PMPH group project: Sorting on GPUs

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November 2024

1 The radix sort algorithm

In order to gain an understanding of radix sort, the sequential implementation will be discussed, and then a discussion of parallelization using basic blocks.

- Radix sort The basic idea of radix sort is to sort an integer into digits consisting of r bits. Then the list is sorted for each digit until all digits have been traversed and the list is fully sorted. The main assumption of radix sort is that all integers in the input list can be split into digits

- Radix sort sequential: The sequential implementation iterates through the digits, r bits at a time, a standard implementation is 1 bit at a time, but more is more efficient. Each digit m, has r bits, such that $m = 2^r$. Radix sort works by performing a counting sort on each set of digits, until all digits have been traversed. Counting sort follows the following syntax:

```
COUNTING-SORT
  HISTOGRAM-KEYS
    do i ← 0 to 2^r − 1
      Bucket[i] ← 0
    do j ← 0 to N − 1
      Bucket[D[j]] ← Bucket[D[j]] + 1
  SCAN-BUCKETS
    Sum ← 0
    do i ← 0 to 2^r − 1
      Val ← Bucket[i]
      Bucket[i] ← Sum
      Sum ← Sum + Val
  RANK-AND-PERMUTE
    do j ← 0 to N − 1
      A ← Bucket[D[j]]
      R[A] ← K[j]
      Bucket[D[j]] ← A + 1
```

Figure 1: Counting Sort (?)

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Where Histogram-Keys counts the number of elements in a bucket for each unique digit. Scan-Buckets, scan the buckets and returns the position at which the first element of a subset having a certain digit is to be placed, that is, the offset. Finally, Rank-and-permute matches the digit value with the bucket offset and iterates through each "sub bucket" to order the elements by their digit. After this, counting sort is repeated on each subset of digits until the whole list is sorted.

- For parallelization of radix sort, similar ideas are used, however, there a few key differences. One such method is discussed in (**?**). With p processors, the input list is split into p sub-lists for each processor to work on. Each processor performs Histogram-keys locally to create local histograms, then scan buckets is run globally to calculate the global offset, Rank and Permute can then be performed locally and written to global memory as Rank and Permute takes in the offset, and therefore doesn't overwrite the work done by other processors (**?**).

  Let us now consider a very simple parallel radix sort implementation going through 1 bit at a time, for 32 bit integers using basic blocks, and let us consider the example of

$$[1, 4, 2, 3]$$

  with a bit representation of:

$$[001, 100, 010, 011]$$

  Please bear in mind that this implementation is a very simplified parallel radix sort implementation in Futhark using basic blocks such as map, scan, reduce and scatter. Due to its syntax, Futhark automatically distributes the processes depending on available processors, however, better approaches are discussed in part 2.

```
-- xs = [1,4,2,3] in integer format
-- xs = [001, 100, 010, 011] in bit format
def radix_bit [n] 't (f: t -> u32) (xs: [n]t) (b: i32): [n]t =
(1)    let bits = map(\x -> (i32.u32 (f x >> u32.i32 b )) & 1) xs -- [1, 0, 0, 1], D = O(1), W = O(n)
(2)    let bits_neg = map (1-) bits                                -- [0, 1, 1, 0], D = O(1), W = O(n)
(3)    let offs = reduce (+) 0 bits_neg                            -- 2,            D = O(log n), W = O(n)
(4)    let idxs0 = map2 (*) bits_neg (scan (+) 0 bits_neg)         -- [0, 1, 2, 0], D = O(log n), W = O(n)
(5)    let idxs1 = map2 (*) bits (map (+offs) (scan (+) 0 bits))   -- [3, 0, 0, 4], D = O(log n), W = O(n)
(6)    let idxs2 = map2 (+) idxs0 idxs1                            -- [3, 1, 2, 4], D = O(1), W = O(n)
(7)    let idxs = map (\x -> x-1) idxs2                            -- [2, 0, 1, 3], D = O(1), W = O(n)
```

```
(8)␣␣␣␣let␣xs' = scatter (copy xs) idxs xs                          -- [4, 2,
    1, 3], D = O(1), W = O(n)
        -- in terms of the bit in question, it becomes        [100, 010, 001,
            011]
(9)     in xs'␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣
    ␣␣␣--␣D␣=␣O(log␣n),␣W␣=␣O(n)

def␣radix␣[n]␣'t (f: t -> u32) (xs: [n]t): [n]t =
(10)     loop xs for i < 32 do radix_bit f xs i                          -- D = O(
    log n), W = O(n)
```

