Volume 3: List of Multi-run Quadratizations

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PRODUCT OF POLYNOMIALS

$$f_1 f_2 \dots f_\kappa = \min\left(f_1,f_2,...,f_\kappa\right), \qquad f_i\left(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}\right) \geq 0 \quad (1)$$

$$f_1 f_2 \dots f_\kappa = \min\left(f_1 f_2 \dots f_{\kappa-1} \max f_\kappa, f_\kappa - \min f_\kappa + f_1 f_2 \dots f_{\kappa-1}\right), \min f_\kappa < 0, \ f_{i<\kappa}\left(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}\right) \geq 0 \quad (2)$$

$b_1b_2b_3b_4 + b_2b_3b_4 - b_3b_4b_5$: (Example of Eq. 2). (3)

$\longrightarrow 2b_3b_4$ 25/32 (78%) (4)

$\longrightarrow b_1b_2 + b_2 - b_5 - b_3b_4 + 1$ 32/32(100%) (5)

MONOMIALS

$$b_1b_2b_3\dots b_k = \min\left(b_1b_2\dots b_{k_1}, b_{k_1+1}b_{k_1+2}\dots b_{k_2}, b_{k_2+1}b_{k_2+2}\dots b_{k_3}, ..., b_{k_n+1}b_{k_n+2}...b_k\right) \quad \text{(Example of Eq. 1).} \quad (6)$$

$$b_1b_2b_3\dots b_k = \min\left(b_1, b_2, b_3, \dots, b_k\right) \quad \text{(Example of Eq. 6: Linearization of a degree-}k\text{ monomial).} \quad (7)$$

$$b_1b_2b_3b_4 = \min\left(b_1b_2, b_3b_4\right) \quad \text{(Example of Eq. 6: Quadratization of a degree-4 monomial).} \quad (8)$$

$b_1b_2b_3b_4b_5b_6b_7b_8$: (9)

$\longrightarrow 3b_a + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - 2b_a(b_1 + b_2 + b_3 + b_4)$ (10)

$\longrightarrow 3b_a + b_5b_6 + b_5b_7 + b_5b_8 + b_6b_7 + b_6b_8 + b_7b_8 - 2b_a(b_5 + b_6 + b_7 + b_8)$ (11)

$$s_1s_2\dots s_k = \min\left(1 + s_1s_2 - s_3s_4\dots s_k, 1 - s_1s_2 + s_3s_4\dots s_k\right), s_i \in \{x, y, z\} \quad (12)$$

$x_1z_2x_3z_4y_5x_6$ (Example of Eq. 12). (13)

$\longrightarrow + x_1z_2 - x_3z_4y_5x_6 + 1$ 48/64 (75%) (14)

$\longrightarrow - x_1z_2 + x_3z_4y_5x_6 + 1$ 64/64(100%) (15)

$z_1z_2x_3 + z_1x_2z_3$: (16)

$\longrightarrow + 2z1 - x2z3 - z2x3 + 2$ 5/8 (63%) (17)

$\longrightarrow - 2z1 + x2z3 + z2x3 + 2$ 8/8(100%) (18)

$z_1z_2x_3 + 2z_1x_2z_3 :$ (19)
$\longrightarrow \ + 3z1 - 2x2z3 - z2x3 + 3$ 5/8 (63%) (20)
$\longrightarrow \ - 3z1 + 2x2z3 + z2x3 + 3$ 8/8(100%) (21)

$z_1z_2x_3 + 3z_1x_2z_3 :$ (22)
$\longrightarrow \ + 4z1 - 3x2z3 - z2x3 + 4$ 5/8 (63%) (23)
$\longrightarrow \ - 4z1 + 3x2z3 + z2x3 + 4$ 8/8(100%) (24)

$z_1z_2x_3 - z_1x_2z_3 :$ (25)
$\longrightarrow \ + 2z1 + x2z3 - z2x3 + 2$ 5/8 (63%) (26)
$\longrightarrow \ - 2z1 - x2z3 + z2x3 + 2$ 8/8(100%) (27)

$z_1z_2x_3 - 2z_1x_2z_3 :$ (28)
$\longrightarrow \ + 3z1 + 2x2z3 - z2x3 + 3$ 5/8 (63%) (29)
$\longrightarrow \ - 3z1 - 2x2z3 + z2x3 + 3$ 8/8(100%) (30)

$z_1(Az_2x_3 + Bx_2z_3) :$ (31)
$\longrightarrow \ + (|A| + |B|)z_1 - (Az_2x_3 + Bx_2z_3) + |A| + |B|$ (32)
$\longrightarrow \ - (|A| + |B|)z_1 + (Az_2x_3 + Bx_2z_3) + |A| + |B|$ (33)

