

Volume 3: List of Multi-run Quadratizations

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DECOMPOSITION OF A MONOMIAL

$$b_1 b_2 b_3 \dots b_k = \min (b_1 b_2 \dots b_{k_1}, b_{k_1+1} b_{k_1+2} \dots b_{k_2}, b_{k_2+1} b_{k_2+2} \dots b_{k_3}, \dots, b_{k_n+1} b_{k_n+2} \dots b_k) \quad (1)$$

$$b_1 b_2 b_3 \dots b_k = \min (b_1, b_2, b_3, \dots, b_k) \quad (\text{Example of Eq. 1: Linearization of a degree-}k \text{ monomial}). \quad (2)$$

$$b_1 b_2 b_3 b_4 = \min (b_1 b_2, b_3 b_4) \quad (\text{Example of Eq. 1: Quadratization of a degree-4 monomial}). \quad (3)$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 : \quad (4)$$

$$\longrightarrow 3b_a + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \quad (5)$$

$$\longrightarrow 3b_a + b_5 b_6 + b_5 b_7 + b_5 b_8 + b_6 b_7 + b_6 b_8 + b_7 b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \quad (6)$$

DECOMPOSITION OF BINOMIALS OF DEGREE- k TERMS

$$b_1 b_2 b_3 b_4 + b_3 b_4 b_5 b_6 = \min (2b_3 b_4, b_1 b_2 + b_5 b_6) \quad (k, n) = (4, 6). \quad (7)$$

$$b_1 b_2 b_3 b_4 + b_3 b_4 b_5 b_6 = \min_{b_a} (b_2 b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3 b_4, b_1 b_2 + b_5 b_6 + b_5 b_a) \quad (k, n) = (4, 6). \quad (8)$$

$$b_1 b_2 b_3 b_4 + b_4 b_5 b_6 b_7 : \quad (k, n) = (4, 7). \quad (9)$$

$$\longrightarrow b_2 b_3 + b_5 b_6 + b_a(1 - b_5 - b_6 + b_7) \quad (10)$$

$$\longrightarrow b_1 b_4 + b_4 + b_a \quad (11)$$

$$\longrightarrow b_5 b_6 + b_1 + b_a(1 - b_5 - b_6 + b_7) \quad (12)$$

$$b_1 b_2 b_3 b_4 + b_4 b_5 b_6 b_7 : \quad (k, n) = (4, 7). \quad (13)$$

$$\longrightarrow b_3 b_4 + b_4 b_6 \quad 89/128 \text{ (70\%)} \quad (14)$$

$$\longrightarrow b_1 b_2 + b_6 b_7 \quad 118/128 \text{ (92\%)} \quad (15)$$

$$\longrightarrow b_2 b_3 - b_5 b_6 + b_5 b_7 + b_5 \quad 127/128 \text{ (99\%)} \quad (16)$$

$$\longrightarrow b_1 b_4 + 2b_5 - b_7 + 1 \quad 128/128(100\%) \quad (17)$$

$$b_1 b_2 b_3 b_4 b_5 + b_3 b_4 b_5 b_6 b_7 : \quad (k, n) = (5, 7). \quad (18)$$

$$\longrightarrow b_2 b_5 + b_5 b_6 + b_5 b_7 + b_6 b_7 + b_a(b_5 + b_6 + b_7 - 1) - b_5 - b_6 - b_7 + 1 \quad (19)$$