This implementation is from radix-sort-key (futhark-lang.org), and applied to each bit of 32 bit unsigned integers, as radix sort requires each digit to be in the range of 0 to m-1. The radix_bit part of the algorithm takes in a bit. The bits are stored in bits (1), bits_neg (2) contains the bits converted to their compliment, offs (3), get the offsets of the bits by calculating how many 0 bits there are, which is what bits_neg is used for. Idxs0 (4) finds the indices of elements with a bit of 0, while idxs1 (5) finds the indices of elements with a bit of 1, and idxs2 (6) concatenate those two lists. Indexes are then stored in idxs (7) assuming zero indexing, by subtracting 1 from every index. Then scatter (8) is used to distribute the elements to their corresponding location.
Let us now consider the work depth complexity, for a list of length n, the work and depth complexity is written next to the code, where scan and reduce both have a depth complexity of O(log n), and all lines have O(n) work complexity, as they traverse n elements. While this work depth complexity might seem good on paper, this is a best case scenario, in which we always have n processors. More advanced methods deal with the distribution of the workload in a fashion more suited to the GPU, and this improving the overall speed and efficiency of the algorithm.

2 Fast parallel radix sort

Although (**?**) was great at the time, our literature search provided several higher performing parallel radix sorts. The first implementation by Satish et. al. provides a significant speed up from previous methods using the following ground principles and distributing 4 bit digits per thread, satish.PNG

Figure 2: Satish steps (?)

. They also proposed parallel local radix counting and pre-sorting as well as global radix ranking and coalesced global shuffling(?)Building off these concepts, Ha. et. al proposed two improvements, implicit counting and a mixed-data structure with 4 elements, that is 2-bit digits.

- Implicit counting
  Uses bit shifting to create a 32-bit register for radix value 0, 1 and 2. With the count of 0 assigned to bits from 0 to 9, the count of 1 assigned to bit values from 10-19, and the count of 2 assigned to bits from 20-29. The implicit count is:

$$impl_{cnt} = cnt_0 + (cnt_1 \ll 10) + (cnt_2 \ll 20)$$

The implicit value is:

$$impl_{val} = (val3) \ll (10 \cdot val)$$

$impl_{cnt}$ is calculated by incrementing by the $impl_{val}$:

$$impl_{cnt} = impl_{cnt} + impl_{val}$$

The count is retrieved by:

$$cnt\left[val\right] = impl_{cnt} \gg (10 \cdot val)$$

and the fourth counting value is retrieved by:

$$cnt\left[3\right] = idx - cnt\left[0\right] - cnt\left[1\right] - cnt\left[2\right]$$

Thus implicit counting reduces the count operations from 4 in Satish. et. al to 2.

- Mixed-data structure

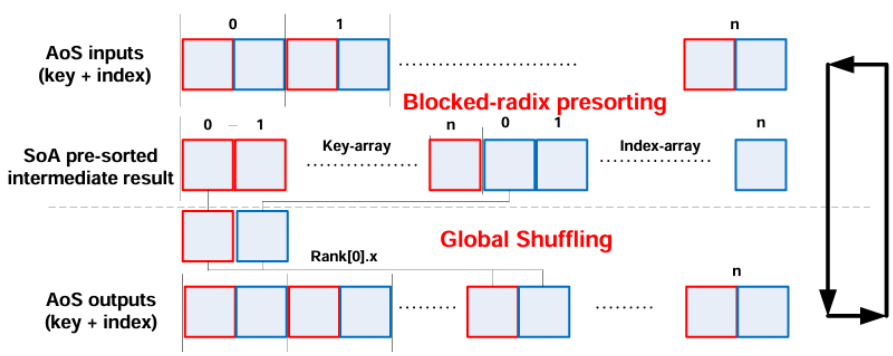


Figure 3: Mixed data method (?)

