Heuristic Quadratization

Nike Dattani*

Harvard-Smithsonian Center for Astrophysics

I. HEURISTIC GADGET TO REDUCE THE SIZE OF THE COEFFICIENTS

Consider the famous gadget of Rosenberg which has been known since 1975:

$$b_1b_2b_3 \to b_1b_a + b_2b_3 - 2b_2b_a - 2b_3b_a + 3b_a. \tag{1}$$

This works because the penalty term is 0 if and only if $b_a = b_2b_3$ which makes the RHS equal the LHS.

Now consider the gadget:

$$b_1b_2b_3 \to b_1b_a + b_2b_3 - b_2b_a - b_3b_a + b_a. \tag{2}$$

The coefficients are much smaller, sometimes double or even triple as small. This makes it much easier to compile onto D-Wave's strict coupling strength limitations. The gadget does not work in 100% of the cases, but the number of cases that fail is small, and often those cases are not found by the annealer anyway because other terms in the overall Hamiltonian cause those terms not to be favored.

II. HEURISTIC GADGET TO REDUCE THE NUMBER OF AUXILIARY VARIABLES

Say we want to quadratize a cubic without using any auxiliary variables:

$$\begin{align}
b_1b_2b_3 &\to \alpha_{12}b_1b_2 + \alpha_{13}b_1b_3 + \alpha_{23}b_2b_3 + \alpha_1b_1 + \alpha_2b_2 + \alpha_3b_3 + \alpha \tag{3}\\
&= \begin{pmatrix} b_1b_2 & b_1b_3 & b_2b_3 & b_1 & b_2 & b_3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} & \alpha_{13} & \alpha_{23} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha \end{pmatrix}^T . \tag{4}
\end{align}$$

We then have 8 equations and 7 unknowns:

$$\begin{pmatrix} 0\times0\times0 \\ 0\times0\times1 \\ 0\times1\times0 \\ 0\times1\times1 \\ 1\times0\times0 \\ 1\times0\times1 \\ 1\times1\times0 \\ 1\times1\times1 \end{pmatrix} = \begin{pmatrix} 0\times0 & 0\times0 & 0\times0 & 0 & 0 & 0 & 1 \\ 0\times0 & 0\times1 & 0\times1 & 0 & 0 & 1 & 1 \\ 0\times1 & 0\times0 & 1\times0 & 0 & 1 & 0 & 1 \\ 0\times1 & 0\times1 & 1\times1 & 0 & 1 & 1 & 1 \\ 1\times0 & 1\times0 & 0\times0 & 1 & 0 & 0 & 1 \\ 1\times0 & 1\times1 & 0\times1 & 1 & 0 & 1 & 1 \\ 1\times1 & 1\times0 & 1\times0 & 1 & 1 & 0 & 1 \\ 1\times1 & 1\times1 & 1\times1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix} \tag{6}$$

* n.dattani@cfa.harvard.edu

We then have the following equations:

$$
\begin{array}{rcc}
0 & = & \alpha \\
0 & = & \alpha_3 + \alpha \\
0 & = & \alpha_2 + \alpha \\
0 & = & \alpha_{23} + \alpha_2 + \alpha_3 + \alpha \\
0 & = & \alpha_1 + \alpha \\
0 & = & \alpha_{13} + \alpha_1 + \alpha_2 + \alpha \\
0 & = & \alpha_{12} + \alpha_1 + \alpha_2 + \alpha \\
1 & = & \alpha_{12} + \alpha_{13} + \alpha_{23} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha
\end{array}
. \tag{7}
$$

Clearly there is no solution because the first 7 equations force all α coefficients to be 0, so the right-hand-side of the last equation cannot be satisfied. Satisfying the first 7 equations yields the function $f(b_1, b_2, b_3) = 0$, which has the right output in 7/8 possible cases, but the wrong behavior (by 1 unit of energy) for the case $(b_1, b_2, b_3) = (1, 1, 1)$. If an excited state (containing $b_1 b_2 b_3 = 1$) is dropped to the same energy as the ground state, then we would have to keep in mind that we can get a "false" ground state. All it means is that if the ground state turns out to have the assignment $(b_1, b_2, b_3) = (1, 1, 1)$, we should be mindful of this (for example we can look for excited states 1 energy higher), but we do not even need to worry about this if we can determine from other terms in the function that $(1, 1, 1)$ will be penalized enough to not creep into the ground state solution. If we instead relax one of the equations (for example, the constant term, since it seems this would have an immediate effect on the smallest number of parameters), we have a different set of equations, starting with:

$$-\alpha_3 = \alpha \tag{8}$$
$$-\alpha_2 = \alpha \tag{9}$$
$$-\alpha_1 = \alpha \tag{10}$$

which leads to:

$$\alpha_1 = \alpha_2 = \alpha_3 = -\alpha. \tag{11}$$

The next equations will be:

$$\alpha_{23} = \alpha \tag{12}$$
$$\alpha_{13} = \alpha \tag{13}$$
$$\alpha_{12} = \alpha \tag{14}$$
$$1 = \alpha. \tag{15}$$