$$\longrightarrow b_1 b_3 + b_3 b_7 + b_a(1 + b_5 - b_7) - b_5 + 1 \quad (20)$$

$$\longrightarrow b_1 b_4 + b_4 b_6 - b_5 b_6 + b_5 b_a - b_5 + b_6 + 1 \quad (21)$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_3 b_4 b_5 b_6 b_7 : & (k, n) = (5, 7). \quad (22) \\
& \longrightarrow b_2 b_3 + b_3 b_7 & 85/128 \quad (66\%) \quad (23) \\
& \longrightarrow 2b_4 b_5 & 121/128 \quad (95\%) \quad (24) \\
& \longrightarrow b_1 b_2 + b_6 b_7 - b_5 + 1 & 128/128(100\%) \quad (25)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 : & (k, n) = (6, 7). \quad (26) \\
& \longrightarrow 2b_3 b_6 & (27) \\
& \longrightarrow 2b_4 b_5 - b_5 b_6 + b_5 & (28) \\
& \longrightarrow b_1 b_4 - b_2 b_5 + b_2 b_6 + b_2 b_7 + b_5 b_7 - b_6 b_7 - b_5 - b_6 + 2 & (29) \\
& \longrightarrow b_1 b_2 - b_1 b_5 + b_1 b_7 + b_2 b_3 + b_3 b_6 - b_3 b_7 - b_4 b_5 - b_5 b_6 - b_3 + b_5 + 2 & (30)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 : & (k, n) = (6, 7). \quad (31) \\
& \longrightarrow 2b_5 b_6 & 97/128 \quad (76\%) \quad (32) \\
& \longrightarrow b_1 b_4 + b_4 b_7 & 119/128 \quad (93\%) \quad (33) \\
& \longrightarrow b_1 b_3 + b_1 b_7 + b_2 b_3 - b_3 b_6 + b_3 b_7 - b_4 b_5 - b_1 - b_7 + 2 & 127/128 \quad (99\%) \quad (34) \\
& \longrightarrow b_1 b_2 + b_2 b_6 & 128/128(100\%) \quad (35)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 : & (k, n) = (6, 7). \quad (36) \\
& \longrightarrow b_5 b_6 + b_5 b_7 - b_5 b_8 + b_6 b_8 - b_7 b_8 + b_8 & (37) \\
& \longrightarrow b_1 b_4 + b_a(b_4 - b_7) + b_7 & (38) \\
& \longrightarrow b_2 b_3 + b_2 b_7 - b_5 b_6 - b_7 b_a + b_5 + b_7 & (39) \\
& \longrightarrow b_2 b_3 + b_7 b_a + b_3 & (40)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_4 b_5 b_6 b_7 b_8 : & (k, n) = (5, 8). \quad (41) \\
& \longrightarrow b_1 b_2 + b_6 b_8 + b_a(1 - b_6 + b_7 - b_8) & (42) \\
& \longrightarrow b_3 b_5 + b_8 b_a + b_5 & (43) \\
& \longrightarrow b_4 b_7 + b_4 & (44) \\
& \longrightarrow b_7 b_8 + b_a(1 + b_6 - b_7 - b_8) + b_3 & (45)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_4 b_5 b_6 b_7 b_8 : & (k, n) = (5, 8). \quad (46) \\
& \longrightarrow b_2 b_5 + b_5 b_8 & 169/256 \quad (66\%) \quad (47) \\
& \longrightarrow b_1 b_4 + b_4 b_7 - b_5 b_8 + b_8 & 233/256 \quad (91\%) \quad (48) \\
& \longrightarrow b_1 b_3 + b_6 b_7 + b_6 b_8 + b_7 b_8 - b_6 - b_7 - b_8 + 1 & 252/256 \quad (98\%) \quad (49) \\
& \longrightarrow b_2 b_3 + b_6 b_7 & 256/256(100\%) \quad (50)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 : & (k, n) = (6, 8). \quad (51) \\
& \longrightarrow b_1 b_6 + b_6 b_8 + b_a(1 - b_6 + b_7 - b_8) & (52) \\
& \longrightarrow b_2 b_5 + b_4 b_5 + b_4 b_a & (53) \\
& \longrightarrow b_3 b_4 + b_3 b_7 - b_a + 1 & (54) \\