. Ha. et. al's hybrid data structure uses AoS and SoA, which reduces suboptimal coalesced scattering effects, and requires only one shuffle pass on the pre sorting data due to the structure (?)
We have decided to attempt to implement the Ha. et. al algorithm

- Pseudocode

```
1. Initialize data structures:
   - Load input keys and values.
     * Work: O(n), Depth: O(1)
   - Assuming the input is integers we can map from 32 to 24 bit by:
   - Determine [a,b] using reduce
```

```
    * Work: O(n), Depth: O(log n)
  - Map input values from a, b to [0, b-a]
    * Work: O(n), Depth: O(1)

2. Iterate through bit shifts:
   For each bit shift (e.g., 4 bits per pass):
     - Map the current bit shift to all elements.
       * Work: O(n), Depth: O(1)
     - Create an array of structures ‘AoS‘ by associating each element’s␣
         index␣and␣key.
␣␣␣␣␣␣␣*␣Work:␣O(n),␣Depth:␣O(1)

3.␣Local␣counting␣and␣presorting␣(within␣each␣block):
␣␣␣-␣For␣each␣block␣of␣block␣size␣B␣*␣4␣elements:
␣␣␣␣␣-␣Compute␣implicit␣counts:
␣␣␣␣␣␣␣*␣Work:␣O(B*4)␣␣per␣block,␣total␣O(n),␣Depth:␣O(log␣B*4)

␣␣␣␣␣-␣Calculate␣the␣local␣rank␣for␣each␣element␣in␣the␣block␣using␣**
    exclusive␣scan**␣on␣the␣histogram:
␣␣␣␣␣␣␣*␣Work:␣O(B*4)␣per␣block,␣total␣O(n),␣Depth:␣O(log␣B*4)

␣␣␣␣␣-␣Scatter␣locally␣(pre␣sort):
␣␣␣␣␣␣␣*␣Work:␣O(B)␣per␣block,␣total␣O(n),␣Depth:␣O(1)

4.␣Global␣ranking␣and␣offset␣calculation:
␣␣␣-␣Compute␣global␣offsets␣using␣**prefix␣sum**␣(scan)␣across␣all␣local␣
    histograms:
␣␣␣␣␣*␣Work:␣O(P)␣=␣O(n/(B))
␣␣␣␣␣*␣Depth:␣O(log␣P)␣=␣O(log␣n/(B))

␣␣␣-␣Adjust␣local␣ranks␣by␣adding␣block␣offsets:
␣␣␣␣␣*␣Work:␣O(n),␣Depth:␣O(1)

5.␣Global␣shuffling␣and␣final␣scatter:
␣␣␣-␣Scatter␣elements␣globally␣using␣computed␣global␣offsets:
␣␣␣␣␣*␣Work:␣O(n),␣Depth:␣O(log␣B)

6.␣Repeat␣until␣all␣bits␣are␣processed.
␣␣␣-␣Number␣of␣passes␣depends␣on␣bit␣width␣(e.g.,␣for␣24-bit␣keys␣(mapping␣
    from␣32␣to␣24)␣with␣2␣bits␣(4␣elements)␣per␣pass,␣we␣need␣16␣passes).
␣␣␣-␣Total:
␣␣␣␣␣*␣Work:␣O(n␣*␣#passes),␣Depth:␣O(#passes␣*␣(log␣B␣+␣log␣P)␣=␣O(#passes␣*
    ␣(log␣B␣+␣log␣(n/B)))
```

Additionally, the algorithm has bank conflict-free access, storing in different arrays to avoid concurrent access, Mapping of integers from [a,b to 0, b-a] for integers, and floats by mapping from [a,b] to [0.5, 1] to change from 32 bit to 24 bit. 30% performance increase. It is an approximation but it is precise enough.

3 In depth cuda implementation

Radix sort can viewed as a sequential application of countsort across different bit intervals, each iteration maintaining the invariant that the array is sorted with respect to the last bit.

We can break countsort into 4 fundamental kernels.

- Histogram kernel
- Transpose kernel
- Scan kernel
- A final rank and permute kernel

We follow the possible good values recommended, thus $Q = 22, lgH = 8, H = 256, B = 256$.

3.1 Histogram kernel

The parallel cuda-implementation of the histogram processes Q elements per thread. We decided to use shared memory while processing the histogram, as otherwise we would have to access global memory many times, which is slower than first using shared memory, and then writing the final histogram to global memory. Unfortunately we use atomic adds for incrementing our histogram buckets. This is done Q times per thread, which is not ideal, as atomic adds impacts performance negatively, which likely affects our results as seen in the last section. Notably for the histogram we sort each element based on each bit, instead of the regular approach of sorting by each digit. This approach is both more memory efficient and faster.

