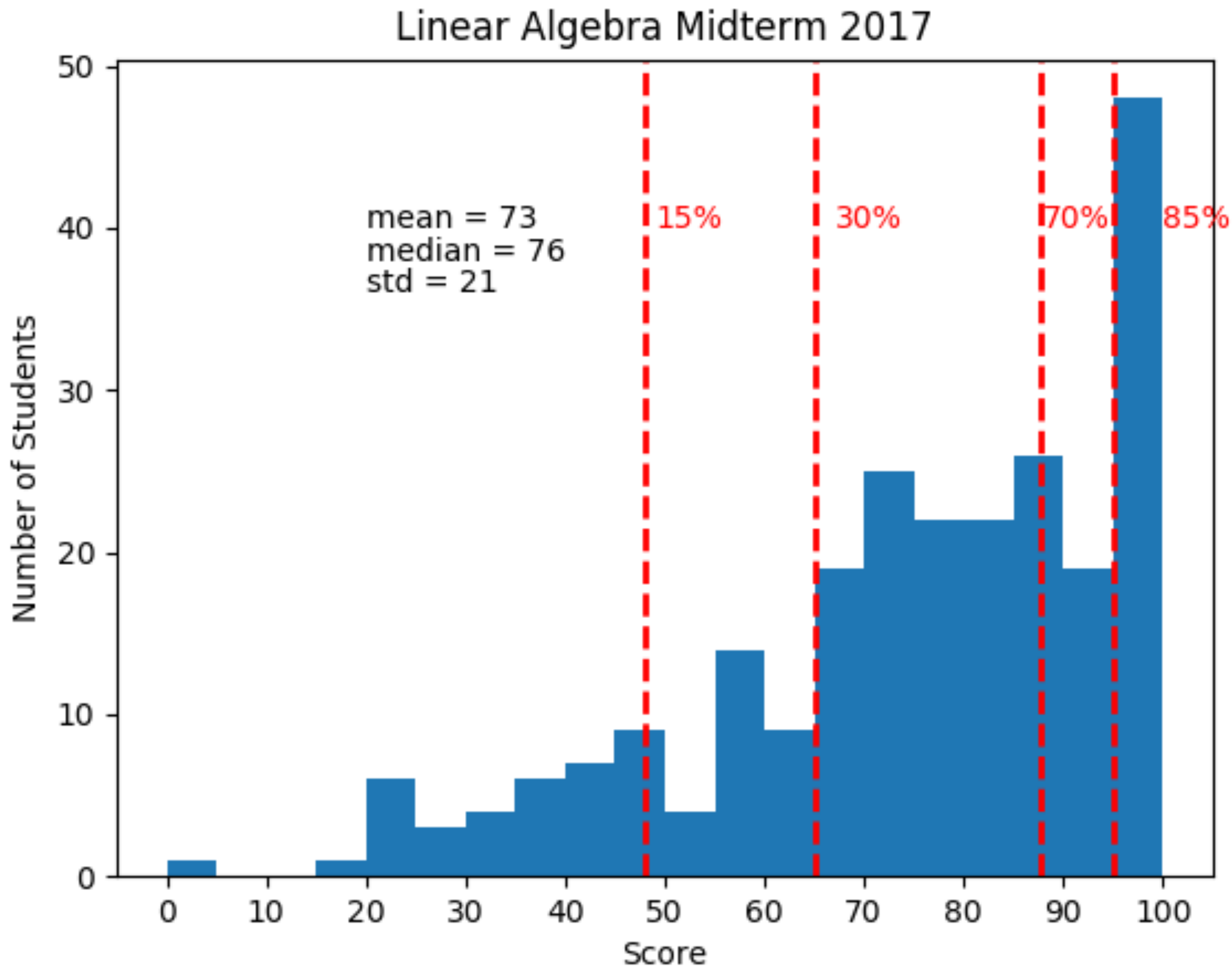
Midterm Recap


Linear Algebra Midterm 2017
mean = 73
median = 76
std = 21
15%
30%
70%
85%
Number of Students
Score
0
10
20
30
40
50
60
70
80
90
100

Question 4, part I

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

Is there a vector $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a unique solution?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

Is there a vector $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a unique solution?

You actually don't have to do any computation for this problem. A row echelon form of A will have at most 3 pivots as there are only 3 rows.

But A has 4 columns. There must be free column.

$A\vec{x} = \vec{b}$ will either be inconsistent, or have infinitely many solutions.

Incorrect Answer

If $A\vec{x} = \vec{b}$ has a unique solution, then A is invertible.

In this case A is 3-by-4 and so is not invertible.

Thus $A\vec{x} = \vec{b}$ cannot have a unique solution.