& \longrightarrow b_2 b_4 + b_7 b_8 & (55) \\
& \longrightarrow b_3 b_4 + b_4 & (56)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 : & (k, n) = (6, 8). \quad (57) \\
& \longrightarrow 2b_5 b_6 & 193/256 \quad (75\%) \quad (58) \\
& \longrightarrow b_1 b_4 + b_4 b_8 & 237/256 \quad (93\%) \quad (59) \\
& \longrightarrow b_2 b_3 + b_3 b_7 - b_4 b_6 + b_4 b_8 - b_5 b_7 - b_5 b_8 + b_6 b_8 - b_6 + b_7 - b_8 + 2 & 254/256 \quad (99\%) \quad (60) \\
& \longrightarrow b_1 b_2 + b_7 b_8 & 256/256(100\%) \quad (61)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 : & (k, n) = (7, 8). \quad (62) \\
& \longrightarrow b_6 b_7 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) & (63) \\
& \longrightarrow b_2 b_3 + b_3 b_4 & (64) \\
& \longrightarrow b_1 b_4 + b_4 b_8 - b_6 b_a + b_6 & (65) \\
& \longrightarrow b_2 b_3 + b_2 & (66) \\
& \longrightarrow b_1 b_5 + b_3 b_5 + b_6 b_a & (67)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 : & (k, n) = (7, 8). \quad (68) \\
& \longrightarrow 2b_5 b_6 & 193/256 \quad (75\%) \quad (69) \\
& \longrightarrow b_1 b_4 + b_4 b_8 & 235/256 \quad (92\%) \quad (70) \\
& \longrightarrow b_2 b_3 + b_2 b_7 - b_5 b_6 + b_6 b_8 + b_5 - b_6 - b_8 + 1 & 250/256 \quad (98\%) \quad (71) \\
& \longrightarrow b_3 b_7 + b_7 b_8 & 254/256 \quad (99\%) \quad (72) \\
& \longrightarrow b_3 b_8 + b_3 & 256/256(100\%) \quad (73)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + b_5 b_6 b_7 b_8 : & (k, n) = (4, 8). \quad (74) \\
& \longrightarrow b_1 b_2 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) & (75) \\
& \longrightarrow b_3 b_4 + b_6 b_8 + 2b_8 b_a & (76) \\
& \longrightarrow b_2 b_3 + b_5 b_7 + b_a (1 - b_6 + b_7) & (77) \\
& \longrightarrow b_1 b_4 + b_5 b_7 - b_6 b_8 + b_7 b_a + b_6 & (78)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + b_5 b_6 b_7 b_8 : & (k, n) = (4, 8). \quad (79) \\
& \longrightarrow b_1 b_2 + b_6 b_7 & 169/256 \quad (66\%) \quad (80) \\
& \longrightarrow b_3 b_4 + b_5 b_8 & 238/256 \quad (93\%) \quad (81) \\
& \longrightarrow b_1 b_4 + b_5 b_6 + b_5 b_7 + b_6 b_7 - b_5 - b_6 - b_7 + 1 & 248/256 \quad (97\%) \quad (82) \\
& \longrightarrow b_2 b_3 + b_6 b_7 + b_6 b_8 + b_7 b_8 - b_6 - b_7 - b_8 + 1 & 254/256 \quad (99\%) \quad (83) \\
& \longrightarrow b_1 b_2 + b_5 b_8 & 256/256(100\%) \quad (84)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (5, 10). \quad (85) \\
& \longrightarrow b_2 b_3 + b_6 b_9 + b_9 b_a & (86) \\
& \longrightarrow b_1 b_4 + b_8 b_{10} + b_9 b_a & (87) \\
& \longrightarrow b_3 b_5 + b_7 b_{10} + b_1 b_a + b_9 b_a & (88) \\
& \longrightarrow b_4 b_5 + b_6 b_9 + b_9 b_a & (89) \\
& \longrightarrow b_1 b_2 + b_7 b_9 + b_9 b_a & (90) \\
& \longrightarrow b_2 b_5 + b_6 b_8 + b_9 b_a & (91) \\
