Dot Product

Reading: Strang 1.2

Learning objective: Be able to compute the dot product and understand its geometrical interpretation.

Example

Let $\vec{a} = (1, 1)$ and $\vec{b} = (1, -1)$.

The dot product or inner product between $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned} \langle \vec{a}, \vec{b} \rangle &= a_1 \cdot b_1 + a_2 \cdot b_2 \\ &= 1 \cdot 1 + 1 \cdot (-1) \\ &= 0 \end{aligned}$$

Note: Strang uses the alternative notation

$$\vec{a} \cdot \vec{b}$$

for dot product. I will use brackets instead to avoid confusion with scalar multiplication.

Example

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The dot product takes as input two vectors of the same dimension and returns a real number.

Geometrical View

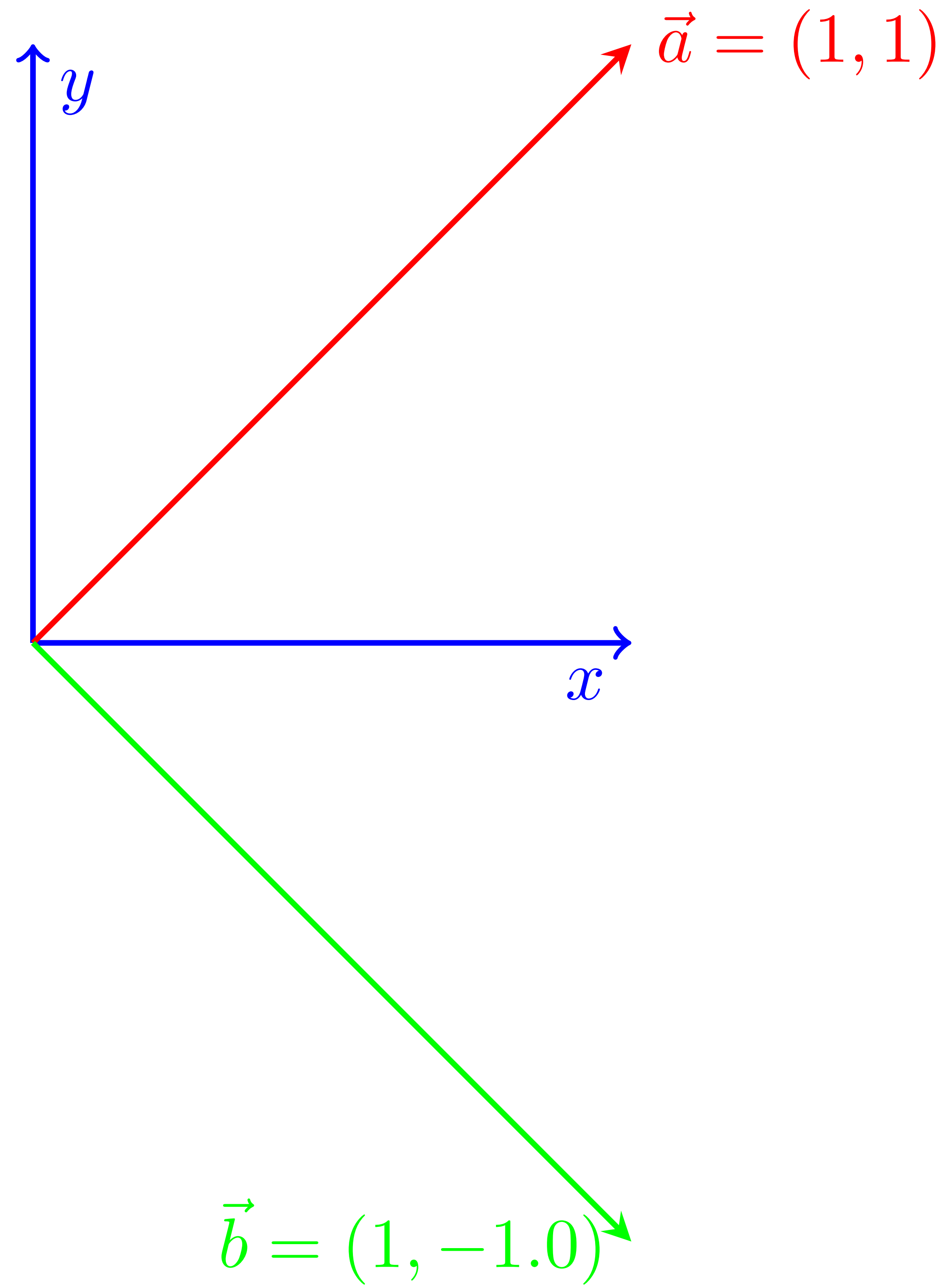
Let $\vec{a} = (1, 1)$ and $\vec{b} = (1, -1)$.

In this case, $\langle \vec{a}, \vec{b} \rangle = 0$.

Let's take a look at the geometry.

The vectors $\vec{a}$ and $\vec{b}$ make a right angle.

They are perpendicular.



Perpendicular Vectors

Two other vectors we know are perpendicular are

$$(1, 0) \quad \text{and} \quad (0, 1)\,.$$

The dot product of these vectors is

$$1 \cdot 0 + 0 \cdot 1 = 0$$

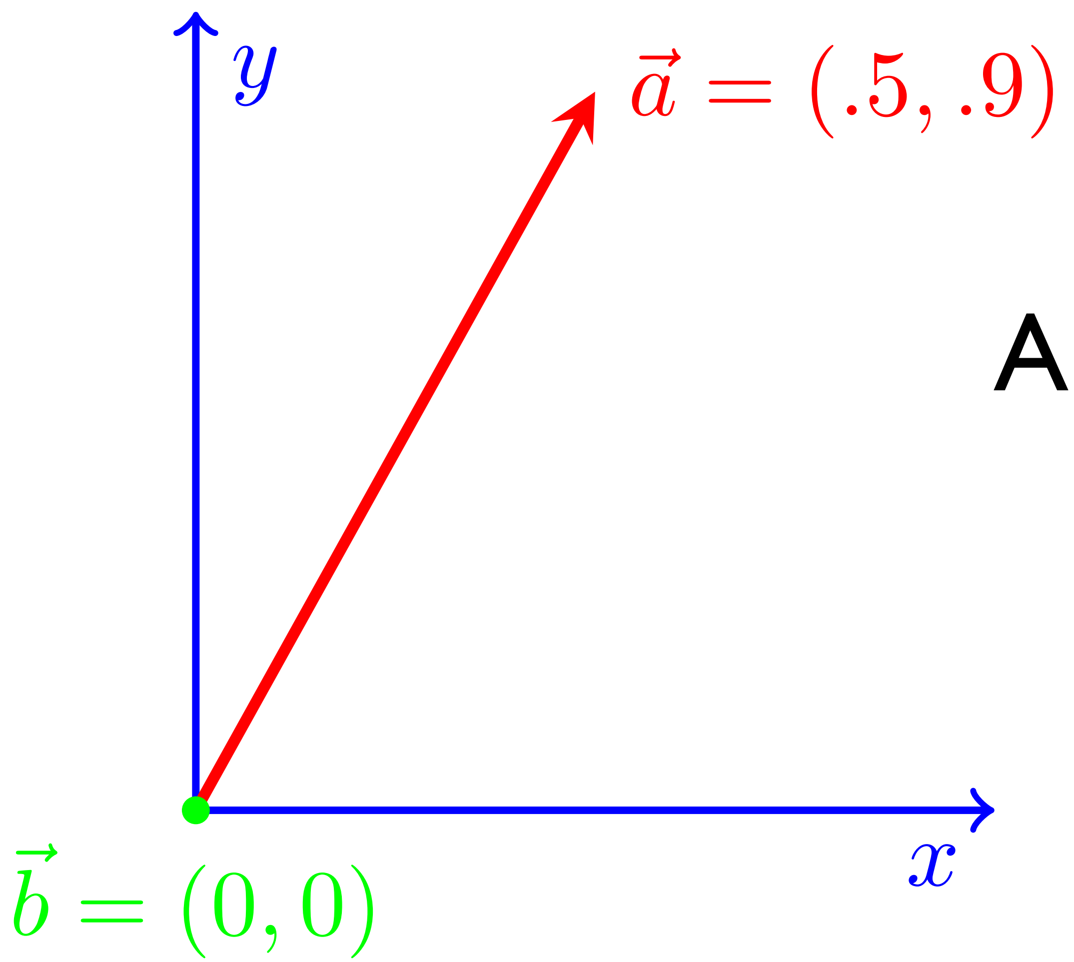
also zero!

