Subspace Review

Vector Space

A vector space is a nonempty set V on which the operations of addition and scalar multiplication are defined.

C1) Closure under addition:

$$x + y \in V \quad \text{for all} \quad x, y \in V$$

C2) Closure under scalar multiplication:

$$c \cdot x \in V \quad \text{for all} \quad x \in V \quad \text{and} \quad c \in \mathbb{R}.$$

Properties of Addition

A1) Addition is commutative

$$x + y = y + x \text{ for all } x, y \in V$$

A2) Addition is associative

$$(x + y) + z = x + (y + z) \text{ for all } x, y, z \in V$$

A3) Existence of a zero element $\mathbf{0} \in V$ such that

$$x + \mathbf{0} = x \text{ for all } x \in V$$

A4) For each x there exists a unique element $-x$ such that

$$x + (-x) = \mathbf{0}$$

Properties of Scalar Mult.

M1) Scalar multiplication is associative

$$a(bx) = (ab)x \text{ for all } a, b \in \mathbb{R}, x \in V$$

M2) Distributivity over addition in V

$$a(x + y) = ax + ay \text{ for all } a \in \mathbb{R}, x, y \in V$$

M3) Distributivity over scalar addition

$$(a + b)x = ax + bx \text{ for all } a, b \in \mathbb{R}, x \in V$$

M4) Identity for scalar multiplication

$$1x = x \text{ for all } x \in V$$

Subspaces

Now suppose that we know V is a vector space.

We want to check if $S \subseteq V$ is also a vector space.

Which conditions do we need to check?

Subspaces

Now suppose that we know V is a vector space.

We want to check if $S \subseteq V$ is also a vector space.

We already know that for any $x, y \in S$ it holds that

$$x + y = y + x$$

because this is true for every $x, y \in V$.

Subspaces

Now suppose that we know V is a vector space.

We want to check if $S \subseteq V$ is also a vector space.

We also already know that for any $x, y \in S, a \in \mathbb{R}$

$$a(x + y) = ax + by$$

because this is true for every $x, y \in V, a \in \mathbb{R}$.

S inherits many of the 10 conditions from V.

Subspaces

To check that $S \subseteq V$ is a subspace, we only need to verify

1. S contains the zero element of V: $\mathbf{0} \in S$.

2. S is closed under vector addition.

3. S is closed under scalar multiplication.

Note: As we have established that $(-x) = (-1) \cdot x$, item 3 guarantees property A4, existence of additive inverses.

Subspaces

What is the difference between a subspace and a vector space?

A subspace is a vector space.

It is a vector space that is a subset of another vector space.

This makes checking that it is a vector space easier.

Subspaces

We have already defined the major vector spaces we will be talking about:

1. $\mathbb{R}^n$

2. Space of m-by-n matrices with real entries.

3. The space of functions $f : \mathbb{R} \to \mathbb{R}$.

From here on we will be looking at subspaces in these three vector spaces.

Examples

Example

Claim: $S = \{t \cdot (1,1,1) : t \in \mathbb{R}\}$ is a subspace.

Proof: Setting $t = 0$ we see that $(0,0,0) \in S$, thus it contains the zero element of $\mathbb{R}^3$.

Closure under addition: Let $\vec{u}, \vec{v} \in S$. Then for some values $t_1, t_2 \in \mathbb{R}$

$$\vec{u} = (t_1, t_1, t_1)$$
$$\vec{v} = (t_2, t_2, t_2)$$

$$\begin{aligned} \vec{u} + \vec{v} &= (t_1 + t_2, t_1 + t_2, t_1 + t_2) \\ &= (t_1 + t_2) \cdot (1,1,1) \end{aligned}$$

Therefore $\vec{u} + \vec{v} \in S$.

Example

Claim: $S = \{t \cdot (1,1,1) : t \in \mathbb{R}\}$ is a subspace.

Proof:

Closure under scalar multiplication: let $\vec{u} \in S$ and $c \in \mathbb{R}$.
Then for some $t \in \mathbb{R}$

$$\vec{u} = t \cdot (1,1,1)$$

$$c \cdot \vec{u} = (c \cdot t) \cdot (1,1,1)$$

Therefore $c \cdot \vec{u} \in S$.

Lines through the origin

$$S = \{t \cdot (1,1,1) : t \in \mathbb{R}\}$$

This set is a line through the origin $(0,0,0)$.

Any line through the origin is a subspace.

A line that does not pass through the origin will not be a subspace.

