More Reductions

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I. THEOREM 4.1 OF ANTHONY-BOROS-CRAMA-GRUBER (ABCG)

Any symmetric fuction can be quadratized with $n-2$ auxiliaries, where α_i comes from Corollary 2.3:

$$f(b_1, b_2, \ldots, b_n) \rightarrow -\alpha_0 - \alpha_0 \sum_i b_i + 2a_2 \sum_{ij} b_i b_j + \tag{1}$$

$$2 \sum_{2i-1} \left(\alpha_{2i-1} - \min\left(\alpha_{2j-1}\right)\right) b_{a_{2i-1}} \left(2i - \frac{3}{2} - \sum_j b_j\right) + \tag{2}$$

$$2 \sum_{2i} \left(\alpha_{2i} - \min\left(\alpha_{2j}\right)\right) b_{a_{2i}} \left(2i - \frac{1}{2} - \sum_j b_j\right) \tag{3}$$

$$\alpha_i = -4 \sum_{j=0}^{i} (-1)^{i-j} f(j) - f(i-1) + 3f(i) \tag{4}$$

Alternatively:

$$f(b_1, b_2, \ldots, b_n) \rightarrow -\alpha_0 - \alpha_0 \sum_i b_i + a_2 \sum_{ij} b_i b_j + 2 \sum_i (\alpha_i - c) b_{a_i} \left(2i - \frac{1}{2} - \sum_j b_j\right) \tag{5}$$

$$c = \begin{cases} \min\left(\alpha_{2j}\right) & , i \in \text{even} \\ \min\left(\alpha_{2j-1}\right) & , i \in \text{odd} \end{cases} \tag{6}$$

$$a_2 = \text{has to be obtained from page 12 of the paper.} \tag{7}$$

They say this is linear in the auxiliary variables, but it doesn't seem to be, because we have $b_{a_i} b_j$ terms where b_{a_i} are auxiliaries.

Pro: quadratization symmetric in all non-auxiliary variables, which isn't true for all quadratizations of symmetric functions. Reproduces the full spectrum.

Con: all quadratic terms of the non-auxiliary variables, are non-submodular. Also very complicated and uses more auxiliaris than simlper methods.

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II. UNPUBLISHED WORK OF ALEXANDER FIX

Any symmetric fuction can be quadratized with $n-1$ auxiliaries. Add a multiple of $E(\sum b_r)$ to each term of Corollary 2.4 of the above paper.

III. ASYMMETRIC REDUCTION FOR NEGATIVE MONOMIALS OF ARBITRARY k

$$-b_1 b_2 \ldots b_k \rightarrow (k-1) b_k b_a - \sum_i b_i (b_a + b_k - 1) \tag{8}$$

$$-b_1 b_2 \ldots b_k \rightarrow -\sum_i b_i - \sum_i b_i b_k - \sum_i b_i b_a + (k-1) b_k b_a \tag{9}$$

Pro: only 1 auxiliary to quadratize k degree term. Only one non-submodular term (and it's quadratic). Reproduces the full spectrum.

Con: Turns symmetric into non-symmetric (but only the b_k is asymmetric).

IV. "A RELATED PAPER BY THE PRESENT AUTHORS [1] GIVES A COMPLETE CHARACTERIZATION OF ALL THE QUADRATIZATIONS OF NEGATIVE MONOMIALS INVOLVING ONE AUXILIARY VARIABLE"

V. ANOTHER REDUCTION FOR NEGATIVE MONOMIALS OF ARBITRARY k

$$-b_1 b_2 \ldots b_k \rightarrow 2 b_a \left(k - \frac{1}{2} - \sum_i b_i \right) \tag{10}$$

$$-b_1 b_2 \ldots b_k \rightarrow (2k-1)\, b_a - 2 \sum_i b_i b_a \tag{11}$$

Pro: only 1 auxiliary to quadratize k degree term. Only one non-submodular term (and it's linear). Symmetric with respect to all non-auxiliary variables. Reproduces the full spectrum.

Con: Coefficients of quadratic terms are twice the size of in the "standard" quadratization for negative monomials, and roughly twice the size for the linear term.