What is wrong with this argument?

Reason

The statement

“If $A\vec{x} = \vec{b}$ has a unique solution, then A is invertible.”

is not correct.

The correct statement is:

If A is square and $A\vec{x} = \vec{b}$ has a unique solution, then A is invertible.

Question 4, part II

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Is there a vector $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a unique solution?

2 pivots → No free column.

$$\begin{bmatrix} 1 & 1 & \mathbf{b_1} \\ 1 & -1 & \mathbf{b_2} \\ 1 & 1 & \mathbf{b_3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{b_1} \\ 0 & -2 & \mathbf{b_2} - \mathbf{b_1} \\ 0 & 0 & \mathbf{b_3} - \mathbf{b_1} \end{bmatrix}$$

Is there a vector $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a unique solution?

2 pivots $\Rightarrow$ No free column.

$$\begin{bmatrix} 1 & 1 & \mathbf{b_1} \\ 1 & -1 & \mathbf{b_2} \\ 1 & 1 & \mathbf{b_3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{b_1} \\ 0 & -2 & \mathbf{b_2} - \mathbf{b_1} \\ 0 & 0 & \mathbf{b_3} - \mathbf{b_1} \end{bmatrix}$$

If $b_1 = b_3$ the system will be consistent and will have a unique solution as there is no free column.

If $b_1 \neq b_3$ the system is inconsistent.

Question 5

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Find the set of $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a solution.

Don't forget to modify the right hand side during Gaussian elimination!

$$A\vec{x} = \vec{b} \implies EA\vec{x} = E\vec{b}$$

Inverse by cofactors

Reading: Strang 5.3

Learning objective: See how to express the inverse of a matrix in terms of the matrix of its cofactors.

Review: Cofactors

Let A be an n-by-n matrix.

The cofactor of the (i,j) entry is $\det(A'_{ij})$, where A'_{ij} is equal to A outside of the i^{th} row and in the i^{th} row is zero everywhere except for the (i,j) entry, which is 1.

Example:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C_{23} = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

cofactor of the $(2,3)$ entry

Cofactors: First Simplification

By adding a multiple of one row to another, which does not change the determinant, we see the (i,j) cofactor is equal to $\det(A''_{ij})$ where A''_{ij} equals A except in the i^{th} row and j^{th} column, where it is all zero except for the (i,j) entry, which is 1.

Example:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C_{23} = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

Cramer's Rule

Let A be an n-by-n matrix with $\det(A) \neq 0$.

The j^{th} component of the solution to $A\vec{x} = \vec{b}$ is given by

$$x_j = \frac{\det(A_{j\leftarrow\vec{b}})}{\det(A)}$$

where $A_{j\leftarrow\vec{b}}$ is the matrix A with the j^{th} column replaced by $\vec{b}$.

Cramer's Rule for Inverses

Let A be an n-by-n matrix with $\det(A) \neq 0$.

Say we want to find the inverse, a matrix X satisfying

$$AX = I$$

How can we use Cramer's rule?

The i^{th} column of the inverse satisfies:

$$AX(:,i) = I(:,i)$$

call this vector $\vec{e_i}$

Cramer's Rule for Inverses

The i^{th} column of the inverse satisfies:

$$AX(:, i) = I(:, i)$$

Applying Cramer's rule, the j^{th} component of the solution, which is $X(j, i)$, satisfies

$$X(j, i) = \frac{\det(A_{j \leftarrow \vec{e_i}})}{\det(A)}$$

Cramer's Rule for Inverses

Applying Cramer's rule, the j^{th} component of the solution, which is $X(j,i)$, satisfies

$$X(j,i) = \frac{\det(A_{j \leftarrow \vec{e_i}})}{\det(A)}$$

Example:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_{32} = \frac{1}{\det(A)} \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\det(A)} \begin{vmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{C_{23}}{\det(A)}$$

Applying Cramer's rule, the j^{th} component of the solution, which is $X(j,i)$, satisfies

$$X(j,i) = \frac{\det(A_{j\leftarrow\vec{e_i}})}{\det(A)} = \frac{C_{ij}}{\det(A)}$$

Define a matrix C of cofactors.

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Theorem: If $\det(A) \neq 0$ then $A^{-1} = \frac{1}{\det(A)} C^T$

Another proof

Let's look at another way to view this result.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{C^T} = \underbrace{\begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}}_{Z}$$

Why is this true?

