Exercises FYS4480, weeks 40 and 41, October 3-14, 2022

Exercise 1

We will continue to study the schematic model (the Lipkin model, Nucl. Phys. **62** (1965) 188) for the interaction among 4 fermions that can occupy two different energy levels. Each levels has degeneration $d = 4$. The two levels have quantum numbers $\sigma = \pm 1$, with the upper level having $\sigma = +1$ and energy $\varepsilon_1 = \varepsilon/2$. The lower level has $\sigma = -1$ and energy $\varepsilon_2 = -\varepsilon/2$. In addition, the substates of each level are characterized by the quantum numbers $p = 1, 2, 3, 4$.

We define the single-particle states

$$|u_{\sigma=-1,p}\rangle = a^\dagger_{-p}|0\rangle \qquad |u_{\sigma=1,p}\rangle = a^\dagger_{+p}|0\rangle .$$

The single-particle states span an orthonormal basis. The Hamiltonian of the system is given by

$$\begin{aligned}
H = &\; H_0 + H_1 + H_2 \\
H_0 = &\; \tfrac{1}{2}\varepsilon \textstyle\sum_{\sigma,p} \sigma a^\dagger_{\sigma,p} a_{\sigma,p} \\
H_1 = &\; \tfrac{1}{2}V \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{\sigma,p'} a_{-\sigma,p'} a_{-\sigma,p} \\
H_2 = &\; \tfrac{1}{2}W \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{-\sigma,p'} a_{\sigma,p'} a_{-\sigma,p}
\end{aligned}$$

where V and W are constants. The operator H_1 can move pairs of fermions as shown earlier while H_2 is a spin-exchange term. The H_2 term moves a pair of fermions from a state $(p\sigma, p' - \sigma)$ to a state $(p - \sigma, p'\sigma)$.

a) Use the quasispin operators to construct the Hamiltonian matrix H for the five-dimensional space that has total spin $J = 2$ and spoin projections $J_z = -2, -1, 0, 1, 2$. Find thereafter the eigenvalues for the following parameter sets:

$$\begin{array}{ll}
(1) & \varepsilon = 2,\ V = -1/3,\ W = -1/4 \\
(2) & \varepsilon = 2,\ V = -4/3, \quad W = -1
\end{array}$$

Which state is the ground state? Comment your results in terms of the coefficients of the various eigenfunctions

b) The single-particle states for the Lipkin model

$$|u_{\sigma=-1,p}\rangle = a^\dagger_{-p}|0\rangle \qquad |u_{\sigma=1,p}\rangle = a^\dagger_{+p}|0\rangle$$

can now be used as basis for a new single-particle state $|\phi_{\alpha,p}\rangle$ via a unitary transformation

$$|\phi_{\alpha,p}\rangle = \sum_{\sigma=\pm 1} C_{\alpha\sigma}|u_{\sigma,p}\rangle$$

with $\alpha = \pm 1$ og $p = 1, 2, 3, 4$. Why is p the same in $|\phi\rangle$ as in $|u\rangle$? Show that the new basis is orthonormal.

c) With the new basis we can construct a new Slater determinant given by $|\Psi\rangle$

$$|\Psi\rangle = \prod_{p=1}^{4} b^\dagger_{\alpha,p}|0\rangle$$

with $b^\dagger_{\alpha,p}|0\rangle = |\phi_{\alpha,p}\rangle$. h) Use the Slater determinanten from the previous exercise to calculate

$$E = \langle\Psi| H |\Psi\rangle ,$$

as a function of the coefficients $C_{\sigma\alpha}$. We assume the coefficients to be real.

d) Show that

$$\frac{\epsilon}{3} > V + W,$$

has to be fulfilled in order to find a minimum in the energy.

Hint: calculate the functional derivative of the energy with respect to the coefficients $C_{\sigma\alpha}$.