Heuristic Quadratization

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I. HEURISTIC GADGET TO REDUCE THE SIZE OF THE COEFFICIENTS

Consider the famous gadget of Rosenberg which has been known since 1975:

$$b_1b_2b_3 \rightarrow b_1b_a + b_2b_3 - 2b_2b_a - 2b_3b_a + 3b_a. \tag{1}$$

This works because the penalty term is 0 if and only if $b_a = b_2b_3$ which makes the RHS equal the LHS.

Now consider the gadget:

$$b_1b_2b_3 \rightarrow b_1b_a + b_2b_3 - b_2b_a - b_3b_a + b_a. \tag{2}$$

The coefficients are much smaller, sometimes double or even triple as small. This makes it much easier to compile onto D-Wave's strict coupling strength limitations. The gadget does not work in 100% of the cases, but the number of cases that fail is small, and often those cases are not found by the annealer anyway because other terms in the overall Hamiltonian cause those terms not to be favored.

II. HEURISTIC GADGET TO REDUCE THE NUMBER OF AUXILIARY VARIABLES

Say we want to quadratize a cubic without using any auxiliary variables:

$$b_1b_2b_3 \rightarrow \alpha_{12}b_1b_2 + \alpha_{13}b_1b_3 + \alpha_{23}b_2b_3 + \alpha_1b_1 + \alpha_2b_2 + \alpha_3b_3 + \alpha \tag{3}$$

$$= \begin{pmatrix} b_1b_2 & b_1b_3 & b_2b_3 & b_1 & b_2 & b_3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} & \alpha_{13} & \alpha_{23} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha \end{pmatrix}^T. \tag{4}$$

We then have 8 equations and 7 unknowns:

$$\begin{pmatrix} 0\times0\times0 \\ 0\times0\times1 \\ 0\times1\times0 \\ 0\times1\times1 \\ 1\times0\times0 \\ 1\times0\times1 \\ 1\times1\times0 \\ 1\times1\times1 \end{pmatrix} = \begin{pmatrix} 0\times0 & 0\times0 & 0\times0 & 0 & 0 & 0 & 1 \\ 0\times0 & 0\times1 & 0\times1 & 0 & 0 & 1 & 1 \\ 0\times1 & 0\times0 & 1\times0 & 0 & 1 & 0 & 1 \\ 0\times1 & 0\times1 & 1\times1 & 0 & 1 & 1 & 1 \\ 1\times0 & 1\times0 & 0\times0 & 1 & 0 & 0 & 1 \\ 1\times0 & 1\times1 & 0\times1 & 1 & 0 & 1 & 1 \\ 1\times1 & 1\times0 & 1\times0 & 1 & 1 & 0 & 1 \\ 1\times1 & 1\times1 & 1\times1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix} \tag{6}$$

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We then have the following equations:

$$\begin{array}{rcc} 0 & = & \alpha \\ 0 & = & \alpha_3 + \alpha \\ 0 & = & \alpha_2 + \alpha \\ 0 & = & \alpha_{23} + \alpha_2 + \alpha_3 + \alpha \\ 0 & = & \alpha_1 + \alpha \\ 0 & = & \alpha_{13} + \alpha_1 + \alpha_2 + \alpha \\ 0 & = & \alpha_{12} + \alpha_1 + \alpha_2 + \alpha \\ 1 & = & \alpha_{12} + \alpha_{13} + \alpha_{23} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha \end{array} . \tag{7}$$

Clearly there is no solution because the first 7 equations force all α coefficients to be 0, so the right-hand-side of the last equation cannot be satisfied. Satisfying the first 7 equations yields the function $f(b_1, b_2, b_3) = 0$, which has the right output in 7/8 possible cases, but the wrong behavior (by 1 unit of energy) for the case $(b_1, b_2, b_3) = (1, 1, 1)$. If an excited state (containing $b_1b_2b_3 = 1$) is dropped to the same energy as the ground state, then we would have to keep in mind that we can get a “false” ground state. All it means is that if the ground state turns out to have the assignment $(b_1, b_2, b_3) = (1, 1, 1)$, we should be mindful of this (for example we can look for excited states 1 energy higher), but we do not even need to worry about this if we can determine from other terms in the function that $(1, 1, 1)$ will be penalized enough to not creep into the ground state solution. If we instead relax one of the equations (for example, the constant term, since it seems this would have an immediate effect on the smallest number of parameters), we have a different set of equations, starting with:

$$-\alpha_3 = \alpha \tag{8}$$

$$-\alpha_2 = \alpha \tag{9}$$

$$-\alpha_1 = \alpha \tag{10}$$

which leads to:

$$\alpha_1 = \alpha_2 = \alpha_3 = -\alpha. \tag{11}$$

The next equations will be:

$$\alpha_{23} = \alpha \tag{12}$$

$$\alpha_{13} = \alpha \tag{13}$$

$$\alpha_{12} = \alpha \tag{14}$$

$$1 = \alpha. \tag{15}$$

The last equation tells us that $\alpha = 1$,so our heuristic quadratization is:

$$b_1b_2b_3 \to b_1b_2 + b_1b_3 + b_2b_3 - b_1 - b_2 - b_3 + 1, \tag{16}$$

and has the following behavior:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{17}$$

This means that the quadratization works except it does not preserve the energy for the case $(b_1, b_2, b_3) = (0, 0, 0)$, and we can also “rule out” (0,0,0) from the minimization process. In the integer factorization and Ramsey number problems, we know the minimum is supposed to be 0, so if we get a minimum of 0 and we have $(b_1, b_2, b_3) \neq (0, 0, 0)$ we do not have to worry at all (especially since in these problems we just need one solution, not every solution). In the unlikely case that we do not get the above, then we may wish to check $(b_1, b_2, b_3) = (0, 0, 0)$ manually somehow (such as by not quadratizing this term, and simply minimizing the function that we get by setting these variables to 0 and checking to see if it's higher or lower than the minimum achieved in the first attempt).

III. HEURISTIC GADGET TO IMPROVE OTHER THINGS (WHAT?)

IV. HEURISTIC VERSIONS OF QUTRIT TO QUBIT GADGETS

V. HEURISTIC EMBEDDING OF K_6 WITH ONE CHIMERA CELL INSTEAD OF THREE

It seems this can only be done by ignoring the two edges that require the extra cells. this corresponds to deleting two quadratic terms from the problem. How big of a consequence will this have on the problem? I don't know, but it seems that "removing quadratic terms" is the *only* way to design a heuristic gadget for minor-embeddings