Mathematicians use another word for perpendicular: orthogonal.

Definition: Two vectors are orthogonal if and only if their dot product is zero.

Orthogonal Example

Which vectors are orthogonal to the zero vector $(0,0)$?



Are these vectors orthogonal?

Yes! Even though they don't make a "right angle" their dot product is zero, so by definition they are orthogonal.

Dot Product

The previous examples are the first clue that the dot product is a useful notion.

Let's step back and look at the general definition.

If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ then

$$\langle \vec{a}, \vec{b} \rangle = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n$$

Notation:

$$\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{n} a_i \cdot b_i$$

The Greek letter "Sigma" stands for sum.

Commutativity

The dot product is commutative:

$$\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle$$

This follows from commutativity of usual multiplication of real numbers

$$a_1 \cdot b_1 = b_1 \cdot a_1$$

Repeatedly applying this gives

$$a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n = b_1 \cdot a_1 + b_2 \cdot a_2 + \cdots + b_n \cdot a_n$$

Scalar Multiplication

The dot product behaves nicely with respect to scalar multiplication:

$$\langle c \cdot \vec{a}, \vec{b} \rangle = c \cdot \langle \vec{a}, \vec{b} \rangle$$

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\begin{aligned} \langle c \cdot \vec{a}, \vec{b} \rangle &= c \cdot a_1 \cdot b_1 + c \cdot a_2 \cdot b_2 + c \cdot a_3 \cdot b_3 \\ &= c \cdot (a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3) \\ &= c \cdot \langle \vec{a}, \vec{b} \rangle \end{aligned}$$

Similarly, $\langle \vec{a}, c \cdot \vec{b} \rangle = c \cdot \langle \vec{a}, \vec{b} \rangle$.

Applications

The dot product is very useful.

What's my grade?

score vector: $(100, 95, 90, 100, 92)$

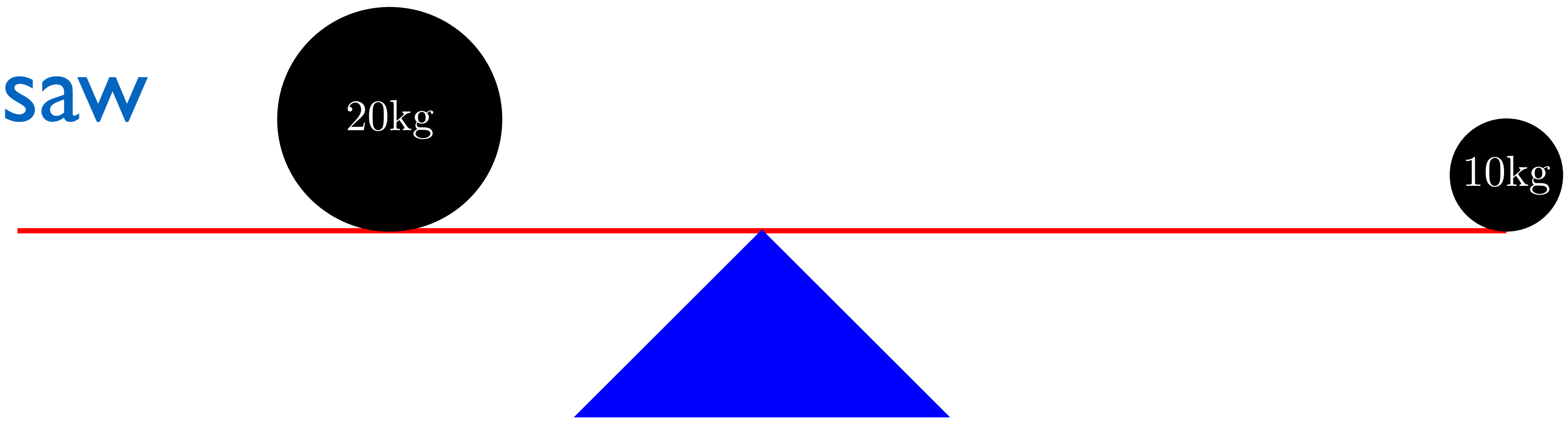
weight vector: $(0.1, 0.1, 0.2, 0.1, 0.5)$

The final grade is the dot product of the score and weight vectors.

Applications

The dot product is very useful.

Balancing the seesaw



distance vector: $(-1, 2)$

mass vector: $(20, 10)$

The seesaw is balanced when the dot product of the distance vector and mass vector is zero.

Applications

How much do you like a movie?

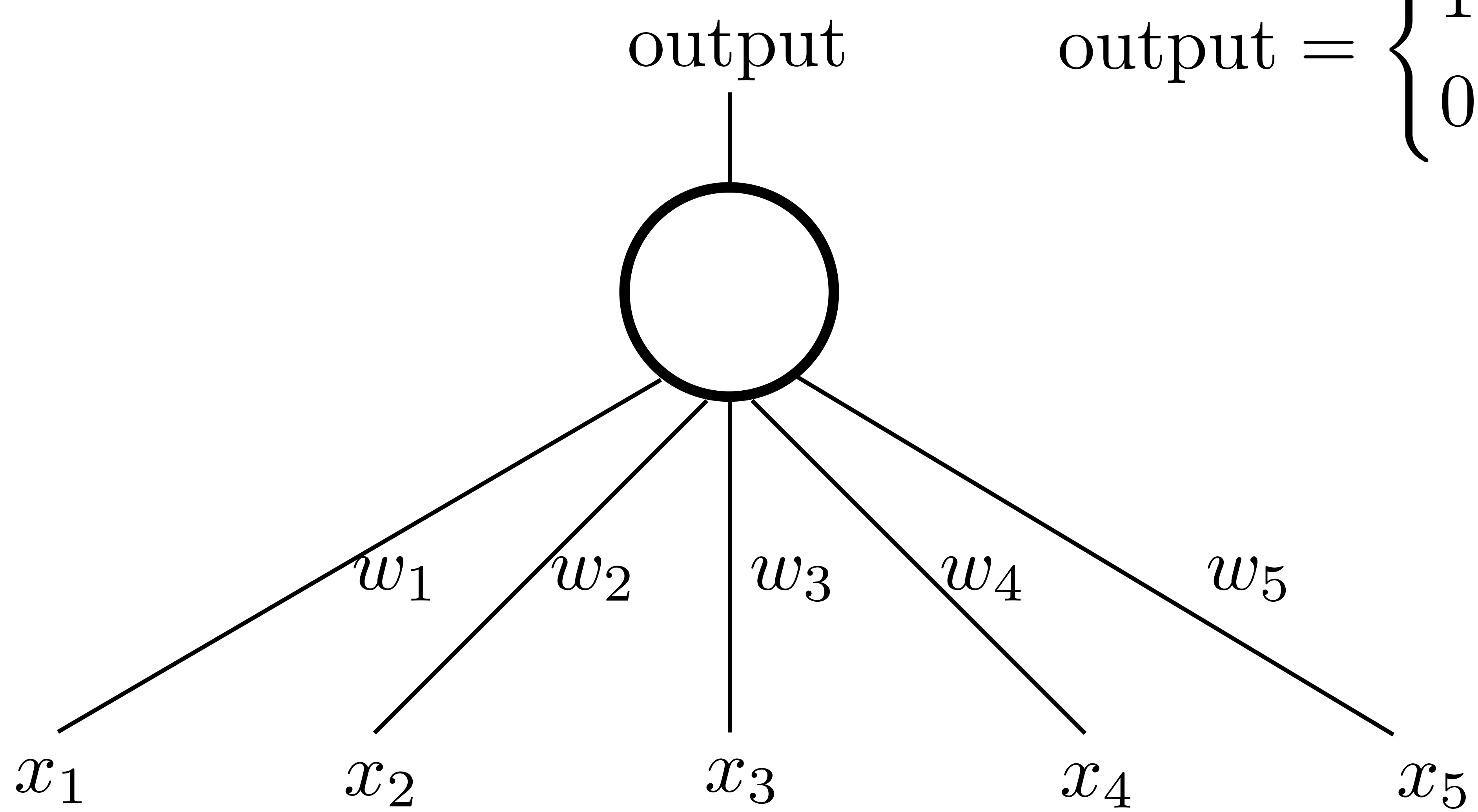

	humor	super heroes	drama	Brad Pitt
Alice	0.7	1	-0.5	1
Wonder Woman	0.3	1	0.8	0

We can model how much a person will like a movie as a dot product.

$$\langle \text{Alice}, \text{WW} \rangle = 0.7 \cdot 0.3 + 1 \cdot 1 + -0.5 \cdot 0.8 + 1 \cdot 0 = 0.81$$