& \longrightarrow b_2 b_3 + b_8 b_{10} + b_9 b_a & (92) \\
& \longrightarrow b_1 b_4 + b_6 b_7 + b_9 b_a & (93)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (5, 10). & (94) \\
\longrightarrow & b_1 b_3 + b_9 b_{10} & 625/1024 \text{ (61\%)} & (95) \\
\longrightarrow & b_2 b_4 + b_7 b_{10} & 851/1024 \text{ (83\%)} & (96) \\
\longrightarrow & b_3 b_5 + b_5 b_{10} + b_8 b_9 & 924/1024 \text{ (90\%)} & (97) \\
\longrightarrow & b_1 b_2 + b_6 & 972/1024 \text{ (95\%)} & (98) \\
\longrightarrow & b_3 b_4 + b_8 b_9 & 997/1024 \text{ (97\%)} & (99) \\
\longrightarrow & b_1 b_5 + b_7 b_{10} & 1010/1024 \text{ (99\%)} & (100) \\
\longrightarrow & -b_1 b_7 - b_1 b_{10} + b_2 b_3 - b_2 b_8 - b_2 b_{10} + b_3 b_5 & & (101) \\
& + b_6 b_9 + b_7 b_{10} - b_8 b_9 + b_9 b_{10} - b_3 - b_7 + b_8 + 3 & 1016/1024 \text{ (99\%)} & (102) \\
\longrightarrow & b_1 b_3 + b_7 b_8 & 1020/1024 \text{ (99\%)} & (103) \\
\longrightarrow & b_2 b_4 + b_2 b_6 - b_2 b_9 - b_3 b_{10} - b_5 b_7 + b_7 b_{10} + b_9 b_{10} - b_{10} + 2 & 1023/1024 \text{ (99\%)} & (104) \\
\longrightarrow & b_2 b_5 + b_2 b_9 + b_6 b_8 & 1024/1024 \text{ (100\%)} & (105)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). & (106) \\
\longrightarrow & b_5 b_6 + b_5 b_7 + b_a(1 - b_{10}) & & (107) \\
\longrightarrow & b_1 b_2 + b_8 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} & & (108) \\
\longrightarrow & b_3 b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10} & & (109) \\
\longrightarrow & b_3 b_6 - b_5 b_{10} + b_6 b_7 + b_a(1 - b_9) + b_{10} & & (110) \\
\longrightarrow & b_1 b_2 + b_7 b_{10} + b_a(1 - b_9 - b_{10}) + b_{10} & & (111) \\
\longrightarrow & b_3 b_4 + b_8 b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1 & & (112)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). & (113) \\
\longrightarrow & 2b_5 b_6 & 769/1024 \text{ (75\%)} & (114) \\
\longrightarrow & b_1 b_3 + b_8 b_9 & 934/1024 \text{ (92\%)} & (115) \\
\longrightarrow & b_2 b_4 + b_7 b_{10} + b_8 b_9 - b_8 - b_9 + 1 & 997/1024 \text{ (97\%)} & (116) \\
\longrightarrow & -b_1 b_3 + b_1 b_9 + b_2 b_4 + b_4 b_9 + b_5 b_8 + b_8 b_9 - b_5 - b_8 - b_9 + 2 & 769/1024 \text{ (99\%)} & (117) \\
\longrightarrow & b_1 b_3 + b_7 b_{10} - b_8 - b_9 + 2 & 1014/1024 \text{ (99\%)} & (118) \\
\longrightarrow & b_2 b_3 + b_8 b_9 & 1024/1024 \text{ (100\%)} & (119)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (7, 10). & (120) \\
\longrightarrow & b_4 b_7 + b_6 b_7 + b_a(1 - b_4 - b_7 + b_{10}) & & (121) \\
\longrightarrow & b_2 b_5 + b_5 b_9 + b_a & & (122) \\
\longrightarrow & b_1 b_4 + b_4 b_8 + b_a & & (123) \\
\longrightarrow & b_1 b_3 + b_6 b_{10} & & (124) \\
\longrightarrow & b_3 b_6 + b_6 b_9 + b_a & & (125) \\