3.2 Transpose kernel

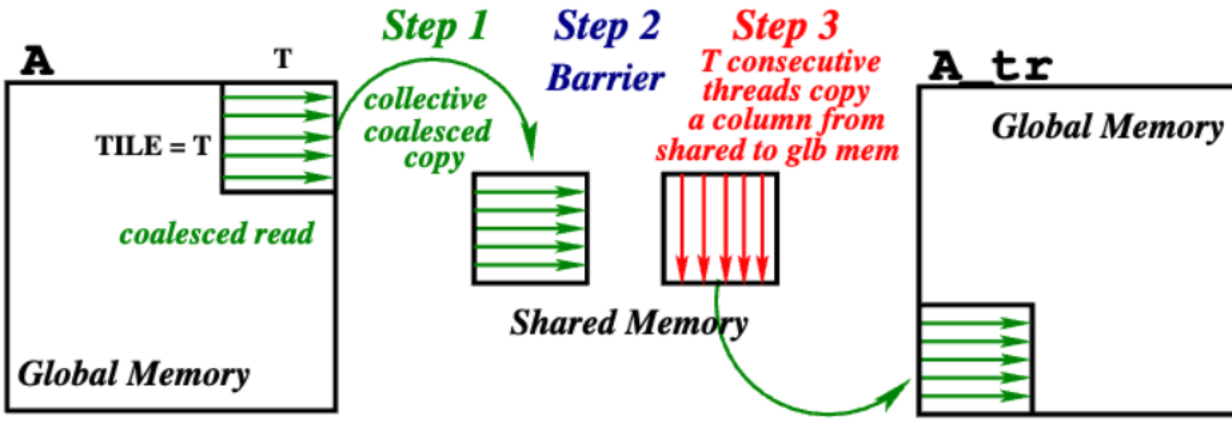
as part of the position computation across buckets we need to compute and update of the ith block and jth key as:

$$Buckets[i,j]_{after} = \sum_{k=0}^{p-1}\sum_{m=0}^{j-1} Buckets[k,m] + \sum_{k=0}^{i-1} Buckets[k,j]$$

doing this naively would lead to uncoalesced access as buckets[i, j] and buckets[i+1, j] will be H apart. As we store our matrix in row-major order. In order to efficiently and in a coalesced manner scan our histograms(one for each block) we transpose our matrix such that buckets[i, j] and buckets[i+1, j] will now only be one element apart. The transpose kernel is relatively simple and is heavily inspired by the one presented in the week 2 assignment. We read coalesced from global memory to shared memory(at miniscule performance cost) then read uncoalesced in shared memory and write back coalesced to global memory.

as per cosmins slides

We use this transpose kernel twice, once before scanning and then again after scanning.



slides/L6-locality.pdf

3.3 Scan kernel

The scan kernel is called on the transposed matrix to comoute the indices for the rand-and-permute kernel. We use inclusive scan that performs scan at both the block and warp level, efficiently explotiting the fact that the work depth of scan over an associate binary operator(+) is log(n).

3.4 Rank-and-permute kernel

This kernel is the most complex and requires deliberate usage of the memory hiearchy to run efficiently on gpus. First each blocks loads Q*BlOCK_SIZE elements into shared memory, from there each thread loads its elements from shared memory into its registers.

next we compute the two-way-partioning sequentially over lgh bits. Each thread computes a sequential reduction over its Q elements checking if the i'th bit of a value is set or unset and stores this in shared memory. next compute an inclusive scan at the block level computing the cumulative sum of unset bits.

each thread then reads from shared memory back to local memory the cumulative sum at their index. In this way each thread obtains the necessary information for correctly ordering its elements in shared memory. At last each thread reads its new elements from shared back to local memory. And this process is repeated lgH times.