Applications

Modelling a neuron!



$$\text{output} = \begin{cases} 1 & \text{if } \langle \vec{w}, \vec{x} \rangle \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

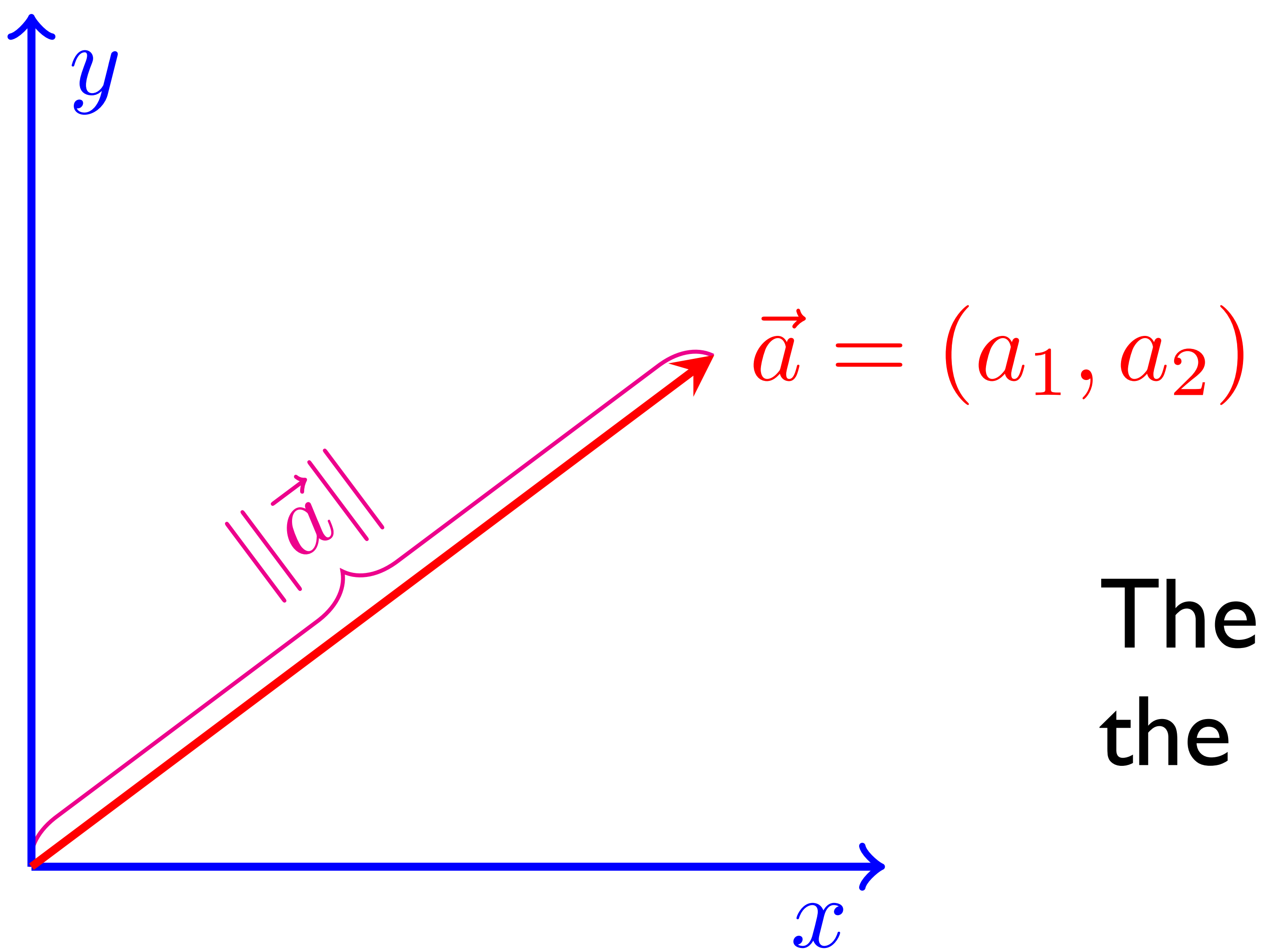
The model computes the dot product of the input and vector of weights. If this is non-negative, the neuron fires.

Length

The length of a vector is the distance from the tail to the head of its arrow.

We denote the length of a vector $\vec{a}$ as $\|\vec{a}\|$.

read as the norm of $\vec{a}$



The dot product can tell us the length of a vector.

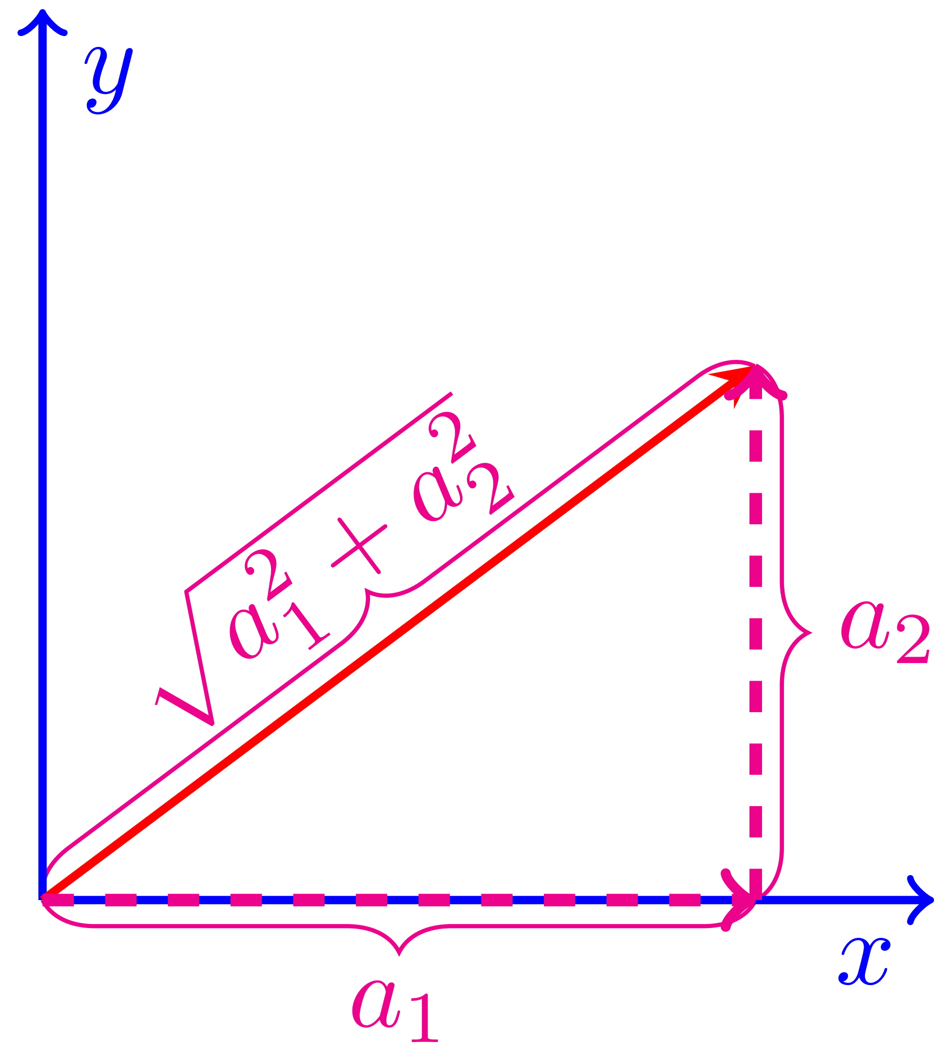
Length

Let's look at the dot product of a vector with itself.

If $\vec{a} = (a_1, a_2)$ then $\langle \vec{a}, \vec{a} \rangle = a_1^2 + a_2^2$.

By the Pythagorean theorem, this is the length of $\vec{a}$ squared!