\longrightarrow & b_2 b_3 + b_8 b_{10} & & (126) \\
\longrightarrow & b_1 b_4 + b_9 & & (127)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (7, 10). \quad (128) \\
& \longrightarrow b_3 b_7 + b_7 b_{10} & 649/1024 \quad (63\%) \quad (129) \\
& \longrightarrow 2b_4 b_6 & 937/1024 \quad (92\%) \quad (130) \\
& \longrightarrow b_1 b_5 + b_5 b_8 & 1001/1024 \quad (98\%) \quad (131) \\
& \longrightarrow b_1 b_2 + b_9 b_{10} & 1019/1024 \quad (99\%) \quad (132) \\
& \longrightarrow b_2 b_3 + b_8 & 1023/1024 \quad (99\%) \quad (133) \\
& \longrightarrow b_3 b_7 + b_9 b_{10} & 1024/1024(100\%) \quad (134)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (135) \\
& \longrightarrow b_2 b_4 + b_4 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} & (136) \\
& \longrightarrow b_1 b_7 + b_7 b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} & (137) \\
& \longrightarrow b_5 b_8 + b_6 b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10} & (138) \\
& \longrightarrow b_3 b_6 + b_a(b_3 - b_{10}) + b_{10} & (139) \\
& \longrightarrow b_1 b_5 + b_5 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} & (140) \\
& \longrightarrow b_6 b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1 & (141) \\
& \longrightarrow b_1 b_2 - b_9 b_a + b_{10} + 1 & (142)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (143) \\
& \longrightarrow 4b_3 b_7 & 768/1024 \quad (75\%) \quad (144) \\
& \longrightarrow b_2 b_8 + b_8 b_9 & 933/1024 \quad (91\%) \quad (145) \\
& \longrightarrow 2b_4 b_6 + b_8 b_9 - b_8 - b_9 + 1 & 1005/1024 \quad (98\%) \quad (146) \\
& \longrightarrow b_1 b_5 + b_5 b_{10} + b_8 b_9 - b_8 - b_9 + 1 & 1022/1024 \quad (99\%) \quad (147) \\
& \longrightarrow b_1 b_2 + b_8 b_9 + b_9 b_{10} - b_8 - b_9 + 1 & 1024/1024(100\%) \quad (148)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (9, 10). \quad (149) \\
& \longrightarrow b_1 b_6 + b_6 b_{10} + b_9 b_a & (150) \\
& \longrightarrow b_4 b_7 + b_7 b_8 & (151) \\
& \longrightarrow b_4 b_9 + b_a(b_9 - b_4) + b_4 & (152) \\
& \longrightarrow b_2 b_3 + b_3 b_8 + b_9 b_a & (153) \\
& \longrightarrow b_1 b_5 + b_2 b_5 + b_9 b_a & (154) \\
& \longrightarrow b_2 b_8 - b_6 b_7 + b_8 b_{10} + b_9 b_a + 1 & (155) \\
& \longrightarrow b_2 b_{10} + b_2 & (156)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (9, 10). \quad (157) \\
& \longrightarrow 2b_2 b_3 - b_8 b_9 + b_9 & 577/1024 \quad (56\%) \quad (158) \\
& \longrightarrow 3b_8 b_9 & 961/1024 \quad (94\%) \quad (159) \\
& \longrightarrow 2b_4 b_6 - b_8 b_9 - b_8 b_{10} + b_{10} + 1 & 1009/1024 \quad (99\%) \quad (160) \\
& \longrightarrow 2b_5 b_7 - b_8 b_{10} + b_{10} & 1021/1024 \quad (99\%) \quad (161) \\
& \longrightarrow b_1 b_6 + b_{10} & 1024/1024(100\%) \quad (162)
\end{aligned}$$