The last equation tells us that $\alpha = 1$, so our heuristic quadratization is:

$$b_1 b_2 b_3 \to b_1 b_2 + b_1 b_3 + b_2 b_3 - b_1 - b_2 - b_3 + 1, \tag{16}$$

and has the following behavior:

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{17}
$$

This means that the quadratization works except it does not preserve the energy for the case $(b_1, b_2, b_3) = (0, 0, 0)$, and we can also "rule out" (0,0,0) from the minimization process. In the integer factorization and Ramsey number problems, we know the minimum is supposed to be 0, so if we get a minimum of 0 and we have $(b_1, b_2, b_3) \neq (0, 0, 0)$ we do not have to worry at all (especially since in these problems we just need one solution, not every solution). In the unlikely case that we do not get the above, then we may wish to check $(b_1, b_2, b_3) = (0, 0, 0)$ manually somehow (such as by not quadratizing this term, and simply minimizing the function that we get by setting these variables to 0 and checking to see if it's higher or lower than the minimum achieved in the first attempt).

III. HEURISTIC GADGET TO IMPROVE OTHER THINGS

The best non-heuristic PTR gadget for cubics uses 1 auxiliary and has 5 quadratic terms (3 of them being sub-modular), 6 total terms, 1 triangle in the gadget graph, and coefficients that range from -1 to 1. Is it possible to do this with the same number of auxiliaries but fewer quadratic terms, or fewer non-sub-modular terms, or fewer triangles, or with a smaller range for the coefficients? Let's try:

$$b_1b_2b_3 = \alpha_{12}b_1b_2 + \alpha_{13}b_1b_3 + \alpha_{1a}b_1b_a + \alpha_{23}b_2b_3 + \alpha_{2a}b_2b_a + \alpha_{3a}b_3b_a + \alpha_1b_1 + \alpha_2b_2 + \alpha_3b_3 + \alpha_ab_a + \alpha \tag{18}$$

$$b_1b_2b_31_a = \begin{pmatrix} b_1b_2 & b_1b_3 & b_1b_a & b_2b_3 & b_2b_a & b_3b_a & b_1 & b_2 & b_3 & b_a & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} & \alpha_{13} & \alpha_{1a} & \alpha_{23} & \alpha_{2a} & \alpha_{3a} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_a & \alpha \end{pmatrix}^T . \tag{19}$$

We then have 16 equations and 11 unknowns:

