Gaussian Elimination

Reading: Strang 2.2

Learning objective: Understand the high-level idea of Gaussian elimination, and carry it out to solve square systems of equations.

Overview

Last week we began talking about linear equations.

We developed some geometric intuition for what the solution set to a system of linear equations looks like.

Now we get down to the business of computing the solution set of a linear system of equations.

We do this through algebraic manipulation: Gaussian Elimination.

Simple Systems

Some systems of linear equations are very easy to solve:

$$\begin{aligned} 2x \qquad\qquad\qquad &= 4 \\ 3y \qquad\quad &= -9 \\ -z &= -1 \end{aligned}$$

diagonal system

For a system like this we can directly write down the solution.

Simple Systems

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 4 \\ & & 3y & - & 6z & = & -9 \\ & & & - & z & = & -1 \end{array}$$

upper triangular system

A system like this we can solve by back substitution.

We start from the bottom and work our way up.

$$z = 1$$

$$3y - 6 = -9 \implies y = -1$$

$$2x - 1 + 1 = 4 \implies x = 2$$

Simple Systems

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 4 \\ & & 3y & - & 6z & = & -9 \\ & & & - & z & = & -1 \end{array}$$

upper triangular system

What makes this system simple to solve?

Variables x and y do not appear in the third equation.

Variable x does not appear in the second equation.

Gaussian Elimination

Gaussian elimination is a systematic way to eliminate variables from equations.

Doing this, we can transform an arbitrary (square) system of linear equations into an upper triangular system with the same solution set.

Then we can solve the upper triangular system by back substitution.

The basic operation to do this transformation is adding a multiple of one row to another (AMORTA).

Example

$$\begin{aligned} x &+ 5y &= 3 \\ 3x &+ 16y &= 10 \end{aligned}$$

We add a multiple of the first equation to the second equation to eliminate the variable x from the second equation.

Here we add -3 times the first equation to the second equation to form a new second equation.

$$\begin{aligned} x + 5y &= 3 \\ y &= 1 \end{aligned}$$

$$R_2' = R_2 - 3R_1$$

the new second equation is original minus 3 times the first

Example

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array}$$

$$\Big\downarrow \quad R_2' = R_2 - 3R_1$$

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Now the system is upper triangular, and we can solve by back substitution.

Key Observation

The key property we are using is that adding a multiple of one equation to another does not change the solution set.

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array}$$

Call the solution set of these equations S_1.

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Call the solution set of these equations S_2.

We want to show that $S_1 = S_2$.

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array}$$

Call the solution set of these equations S_1.

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Call the solution set of these equations S_2.

We want to show that $S_1 = S_2$.

We do this in two steps:

§ First step: $S_1 \subseteq S_2$

Every element of S_1 is also an element of S_2.

§ Second step: $S_2 \subseteq S_1$

Every element of S_2 is also an element of S_1.

$$S_1 \subseteq S_2$$

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array} \qquad \begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Let (a, b) be a solution to the first set of equations:

$$a + 5b = 3$$

$$3a + 16b = 10$$

To go from the first set of eqns. to the second, we did the operation $R_2' = R_2 - 3R_1$.

Following these steps:

$$-3 \cdot (a + 5b) = -3 \cdot 3 \qquad -3R_1$$

$$3a + 16b - 3 \cdot (a + 5b) = 10 - 3 \cdot 3 \qquad R_2 - 3R_1$$

$$S_1 \subseteq S_2$$

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array} \qquad \begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Let (a, b) be a solution to the first set of equations:

$$a + 5b = 3$$

$$3a + 16b = 10$$

$$3a + 16b - 3 \cdot (a + 5b) = 10 - 3 \cdot 3 \qquad R_2 - 3R_1$$

Simplifying gives $b = 1$.

Thus (a, b) is also a solution to the second set of eqns.

$S_2 \subseteq S_1$

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array} \qquad \begin{array}{rcrcr} x & + & 5y & = & 3 \\ & & y & = & 1 \end{array}$$

Now let (a, b) be a solution to the second set of equations:

$$a + 5b = 3$$

$$b = 1$$

We reverse the operation we did to arrive at the second set of equations:

$$R_2' = R_2 - 3R_1 \implies R_2 = R_2' + 3R_1$$

$$b + 3 \cdot (a + 5b) = 1 + 3 \cdot 3$$

$$3a + 16b = 10$$

Thus (a, b) is also a solution to the first set of eqns.