DECOMPOSITION OF DEGREE- k , EXACT- k -OF- n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 :$	$(k, n) = (4, 6). \quad (163)$
$\longrightarrow b_2b_4 + 2b_4b_5$	$43/64 \quad (67\%) \quad (164)$
$\longrightarrow b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$	$60/64 \quad (94\%) \quad (165)$
$\longrightarrow b_1b_2 + b_2b_5 + b_5b_6$	$64/64(100\%) \quad (166)$
$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 :$	$(k, n) = (4, 8). \quad (167)$
$\longrightarrow b_1b_4 + 2b_5b_6$	$159/256 \quad (62\%) \quad (168)$
$\longrightarrow b_2b_3 + b_3b_5 + b_7b_8$	$225/256 \quad (88\%) \quad (169)$
$\longrightarrow b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1$	$244/256 \quad (95.3\%) \quad (170)$
$\longrightarrow b_2b_3 + b_6b_8 + b_6$	$253/256 \quad (98.8\%) \quad (171)$
$\longrightarrow b_2b_3 + b_5b_7 + b_5$	$256/256 \quad (100\%) \quad (172)$
$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 :$	$(k, n) = (4, 8). \quad (173)$
$\longrightarrow b_2b_4 + 2b_5b_6$	$159/256 \quad (62\%) \quad (174)$
$\longrightarrow b_3b_6 + b_7b_8 + b_3$	$212/256 \quad (83\%) \quad (175)$
$\longrightarrow b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7$	$234/256 \quad (91\%) \quad (176)$
$\longrightarrow b_1b_3 + 2b_5b_6$	$253/256 \quad (99\%) \quad (177)$
$\longrightarrow b_7b_8 + b_1 + b_6$	$256/256(100\%) \quad (178)$
$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 :$	$(k, n) = (5, 7). \quad (179)$
$\longrightarrow b_1b_5 + 2b_5b_6$	(180)
$\longrightarrow b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1$	(181)
$\longrightarrow b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1)$	(182)
$\quad + b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5$	(183)
$\longrightarrow b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1$	(184)
$\longrightarrow b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2$	(185)
$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 :$	$(k, n) = (5, 7). \quad (186)$
$\longrightarrow 2b_4b_5 + b_4b_6$	$81/128 \quad (63\%) \quad (187)$
\longrightarrow	$111/128 \quad (87\%) \quad (188)$
\longrightarrow	$122/128 \quad (95\%) \quad (189)$
\longrightarrow	$128/128(100\%) \quad (190)$
$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 :$	$(k, n) = (6, 8). \quad (191)$
$\longrightarrow b_1b_3 + b_3b_5 + b_3b_8 + b_a(1 + b_6 - b_7)$	(192)
$\longrightarrow b_2b_6 + b_6b_7 + b_a(-b_6 + b_7) + b_6$	(193)
$\longrightarrow b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4$	(194)
$\longrightarrow -b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8$	(195)
$\quad + b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3$	(196)
$\longrightarrow b_1b_4 + b_6b_7 + b_7b_8$	(197)
$\longrightarrow b_2b_7 + b_7b_8 + b_8b_a + b_2$	(198)