$z_1x_2z_3 + z_1z_2x_3 + z_1x_2x_3 :$ (34)
$\longrightarrow \ + 3z1 - x2z3 - z2x3 - x2x3 + 3$ 4/8 (50%) (35)
$\longrightarrow \ - 3z1 + x2z3 + z2x3 + x2x3 + 3$ 8/8(100%) (36)

$z_1x_2z_3 + z_1z_2x_3 + z_1x_2x_3 + z_1z_2z_3 :$ (37)
$\longrightarrow \ + 4z1 - x2z3 - z2x3 - x2x3 - z2z3 + 4$ 4/8 (50%) (38)
$\longrightarrow \ - 4z1 + x2z3 + z2x3 + x2x3 + z2z3 + 4$ 8/8(100%) (39)

$z_1x_2z_3 + z_1z_2x_3 + z_1x_2x_3 + z_1z_2z_3 :$ (40)
$\longrightarrow \ + 2z1 - x2z3 - z2x3 - x2x3 - z2z3 + 2$ 6/8 (75%) (41)
$\longrightarrow \ - 2z1 + x2z3 + z2x3 + x2x3 + z2z3 + 2$ 8/8(100%) (42)

$z_1x_2z_3 + z_1z_2x_3 + z_1x_2x_3 + z_1z_2z_3 + z_1x_2y_3 :$ (43)
$\longrightarrow \ + 3z1 - x2x3 - x2z3 - z2x3 - z2z3 - x2y3 + 3$ 4/8 (50%) (44)
$\longrightarrow \ - 3z1 + x2x3 + x2z3 + z2x3 + z2z3 + x2y3 + 3$ 8/8(100%) (45)

$z_1x_2z_3 + z_1z_2x_3 + z_1x_2x_3 + z_1z_2z_3 + z_1x_2y_3 + z_1y_2x_3 + z_1y_2y_3 :$ (46)
$\longrightarrow \ - 3z1 + x2y3 + x2z3 + y2x3 + z2x3 + x2x3 + y2y3 + z2z3 + 3$ 5/8 (63%) (47)
$\longrightarrow \ + 4z1 - x2y3 - x2z3 - y2x3 - z2x3 - x2x3 - y2y3 - z2z3 + 4$ 8/8(100%) (48)

$$(A_1x_1z_2 + A_2y_1x_2 + A_3x_1x_2 + A_4y_1y_2)z_3y_4 : \tag{49}$$
$$\longrightarrow \ -(A_1x_1z_2 + A_2y_1x_2 + A_3x_1x_2 + A_4y_1y_2) + \sum_i |A_i|\, z_3y_4 + \sum_i |A_i| \tag{50}$$
$$\longrightarrow \ +(A_1x_1z_2 + A_2y_1x_2 + A_3x_1x_2 + A_4y_1y_2) - \sum_i |A_i|\, z_3y_4 + \sum_i |A_i| \tag{51}$$

HEURISTIC GADGETS

$$z_1z_2x_3 \tag{52}$$
$$\longrightarrow \ z_1z_2 + z_1x_3 + z_2x_3 - z_1 - z_2 - x_3 + 1 \qquad 7/8 \ (88\%) \tag{53}$$

BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min\left(2b_3b_4, b_1b_2 + b_5b_6\right) \qquad (k,n) = (4,6). \tag{54}$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min_{b_a}\left(b_2b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3b_4, b_1b_2 + b_5b_6 + b_5b_a\right) \qquad (k,n) = (4,6). \tag{55}$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad (k,n) = (4,7). \tag{56}$$
$$\longrightarrow \ b_3b_4 + b_4b_6 + b_a(b_5 + b_7) \qquad 89/128 \ (70\%) \tag{57}$$
$$\longrightarrow \ b_1b_2 + b_5b_7 + b_a(1 - b_5 + b_6 - b_7) \qquad 125/128 \ (98\%) \tag{58}$$
$$\longrightarrow \ b_5b_7 + b_3 \qquad 128/128(100\%) \tag{59}$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad (k,n) = (4,7). \tag{60}$$
$$\longrightarrow \ b_3b_4 + b_4b_6 \qquad 89/128 \ (70\%) \tag{61}$$
$$\longrightarrow \ b_1b_2 + b_6b_7 \qquad 118/128 \ (92\%) \tag{62}$$
$$\longrightarrow \ b_2b_3 - b_5b_6 + b_5b_7 + b_5 \qquad 127/128 \ (99\%) \tag{63}$$
$$\longrightarrow \ b_1b_4 + 2b_5 - b_7 + 1 \qquad 128/128(100\%) \tag{64}$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \tag{65}$$
$$\longrightarrow \ b_1b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(1 - b_5 - 2b_6 - b_7) + b_6 \qquad 188/256 \ (73\%) \tag{66}$$
$$\longrightarrow \ b_3b_4 + b_a(b_4 - b_6) + b_6 \qquad 236/256 \ (92\%) \tag{67}$$
$$\longrightarrow \ b_2b_3 + b_3b_6 - b_4b_6 + b_6b_a + b_6 \qquad 254/256 \ (99\%) \tag{68}$$
$$\longrightarrow \ b_2b_5 + b_5b_7 \qquad 256/256(100\%) \tag{69}$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \tag{70}$$
$$\longrightarrow \ b_2b_3 + b_3b_7 \qquad 85/128 \ (66\%) \tag{71}$$
$$\longrightarrow \ 2b_4b_5 \qquad 121/128 \ (95\%) \tag{72}$$
$$\longrightarrow \ b_1b_2 + b_6b_7 - b_5 + 1 \qquad 128/128(100\%) \tag{73}$$