$$\begin{pmatrix} 0\times0\times0\times1 \\ 0\times0\times0\times1 \\ 0\times0\times1\times1 \\ 0\times0\times1\times1 \\ 0\times1\times0\times1 \\ 0\times1\times0\times1 \\ 0\times1\times1\times1 \\ 0\times1\times1\times1 \\ 1\times0\times0\times1 \\ 1\times0\times0\times1 \\ 1\times0\times1\times1 \\ 1\times0\times1\times1 \\ 1\times1\times0\times1 \\ 1\times1\times0\times1 \\ 1\times1\times1\times1 \\ 1\times1\times1\times1 \end{pmatrix} = \begin{pmatrix} 0\times0 & 0\times0 & 0\times0 & 0\times0 & 0\times0 & 0\times0 & 0 & 0 & 0 & 0 & 1 \\ 0\times0 & 0\times0 & 0\times1 & 0\times0 & 0\times1 & 0\times1 & 0 & 0 & 0 & 1 & 1 \\ 0\times0 & 0\times1 & 0\times0 & 0\times1 & 0\times0 & 1\times0 & 0 & 0 & 1 & 0 & 1 \\ 0\times0 & 0\times1 & 0\times1 & 0\times1 & 0\times1 & 1\times1 & 0 & 0 & 1 & 1 & 1 \\ 0\times1 & 0\times0 & 0\times0 & 1\times0 & 1\times0 & 0\times0 & 0 & 1 & 0 & 0 & 1 \\ 0\times1 & 0\times0 & 0\times1 & 1\times0 & 1\times1 & 0\times1 & 0 & 1 & 0 & 1 & 1 \\ 0\times1 & 0\times1 & 0\times0 & 1\times1 & 1\times0 & 1\times0 & 0 & 1 & 1 & 0 & 1 \\ 0\times1 & 0\times1 & 0\times1 & 1\times1 & 1\times1 & 1\times1 & 0 & 1 & 1 & 1 & 1 \\ 1\times0 & 1\times0 & 1\times0 & 0\times0 & 0\times0 & 0\times0 & 1 & 0 & 0 & 0 & 1 \\ 1\times0 & 1\times0 & 1\times1 & 0\times0 & 0\times1 & 0\times1 & 1 & 0 & 0 & 1 & 1 \\ 1\times0 & 1\times1 & 1\times0 & 0\times1 & 0\times0 & 1\times0 & 1 & 0 & 1 & 0 & 1 \\ 1\times0 & 1\times1 & 1\times1 & 0\times1 & 0\times1 & 1\times1 & 1 & 0 & 1 & 1 & 1 \\ 1\times1 & 1\times0 & 1\times0 & 1\times0 & 1\times0 & 0\times0 & 1 & 1 & 0 & 0 & 1 \\ 1\times1 & 1\times0 & 1\times1 & 1\times0 & 1\times1 & 0\times1 & 1 & 1 & 0 & 1 & 1 \\ 1\times1 & 1\times1 & 1\times0 & 1\times1 & 1\times0 & 1\times0 & 1 & 1 & 1 & 0 & 1 \\ 1\times1 & 1\times1 & 1\times1 & 1\times1 & 1\times1 & 1\times1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{1a} \\ \alpha_{23} \\ \alpha_{2a} \\ \alpha_{3a} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_a \\ \alpha \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{1a} \\ \alpha_{23} \\ \alpha_{2a} \\ \alpha_{3a} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_a \\ \alpha \end{pmatrix} . \tag{21}$$

We then have the following equations:

$$0 = \alpha \tag{22}$$
$$0 = \alpha_a + \alpha \tag{23}$$
$$0 = \alpha_3 + \alpha \tag{24}$$
$$0 = \alpha_{3a} + \alpha_3 + \alpha_a + \alpha \tag{25}$$
$$0 = \alpha_2 + \alpha \tag{26}$$
$$0 = \alpha_{2a} + \alpha_2 + \alpha_a + \alpha \tag{27}$$
$$0 = \alpha_{23} + \alpha_2 + \alpha_3 + \alpha \tag{28}$$
$$0 = \alpha_{23} + \alpha_{2a} + \alpha_{3a} + \alpha_2 + \alpha_3 + \alpha_a + \alpha \tag{29}$$
$$0 = \alpha_1 + \alpha \tag{30}$$
$$0 = \alpha_{1a} + \alpha_1 + \alpha_a + \alpha \tag{31}$$
$$0 = \alpha_{13} + \alpha_1 + \alpha_3 + \alpha \tag{32}$$
$$0 = \alpha_{13} + \alpha_{1a} + \alpha_{3a} + \alpha_1 + \alpha_3 + \alpha_a + \alpha \tag{33}$$
$$0 = \alpha_{12} + \alpha_1 + \alpha_2 + \alpha \tag{34}$$
$$0 = \alpha_{12} + \alpha_{1a} + \alpha_{2a} + \alpha_1 + \alpha_2 + \alpha_a + \alpha \tag{35}$$
$$1 = \alpha_{12} + \alpha_{13} + \alpha_{23} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha \tag{36}$$
$$1 = \alpha_{12} + \alpha_{13} + \alpha_{1a} + \alpha_{23} + \alpha_{2a} + \alpha_{3a} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_a + \alpha. \tag{37}$$