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 + b_3 b_4 b_5 b_6 b_7 b_8 : & (k, n) = (6, 8). \quad (199) \\
\longrightarrow & b_1 b_6 + 2b_6 b_7 & 164/256 \quad (64\%) \quad (200) \\
\longrightarrow & b_1 b_5 + b_2 b_5 - b_3 b_6 + b_5 b_8 + b_3 & 219/256 \quad (86\%) \quad (201) \\
\longrightarrow & b_2 b_4 + b_4 b_7 + b_4 b_8 - b_6 + 1 & 243/256 \quad (95\%) \quad (202) \\
\longrightarrow & & 253/256 \quad (99\%) \quad (203) \\
\longrightarrow & b_1 b_2 + b_2 b_6 + b_5 b_7 - b_6 b_7 + b_7 b_8 - b_5 + 1 & 256/256(100\%) \quad (204)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (205) \\
\longrightarrow & b_3 b_5 + b_5 b_9 + b_5 b_{10} & (206) \\
\longrightarrow & b_1 b_4 + b_4 b_7 + b_4 b_9 & (207) \\
\longrightarrow & b_1 b_6 + b_2 b_6 - b_5 b_6 + b_6 b_{10} + b_6 & (208) \\
\longrightarrow & b_4 b_8 + 2b_8 & (209) \\
\longrightarrow & b_1 b_7 + b_2 b_7 - b_a(b_4 + b_6) + b_7 b_9 + b_4 + 1 & (210) \\
\longrightarrow & b_2 b_3 + b_3 b_5 + b_3 & (211) \\
\longrightarrow & b_1 b_2 + b_2 b_9 - b_5 b_6 - b_5 b_a + b_6 b_9 + 2 & (212) \\
\longrightarrow & b_2 b_7 + b_2 + b_{10} & (213)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (214) \\
\longrightarrow & 3b_5 b_8 & 769/1024 \quad (75\%) \quad (215) \\
\longrightarrow & 2b_2 b_6 + b_4 b_6 & 931/1024 \quad (91\%) \quad (216) \\
\longrightarrow & b_1 b_7 - b_5 b_{10} + b_7 b_9 + b_9 b_{10} - b_6 + b_{10} + 1 & 984/1024 \quad (96\%) \quad (217) \\
\longrightarrow & 3b_2 b_3 + b_3 b_{10} - b_6 b_8 + 1 & 1011/1024 \quad (99\%) \quad (218) \\
\longrightarrow & b_4 b_7 + b_4 b_8 - b_3 + b_4 - b_8 + 2 & 1019/1024 \quad (99\%) \quad (219) \\
\longrightarrow & b_2 b_3 - b_2 b_4 - b_3 b_4 - b_3 b_8 - b_5 b_{10} - b_6 b_9 + b_7 b_8 + b_7 b_9 + b_8 b_9 + b_7 + 3 & 1023/1024 \quad (99\%) \quad (220) \\
\longrightarrow & b_2 b_8 + 2b_8 b_9 & 1024/1024(100\%) \quad (221)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). \quad (222) \\
\longrightarrow & 2b_3 b_4 + b_7 b_{10} & 591/1024 \quad (58\%) \quad (223) \\
\longrightarrow & 2b_3 b_5 + b_5 b_6 & 847/1024 \quad (83\%) \quad (224) \\
\longrightarrow & b_1 b_2 + b_7 b_8 + b_8 b_9 & 951/1024 \quad (93\%) \quad (225) \\
\longrightarrow & 3b_5 b_6 & 995/1024 \quad (97\%) \quad (226) \\
\longrightarrow & b_1 b_3 + b_3 b_4 + b_9 b_{10} & 1009/1024 \quad (99\%) \quad (227) \\
\longrightarrow & b_1 b_2 + b_5 b_7 + b_7 b_{10} & 1018/1024 \quad (99\%) \quad (228) \\
\longrightarrow & 2b_1 b_4 - b_1 b_{10} + b_2 b_4 + b_4 b_5 + b_4 b_{10} + b_5 b_8 - b_6 b_8 + b_8 b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3 & 1023/1024 \quad (99\%) \quad (229) \\
\longrightarrow & b_2 b_8 + b_3 b_6 + b_6 b_8 & 1024/1024(100\%) \quad (230)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10} : & (k, n) = (4, 10). \quad (231) \\
\longrightarrow & b_3b_4 + b_4b_6 + b_9b_{10} & 581/1024 \quad (57\%) \quad (232) \\
\longrightarrow & b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9 & 759/1024 \quad (74\%) \quad (233) \\
\longrightarrow & b_5b_6 + b_8b_9 + b_1 + b_8 & 842/1024 \quad (82\%) \quad (234) \\
\longrightarrow & b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8 & 935/1024 \quad (91\%) \quad (235) \\
\longrightarrow & b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1 & 969/1024 \quad (95\%) \quad (236) \\
\longrightarrow & b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9 & 992/1024 \quad (97\%) \quad (237) \\
\longrightarrow & b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1 & 1004/1024 \quad (98\%) \quad (238) \\
\longrightarrow & b_1b_3 + b_6b_7 + b_9b_{10} & 1013/1024 \quad (99\%) \quad (239) \\
\longrightarrow & b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9 & 1019/1024 \quad (99\%) \quad (240) \\
\longrightarrow & b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9 & 1022/1024 \quad (99\%) \quad (241) \\
\longrightarrow & -b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1 & 1023/1024 \quad (99\%) \quad (242) \\
\longrightarrow & b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2 & 1024/1024(100\%) \quad (243)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : & (k, n) = (5, 8). \quad (244) \\
\longrightarrow & b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8) & (245) \\
\longrightarrow & b_1b_3 + b_3b_6 + b_6b_7 & (246) \\
\longrightarrow & b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7) & (247) \\
\longrightarrow & b_2b_5 + b_6b_8 + b_6 & (248) \\
\longrightarrow & b_1b_3 + b_3 + b_8 & (249)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : & (k, n) = (5, 8). \quad (250) \\
\longrightarrow & b_4b_5 + 2b_5b_6 & 165/256 \quad (64\%) \quad (251) \\
\longrightarrow & b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7 & 215/256 \quad (84\%) \quad (252) \\
\longrightarrow & b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7 & 242/256 \quad (95\%) \quad (253) \\
\longrightarrow & b_1b_3 + b_5b_6 + b_6b_7 & 254/256 \quad (99\%) \quad (254) \\
\longrightarrow & b_1b_2 + b_5b_6 + b_6b_8 & 256/256(100\%) \quad (255)
\end{aligned}$$

