FYS 4480 Sept 9

$$N[a_1 a_2 a_3 a_4^\dagger] = (-)^p a_4^\dagger a_1 a_2 a_3$$

$$= -1 \times a_4^\dagger a_1 a_2 a_3$$

$N[XYZ \cdots W] = (-)^p$ [creation operators] [annihilation operators]

Contraction

$$\overbrace{a_1 a_2^\dagger}$$

$$\langle 0|a_1 a_2^\dagger|0\rangle = \langle 0|\delta_{12}|0\rangle - \langle 0|a_2^\dagger a_1|0\rangle$$

$$= \overbrace{a_1 a_2^\dagger} + N[a_1 a_2^\dagger]$$

$$a_\alpha|0\rangle = 0 \qquad a_\alpha^\dagger|0\rangle = |\alpha\rangle$$

$$\overbrace{a_1^\dagger a_2} = \overbrace{a_1^\dagger a_2^\dagger} = \overbrace{a_1 a_2} = 0$$

4-operators

$$\langle 0| a_1 a_2 a_3^{+} a_4^{+} |0\rangle$$

$$= N[a_1 a_2 a_3^{+} a_4^{+}]$$

$$+ \sum_{(1)} N[a_1 a_2 a_3^{+} a_4^{+}]$$

$$+ \sum_{(2)} N[a_1 a_2 a_3^{+} a_4^{+}]$$

<u>Theorem</u>

$$\langle 0| x y z \cdots w |0\rangle$$

$$= x y z \cdots w$$

$$= N[x y z \cdots w]$$

$$+ \sum_{(1)} N[x y z \cdots w]$$

$$+ \sum_{(2)} N[x y z \cdots w]$$

$$+ \cdots$$

$$+ \cdots \sum_{([\frac{N}{2}])} N[x y z \cdots w]$$

$$= X\,Y\,Z \cdots W$$

<u>Example</u>

$$H_I = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | v | \gamma\delta \rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$

$$\langle \alpha_1 \alpha_2 | H_1 | \alpha_1 \alpha_2 \rangle =$$

$$\left(|\phi\rangle = |\alpha_1 \alpha_2\rangle = a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle \right)$$

$$\frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | v | \gamma\delta \rangle \times$$

$$\langle 0 | a_{\alpha_2} a_{\alpha_1} a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle$$

$$a_{\alpha_2} a_{\alpha_1} a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma a^\dagger_{\alpha_1} a^\dagger_{\alpha_2}$$

$$\Big(\delta_{\alpha_1 \alpha} \delta_{\alpha_2 \beta} \qquad \delta_{\delta \alpha_2} \delta_{\alpha_1 \gamma}$$

$$\langle \alpha_1 \alpha_2 | v | \alpha_1 \alpha_2 \rangle \Big)$$

$$+ \quad \Big($$

$$\Big(\delta_{\alpha_1\alpha}\,\delta_{\alpha_2\beta} - \delta_{\delta\alpha_1}\,\delta_{\gamma\alpha_2}$$

$$- \langle \alpha_1\alpha_2 | v | \alpha_2\alpha_1 \rangle$$

$$a_{\alpha_2} a_{\alpha_1} a^\dagger_{\alpha} a^\dagger_{\beta} a_{\delta} a_{\gamma} a^\dagger_{\alpha_1} a^\dagger_{\alpha_2}$$

$$+ \Big(- \delta_{\alpha_2\alpha}\,\delta_{\alpha_1\beta} \quad \delta_{\delta\alpha_2}\,\delta_{\gamma\alpha_1}$$

$$- \langle \alpha_2\alpha_1 | v | \alpha_1\alpha_2 \rangle \Big)$$

$$+ \Big(\delta_{\alpha_2\alpha}\,\delta_{\alpha_1\beta} \quad \delta_{\delta\alpha_1}\,\delta_{\gamma\alpha_2}$$

$$\langle \alpha_2\alpha_1 | v | \alpha_2\alpha_1 \rangle \Big)$$

$$= \frac{1}{2}\Big[\langle \alpha_1\alpha_2 | v | \alpha_1\alpha_2 \rangle - \langle \alpha_2\alpha_1 | v | \alpha_1\alpha_2 \rangle$$

$$- \underbrace{\langle \alpha_1\alpha_2 | v | \alpha_2\alpha_1 \rangle}_{\alpha_2 \leftrightarrow \alpha_1}$$

$$+ \underbrace{\langle \alpha_2\alpha_1 | v | \alpha_2\alpha_1 \rangle}_{\alpha_2 \leftrightarrow \alpha_1} \Big]$$

$$v(1,2) = v(2,1)$$

$$= \langle \alpha_1\alpha_2 | v | \alpha_1\alpha_2 \rangle - \langle \alpha_2\alpha_1 | v | \alpha_1\alpha_2 \rangle$$

Proof of Wick's theorem

assume valid for N-operators, want it to be valid for $N+1$ operators.
Proof by induction.

