Exercises FYS4480, week 38, September 19-23, 2022

Feel free to continue working on the Lipkin model from last week. See also the additional challenge to last week under exercise 3 here.

Exercise 1

We define the one-particle operator

$$\hat{T} = \sum_{\alpha\beta} \langle\alpha| t |\beta\rangle a_\alpha^\dagger a_\beta,$$

and the two-particle operator

$$\hat{V} = \frac{1}{2}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta| v |\gamma\delta\rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma.$$

We have defined a single-particle basis with quantum numbers given by the set of greek letters $\alpha, \beta, \gamma, \dots$

a) Show that the form of these operators remain unchanged under a transformation of the single-particle basis given by

$$|i\rangle = \sum_\lambda |\lambda\rangle \langle\lambda|i\rangle,$$

with $\lambda \in \{\alpha, \beta, \gamma, \dots\}$. Show also that $a_i^\dagger a_i$ is the number operator for the orbital $|i\rangle$.

b) Find also the expressions for the operators T and V when T is diagonal in the representation i.

Exercise 2

Consider the Hamilton operator for a harmonic oscillator ($c = \hbar = 1$)

$$\hat{H} = \frac{1}{2m}p^2 + \frac{1}{2}kx^2, \qquad k = m\omega^2$$

a) Define the operators

$$a^\dagger = \frac{1}{\sqrt{2m\omega}}(p + im\omega x), \qquad a = \frac{1}{\sqrt{2m\omega}}(p - im\omega x)$$

and find the commutation relations for these operators by using the corresponding relations for p and x.

b) Show that

$$H = \omega(a^\dagger a + \frac{1}{2})$$

c) Show that if for a state $|0\rangle$ which satisfies $\hat{H}|0\rangle = \frac{1}{2}\omega|0\rangle$, then we have

$$\hat{H}|n\rangle = \hat{H}(a^\dagger)^n|0\rangle = (n + \frac{1}{2})\omega|n\rangle$$

d) Show that the state $|0\rangle$ from c), with the property $a|0\rangle = 0$, must exist.

Exercise 3, Challenge

In the previous exercise set from week 37 we considered a state with all fermions in the lowest single-particle state

$$|\Phi_{J_z=-2}\rangle = a^\dagger_{1-}a^\dagger_{2-}a^\dagger_{3-}a^\dagger_{4-}|0\rangle .$$

This state has $J_z=-2$ and belongs to the set of projections for $J=2$. We will use the shorthand notation $|J,J_z\rangle$ for states with different spon J and spin projection J_z. The other possible states have $J_z=-1$, $J_z=0$, $J_z=1$ and $J_z=2$.

Use the raising or lowering operators J_+ and J_- in order to construct the states for spin $J_z=-1$ $J_z=0$, $J_z=1$ and $J_z=2$. The action of these two operators on a given state with spin J and projection J_z is given by ($\hbar=1$) by $J_+|J,J_z\rangle = \sqrt{J(J+1)-J_z(J_z+1)}\,|J,J_z+1\rangle$ and $J_-|J,J_z\rangle = \sqrt{J(J+1)-J_z(J_z-1)}\,|J,J_z-1\rangle$.