VI. ABCG VERSION OF ISHIKAWA:

$$b_1 b_2 \dots b_k \to \sum_i b_i + \sum_{ij} b_i b_j + \sum_{2i-1} b_{a_{2i-1}} \left(4i - 3 - \sum_j b_j \right) \tag{12}$$

$$b_1 b_2 \dots b_k \to \sum_i b_i + (4i-3) \sum_{2i-1} b_{a_{2i-1}} + \sum_{ij} b_i b_j - \sum_{2i-1,j} b_j b_{a_{2i-1}} \tag{13}$$

Pro. Same number of auxiliaries as Ishikawa. Reproduces the full spectrum.

Con. Only works for odd k, but when k is even we can use Ishikawa, so no big loss.

VII. ANOTHER ABCG VERSION OF ISHIKAWA:

$$b_1 b_2 \dots b_k \to \prod_{i=1}^{k-1} b_i - \prod_{i=1}^{k-1} b_i (1 - b_k) \tag{14}$$

Now quadratize the first term using Ishikawa, and use a negative monomial method for the second term.

VIII. COROLLARY 4.4 OF ABCG

IX. COROLLARY 4.5 OF ABCG

X. QUADRATIZATION OF "PARITY" FUNCTION ON PAGE 17 OF ABCG (THEOREM 4.6)

XI. THEOREM 5.6 OF ABCG

XII. THEOREM 1.1 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $n/2 \le m \le n$

$$f\left(b_1,b_2,\ldots,b_n\right) \to \left(\sum_i b_i - \left(m-2^{\lceil \log m\rceil}\right) b_{a_{\lceil \log m\rceil+1}} - (m+1)\left(1-b_{a_{\lceil \log m\rceil+1}}\right) - \sum_i^{\lceil \log m\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{15}$$

$$= \left(\sum_i b_i - \left(m-2^{\lceil \log m\rceil}\right) b_{a_{\lceil \log m\rceil+1}} - (m+1) + (m+1) b_{a_{\lceil \log m\rceil+1}} - \sum_i^{\lceil \log m\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{16}$$

$$= \left(-(m+1) + \sum_i b_i - \left(2m-2^{\lceil \log m\rceil}+1\right) b_{a_{\lceil \log m\rceil+1}} - \sum_i^{\lceil \log m\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{17}$$

$$= (m+1)^2 - 2(m+1)\sum_i b_i + 2(m+1)\left(2m-2^{\lceil \log m\rceil}+1\right) b_{a_{\lceil \log m\rceil+1}} + 2(m+1)\sum_i^{\lceil \log m\rceil} 2^{i-1} b_{a_i} \tag{18}$$

$$+ \sum_{ij} b_i b_j - 2\sum_i \left(2m-2^{\lceil \log m\rceil}+1\right) b_i b_{a_{\lceil \log m\rceil+1}} - 2\sum_i \sum_j^{\lceil \log m\rceil} 2^{i-1} b_i b_{a_j} + \sum_{i,j}^{\lceil \log m\rceil} 2^{i+j-2} b_{a_i} b_{a_j} \tag{19}$$

$$= \alpha^I + \alpha^b \sum_i b_i + \alpha^{b_{a,1}} \sum_i^{\lceil \log m\rceil} b_{a_i} + \alpha^{b_{a,2}} b_{a_{\lceil \log m\rceil+1}} + \alpha^{bb}\sum_{ij} b_i b_j + \alpha^{bb_{a,1}} \sum_i \sum_j^{\lceil \log m\rceil} b_i b_{a_j} \tag{20}$$

$$+ \alpha^{bb_{a,2}} \sum_i b_i b_{a_{\lceil \log m\rceil+1}} + \alpha^{b_a b_a} \sum_{i,j}^{\lceil \log m\rceil} b_{a_i} b_{a_j} \tag{21}$$

The number of auxiliary varibales is $\lceil \log m\rceil + 1$.