Example 2

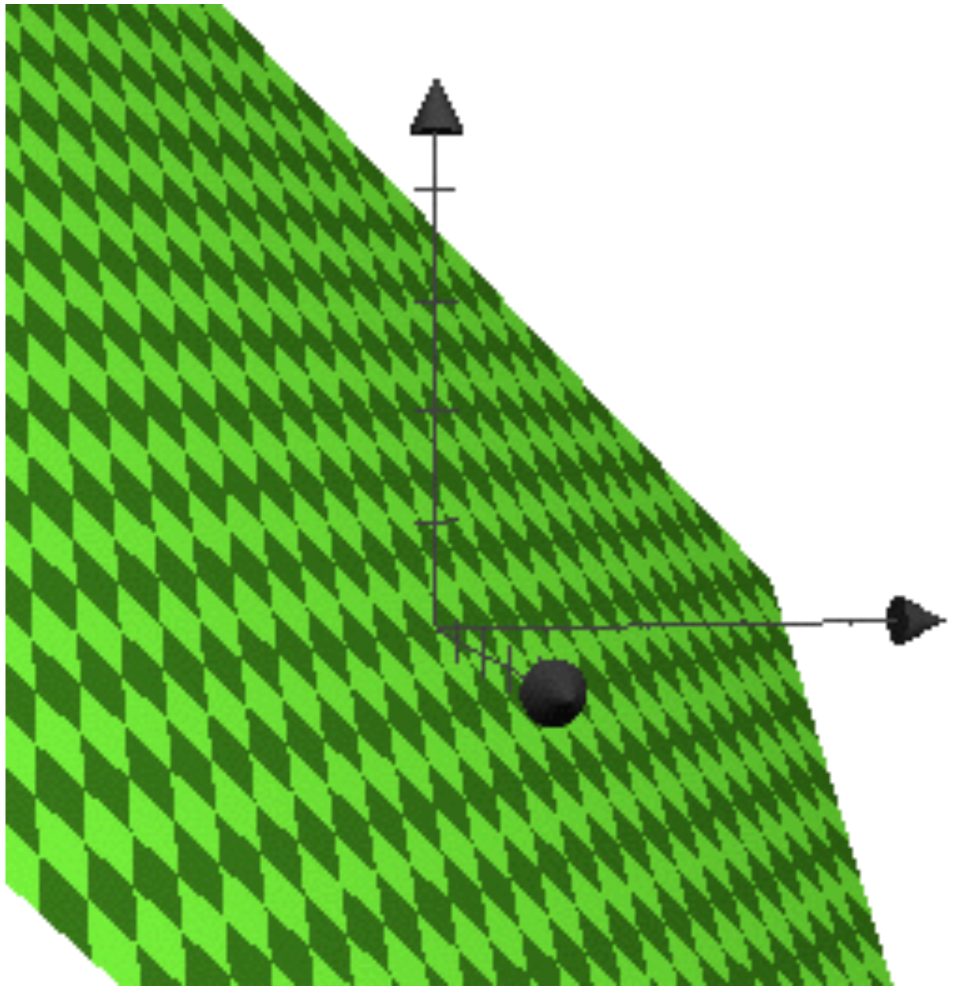
Let

$$S = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

This is a subspace of $\mathbb{R}^3$.

It is the set of all linear combinations of two vectors.

It is a plane passing through the origin.



Let

$$S = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Setting $s = t = 0$ we see that $(0,0,0) \in S$.

Closure under addition: let's add together two arbitrary elements of S .

$$s_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s_2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= (s_1 + s_2) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (t_1 + t_2) \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in S$$

Let

$$S = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Closure under scalar multiplication: let's take an arbitrary element of S and multiply it by $c \in \mathbb{R}$.

$$c \cdot \left(s \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = (c \cdot s) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (c \cdot t) \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in S$$

Planes through the origin

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. The set $\{s \cdot \vec{u} + t \cdot \vec{v} : s, t \in \mathbb{R}\}$ is the set of all linear combinations of $\vec{u}$ and $\vec{v}$.

This set will always be a subspace.

Any plane through the origin is a set of this form.

Planes through the origin are subspaces.

Column Space

Reading: Strang 3.1

Learning objective: Understand the definition of the column space and its relation to solving systems of linear equations.

Example 2 revisited

Let

$$S = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Let us view this set in a slightly different way.

Make a matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

with these two vectors as its columns.

Column Space

Make a matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The set of all linear combinations of the columns of A , denoted $C(A)$, is called the column space of A .

This is a subspace.

Column Space

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The set of all linear combinations of the columns of A, denoted $C(A)$, is called the column space of A.

The matrix-vector product

$$A\vec{u} = u_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in C(A)$$

is a linear combination of the columns of A, and therefore lies in the column space of A.