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 :$ $(k,n) = (6,7).$ (74)
$\longrightarrow\ b_5b_6 + b_6b_7 + b_a(1 + b_5 - b_6 - b_7)$ 196/256 (77%) (75)
$\longrightarrow\ b_1b_4 + b_2b_4 + b_7b_a$ 238/256 (93%) (76)
$\longrightarrow\ b_1b_3 + b_3b_7 - b_4b_6 + 2b_5b_a - b_6b_7 - b_5 + b_6 + b_7 + b_a + 1$ 252/256 (98%) (77)
$\longrightarrow\ b_2b_6 + b_2 - b_6 + 1$ 256/256(100%) (78)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 :$ $(k,n) = (6,7).$ (79)
$\longrightarrow\ 2b_5b_6$ 97/128 (76%) (80)
$\longrightarrow\ b_1b_4 + b_4b_7$ 119/128 (93%) (81)
$\longrightarrow\ b_1b_3 + b_1b_7 + b_2b_3 - b_3b_6 + b_3b_7 - b_4b_5 - b_1 - b_7 + 2$ 127/128 (99%) (82)
$\longrightarrow\ b_1b_2 + b_2b_6$ 128/128(100%) (83)

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 :$ $(k,n) = (5,8).$ (84)
$\longrightarrow\ b_3b_5 + b_7b_8 + b_a(-1 - b_6 + b_7 + b_8) + b_6 - b_7 - b_8 + 1$ 360/512 (70%) (85)
$\longrightarrow\ b_1b_4 + b_4b_8 + b_a(b_4 + b_6)$ 468/512 (91%) (86)
$\longrightarrow\ b_1b_2 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8)$ 496/512 (97%) (87)
$\longrightarrow\ b_3b_5 + b_5$ 512/512(100%) (88)

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 :$ $(k,n) = (5,8).$ (89)
$\longrightarrow\ b_2b_5 + b_5b_8$ 169/256 (66%) (90)
$\longrightarrow\ b_1b_4 + b_4b_7 - b_5b_8 + b_8$ 233/256 (91%) (91)
$\longrightarrow\ b_1b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 252/256 (98%) (92)
$\longrightarrow\ b_2b_3 + b_6b_7$ 256/256(100%) (93)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 :$ $(k,n) = (6,8).$ (94)
$\longrightarrow\ b_1b_6 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8)$ 364/512 (71%) (95)
$\longrightarrow\ b_2b_3 + b_5b_8 - b_6b_8 + b_7b_a - b_7 + b_8 - b_a + 1$ 450/512 (88%) (96)
$\longrightarrow\ b_1b_4 + b_4$ 488/512 (95%) (97)
$\longrightarrow\ b_2b_3 + b_3b_7 - b_6b_8 + b_8 - b_a + 1$ 502/512 (98%) (98)
$\longrightarrow\ b_2b_5 + b_5$ 512/512(100%) (99)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 :$ $(k,n) = (6,8).$ (100)
$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (101)
$\longrightarrow\ b_1b_4 + b_4b_8$ 237/256 (93%) (102)
$\longrightarrow\ b_2b_3 + b_3b_7 - b_4b_6 + b_4b_8 - b_5b_7 - b_5b_8 + b_6b_8 - b_6 + b_7 - b_8 + 2$ 254/256 (99%) (103)
$\longrightarrow\ b_1b_2 + b_7b_8$ 256/256(100%) (104)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 :$ $(k,n) = (7,8).$ (105)
$\longrightarrow\ b_6b_7 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 388/512 (76%) (106)
$\longrightarrow\ b_1b_3 + b_3b_8 + b_a(1 + b_8)$ 470/512 (92%) (107)
$\longrightarrow\ b_2b_4 - b_3b_8 + b_4b_5 + b_a(1 - b_7) + b_8$ 500/512 (98%) (108)
$\longrightarrow\ b_2b_5 + b_2b_8 - b_4b_8 - b_6b_7 + b_6b_8 + b_a(-1 - b_4 - b_7 + b_8) - b_3 + b_7 - b_8 + 4$ 508/512 (99%) (109)
$\longrightarrow\ b_2b_5 - b_7b_8 + b_5 + 1$ 512/512(100%) (110)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8$: $(k,n) = (7,8)$. (111)