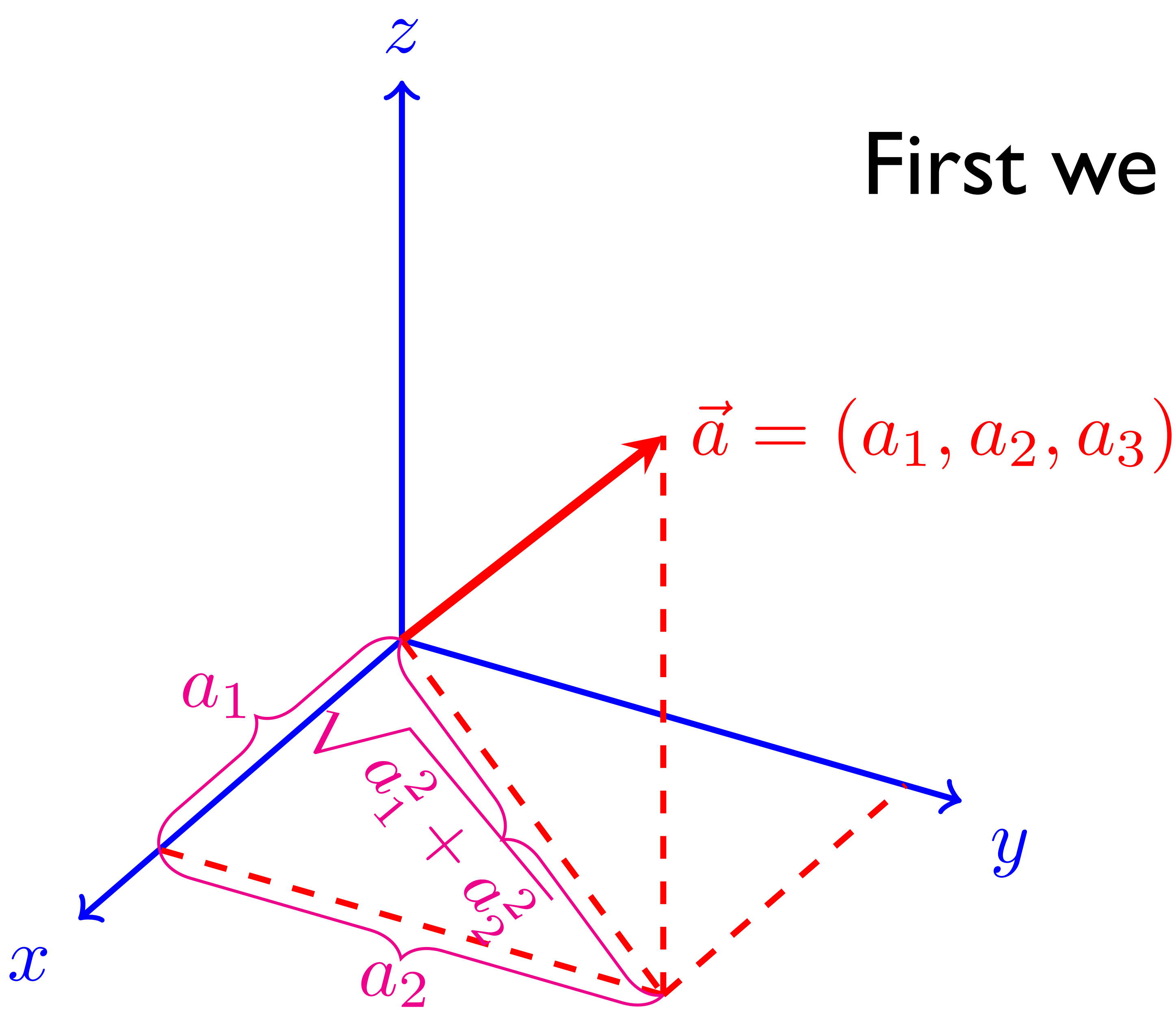
$$\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$



3D Example

What about in three dimensions?

If $\vec{a} = (a_1, a_2, a_3)$ then $\langle \vec{a}, \vec{a} \rangle = a_1^2 + a_2^2 + a_3^2$.



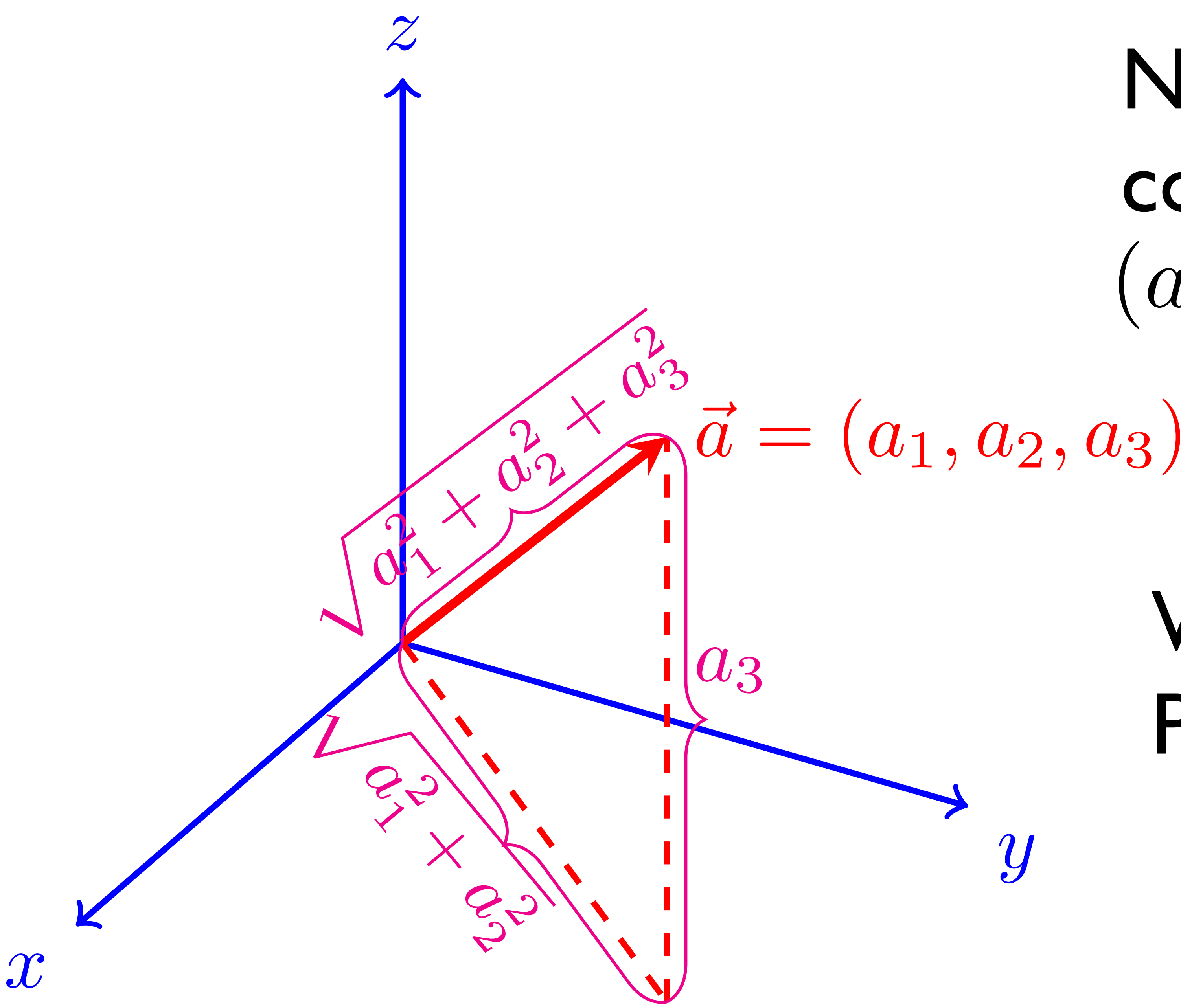
First we work in the x-y plane.

The length of the vector $(a_1, a_2, 0)$ is $\sqrt{a_1^2 + a_2^2}$.

3D Example

What about in three dimensions?

If $\vec{a} = (a_1, a_2, a_3)$ then $\langle \vec{a}, \vec{a} \rangle = a_1^2 + a_2^2 + a_3^2$.