Let's look at the diagonals of the product first.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{C^T} = \underbrace{\begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}}_{Z}$$

Let's look at the diagonals first.

$$Z_{11} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This is cofactor expansion in the first row!

$$Z_{11} = \det(A)$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{C^T} = \underbrace{\begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}}_{Z}$$

The other diagonals are similar.

$$\begin{aligned} Z_{22} &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= \det(A) \end{aligned}$$

cofactor expansion along the second row.

$$\begin{aligned} Z_{33} &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= \det(A) \end{aligned}$$

cofactor expansion along the third row.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$

$$Z$$

Now let's look at an off diagonal entry, say Z_{12}

$$Z_{12} = a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

We can also think of this as a determinant,
but it's not of the matrix A.

Off diagonal entries

Now let's look at an off diagonal entry, say Z_{12}

$$Z_{12} = a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

Claim: $$Z_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Do cofactor expansion in the second row.

$$\longrightarrow \quad Z_{12} = 0$$

Off diagonal entries

A similar thing happens for all other off diagonal entries:

$$Z_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3}$$

This is the determinant of the matrix with i^{th} row of A repeated in row j

This will be zero if $i \neq j$.

Vector Spaces

Reading: Strang 3.1

Learning objective: Make the leap from $\mathbb{R}^n$ to vector spaces.

"We now come to the decisive step of mathematical abstraction: we forget about what the symbols stand for..."

Hermann Weyl, The Mathematical Way of Thinking

Vector Spaces

Thus far "vectors" have denoted n-dimensional column vectors $\vec{u} \in \mathbb{R}^n$.

We have seen how to add vectors and multiply them by scalars.

We have seen rules for how these actions behave, for example:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

$$c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$$

Vector Spaces

A vector space is an abstraction that forgets about what $\vec{u}$ stands for.

There are other objects that we can add together and multiply by scalars, for example matrices.

Moreover, for matrices these operations obey similar rules:

$$A + B = B + A$$

$$c \cdot (A + B) = c \cdot A + c \cdot B$$

With the generalization of vector spaces, we can prove statements about any system that obeys certain rules.

Definition

A vector space is a nonempty set V on which the operations of addition and scalar multiplication are defined.

To be a vector space, there are 10 conditions which must be satisfied.

We divide these conditions into 3 groups.

1) Closure conditions

2) Properties of addition

3) Properties of scalar multiplication

see the beginning of problem set 3.1 in the Strang book

Closure Conditions

A vector space is a nonempty set V on which the operations of addition and scalar multiplication are defined.

C1) Closure under addition:

$x + y \in V$ for all $x, y \in V$

C2) Closure under scalar multiplication:

$c \cdot x \in V$ for all $x \in V$ and $c \in \mathbb{R}$.

Properties of Addition

A1) Addition is commutative

$$x + y = y + x \text{ for all } x, y \in V$$

A2) Addition is associative

$$(x + y) + z = x + (y + z) \text{ for all } x, y, z \in V$$

A3) Existence of a zero element $\mathbf{0} \in V$ such that

$$x + \mathbf{0} = x \text{ for all } x \in V$$

A4) For each x there exists a unique element $-x$ such that

$$x + (-x) = \mathbf{0}$$

Properties of Scalar Mult.

M1) Scalar multiplication is associative

$$a(bx) = (ab)x \text{ for all } a, b \in \mathbb{R}, x \in V$$

M2) Distributivity over addition in V

$$a(x + y) = ax + ay \text{ for all } a \in \mathbb{R}, x, y \in V$$

M3) Distributivity over scalar addition

$$(a + b)x = ax + bx \text{ for all } a, b \in \mathbb{R}, x \in V$$

M4) Identity for scalar multiplication

$$1x = x \text{ for all } x \in V$$

Definition

A vector space is a nonempty set V on which the operations of addition and scalar multiplication are defined and satisfy the 10 conditions C1-C2, A1-A4, M1-M4.

Checking all these conditions is quite tedious.

I won't go through these checks in lecture, and also won't ask you to do it on a quiz or exam.

Instead, let's look at some examples of vector spaces.

Primary Example: $\mathbb{R}^n$

$\mathbb{R}^n$ is a vector space.

The zero element is $\vec{0}$, the all zero vector.

Most of the conditions A1-A4, M1-M4 we already discussed at the beginning of the semester.

Space of Matrices

The set $M_{m,n}$ of all m-by-n matrices with real entries is a vector space.

The zero element of $M_{m,n}$ is $\mathbf{0}_{m \times n}$, the all zero matrix.

We have also already seen that matrix addition and multiplying a matrix by a scalar obeys many of the rules A1-A4, M1-M4.

Space of real valued functions

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space, where addition and scalar multiplication are defined as

$$(f + g)(x) = f(x) + g(x)$$

$$(c \cdot f)(x) = c \cdot f(x)$$

The zero element is the constant zero function:

$$\mathbf{0}(x) = 0 \text{ for all } x \in \mathbb{R}$$