$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (112)

$\longrightarrow\ b_1b_4 + b_4b_8$ 235/256 (92%) (113)

$\longrightarrow\ b_2b_3 + b_2b_7 - b_5b_6 + b_6b_8 + b_5 - b_6 - b_8 + 1$ 250/256 (98%) (114)

$\longrightarrow\ b_3b_7 + b_7b_8$ 254/256 (99%) (115)

$\longrightarrow\ b_3b_8 + b_3$ 256/256(100%) (116)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (117)

$\longrightarrow\ b_2b_3 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 390/512 (76%) (118)

$\longrightarrow\ b_1b_4 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 480/512 (94%) (119)

$\longrightarrow\ b_2b_4 + b_5 - b_a + 1$ 506/512 (99%) (120)

$\longrightarrow\ b_1b_3 - b_6b_a + b_5 + 1$ 512/512(100%) (121)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (122)

$\longrightarrow\ b_1b_2 + b_6b_7$ 169/256 (66%) (123)

$\longrightarrow\ b_3b_4 + b_5b_8$ 238/256 (93%) (124)

$\longrightarrow\ b_1b_4 + b_5b_6 + b_5b_7 + b_6b_7 - b_5 - b_6 - b_7 + 1$ 248/256 (97%) (125)

$\longrightarrow\ b_2b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 254/256 (99%) (126)

$\longrightarrow\ b_1b_2 + b_5b_8$ 256/256(100%) (127)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (128)

$\longrightarrow\ b_1b_4 + b_7b_9$ 625/1024 (61%) (129)

$\longrightarrow\ b_3b_5 + b_6b_8$ 889/1024 (87%) (130)

$\longrightarrow\ b_2b_5 + b_7b_{10}$ 972/1024 (95%) (131)

$\longrightarrow\ b_2b_4 + b_6b_8$ 999/1024 (98%) (132)

$\longrightarrow\ b_1b_3 + b_9b_{10} + b_9b_a$ 1016/1024 (99%) (133)

$\longrightarrow\ b_1b_5 + b_6b_9$ 1020/1024 (99%) (134)

$\longrightarrow\ b_1b_4 + b_8b_{10}$ 1022/1024 (99%) (135)

$\longrightarrow\ b_2b_3 - b_4b_{10} + b_7b_9 + b_9b_a + 1$ 1024/1024(100%) (136)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (137)

$\longrightarrow\ b_1b_3 + b_9b_{10}$ 625/1024 (61%) (138)

$\longrightarrow\ b_2b_4 + b_7b_{10}$ 851/1024 (83%) (139)

$\longrightarrow\ b_3b_5 + b_5b_{10} + b_8b_9$ 924/1024 (90%) (140)

$\longrightarrow\ b_1b_2 + b_6$ 972/1024 (95%) (141)

$\longrightarrow\ b_3b_4 + b_8b_9$ 997/1024 (97%) (142)

$\longrightarrow\ b_1b_5 + b_7b_{10}$ 1010/1024 (99%) (143)

$\longrightarrow\ b_2b_3 - b_1b_7 - b_1b_{10} - b_2b_8 - b_2b_{10} + b_3b_5 + b_6b_9 + b_7b_{10} - b_8b_9 + b_9b_{10} - b_3 - b_7 + b_8 + 3$ 1016/1024 (99%) (144)

$\longrightarrow\ b_1b_3 + b_7b_8$ 1020/1024 (99%) (145)

$\longrightarrow\ b_2b_4 + b_2b_6 - b_2b_9 - b_3b_{10} - b_5b_7 + b_7b_{10} + b_9b_{10} - b_{10} + 2$ 1023/1024 (99%) (146)