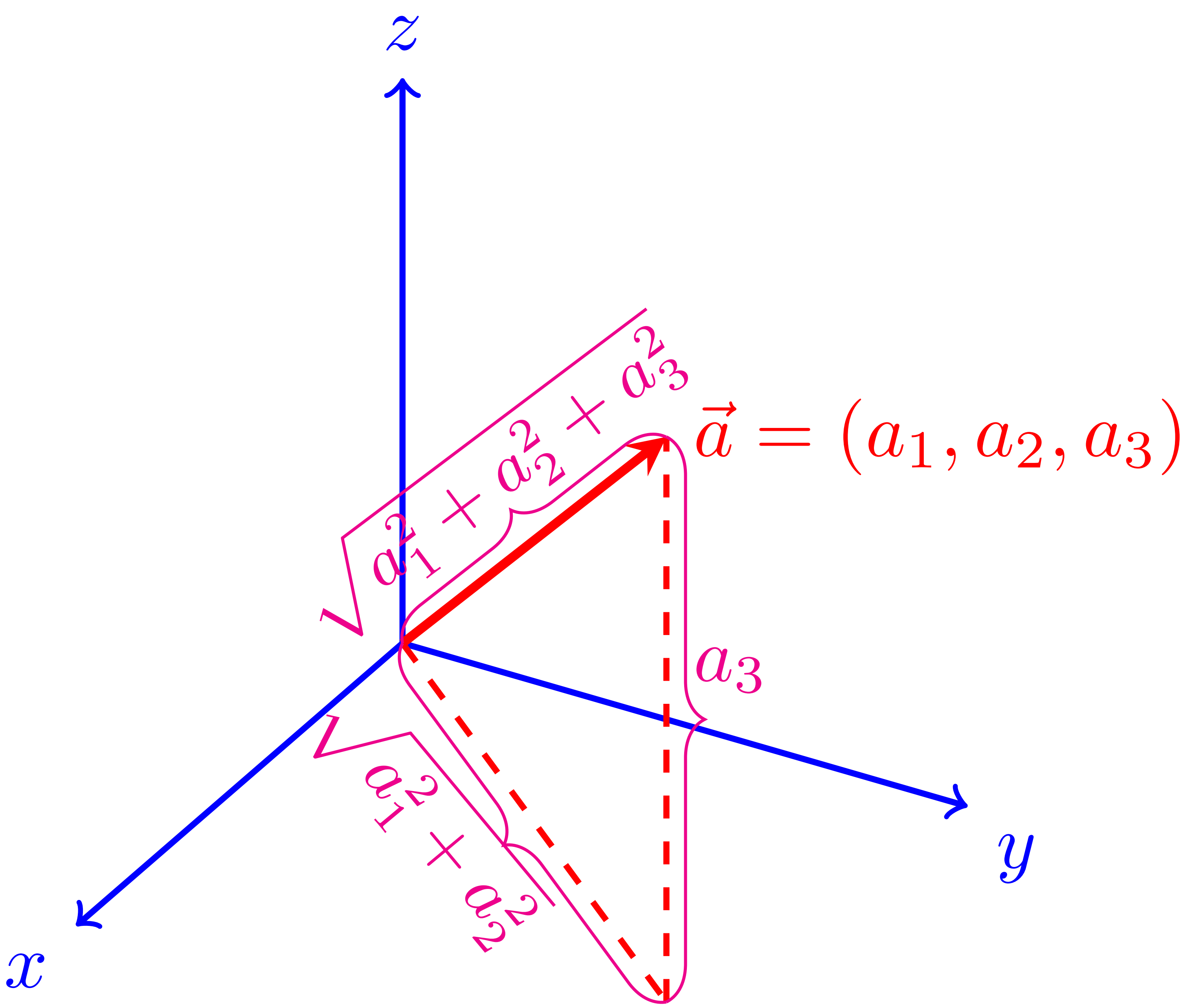
Next we work in the plane containing $(a_1, a_2, 0)$ and (a_1, a_2, a_3).

We can again apply the Pythagorean Theorem.

3D Example

What about in three dimensions?

If $\vec{a} = (a_1, a_2, a_3)$ then $\langle \vec{a}, \vec{a} \rangle = a_1^2 + a_2^2 + a_3^2$.



The length of $\vec{a}$ is

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$= \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

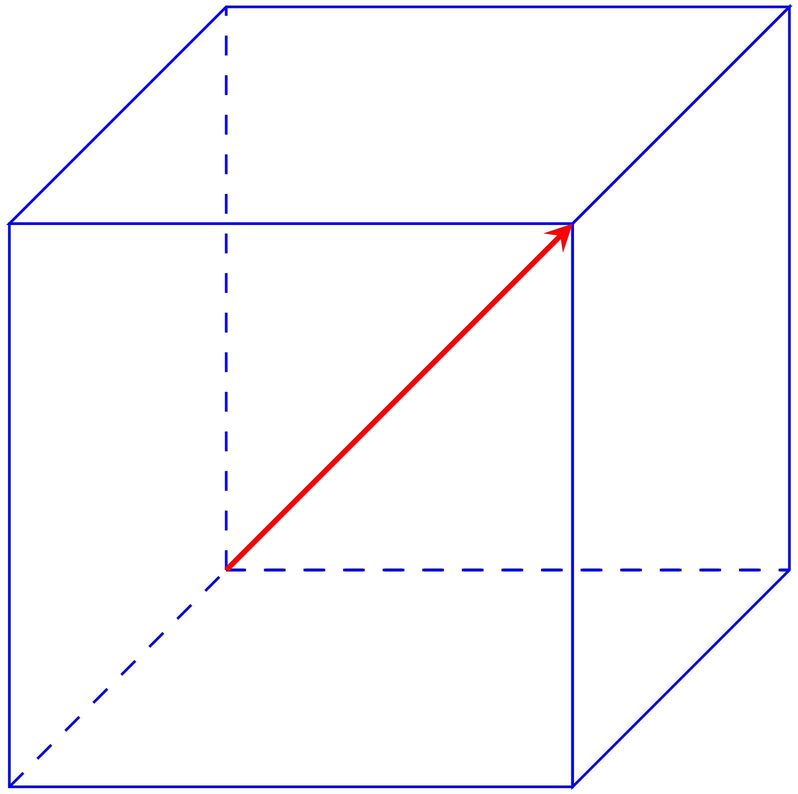
Length

The connection between length and dot product holds in any number of dimensions.

Thm: Let $\vec{a} = (a_1, \ldots, a_n)$. Then $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$.

You can prove this as we did in the 3D case, building up one dimension at a time, each time applying the Pythagorean Theorem.

Examples



The length of the vector $(1, 1, 1)$ is $\sqrt{3}$.

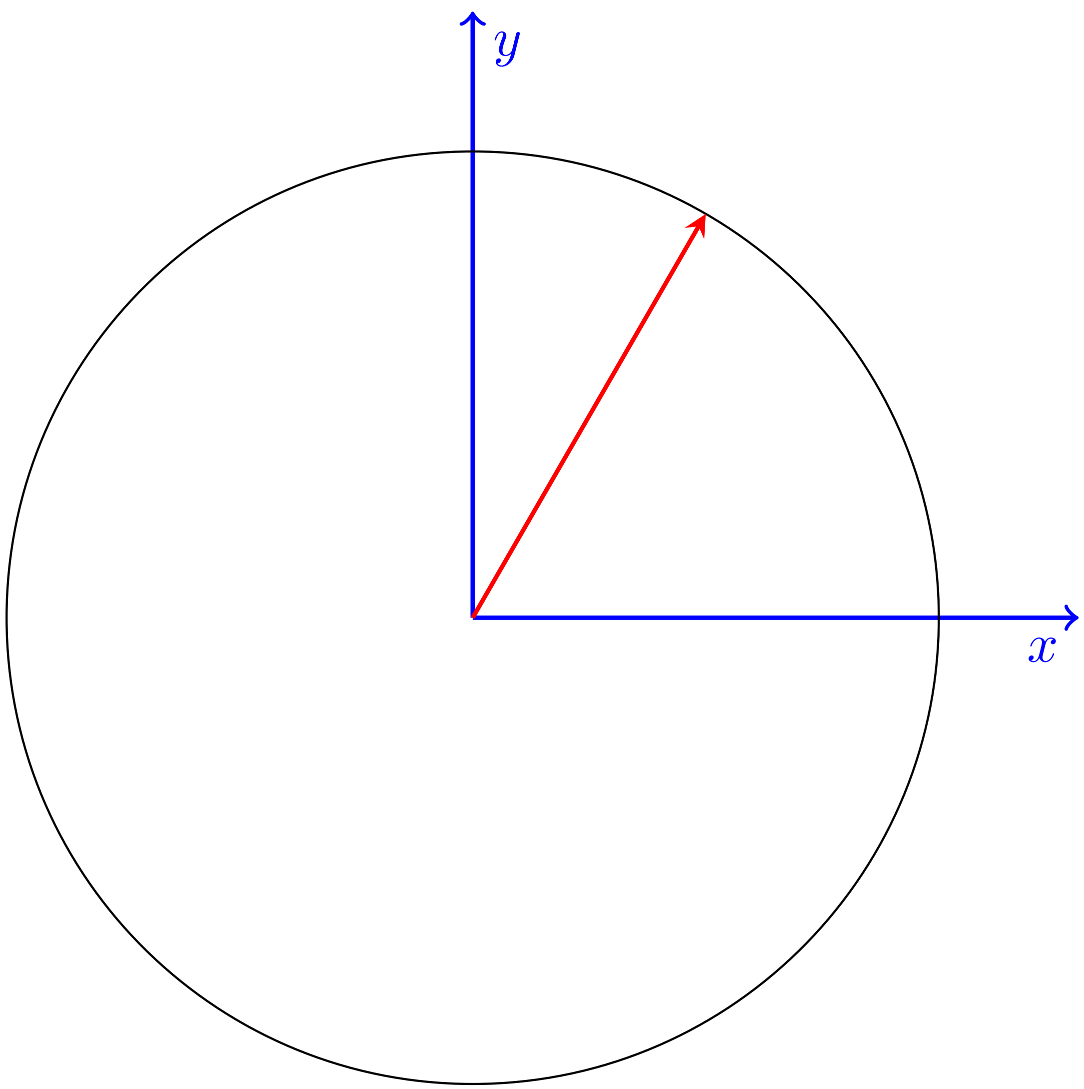
The length of the vector $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is $1.$

A vector of length 1 we call a unit vector.