Need a Lemma:

We have $N[xyz\ldots w]$
multiply with Ω

$$N[xyz\ldots w]\Omega =$$

$$N[xyz\ldots w\Omega] +$$

$$\sum_{(i)\Omega} N[xy\overline{z\ldots w\Omega}]$$

To show this note the following:

(i) $N[a_1^{\dagger}a_2^{\dagger}\ldots a_N^{\dagger}] = a_1^{\dagger}a_2^{\dagger}\ldots a_N^{\dagger}$

$N[a_1 a_2 \ldots a_N] = a_1 a_2 \ldots a_N$

(ii) The lemma is valid if

Ω is an annihilation operator

Since $N[XYZ \cdots W]\, a_Z$ is normal ordered

$$= (-1)^P N[a_1^{+} \cdots a_l^{+}]\, N[a_m \cdots a_N]\, a_Z$$

Second term

$$\underbrace{a_1^{+} a_2^{+} \cdots a_l^{+}}\; \underbrace{a_m a_n \cdots a_N}\; a_Z$$

$$a_i^{+} a_j = 0$$

(iii) if $N[XYZ \cdots W]$ is not normal ordered, then normal ordering it brings the product to the form

$$(-1)^P [a_1^{+} \cdots a_l^{+}]\, [a_m \cdots a_N]$$

(iv) if Ω is a creation operator we only need to prove the lemma if all $XYZ \cdots W$ are annihilation operator since $a_i^{+} a_j^{+} = 0$

(v) in (iv) we anticommute Ω through all $[xyz \ldots w]$

<u>Example</u>

$$N[a_1 a_2 \ldots a_k] a_l^{\dagger}$$
$$= N[a_1 a_2 \ldots a_k a_l^{\dagger}]$$
$$+ \sum_{(1)} N[a_1 a_2 \ldots a_k a_l^{\dagger}]$$

$k = 1$

$$N[a_1] a_l^{\dagger} = N[a_1 a_l^{\dagger}]$$
$$+ N[a_1 a_l^{\dagger}]$$
$$= -a_l^{\dagger} a_1 + \delta_{1l}$$

if we have for $k = N$, this is valid, want to show that it is valid for $k = N+1$

$$a_0 \times N[a_1 \ldots a_N] a_l^{\dagger}$$

$$= a_0 N[a_1 \ldots a_N a_l^{\dagger}]$$

$$+ \sum_{(i)} a_0 \, N[a_1 \cdots \overline{\cdots a_N a_e^{+}}]$$

first term:

$$a_0 \, a_1 \, a_2 \cdots a_N \, a_e^{+}$$

$$= N[a_0 a_1 a_2 \cdots a_N] \, a_e^{+}$$

2nd term

$$\sum_{(1)} a_0 \, N[a_1 \cdots \overline{\cdots a_N a_e^{+}}]$$

$$= \sum_{(1)} N[a_0 a_1 \cdots \overline{\cdots a_N a_e^{+}}]$$

$$a_0 \, N[a_1 \cdots a_N \, a_e^{+}]$$

$$= (-)^N \, \underline{a_0 a_e}^{+} N[a_1 \cdots a_N]$$

$$= (-1)^N \left[N[a_0 a_e^{+}] + \overline{a_0 a_e}^{+} \right] N[a_1 \cdots a_N]$$

$$N[a_0 a_e^{+}] = - N[a_e^{+} a_0]$$

$$\overline{a_0 a_e}^{+} N[a_1 \cdots a_N] =$$

$$(-)^N N[a_1 a_2 \cdots a_N a_e^{+}]$$

Bringing back into

$$a_0 N[a_1 \dots a_N a_e^+] =$$

$$= (-)^{N+1} N[a_e^+ a_0 \dots a_N]$$

$$+ (-)^{2N} N[a_0 a_1 \dots a_N a_e^+]$$

$$\Rightarrow$$

$$N[a_0 a_1 \dots a_N] a_e^+$$

$$= N[a_0 a_1 \dots a_e^+]$$

$$+ \sum_{(1)} N[a_0 a_1 \dots a_N a_e^+]$$

and concludes proof of the Lemma

Wick's theorem

assume that holds

$$XYZ \dots W$$

with Lemma we can show that it holds for

$$x \cdot y \cdot z \cdots w \cdot \Omega =$$

$$N[x y z \cdots w \Omega]$$

$$+ \sum_{(1)\Omega} N[x y z \overbrace{\cdots w \Omega}]$$

$$+ \sum_{(1)\neq\Omega} N[x y z \overbrace{\cdots w} \Omega]$$

$$+ \sum_{(2)\Omega} N[x \underbrace{y z \cdots \overbrace{w \Omega}}]$$

$$+ \sum_{(2)\neq\Omega} N[x \underbrace{y} z \overbrace{\cdots w} \Omega]$$

$$\vdots$$

$$+ \sum_{(\frac{N}{2})\Omega} N[\underbrace{x \overbrace{y z}} \underbrace{\cdots \overbrace{w \Omega}}]$$

$$+ \sum_{(\frac{N}{2})\neq\Omega} N[x \underbrace{y \overbrace{z \cdots w}} \Omega]$$

$$\sum_{(i)\Omega} + \sum_{(i)\neq\Omega} = \sum_{(i)}$$

$$= N[\underbrace{x} \underbrace{y z} \overbrace{\cdots \underbrace{w \Omega}}]$$