$\longrightarrow\ b_2b_5 + b_2b_9 + b_6b_8$ 1024/1024(100%) (147)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (148)
$\longrightarrow\ b_4b_5 + b_5b_9$ 657/1024 (64%) (149)
$\longrightarrow\ b_2b_6 + b_6b_8$ 905/1024 (88%) (150)
$\longrightarrow\ b_1b_3 + b_7b_8$ 982/1024 (96%) (151)
$\longrightarrow\ b_2b_3 + b_a(b_{10} - b_9) + b_9$ 1011/1024 (99%) (152)
$\longrightarrow\ b_2b_4 + b_7b_{10}$ 1020/1024 (99%) (153)
$\longrightarrow\ b_9b_{10} + b_1$ 1023/1024 (99%) (154)
$\longrightarrow\ b_7b_8 + b_4$ 1024/1024(100%) (155)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (156)
$\longrightarrow\ 2b_5b_6$ 769/1024 (75%) (157)
$\longrightarrow\ b_1b_3 + b_8b_9$ 934/1024 (92%) (158)
$\longrightarrow\ b_2b_4 + b_7b_{10} + b_8b_9 - b_8 - b_9 + 1$ 997/1024 (97%) (159)
$\longrightarrow\ -b_1b_3 + b_1b_9 + b_2b_4 + b_4b_9 + b_5b_8 + b_8b_9 - b_5 - b_8 - b_9 + 2$ 769/1024 (99%) (160)
$\longrightarrow\ b_1b_3 + b_7b_{10} - b_8 - b_9 + 2$ 1014/1024 (99%) (161)
$\longrightarrow\ b_2b_3 + b_8b_9$ 1024/1024(100%) (162)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (163)
$\longrightarrow\ b_3b_5 + b_5b_8$ 649/1024 (63%) (164)
$\longrightarrow\ b_2b_4 + b_4b_9$ 893/1024 (87%) (165)
$\longrightarrow\ b_1b_7 + b_7b_{10}$ 985/1024 (96%) (166)
$\longrightarrow\ b_1b_6 + b_6b_9 + b_a$ 1015/1024 (99%) (167)
$\longrightarrow\ b_2b_3 + b_8b_{10} + b_a$ 1022/1024 (99%) (168)
$\longrightarrow\ b_1b_3 + b_8b_9$ 1024/1024(100%) (169)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (170)
$\longrightarrow\ b_3b_7 + b_7b_{10}$ 649/1024 (63%) (171)
$\longrightarrow\ 2b_4b_6$ 937/1024 (92%) (172)
$\longrightarrow\ b_1b_5 + b_5b_8$ 1001/1024 (98%) (173)
$\longrightarrow\ b_1b_2 + b_9b_{10}$ 1019/1024 (99%) (174)
$\longrightarrow\ b_2b_3 + b_8$ 1023/1024 (99%) (175)
$\longrightarrow\ b_3b_7 + b_9b_{10}$ 1024/1024(100%) (176)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (177)
$\longrightarrow\ b_2b_8 + b_8b_9$ 645/1024 (63%) (178)
$\longrightarrow\ b_1b_3 + b_3b_{10} + b_9b_a$ 887/1024 (87%) (179)
$\longrightarrow\ b_4b_6 + b_5b_6$ 977/1024 (95%) (180)
$\longrightarrow\ b_2b_7 + b_7b_{10}$ 1007/1024 (98%) (181)
$\longrightarrow\ b_1b_4 + b_4b_5 + 2b_9b_a$ 1018/1024 (99%) (182)
$\longrightarrow\ b_1b_5 + b_5b_9$ 1024/1024(100%) (183)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (184)
$\longrightarrow\ 4b_3b_7$ 768/1024 (75%) (185)
$\longrightarrow\ b_2b_8 + b_8b_9$ 933/1024 (91%) (186)
$\longrightarrow\ 2b_4b_6 + b_8b_9 - b_8 - b_9 + 1$ 1005/1024 (98%) (187)
$\longrightarrow\ b_1b_5 + b_5b_{10} + b_8b_9 - b_8 - b_9 + 1$ 1022/1024 (99%) (188)
$\longrightarrow\ b_1b_2 + b_8b_9 + b_9b_{10} - b_8 - b_9 + 1$ 1024/1024(100%) (189)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (190)
$\longrightarrow\ b_1b_9 + b_9b_{10} + b_{10}b_a$ 643/1024 (63%) (191)
$\longrightarrow\ b_2b_4 + b_4b_5$ 883/1024 (86%) (192)
$\longrightarrow\ b_3b_7 + b_3b_8$ 973/1024 (95%) (193)
$\longrightarrow\ b_2b_6 + b_6b_8$ 1003/1024 (98%) (194)
$\longrightarrow\ b_2b_5 + b_5b_7 - b_{10}b_a + b_{10}$ 1015/1024 (99%) (195)
$\longrightarrow\ b_1b_8 + b_7b_8$ 1019/1024 (99%) (196)
$\longrightarrow\ b_2b_7 + b_2b_{10} - b_4b_5 - b_{10}b_a + b_{10} + 1$ 1023/1024 (99%) (197)
$\longrightarrow\ b_4b_7 + b_7$ 1024/1024(100%) (198)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (199)
$\longrightarrow\ 2b_2b_3 - b_8b_9 + b_9$ 577/1024 (56%) (200)
$\longrightarrow\ 3b_8b_9$ 961/1024 (94%) (201)
$\longrightarrow\ 2b_4b_6 - b_8b_9 - b_8b_{10} + b_{10} + 1$ 1009/1024 (99%) (202)
$\longrightarrow\ 2b_5b_7 - b_8b_{10} + b_{10}$ 1021/1024 (99%) (203)
$\longrightarrow\ b_1b_6 + b_{10}$ 1024/1024(100%) (204)