Angle

Now we look more in depth at the relationship of the dot product to the angle between vectors.

First, let's consider two unit vectors in two dimensions.

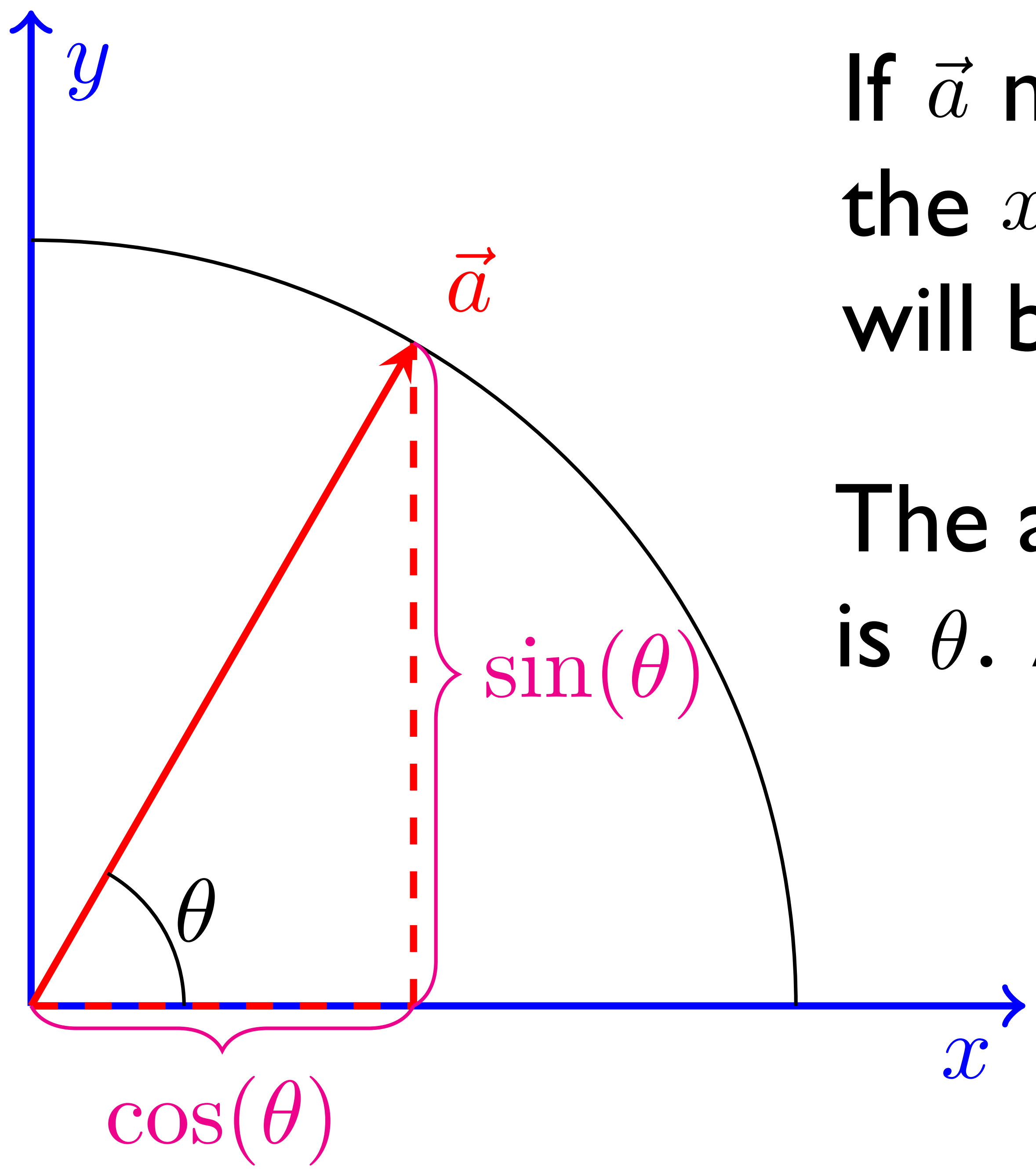


Any unit vector will lie on the unit circle.

Let's further take one of the unit vectors to be $(1, 0)$.

Angle

Let the other unit vector be $\vec{a}$.

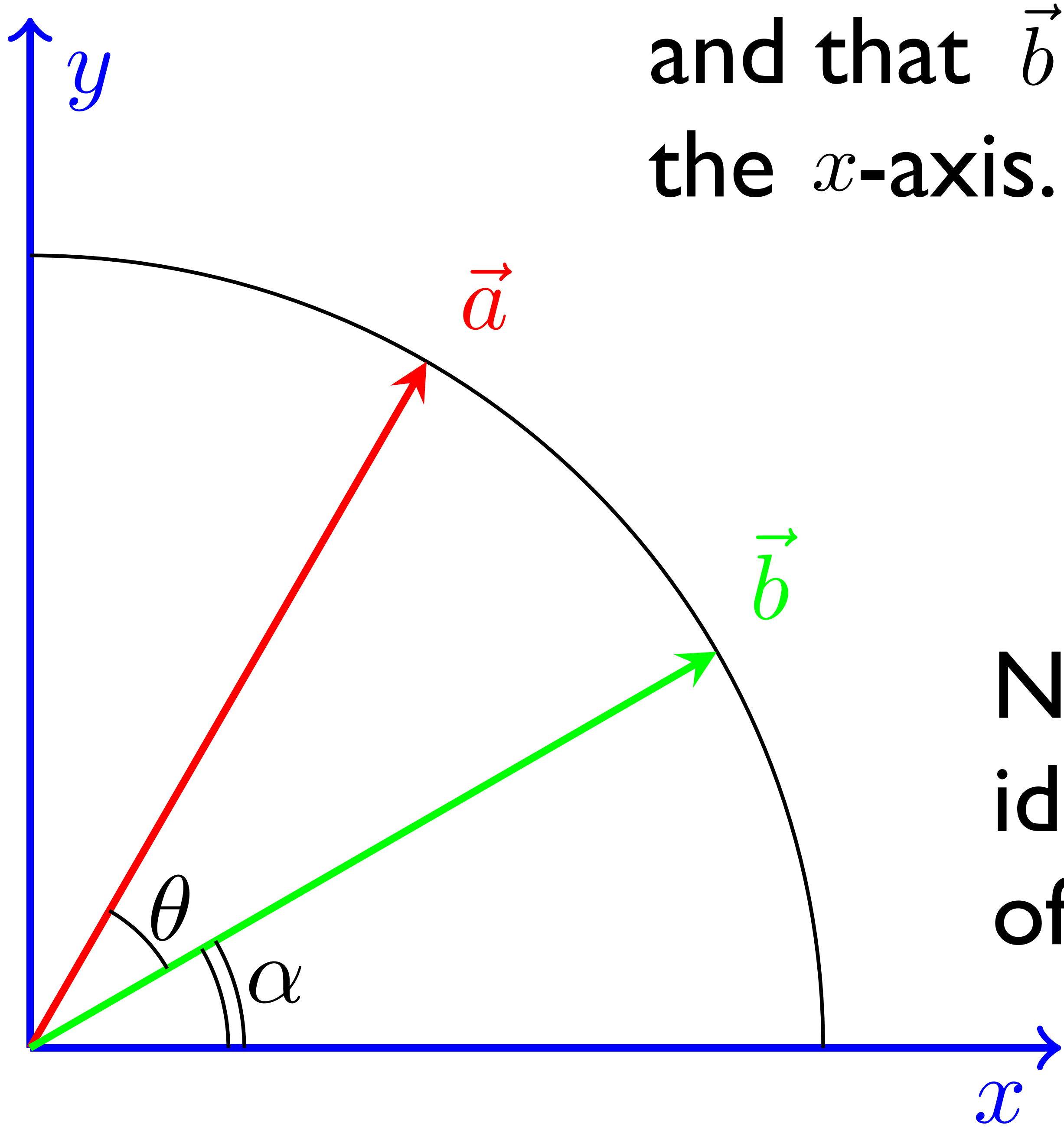


If $\vec{a}$ makes an angle θ with the x-axis, then the coordinates will be $(\cos(\theta), \sin(\theta))$.