Reversible Operation

Adding a multiple of one equation to another does not change the solution set.

The reason is because adding a multiple of one equation to another is a reversible operation.

$$R_2' = R_2 + a \cdot R_1 \implies R_2 = R_2' - a \cdot R_1$$

What is not allowed

We disallow adding a multiple of one equation to itself.

There is a problem that can occur with this:

$$\begin{array}{rcrcr} x & + & 5y & = & 3 \\ 3x & + & 16y & = & 10 \end{array}$$

$$\Big\downarrow \qquad R_1' = R_1 - R_1$$

$$0 = 0$$

$$3x + 16y = 10$$

These two systems do not have the same solution set!

Larger Example

Let's try a bigger example: three equations, three unknowns.

$$x - 2y + 3z = 7$$
$$2x + y + z = 4$$
$$3x - 2y + 2z = 10$$

We add appropriate multiples of the first equation to the second and third equations to eliminate x from them.

$$x - 2y + 3z = 7$$
$$5y - 5z = -10 \qquad R_2' = R_2 - 2R_1$$
$$4y - 7z = -11 \qquad R_3' = R_3 - 3R_1$$

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ 3x - 2y + 2z &= 10 \end{aligned} \xrightarrow{\begin{subarray}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{subarray}} \begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 4y - 7z &= -11 \end{aligned}$$

The coefficient of x in the first equation is called a pivot. In this case, the pivot is 1.

To eliminate x from the second equation, the multiple of the first equation we add to it is

$$-(\text{entry to eliminate}) \text{ divided by the pivot } = -\frac{2}{1}$$

Multiplier $= -\dfrac{}{}$

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ 3x - 2y + 2z &= 10 \end{aligned} \quad \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \quad \begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 4y - 7z &= -11 \end{aligned}$$

The coefficient of x in the first equation is called a pivot. In this case, the pivot is 1.

To eliminate x from the third equation, the multiple of the first equation we add to it is

$$-\text{(entry to eliminate) divided by the pivot} = -\frac{3}{1}$$

Multiplier $= -$

We have arrived at the system.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 4y - 7z &= -11 \end{aligned}$$

The solution set of this system is the same as our original system.

Is this system upper triangular?

$$
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
4y - 7z &= -11
\end{aligned}
$$

The next pivot is the coefficient of y in the second equation. The next pivot is 5.

What multiple of the second equation should we add to the third equation to eliminate y?

$$\text{Multiplier } = -\frac{\bigcirc}{\bigcirc} = -\frac{4}{5}$$

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 4y - 7z &= -11 \end{aligned} \xrightarrow[R_3' = R_3 - \frac{4}{5}R_2]{} \begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ -3z &= -3 \end{aligned}$$

Is this system upper triangular?

Gaussian elimination has reached its endpoint.

The pivots lie on the diagonal.

Now we can solve by back substitution.

$$z = 1$$

$$5y - 5 = -10 \implies y = -1$$

$$x + 2 + 3 = 7 \implies x = 2$$

What can go wrong?

Can we always reach an upper triangular system by adding a multiple of one equation to another below it?

$$x - 2y + 3z = 7$$
$$2x - 4y + z = 4$$
$$3x - 2y + 2z = 10$$

Here I have changed the coefficient of y in the second equation.

Let's try Gaussian elimination again.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x - 4y + z &= 4 \\ 3x - 2y + 2z &= 10 \end{aligned} \quad \xrightarrow[\substack{R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1}]{} \quad \begin{aligned} x - 2y + 3z &= 7 \\ -5z &= -10 \\ 4y - 7z &= -11 \end{aligned}$$

Now the coefficient of y in the second equation is zero.

We can't eliminate y from the third equation using the second equation.

What to do now?

Row Swap

We allow another operation in Gaussian elimination: exchanging the order of two equations.

$$\begin{aligned} x - 2y + 3z &= 7 \\ -5z &= -10 \\ 4y - 7z &= -11 \end{aligned} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{aligned} x - 2y + 3z &= 7 \\ 4y - 7z &= -11 \\ -5z &= -10 \end{aligned}$$

Does changing the order of equations affect the solution set?

