Exercises FYS4480, week 37, September 12-16, 2022

Exercise 1

We will study a schematic model (the Lipkin model, Nucl. Phys. **62** (1965) 188) for the interaction among 4 fermions that can occupy two different energy levels. Each levels has degeneration $d = 4$. The two levels have quantum numbers $\sigma = \pm 1$, with the upper level having $\sigma = +1$ and energy $\varepsilon_1 = \varepsilon/2$. The lower level has $\sigma = -1$ and energy $\varepsilon_2 = -\varepsilon/2$. In addition, the substates of each level are characterized by the quantum numbers $p = 1, 2, 3, 4$.

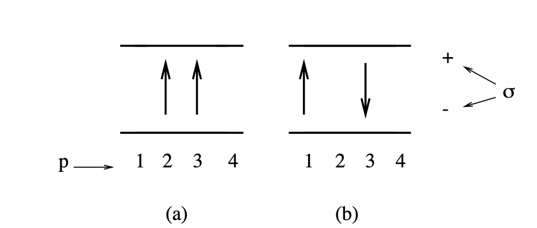
We define the single-particle states

$$|u_{\sigma=-1,p}\rangle = a^\dagger_{-p}|0\rangle \qquad |u_{\sigma=1,p}\rangle = a^\dagger_{+p}|0\rangle .$$

The single-particle states span an orthonormal basis. The Hamiltonian of the system is given by

$$\begin{aligned} H =& \ H_0 + H_1 + H_2 \\ H_0 =& \ \tfrac{1}{2}\varepsilon \textstyle\sum_{\sigma,p} \sigma a^\dagger_{\sigma,p} a_{\sigma,p} \\ H_1 =& \ \tfrac{1}{2}V \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{\sigma,p'} a_{-\sigma,p'} a_{-\sigma,p} \\ H_2 =& \ \tfrac{1}{2}W \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{-\sigma,p'} a_{\sigma,p'} a_{-\sigma,p} \end{aligned}$$

where V and W are constants. The operator H_1 can move pairs of fermions as shown in Fig. (a) while H_2 is a spin-exchange term. As shown in Fig. (b), H_2 moves a pair of fermions from a state $(p\sigma, p' - \sigma)$ to a state $(p - \sigma, p'\sigma)$.



We will encounter this model again in our analysis of the Hartree-fock method.

a. *Part a)* Introduce the quasispin operators

$$\begin{aligned} J_+ =& \ \textstyle\sum_p a^\dagger_{p+} a_{p-} \\ J_- =& \ \textstyle\sum_p a^\dagger_{p-} a_{p+} \\ J_z =& \ \tfrac{1}{2} \textstyle\sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} \\ J^2 =& \ J_+J_- + J_z^2 - J_z \end{aligned}$$

Show that these operators obey the commutation relations for angular momentum.

b. *Part b)* Express H in terms of the above quasispin operators and the number operator

$$N = \sum_{p\sigma} a^\dagger_{p\sigma} a_{p\sigma} .$$

c. *Part c)* Show that H commutes with J^2, viz., J is a good quantum number.

d. Part d) Consider thereafter a state with all four fermions in the lowest level (see the above figure). We can write this state as

$$|\Phi_{J_z=-2}\rangle = a^\dagger_{1-}a^\dagger_{2-}a^\dagger_{3-}a^\dagger_{4-}|0\rangle \,.$$

This state has $J_z = -2$ and belongs to the set of possible projections of $J = 2$. We introduce the shorthand notation $|J, J_z\rangle$ for states with different values of spin J and its projection J_z.

The other possible values are $J_z = -1$, $J_z = 0$, $J_z = 1$ and $J_z = 2$. Use the ladder operators J_+ and J_- to set up the states with spin $J_z = -1$ $J_z = 0$, $J_z = 1$ and $J_z = 2$. The action of these operators on a state with given spin J and J_z is (with $\hbar = 1$) $J_+|J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z+1)}\,|J, J_z+1\rangle$ and $J_-|J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z-1)}\,|J, J_z-1\rangle$, respectively.

e. Part e) Use thereafter the quasispin operators to construct the Hamiltonian matrix H for this five-dimensional space. Find thereafter the eigenvalues (numerically using for example Octave or Matlab or python) for the following parameter sets: sett av verdier:

$$\begin{array}{ll} (1) & \varepsilon = 2,\ V = -1/3,\ W = -1/4 \\ (2) & \varepsilon = 2,\ V = -4/3,\quad W = -1 \end{array}$$

Which state is the ground state? Comment your results.