Column Space

The matrix vector product

$$A\vec{u} = u_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in C(A)$$

is a linear combination of the columns of A, and therefore lies in the column space of A.

$$\boxed{\text{The equation } A\vec{x} = \vec{b} \text{ has a solution iff } \vec{b} \in C(A)}$$

Span

Reading: Strang 3.1

Learning objective: Understand and be able to apply the definition of the span of vectors.

Definition

Let V be a vector space and $v_1, \ldots, v_k \in V$.

The span of $\{v_1, \ldots, v_k\}$, written as $\text{span}(\{v_1, \ldots, v_k\})$, is the set of all linear combinations of $v_1, \ldots, v_k$.

$$\text{span}(\{v_1, \ldots, v_k\}) = \{a_1 v_1 + a_2 v_2 + \cdots + a_k v_k : a_1, \ldots, a_k \in \mathbb{R}\}$$

We define the span of the empty set to be the set consisting of the zero element of V.

$$\text{span}(\emptyset) = \{\mathbf{0}\}$$

Notes

The argument to span is a set of vectors and the result is also a set of vectors.

We will often informally say "the span of $v_1, \ldots, v_k$".

For a matrix A, it is fine to say "the column space of A is the span of the columns of A".

It does not make sense to say "the span of A".

Notes

For a set S the textbook uses the notation SS to denote the span of S.

I will use the more standard notation $\text{span}(S)$.

So far we have just defined the span of a finite set of vectors.

We will almost always use this case, but later we will see a more general definition.

Example 1

What is the span of the vectors $(1,0,0),(0,1,0),(0,0,1)$?

$$\{a\cdot(1,0,0)+b\cdot(0,1,0)+c\cdot(0,0,1) : a,b,c\in\mathbb{R}\}$$

$$=\{(a,b,c) : a,b,c\in\mathbb{R}\}$$

$$=\mathbb{R}^3$$

Example 2

What is the span of the following matrices?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This will be

$$\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} : a_1, \ldots, a_6 \in \mathbb{R} \right\}$$

the set of upper triangular 3-by-3 matrices.

Example 3

What is the span of the functions $1, x, x^2$?

This will be $\{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ the set of all polynomials of degree at most two.

In all 3 examples, the span turned out to be a subspace.

This is a general fact.

Span is a Subspace

Theorem: Let V be a vector space and $v_1, \ldots, v_k \in V$. $\operatorname{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

Proof: Recall that $0 \cdot v = \mathbf{0}$ for any vector $v \in V$.

Thus $$\begin{aligned} 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_k &= \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

This shows that $\mathbf{0} \in \operatorname{span}(\{v_1, \ldots, v_k\})$.

Theorem: Let V be a vector space and $v_1, \ldots, v_k \in V$. $\mathrm{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

Proof: Closure under vector addition.

Let $x, y \in \mathrm{span}(\{v_1, \ldots, v_k\})$.

$$x = a_1 \cdot v_1 + \cdots + a_k \cdot v_k$$
$$y = b_1 \cdot v_1 + \cdots + b_k \cdot v_k$$

Using commutativity and distributivity we see

$$x + y = (a_1 + b_1) \cdot v_1 + \cdots + (a_k + b_k) \cdot v_k$$

is also a linear combination of $v_1, \ldots, v_k$.

Theorem: Let V be a vector space and $v_1, \ldots, v_k \in V$. $\mathrm{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

Proof: Closure under scalar multiplication.

Let $x \in \mathrm{span}(\{v_1, \ldots, v_k\})$ and $c \in \mathbb{R}$.

Then for some $a_1, \ldots, a_k \in \mathbb{R}$

$$x = a_1 \cdot v_1 + \cdots + a_k \cdot v_k$$

We have

$$\begin{aligned} c \cdot x &= c \cdot (a_1 \cdot v_1 + \cdots + a_k \cdot v_k) \\ &= (ca_1) \cdot v_1 + \cdots + (ca_k) \cdot v_k \end{aligned}$$

by distributivity and associativity of scalar multiplication. This is a linear combination of $v_1, \ldots, v_k$.

Span is a Subspace

Theorem: Let V be a vector space and $v_1, \ldots, v_k \in V$. $\operatorname{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

Remark 1: Recall we defined $\operatorname{span}(\emptyset) = \{\mathbf{0}\}$ thus the span is a subspace in this case as well.

Remark 2: Any subspace that contains $v_1, \ldots, v_k$ must also contain $\operatorname{span}(\{v_1, \ldots, v_k\})$.

$\operatorname{span}(\{v_1, \ldots, v_k\})$ is the smallest subspace containing $v_1, \ldots, v_k$.