot \vec{b}) + u_2 \cdot (d_1 \cdot \vec{a} + d_2 \cdot \vec{b})\end{aligned}$$

Now use distributivity over vector addition

$$= (u_1 c_1) \cdot \vec{a} + (u_1 c_2) \cdot \vec{b} + (u_2 d_1) \cdot \vec{a} + (u_2 d_2) \cdot \vec{b}$$

$$
\begin{aligned}
(u_1, u_2) &= u_1 \cdot (1,0) + u_2 \cdot (0,1) \\
&= u_1 \cdot (c_1 \cdot \vec{a} + c_2 \cdot \vec{b}) + u_2 \cdot (d_1 \cdot \vec{a} + d_2 \cdot \vec{b})
\end{aligned}
$$

Now use distributivity over vector addition

$$= (u_1 c_1) \cdot \vec{a} + (u_1 c_2) \cdot \vec{b} + (u_2 d_1) \cdot \vec{a} + (u_2 d_2) \cdot \vec{b}$$

Use commutativity

$$= (u_1 c_1) \cdot \vec{a} + (u_2 d_1) \cdot \vec{a} + (u_1 c_2) \cdot \vec{b} + +(u_2 d_2) \cdot \vec{b}$$

Use distributivity of scalar addition

$$= (u_1 c_1 + u_2 d_1) \cdot \vec{a} + (u_1 c_2 + u_2 d_2) \cdot \vec{b}$$

Done!