DEGREE-k, EXACT-k-OF-n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $(k,n) = (4,6)$. (205)
$\longrightarrow\ b_1b_4 + 2b_4b_5 + b_7$ 44/64 (69%) (206)
$\longrightarrow\ b_1b_3 + b_2b_3 + b_3b_6 + b_6b_7$ 60/64 (94%) (207)
$\longrightarrow\ b_2b_4 + b_5b_6 + b_2$ 64/64(100%) (208)

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $(k,n) = (4,6)$. (209)
$\longrightarrow\ b_2b_4 + 2b_4b_5$ 43/64 (67%) (210)
$\longrightarrow\ b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$ 60/64 (94%) (211)
$\longrightarrow\ b_1b_2 + b_2b_5 + b_5b_6$ 64/64(100%) (212)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (213)
$\longrightarrow\ b_1b_4 + 2b_5b_6$ 159/256 (62%) (214)
$\longrightarrow\ b_2b_3 + b_3b_5 + b_7b_8$ 225/256 (88%) (215)
$\longrightarrow\ b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1$ 244/256 (95.3%) (216)
$\longrightarrow\ b_2b_3 + b_6b_8 + b_6$ 253/256 (98.8%) (217)
$\longrightarrow\ b_2b_3 + b_5b_7 + b_5$ 256/256 (100%) (218)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (219)
$\longrightarrow\ b_2b_4 + 2b_5b_6$ 159/256 (62%) (220)
$\longrightarrow\ b_3b_6 + b_7b_8 + b_3$ 212/256 (83%) (221)
$\longrightarrow\ b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7$ 234/256 (91%) (222)
$\longrightarrow\ b_1b_3 + 2b_5b_6$ 253/256 (99%) (223)
$\longrightarrow\ b_7b_8 + b_1 + b_6$ 256/256(100%) (224)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (225)
$\longrightarrow\ b_1b_5 + b_5b_6 + b_6b_7 + b_a(-2 - b_5 + 2b_6 + b_7) + b_5 - 2b_6 - b_7 + 2$ 86/128 (67%) (226)
$\longrightarrow\ b_1b_3 + b_3b_4 - b_3b_6 + b_3b_7 + b_a(b_5 + b_7) + b_3$ 112/128 (88%) (227)
$\longrightarrow\ b_1b_4 + b_2b_4 + b_4b_7 + b_5b_7 + b_a(-1 - b_6 - b_7) - b_5 + b_6 + 2$ 124/128 (97%) (228)
$\longrightarrow\ b_2b_4 - 2b_5b_a + b_6b_7 + b_2 + b_5 + 1$ 128/128(100%) (229)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (230)
$\longrightarrow\ 2b_4b_5 + b_4b_6$ 81/128 (63%) (231)
$\longrightarrow\ b_1b_3 + b_3b_6 + b_3b_7 - b_4b_5 + b_5$ 111/128 (87%) (232)
$\longrightarrow\ b_1b_2 + b_2b_6 - b_4b_5 + b_6b_7 + b_5$ 122/128 (95%) (233)
$\longrightarrow\ 2b_4b_5 + b_5$ 128/128(100%) (234)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (235)
$\longrightarrow\ b_1b_6 + 2b_6b_7$ 164/256 (64%) (236)
$\longrightarrow\ b_1b_5 + b_2b_5 - b_3b_6 + b_5b_8 + b_3$ 219/256 (86%) (237)
$\longrightarrow\ b_2b_4 + b_4b_7 + b_4b_8 - b_6 + 1$ 243/256 (95%) (238)
$\longrightarrow\ b_2b_3 + b_3b_8 - b_5b_6 + b_3 + b_6$ 253/256 (99%) (239)
$\longrightarrow\ b_1b_2 + b_2b_6 + b_5b_7 - b_6b_7 + b_7b_8 - b_5 + 1$ 256/256(100%) (240)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (241)
$\longrightarrow\ 3b_5b_8$ 769/1024 (75%) (242)
$\longrightarrow\ 2b_2b_6 + b_4b_6$ 931/1024 (91%) (243)
$\longrightarrow\ b_1b_7 - b_5b_{10} + b_7b_9 + b_9b_{10} - b_6 + b_{10} + 1$ 984/1024 (96%) (244)
$\longrightarrow\ 3b_2b_3 + b_3b_{10} - b_6b_8 + 1$ 1011/1024 (99%) (245)
$\longrightarrow\ b_4b_7 + b_4b_8 - b_3 + b_4 - b_8 + 2$ 1019/1024 (99%) (246)
$\longrightarrow\ b_2b_3 - b_2b_4 - b_3b_4 - b_3b_8 - b_5b_{10} - b_6b_9 + b_7b_8 + b_7b_9 + b_8b_9 + b_7 + 3$ 1023/1024 (99%) (247)
$\longrightarrow\ b_2b_8 + 2b_8b_9$ 1024/1024(100%) (248)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (249)
$\longrightarrow\ b_5b_6 + b_5b_8 + b_5b_9 + b_8b_{11} + b_9b_{11} + b_{10}b_{11}$ 583/1024 (57%) (250)
$\longrightarrow\ b_1b_2 + b_4b_7 + b_7b_{10} + b_9b_{11} - b_9 - b_{11} + 1$ 815/1024 (80%) (251)
$\longrightarrow\ b_1b_6 + b_5b_6 + b_6 - b_{11} + 1$ 917/1024 (90%) (252)
$\longrightarrow\ b_3b_4 + b_3b_7 + b_8b_9 + b_9b_{11}$ 979/1024 (96%) (253)
$\longrightarrow\ b_2b_4 + b_4b_8 + b_8b_9 - b_9b_{11} + b_9 - b_{11} + 1$ 1007/1024 (98%) (254)
$\longrightarrow\ b_1b_3 + b_7b_{10} + b_{10}b_{11} + b_3$ 1016/1024 (99%) (255)
$\longrightarrow\ b_1b_4 + b_4b_8 + b_7b_{10} + b_9b_{11} + b_{10}b_{11} - b_9 - b_{11} + 1$ 1021/1024 (99%) (256)
$\longrightarrow\ b_1b_3 - b_2b_{11} + b_7b_8 + b_8b_9 - b_{10}b_{11} - b_{11} + 3$ 1024/1024(100%) (257)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (258)