The angle between $\vec{a}$ and $(1, 0)$ is θ. And their dot product is $\cos(\theta)$.

Angle

Let's consider two arbitrary unit vectors $\vec{a}, \vec{b}$ in two dimensions.



Say that the angle between them is θ and that $\vec{b}$ makes an angle of α with the x-axis.

$$\vec{b} = (\cos(\alpha), \sin(\alpha))$$

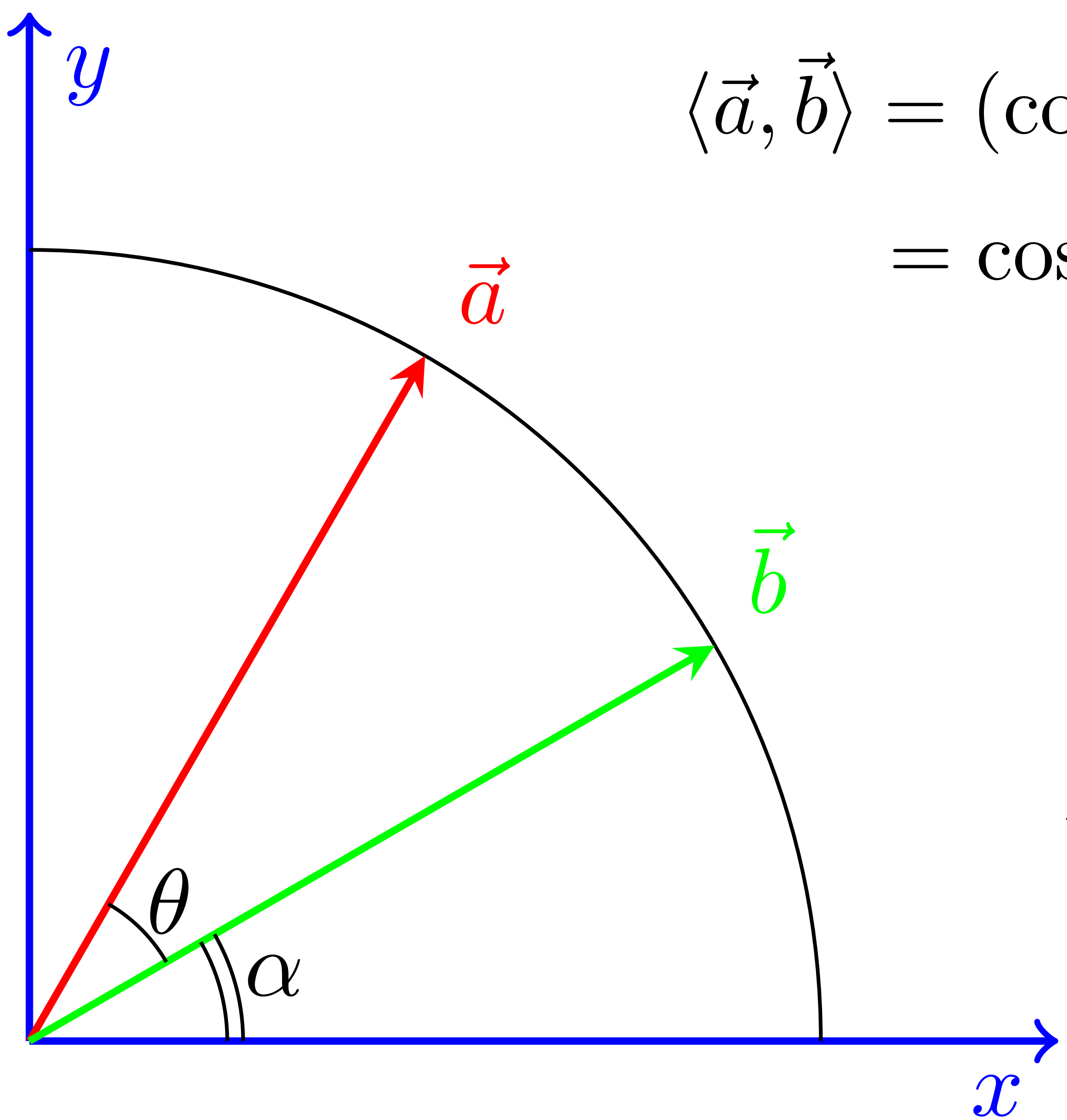
$$\vec{a} = (\cos(\theta + \alpha), \sin(\theta + \alpha))$$

Now we can use trigonometric identities on the coordinates of $\vec{a}$.

$$\vec{b} = (\cos(\alpha), \sin(\alpha))$$

$$\vec{a} = (\cos(\theta + \alpha), \sin(\theta + \alpha))$$

$$= (\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha), \sin(\theta)\cos(\alpha) + \sin(\alpha)\cos(\theta))$$



$$\langle \vec{a}, \vec{b} \rangle = (\cos(\alpha))^2 \cos(\theta) + (\sin(\alpha))^2 \cos(\theta)$$

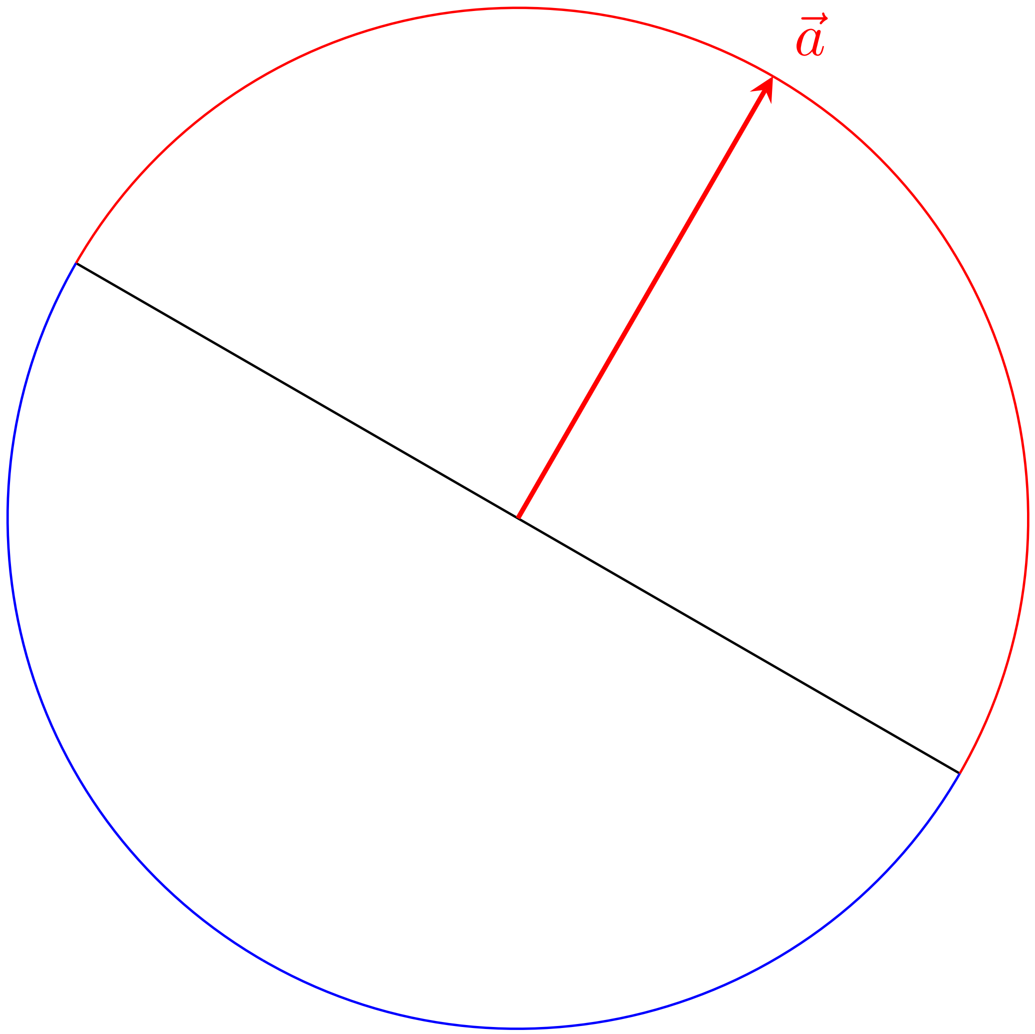
$$= \cos(\theta)$$

Later we will see how to use matrices to remember these trig identities.