DECOMPOSITION OF DEGREE- k , EXACT- k -OF- n QUADRINOMIALS

$$\begin{aligned}
& b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 : & (k, n) = (3, 4). \quad (256) \\
\longrightarrow & 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3 & 13/16 \quad (81\%) \quad (257) \\
\longrightarrow & 2b_1b_3 + b_2b_3 + b_2 & 16/16(100\%) \quad (258)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} : & (k, n) = (7, 10). \quad (259) \\
\longrightarrow & 4b_4b_5 & 769/1024 \quad (75\%) \quad (260) \\
\longrightarrow & b_2b_6 + 2b_3b_6 + b_6b_9 & 915/1024 \quad (89\%) \quad (261) \\
\longrightarrow & b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10} & 974/1024 \quad (95\%) \quad (262) \\
\longrightarrow & b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10} & 995/1024 \quad (97\%) \quad (263) \\
\longrightarrow & b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10} & 1008/1024 \quad (98\%) \quad (264) \\
\longrightarrow & b_1b_2 + b_2b_4 + b_9b_{10} + b_9 & 1016/1024 \quad (99\%) \quad (265) \\
\longrightarrow & b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8 & 1023/1024 \quad (99\%) \quad (266) \\
\longrightarrow & b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2 & 1024/1024(100\%) \quad (267)
\end{aligned}$$

DECOMPOSITION OF DEGREE- k , NOT EXACT- k -OF- n QUADRINOMIALS

$$\begin{aligned}
 & b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (268) \\
 \longrightarrow & b_1 b_2 + 4b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - b_1 - b_2 - b_3 - b_4 + 1 & 12/16 \quad (75\%) \quad (269) \\
 \longrightarrow & b_1 b_2 + b_1 b_3 + 4b_1 b_4 + b_2 b_4 & 16/16(100\%) \quad (270)
 \end{aligned}$$

$$\begin{aligned}
 & b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (271) \\
 \longrightarrow & b_1 b_2 + 4b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - b_1 - b_2 - b_3 - b_4 + 1 & 12/16 \quad (75\%) \quad (272) \\
 \longrightarrow & 2b_2 b_3 + 3b_2 b_4 + b_3 b_4 & 16/16(100\%) \quad (273)
 \end{aligned}$$

$$\begin{aligned}
 & b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + 3b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (274) \\
 \longrightarrow & 2b_1 b_2 + 5b_1 b_4 + b_3 b_4 & (275) \\
 \longrightarrow & -b_1 b_2 + 3b_1 b_3 + 4b_2 b_3 + 2b_2 b_4 - 4b_3 b_4 + 4b_3 - b_4 + 1 & (276)
 \end{aligned}$$

$$\begin{aligned}
 & b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_3 b_4 : & (k, n) = (4, 4). \quad (277) \\
 \longrightarrow & 4b_1 b_3 & (278) \\
 \longrightarrow & 2b_1 b_2 + b_1 b_4 + b_2 b_4 & (279)
 \end{aligned}$$

$$\begin{aligned}
 & b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_3 b_4 : & (k, n) = (4, 4). \quad (280) \\
 \longrightarrow & 2b_1 b_3 + 2b_3 b_4 & 12/16 \quad (75\%) \quad (281) \\
 \longrightarrow & 3b_1 b_2 + b_1 b_4 & 16/16(100\%) \quad (282)
 \end{aligned}$$