After the two-way partitioning each thread write its q elements to shared memory and then each block writes its QB elements from shared to global memory. Note that we do not have to perform any inter-block communication as all position information has been communicated via the scan across blocks. We include the inner loop over lgh, the most intersting aspects of the kernel below.

```
// Sequentially reduce Q elements
      uint16_t accum = 0;
      for (int q_idx = 0; q_idx < P::Q; q_idx++) {
          // we shift by bit to get the bit value and we mask it with 1 to
              get the boolean value
          // we first shift by the bit_offs offset and then by the current
              bit from 0 to lgH-1
          uint16_t res = isBitUnset<uint>(bit_offs + bit, reg[q_idx]);
          accum += res;
      }

      // the thread local count is stored to shared memory
      local_histo[tid] = accum;

      // we compute the inclusive scan across shared memory
      // to get the cumulative sum of unset bits up to our threadIdx
      uint16_t res = scanIncBlock<Add<uint16_t>>(local_histo, tid);
      __syncthreads();
      local_histo[tid] = res;
      __syncthreads();

      // we get the prefix sum for the entire block
      // which is the cumulative sum of unset bits up to the last thread in
           the block
      int16_t prefix_sum = local_histo[P::BLOCK_SIZE-1];

      // each thread loads to its registers from shared memory
      // the prefix sum of the previous thread
      if (tid == 0) {
          accum = 0;
      } else {
          accum = local_histo[tid-1];
      }
      __syncthreads();


      // each thread sequentially rearranges its Q elements
      // and write them to their new position in shared memory
      for (int q_idx = 0; q_idx < P::Q; q_idx++) {
          uint val = reg[q_idx];
          uint16_t bit_val = (uint16_t) isBitUnset<uint>(bit_offs + bit,
              val);

          accum += bit_val;

          int newpos;

          if (bit_val == uint(1)) {
              newpos = accum -1 ;
          } else {
              // we add the prefix of the thread to the local histogram
                  position
              // and we subtract the accumumulator to get the new position
              // we offset by threadIdx.x*Q
              newpos = prefix_sum + thread_offset + q_idx - accum;
          }
          // we write the element to the new position
```

```
            shmem[newpos] = val;
        }
        // wait for all threads to have written their elements
        __syncthreads();


        // each thread loads its new elements from shared memory to its
            registers
        if (bit < P::lgH-1) {
            // if we are not at the last bit
            // in the new position
            for (int q_idx = 0; q_idx < P::Q; q_idx++) {
                reg[q_idx] = shmem[thread_offset + q_idx];
            }
        } else {
            // if we are at the last bit
            for (int q_idx = 0; q_idx < P::Q; q_idx++) {
                // we iterate with a stride of BLOCK_SIZE
                // but this is not a big deal as uncoalesced
                // access in shared memory does not affect performance
                uint local_pos = q_idx*P::BLOCK_SIZE + tid;
                reg[q_idx] = shmem[local_pos];
            }
        }
        __syncthreads();
```

3.5 Sorting the whole array

In order to sort the entire array we apply countsort in a loop. This means our sorting algorithm first takes in a completely unsorted array, then applies all the kernels to it. After the rank-and-permute kernel, we overwrite the input array we gave to the histogram and build a new array on the result of the rank-and-permute kernel. This process is then repeated, where the histogram is build on the rank-and-permute output, for however many bits our elements in our array have. As we are using 32 bit numbers, it means we launch the kernels 32 times to obtain a fully sorted array. Consequently, this can have an effect on the performance of our sorting and some papers like **?** suggest that for instead of building a histogram for each bit, it was more beneficial if we processed four bits at a time, which would lead to only launching the kernels 8 times for 32 bit number. at each iteration except for the last one, we swap our input and output buffer.

4 Performance evaluation

We benchmarked cuda, futhark and our own implementation for data types u8, u16, u32, u64. We use GB/S as a measure of performance(JUSTIFICATION).

with more time, we would have liked to also benchmark the performance for a fixed array size as a function of entropy of the keys. Another thing that would be interesting to study is the impact of lgH on our performance, on the one hand increasing lgh would lead to fewer steps of countsort(which would have to be performed sequentially to maintain the invariant) Additionally we would increase.

4.1 Discussion of results

The results show that our implementation is comparable to cub's implementation, and is definitely better than the futhark baseline, which was the goal of this project.