Angle

This result actually holds for two unit vectors in any number of dimensions.

Thm: If $\vec{a}, \vec{b} \in \mathbb{R}^n$ are two unit vectors then $\langle \vec{a}, \vec{b} \rangle = \cos(\theta)$.

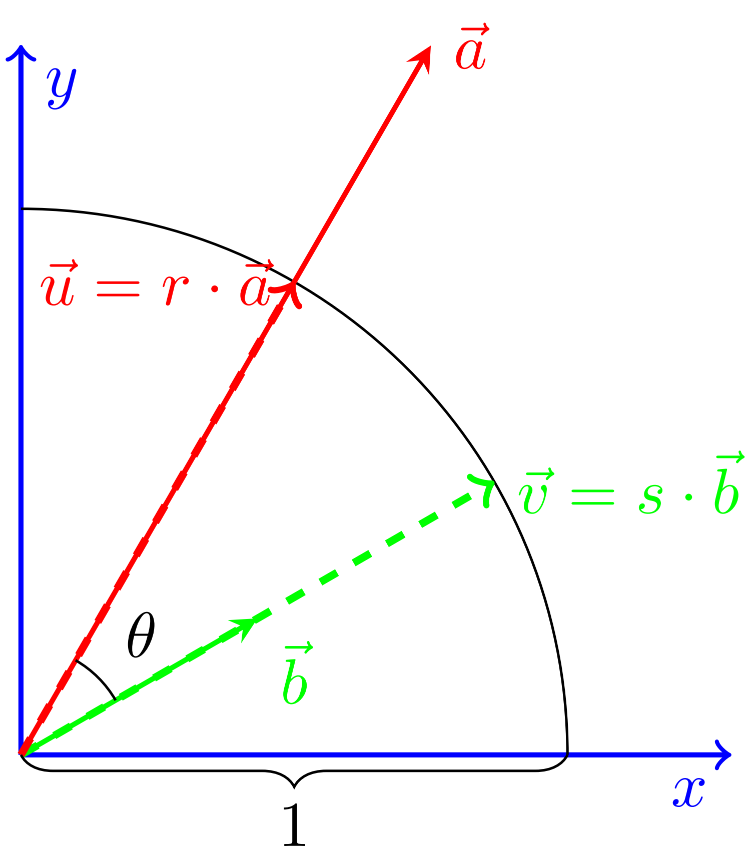


Angle

What about non-unit vectors?

As we said, multiplying a vector by a positive scalar does not change its direction.

If $\vec{a}, \vec{b}$ are nonzero vectors, then the angle between them is the same as the angle between $r \cdot \vec{a}, s \cdot \vec{b}$ for $r, s > 0$.



For what r is $r \cdot \vec{a}$ a unit vector?

Check it's a unit vector

We should take $r = \dfrac{1}{\|\vec{a}\|}$.

Recall that $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$.

If $\vec{v} = \dfrac{\vec{a}}{\|\vec{a}\|}$ then

$$
\begin{aligned}
\|\vec{v}\|^2 &= \left\langle \frac{\vec{a}}{\|\vec{a}\|}, \frac{\vec{a}}{\|\vec{a}\|} \right\rangle \\
&= \frac{1}{\|\vec{a}\|} \langle \vec{a}, \frac{\vec{a}}{\|\vec{a}\|} \rangle \\
&= \frac{1}{\|\vec{a}\|^2} \langle \vec{a}, \vec{a} \rangle \\
&= 1
\end{aligned}
$$

Angle

If $\vec{a}, \vec{b}$ are nonzero vectors, then

$$\cos(\theta) = \frac{\langle \vec{a}, \vec{b} \rangle}{||\vec{a}|| \, ||\vec{b}||}$$

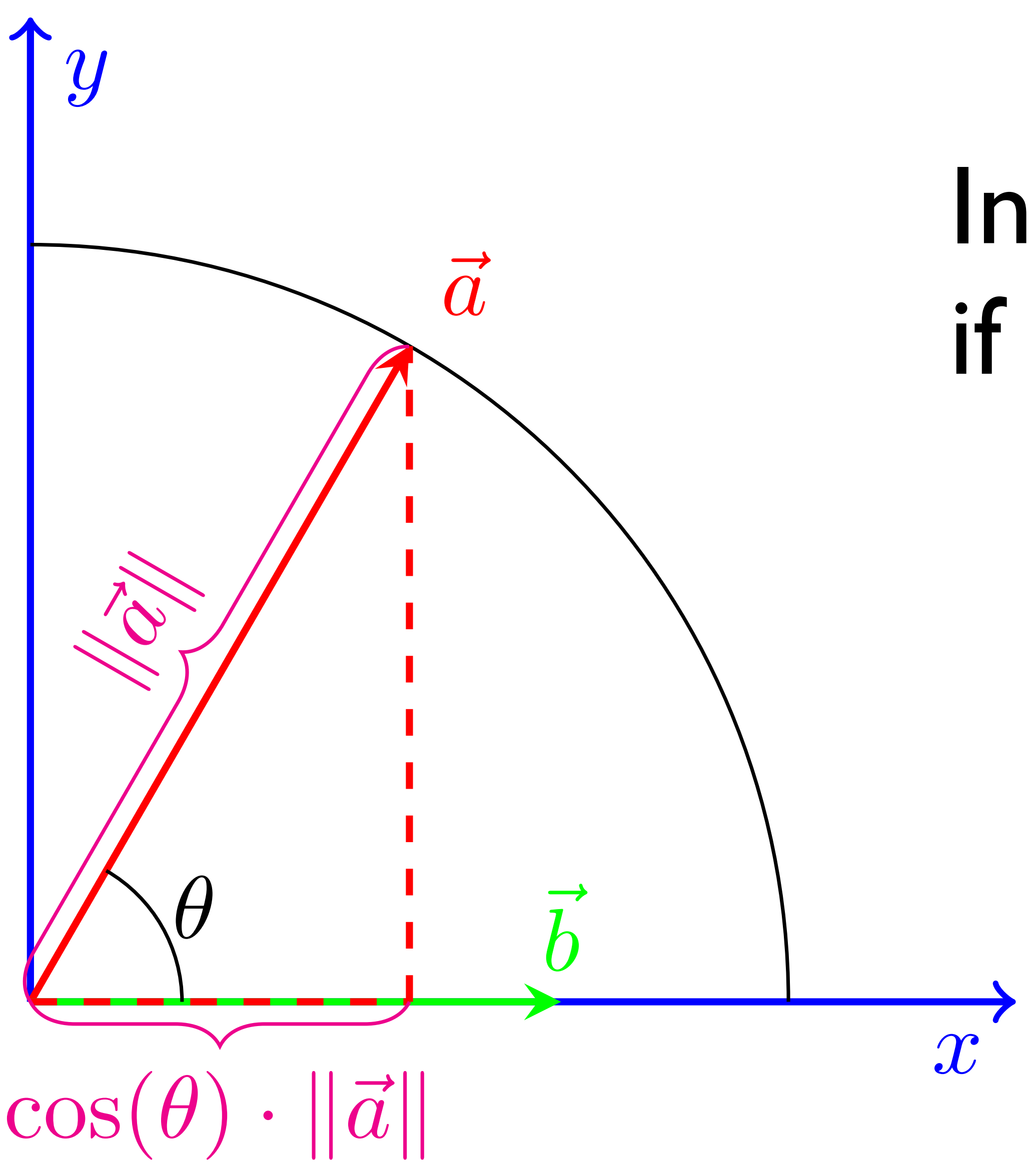
Thm: for any two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$

$$\langle \vec{a}, \vec{b} \rangle = \cos(\theta) ||\vec{a}|| \, ||\vec{b}||$$

Cauchy-Schwarz Inequality

From this theorem we can deduce one of the most important inequalities in mathematics.

As $|\cos(\theta)| \leq 1$ we have $|\langle \vec{a}, \vec{b} \rangle| = |\cos(\theta)| \cdot \|a\| \|b\| \leq \|a\| \|b\|$.



In particular, $|\langle \vec{a}, \vec{b} \rangle| = \|\vec{a}\| \|\vec{b}\|$ if and only if $\vec{a} = c \cdot \vec{b}$.

That is, when $\vec{a}, \vec{b}$ lie on the same line.