$\longrightarrow$ $2b_3b_4 + b_7b_{10}$ 591/1024 (58%) (259)

$\longrightarrow$ $2b_3b_5 + b_5b_6$ 847/1024 (83%) (260)

$\longrightarrow$ $b_1b_2 + b_7b_8 + b_8b_9$ 951/1024 (93%) (261)

$\longrightarrow$ $3b_5b_6$ 995/1024 (97%) (262)

$\longrightarrow$ $b_1b_3 + b_3b_4 + b_9b_{10}$ 1009/1024 (99%) (263)

$\longrightarrow$ $b_1b_2 + b_5b_7 + b_7b_{10}$ 1018/1024 (99%) (264)

$\longrightarrow$ $2b_1b_4 - b_1b_{10} + b_2b_4 + b_4b_5 + b_4b_{10} + b_5b_8 - b_6b_8 + b_8b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3$ 1023/1024 (99%) (265)

$\longrightarrow$ $b_2b_8 + b_3b_6 + b_6b_8$ 1024/1024(100%) (266)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $(k,n) = (4,10)$. (267)

$\longrightarrow$ $b_2b_3 + b_6b_7 + b_7b_9 + 2b_9b_a$ 581/1024 (57%) (268)

$\longrightarrow$ $b_2b_4 + b_4b_6 + b_a(b_9 - b_{10}) + b_{10}$ 823/1024 (80%) (269)

$\longrightarrow$ $b_1b_3 + b_5b_6 + b_8b_9 + b_a(b_9 - b_{10}) - b_9 + 1$ 930/1024 (91%) (270)

$\longrightarrow$ $b_1b_4 + b_4b_5 + b_4b_{10} + b_8b_{10} + b_a(1 - b_7 + b_9)$ 978/1024 (96%) (271)

$\longrightarrow$ $b_1b_4 + b_7b_8 + b_a(1 + b_9) + b_7$ 1000/1024 (98%) (272)

$\longrightarrow$ $b_2b_3 + b_a(b_9 - b_{10}) + b_5 + b_{10}$ 1015/1024 (99%) (273)

$\longrightarrow$ $b_1b_3 + b_6 + b_{10}$ 1020/1024 (99%) (274)

$\longrightarrow$ $b_5b_6 + b_2 + b_8 + b_a$ 1024/1024(100%) (275)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $(k,n) = (4,10)$. (276)

$\longrightarrow$ $b_3b_4 + b_4b_6 + b_9b_{10}$ 581/1024 (57%) (277)