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (m+1)^2 & 1 \\ -2(m+1) & -2^i \\ 2(m+1) & -2\left(2m-2^{\lceil \log m\rceil}+1\right) \\ 2(m+1)\left(2m-2^{\lceil \log m\rceil}+1\right) & 2^{i+j-2} \end{pmatrix}. \tag{22}$$

XIII. THEOREM 1.2 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $0 \le m \le n/2$.

$$f\left(b_1,b_2,\ldots,b_n\right) \to \left(n - \sum_i b_i - \left(n-m-2^{\lceil \log(n-m)\rceil}\right) b_{a_{\lceil \log(n-m)\rceil+1}} - (n-m+1)\left(1-b_{a_{\lceil \log(n-m)\rceil+1}}\right) - \sum_i^{\lceil \log(n-m)\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{23}$$

$$= \left(n - \sum_i b_i - \left(n-m-2^{\lceil \log(n-m)\rceil}\right) b_{a_{\lceil \log(n-m)\rceil+1}} - (n-m+1) + (n-m+1) b_{a_{\lceil \log(n-m)\rceil+1}} - \sum_i^{\lceil \log(n-m)\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{24}$$

$$= \left((m-1) - \sum_i b_i - \left(2(n-m)-2^{\lceil \log(n-m)\rceil}+1\right) b_{a_{\lceil \log(n-m)\rceil+1}} - \sum_i^{\lceil \log(n-m)\rceil} 2^{i-1} b_{a_i}\right)^2 \tag{25}$$

The number of auxiliary varibales is $\lceil \log(n-m)\rceil + 1$.

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}. \tag{26}$$

XIV. COROLLARY 1 OF BCR

transformation not explicitly given, but the function can be more general than in Theorem 1, but requires a factor of μ more variables.

XV. THEOREM 2 OF BCR

Once again requires typing out a nasty function [should really be done in mathematica rather than by hand)

XVI. THEOREM 4 OF BCR

This is a special case of Theorem 1, for the specific function $f = b_1 b_2 \ldots b_k$. For som p such that $k \leq 2^p$, we have:

$$b_1 b_2 \ldots b_k \to \left(2^p - k + \sum_i b_i - \sum_i 2^{i-1} b_{a_i}\right)^2 \tag{27}$$

$$= \left(2^p - k\right)^2 + 2\left(2^p - k\right)\sum_i b_i - 2\left(2^p - k\right)\sum_i 2^{i-1} b_{a_i} + \sum_{ij} b_i b_j - \sum_{ij} 2^{j-1} b_i b_{a_j} + \sum_{ij} 2^{i+-2} b_{a_i} b_{a_j} \tag{28}$$

$$= \alpha^I + \alpha^b \sum_i b_i + \alpha^{b_{a_i}} \sum_i 2^{i-1} b_{a_i} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_a} \sum_{ij} b_i b_{a_j} + \alpha^{b_{a_i} b_{a_j}} b_{a_i} b_{a_j} \tag{29}$$

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_a} \\ \alpha^{b_a} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} \left(2^p - k\right)^2 & 1 \\ 2\left(2^p - k\right) & 2^{j-1} \\ -2\left(2^p - k\right) & 2^{i+-2} \end{pmatrix}. \tag{30}$$

Pro: only requires $\log k$ auxiliaries.

Con: All terms non-submodular except for the term linear in auxiliaries.

XVII. THEOREM 5 OF BCR

(already written up by Richard)

Pro: only requires $\log k$-1 auxiliaries.

XVIII. THEOREM 7 OF BCR

(already written up by Richard)

XIX. THEOREM 9 OF BCR

For the symmetric function which is a function of the sum of all n variables, for some huge integer λ such that $\lambda > \max(f)$, we have:

$$f\left(\sum b_i\right) \to \sum_{ij}^{\sqrt{n+1}} f\left((i-1)\left(\lceil\sqrt{n+1}\rceil + 1\right) + (j-1)\right) b_{a_i} b_{a_{\sqrt{n+1}+j}} + \tag{31}$$

$$+ \lambda\left(\left(1 - \sum_i^{\sqrt{n+1}} b_{a_i}\right)^2 + \left(1 - \sum_i^{\sqrt{n+1}} b_{a_{\sqrt{n+1}+i}}\right)^2 + \right. \tag{32}$$

$$\left. + \left(\sum_i b_i - \left(\left(\lceil\sqrt{n+1}\rceil + 1\right) \sum_i^{\sqrt{n+1}} (i-1) y_{a_i} + \sum_i^{\sqrt{n+1}} (i-1) b_{a_{\sqrt{n+1}+i}}\right)\right)^2\right) \tag{33}$$

XX. THEOREM 10 OF BCR

Works on a generalization of $f\left(\sum b_i\right)$ but instead we have a weighted sum.