4.2 Suggestions for improving our radix-sort implementation

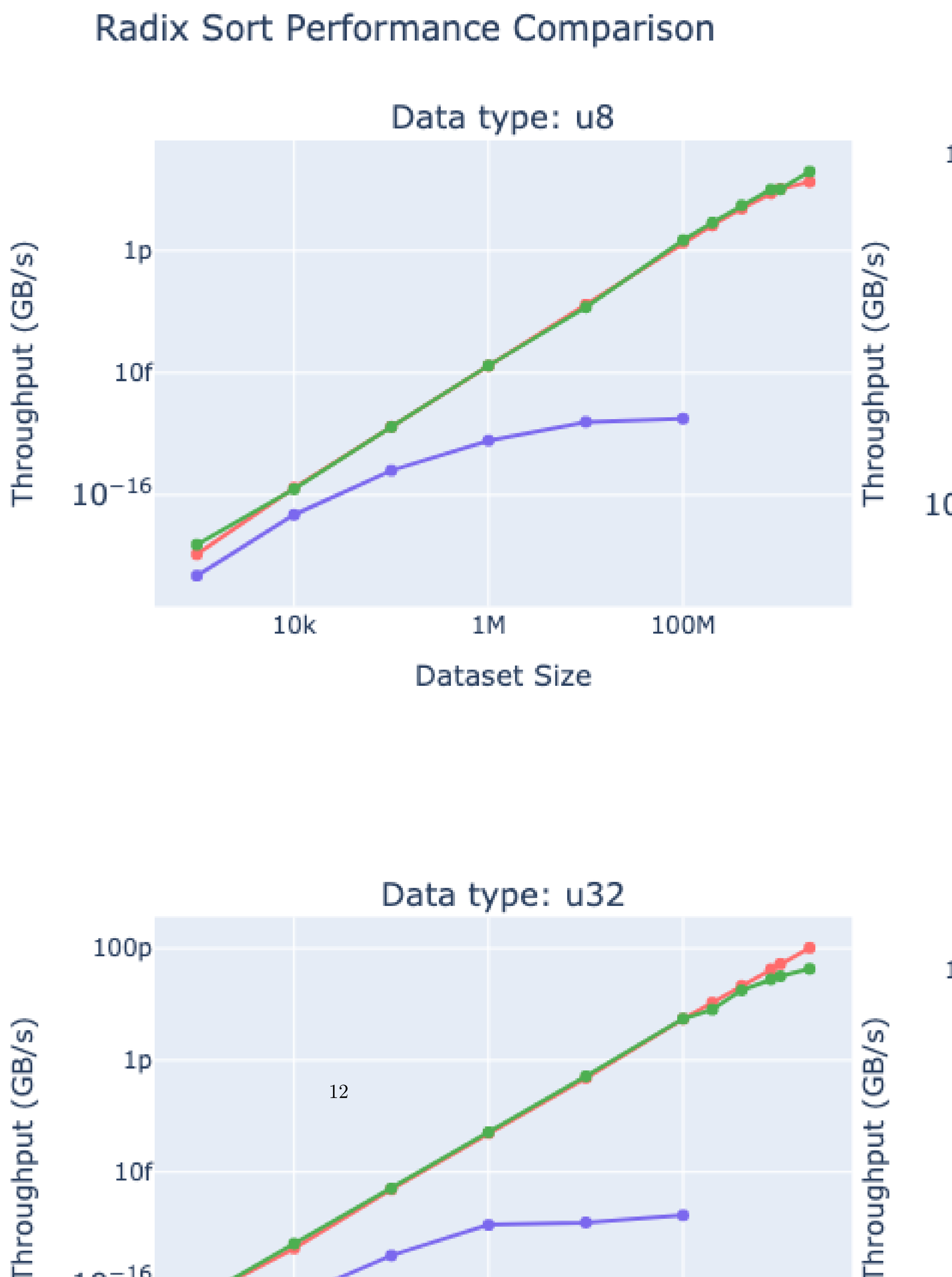
As alluded to earlier, we could have processed more bits for our histogram, which could have lead to us launching oour kernels fewer times, thus increasing performance.

4.3 Usage

for interacting with

References

Linh Ha, Jens Krüger, and Cláudio T Silva. Implicit radix sorting on gpus. *arXiv preprint arXiv:1010.2016*, 2010.


Radix Sort Performance Comparison
Data type: u8
Throughput (GB/s)
1p
10f
10^−16
10k
1M
100M
Dataset Size
Data type: u32
100p
1p
10f
10^−16
Throughput (GB/s)