No, the condition to be a solution does not depend on the order of the equations.

Row Swap

$$\begin{aligned} x - 2y + 3z &= 7 \\ -5z &= -10 \\ 4y - 7z &= -11 \end{aligned} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{aligned} x - 2y + 3z &= 7 \\ 4y - 7z &= -11 \\ -5z &= -10 \end{aligned}$$

After exchanging the order of equations two and three we again have an upper triangular system.

Gaussian elimination again terminates with an upper triangular system.

What else can happen?

The examples we have seen so far have had a full set of pivots.

All coefficients on the diagonal of the final triangular system were nonzero.

Let's see an example where this is not the case.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ 4x - 3y + 7z &= 18 \end{aligned}$$

I've gone back to our first example and changed the last equation.

$$
\begin{aligned}
x - 2y + 3z &= 7 \\
2x + y + z &= 4 \\
4x - 3y + 7z &= 18
\end{aligned}
\quad
\xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1}
\quad
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
5y - 5z &= -10
\end{aligned}
$$

In the next step we do:

$$
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
5y - 5z &= -10
\end{aligned}
\quad
\xrightarrow{R_3' = R_3 - R_2}
\quad
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
0 &= 0
\end{aligned}
$$

We have still arrived at an upper triangular system.

However, there are only two pivots.

Back substitution

Let's see what happens in the case of back substitution with an incomplete set of pivots.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 0 &= 0 \end{aligned}$$

The third column does not contain a pivot. The third column is a free column.

Variables corresponding to free columns we can let take any value—these are free variables.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 0 &= 0 \end{aligned}$$

As usual in back substitution, we start from the bottom.

The first nonzero equation says

$$5y - 5z = -10$$

This does not determine the value of y and z.

No matter what value $t \in \mathbb{R}$ we give to z, we can find a value of y to make the equation hold.

$$\begin{aligned} z &= t \\ y &= -2 + t \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 0 &= 0 \end{aligned}$$

Now we use the first equation to solve for x.

$$\begin{aligned} &z = t \\ &y = -2 + t \\ &x - 2(-2 + t) + 3t = 7 \implies x = 3 - t \end{aligned}$$

We write the solution set like this:

$$\{(3 - t, -2 + t, t) : t \in \mathbb{R}\}$$

$$
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
0 &= 0
\end{aligned}
$$

We write the solution set like this:

$$\{(3 - t, -2 + t, t) : t \in \mathbb{R}\}$$

The solution set is the line

$$(3, -2, 0) + t \cdot (-1, 1, 1)$$

In this case, we had an incomplete set of pivots and infinitely many solutions.

Inconsistent case

There is one more possibility. Here is the same example with the right hand side of the third equation changed.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ 4x - 3y + 7z &= 19 \end{aligned} \quad \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1} \quad \begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 5y - 5z &= -9 \end{aligned}$$

In the next step:

$$\begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 5y - 5z &= -9 \end{aligned} \quad \xrightarrow[R_3' = R_3 - R_2]{} \quad \begin{aligned} x - 2y + 3z &= 7 \\ 5y - 5z &= -10 \\ 0 &= 1 \end{aligned}$$

$$
\begin{aligned}
x - 2y + 3z &= 7 \\
5y - 5z &= -10 \\
0 &= 1
\end{aligned}
$$

This system of equations has no solution.

Thus our original system also has no solution.

The original system is inconsistent.

In this case, we had an incomplete set of pivots and the system was inconsistent.

Summary

(systems of equations where #equations = #unknowns)

Gaussian elimination terminates in an upper triangular system.

If all diagonal coefficients are nonzero (full set of pivots) then the system is consistent and has a unique solution.

If some diagonal coefficients are zero then:

if there is an all zero LHS paired with a nonzero RHS

$\longrightarrow$ no solution

otherwise, there are infinitely many solutions.

To think about

Why does Gaussian elimination always terminate with an upper triangular system?

How many steps can it take?

Why is there always a solution as long as there is not an equation like $0 = 1$?

What happens for non-square systems?