But we only care about the subspace in which we have minimized over b_a, this means it is really 8 (not necessarily linear) equations and 11 unknowns:

$$0 = \min(\alpha, \alpha_a + \alpha) \tag{38}$$
$$0 = \min(\alpha_3 + \alpha, \alpha_{3a} + \alpha_3 + \alpha_a + \alpha) \tag{39}$$
$$0 = \min(\alpha_2 + \alpha, \alpha_{2a} + \alpha_2 + \alpha_a + \alpha) \tag{40}$$
$$0 = \min(\alpha_{23} + \alpha_2 + \alpha_3 + \alpha, \alpha_{23} + \alpha_{2a} + \alpha_{3a} + \alpha_2 + \alpha_3 + \alpha_a + \alpha) \tag{41}$$
$$0 = \min(\alpha_1 + \alpha, \alpha_{1a} + \alpha_1 + \alpha_a + \alpha) \tag{42}$$
$$0 = \min(\alpha_{13} + \alpha_1 + \alpha_3 + \alpha, \alpha_{13} + \alpha_{1a} + \alpha_{3a} + \alpha_1 + \alpha_3 + \alpha_a + \alpha) \tag{43}$$
$$0 = \min(\alpha_{12} + \alpha_1 + \alpha_2 + \alpha, \alpha_{12} + \alpha_{1a} + \alpha_{2a} + \alpha_1 + \alpha_2 + \alpha_a + \alpha) \tag{44}$$
$$1 = \min(\alpha_{12} + \alpha_{13} + \alpha_{23} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha, \alpha_{12} + \alpha_{13} + \alpha_{1a} + \alpha_{23} + \alpha_{2a} + \alpha_{3a} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_a + \alpha) \tag{45}$$

A. Removing the triangle

Simply removing one of the edges of the triangle would not do extreme damage, but we can do better by choosing the other coefficients optimally to make the gadget reproduce the desired behavior in as many cases as possible. The most general ansatz for a degree-4 gadget without a triangle is:

$$b_1b_2b_3 = \alpha_{12}b_1b_2 + \alpha_{13}b_1b_3 + \alpha_{2a}b_2b_a + \alpha_{3a}b_3b_a + \alpha_1b_1 + \alpha_2b_2 + \alpha_3b_3 + \alpha_ab_a + \alpha \tag{46}$$

We now have only 9 unknowns and still 8 (not necessarily linear) equations:

$$0 = \min(\alpha, \alpha_a + \alpha) \tag{47}$$
$$0 = \min(\alpha_3 + \alpha, \alpha_{3a} + \alpha_3 + \alpha_a + \alpha) \tag{48}$$
$$0 = \min(\alpha_2 + \alpha, \alpha_{2a} + \alpha_2 + \alpha_a + \alpha) \tag{49}$$
$$0 = \min(\alpha_{23} + \alpha_2 + \alpha_3 + \alpha, \alpha_{2a} + \alpha_{3a} + \alpha_2 + \alpha_3 + \alpha_a + \alpha) \tag{50}$$
$$0 = \min(\alpha_1 + \alpha, \alpha_1 + \alpha_a + \alpha) \tag{51}$$
$$0 = \min(\alpha_{13} + \alpha_1 + \alpha_3 + \alpha, \alpha_{13} + \alpha_{3a} + \alpha_1 + \alpha_3 + \alpha_a + \alpha) \tag{52}$$
$$0 = \min(\alpha_{12} + \alpha_1 + \alpha_2 + \alpha, \alpha_{12} + \alpha_{2a} + \alpha_1 + \alpha_2 + \alpha_a + \alpha) \tag{53}$$
$$1 = \min(\alpha_{12} + \alpha_{13} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha, \alpha_{12} + \alpha_{13} + \alpha_{2a} + \alpha_{3a} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_a + \alpha) \tag{54}$$

1. Attempt 1: Solve a system of 16 inequalities

These equations tell us the following, which might help, but don't give a complete description of the problem, because for example the first inequality does not need to be satisfied, if for example $\alpha = -\alpha_a$:

$$\alpha \geq 0 \tag{55}$$
$$\alpha_a + \alpha \geq 0 \tag{56}$$
$$\alpha_3 + \alpha \geq 0 \tag{57}$$
$$\alpha_{3a} + \alpha_3 + \alpha_a + \alpha \geq 0 \tag{58}$$
$$\alpha_2 + \alpha \geq 0 \tag{59}$$
$$\alpha_{2a} + \alpha_2 + \alpha_a + \alpha \geq 0 \tag{60}$$
$$\alpha_{23} + +\alpha_2 + \alpha_3 + \alpha \geq 0 \tag{61}$$
$$\alpha_{2a} + \alpha_{3a} + \alpha_2 + \alpha_3 + \alpha_a + \alpha \geq 0 \tag{62}$$
$$\alpha_1 + \alpha \geq 0 \tag{63}$$
$$\alpha_1 + \alpha_a + \alpha \geq 0 \tag{64}$$
$$\alpha_{13} + \alpha_1 + \alpha_3 + \alpha \geq 0 \tag{65}$$
$$\alpha_{13} + \alpha_{3a} + \alpha_1 + \alpha_3 + \alpha_a + \alpha \geq 0 \tag{66}$$
$$\alpha_{12} + \alpha_1 + \alpha_2 + \alpha \geq 0 \tag{67}$$
$$\alpha_{12} + \alpha_{2a} + \alpha_1 + \alpha_2 + \alpha_a + \alpha \geq 0 \tag{68}$$
$$\alpha_{12} + \alpha_{13} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha \geq 1 \tag{69}$$
$$\alpha_{12} + \alpha_{13} + \alpha_{2a} + \alpha_{3a} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_a + \alpha \geq 1 \tag{70}$$