$\longrightarrow$ $b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9$ 759/1024 (74%) (278)

$\longrightarrow$ $b_5b_6 + b_8b_9 + b_1 + b_8$ 842/1024 (82%) (279)

$\longrightarrow$ $b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8$ 935/1024 (91%) (280)

$\longrightarrow$ $b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1$ 969/1024 (95%) (281)

$\longrightarrow$ $b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9$ 992/1024 (97%) (282)

$\longrightarrow$ $b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1$ 1004/1024 (98%) (283)

$\longrightarrow$ $b_1b_3 + b_6b_7 + b_9b_{10}$ 1013/1024 (99%) (284)

$\longrightarrow$ $b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9$ 1019/1024 (99%) (285)

$\longrightarrow$ $b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9$ 1022/1024 (99%) (286)

$\longrightarrow$ $-b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1$ 1023/1024 (99%) (287)

$\longrightarrow$ $b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2$ 1024/1024(100%) (288)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (289)

$\longrightarrow$ $b_1b_3 + b_3b_4 + b_6b_8 - b_6b_9 + b_7b_9 - b_8b_9 + b_9$ 156/256 (61%) (290)

$\longrightarrow$ $b_1b_5 + b_5b_7 + b_7b_9 + b_8b_9 + b_5 - b_7 - b_9 + 1$ 202/256 (79%) (291)

$\longrightarrow$ $b_2b_4 + b_6b_8 + b_6b_9 - b_7b_9 + b_8b_9 + b_7 - b_8 - b_9 + 1$ 230/256 (90%) (292)

$\longrightarrow$ $b_2b_4 + b_4b_8 + b_4 - b_9 + 1$ 246/256 (96%) (293)

$\longrightarrow$ $b_1b_5 + 2b_6$ 252/256 (98%) (294)

$\longrightarrow$ $b_2b_5 + b_7b_8 + b_5$ 256/256(100%) (295)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 :$ $(k,n) = (5,8).$ (296)
$\longrightarrow\ b_4b_5 + 2b_5b_6$ 165/256 (64%) (297)
$\longrightarrow\ b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7$ 215/256 (84%) (298)
$\longrightarrow\ b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7$ 242/256 (95%) (299)
$\longrightarrow\ b_1b_3 + b_5b_6 + b_6b_7$ 254/256 (99%) (300)
$\longrightarrow\ b_1b_2 + b_5b_6 + b_6b_8$ 256/256(100%) (301)

DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (3,4).$ (302)
$\longrightarrow\ 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3$ 13/16 (81%) (303)
$\longrightarrow\ 2b_1b_3 + b_2b_3 + b_2$ 16/16(100%) (304)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} :$ $(k,n) = (7,10).$ (305)
$\longrightarrow\ 4b_4b_5$ 769/1024 (75%) (306)
$\longrightarrow\ b_2b_6 + 2b_3b_6 + b_6b_9$ 915/1024 (89%) (307)
$\longrightarrow\ b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10}$ 974/1024 (95%) (308)
$\longrightarrow\ b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10}$ 995/1024 (97%) (309)
$\longrightarrow\ b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10}$ 1008/1024 (98%) (310)
$\longrightarrow\ b_1b_2 + b_2b_4 + b_9b_{10} + b_9$ 1016/1024 (99%) (311)
$\longrightarrow\ b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8$ 1023/1024 (99%) (312)
$\longrightarrow\ b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2$ 1024/1024(100%) (313)

DEGREE-k, NOT EXACT-k-OF-n MULTINOMIALS

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (314)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (315)
$\longrightarrow\ b_1b_2 + b_1b_3 + 4b_1b_4 + b_2b_4$ 16/16(100%) (316)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (317)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (318)
$\longrightarrow\ 2b_2b_3 + 3b_2b_4 + b_3b_4$ 16/16(100%) (319)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + 3b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (320)
$\longrightarrow\ 2b_1b_2 + 5b_1b_4 + b_3b_4$ 11/16 (69%) (321)
$\longrightarrow\ -b_1b_2 + 3b_1b_3 + 4b_2b_3 + 2b_2b_4 - 4b_3b_4 + 4b_3 - b_4 + 1$ 16/16(100%) (322)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 :$ $(k,n) = (4,4).$ (323)
$\longrightarrow\ 4b_1b_3$ 13/16 (81%) (324)
$\longrightarrow\ 2b_1b_2 + b_1b_4 + b_2b_4$ 16/16(100%) (325)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 :$ $(k, n) = (4, 4).$ (326)

$\longrightarrow\ 2b_1b_3 + 2b_3b_4$ $12/16\ (75\%)$ (327)

$\longrightarrow\ 3b_1b_2 + b_1b_4$ $16/16(100\%)$ (328)