2. Attempt 2: Set $b_a = 0$ or 1:

Alternatively we can just guess whether $b_a = 0$ or $b_a = 1$ minimizes the quadratic function. If it is $b_a = 0$, we get:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix}. \tag{71}$$

The least-squares solution is:

$$\begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ 1/8 \end{pmatrix} \tag{72}$$

Multiplying this solution by 8 and adding 1, we get:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 2 \\ 8 \end{pmatrix}. \tag{73}$$

We still have a decent sized gap between the low-lying states and the single maximum, and the states in the low-lying energy space and the high-lying space are correct, however 4/7 of the states which in theory could constitute part of the global ground state, have been raised in energy by 2 units. If the true global minimum is at $(b_1, b_2, b_3) = (0, 0, 1) = 001$ or 010 or 110, we are not only happy that the annealer will find the right state, but we have also pushed the excited states up (in one case by a factor of 8), making it easier for the annealer to find the ground state.

Let's try $b_a = 1$:

$$\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&1&1\\0&0&0&0&1&0&1\\0&0&0&1&1&1&1\\0&0&1&0&0&0&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\\1&1&1&1&1&1&1\end{pmatrix} \begin{pmatrix}\alpha_{12}\\\alpha_{13}\\(\alpha_{1a}+\alpha_1)/2\\\alpha_{23}\\(\alpha_{2a}+\alpha_2)/2\\(\alpha_{3a}+\alpha_3)/2\\\alpha+\alpha_a\end{pmatrix}. \tag{74}$$

The least-squares solution is:

$$\begin{pmatrix}\alpha_{12}\\\alpha_{13}\\(\alpha_{1a}+\alpha_1)/2\\\alpha_{23}\\(\alpha_{2a}+\alpha_2)/2\\(\alpha_{3a}+\alpha_3)/2\\\alpha+\alpha_a\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1/4\\1/2\\-1/4\\-1/4\\1/8\end{pmatrix} \tag{75}$$

Multiplying this solution by 8 and adding 1, we get:

$$\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&1&1\\0&0&0&0&1&0&1\\0&0&1&0&1&1&1\\0&0&0&1&0&0&1\\0&1&0&1&0&1&1\\1&0&0&1&1&0&1\\1&1&1&1&1&1&1\end{pmatrix} \begin{pmatrix}4\\4\\-2\\4\\-2\\-2\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\0\\2\\0\\2\\2\\8\end{pmatrix}. \tag{76}$$

This is the same as what we got for the case $b_a = 0$, probably because we "eliminated" b_a out of the process, which was the wrong thing to do.

3. *Attempt 3: Least-squares solution of the full system:*

Let's try to combine the two cases for b_a instead of treating them separately as we tried in "Attempt 2". First we will try without the restriction that there's no triangles (note that the second last element in the vector on the left should be 1 but I made a typo while doing this and it still worked out well it seems):

$$\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0&1&0&1&1\\ 0&0&0&1&0&0&0&1&1&0&1\\ 0&0&0&1&1&1&0&1&1&1&1\\ 0&0&0&0&0&0&1&0&0&0&1\\ 0&0&1&0&0&0&1&0&0&1&1\\ 0&1&0&0&0&0&1&0&1&0&1\\ 0&1&1&0&0&1&1&0&1&1&1\\ 1&0&0&0&0&0&1&1&0&0&1\\ 1&0&1&0&1&0&1&1&0&1&1\\ 1&1&0&1&0&0&1&1&1&0&1\\ 1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix} \begin{pmatrix} \alpha_{12}\\ \alpha_{13}\\ \alpha_{1a}\\ \alpha_{23}\\ \alpha_{2a}\\ \alpha_{3a}\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_a\\ \alpha \end{pmatrix} \tag{77}$$

The least-squares solution is:

$$\begin{pmatrix} \alpha_{12}\\ \alpha_{13}\\ \alpha_{1a}\\ \alpha_{23}\\ \alpha_{2a}\\ \alpha_{3a}\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_a\\ \alpha \end{pmatrix} = \begin{pmatrix} 1/4\\ 1/4\\ 1/4\\ 1/4\\ 1/4\\ 1/4\\ -1/4\\ -1/4\\ -1/4\\ -1/4\\ 3/16 \end{pmatrix}. \tag{78}$$

Multiplying this solution by 4 and adjusting the constant term we get:

$$\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1&0&1\\0&0&0&0&0&1&0&0&1&1&1\\0&0&0&0&0&0&0&1&0&0&1\\0&0&0&0&1&0&0&1&0&1&1\\0&0&0&1&0&0&0&1&1&0&1\\0&0&0&1&1&1&0&1&1&1&1\\0&0&0&0&0&0&1&0&0&0&1\\0&0&1&0&0&0&1&0&0&1&1\\0&1&0&0&0&0&1&0&1&0&1\\0&1&1&0&0&1&1&0&1&1&1\\1&0&0&0&0&0&1&1&0&0&1\\1&0&1&0&1&0&1&1&0&1&1\\1&1&0&1&0&0&1&1&1&0&1\\1&1&1&1&1&1&1&1&1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\\-1\\-1\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\1\\0\\1\\1\\3\end{pmatrix} \tag{79}$$

$$\min_{b_a}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\1\\0\\1\\1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}. \tag{80}$$

Now let's remove b_{1a} and b_{23} so that there's no triangles:

$$\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&0&1\\0&0&0&1&0&0&1&1&1\\0&0&0&0&0&1&0&0&1\\0&0&1&0&0&1&0&1&1\\0&0&0&0&0&1&1&0&1\\0&0&1&1&0&1&1&1&1\\0&0&0&0&1&0&0&0&1\\0&0&0&0&1&0&0&1&1\\0&1&0&0&1&0&1&0&1\\0&1&0&1&1&0&1&1&1\\1&0&0&0&1&1&0&0&1\\1&0&1&0&1&1&0&1&1\\1&1&0&0&1&1&1&0&1\\1&1&1&1&1&1&1&1&1\end{pmatrix}\begin{pmatrix}\alpha_{12}\\\alpha_{13}\\\alpha_{2a}\\\alpha_{3a}\\\alpha_1\\\alpha_2\\\alpha_3\\\alpha_a\\\alpha\end{pmatrix}. \tag{81}$$

The least-squares solution is:

$$\begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{2a} \\ \alpha_{3a} \\ \alpha_{1} \\ \alpha_{2} \\ \alpha_{3} \\ \alpha_{a} \\ \alpha \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \\ -1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{82}$$

This means we have the very simple heuristic quadratization (when multiplying by 4 and adding 1):

$$b_1b_2b_3 \rightarrow 2b_1b_2 + 2b_1b_3 - b_1 + 1. \tag{83}$$

It's output is as follows:

$$000 \rightarrow 1 \tag{84}$$
$$001 \rightarrow 1 \tag{85}$$
$$010 \rightarrow 1 \tag{86}$$
$$011 \rightarrow 1 \tag{87}$$
$$100 \rightarrow 0 \tag{88}$$
$$101 \rightarrow 2 \tag{89}$$
$$110 \rightarrow 2 \tag{90}$$
$$111 \rightarrow 4 \tag{91}$$

Still the maximum of $b_1b_2b_3$ is the maximum attained by the quadratic function but we better hope the global minimum has 100 because that's what the annealer will find if it works perfectly. In reality it's likely that the annealer won't find the global minimum anyway, so any one of the low-lying states seems to be at least twice as likely to achieve than the maximum at 111 which we want to avoid.

IV. HEURISTIC GADGET TO QUADRATIZE MULTIPLE CUBIC TERMS WITH 1 AUXILIARY

V. HEURISTIC VERSIONS OF QUTRIT TO QUBIT GADGETS

VI. HEURISTIC EMBEDDING OF K_6 WITH ONE CHIMERA CELL INSTEAD OF THREE

It seems this can only be done by ignoring the two edges that require the extra cells. this corresponds to deleting two quadratic terms from the problem. How big of a consequence will this have on the problem? I don't know, but it seems that "removing quadratic terms" is the *only* way to design a heuristic gadget for minor-embeddings