Volume 3: List of Multi-run Quadratizations

Nike Dattani and Andreas Soteriou
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DECOMPOSITION OF A MONOMIAL

$$b_1 b_2 b_3 \ldots b_k = \min\left(b_1 b_2 \ldots b_{k_1}, b_{k_1+1} b_{k_1+2} \ldots b_{k_2}, b_{k_2+1} b_{k_2+2} \ldots b_{k_3}, \ldots, b_{k_n+1} b_{k_n+2} \ldots b_k\right) \tag{1}$$

$$b_1 b_2 b_3 \ldots b_k = \min\left(b_1, b_2, b_3, \ldots, b_k\right) \qquad \text{(Example of Eq. 1: Linearization of a degree-}k\text{ monomial).} \tag{2}$$

$$b_1 b_2 b_3 b_4 = \min\left(b_1 b_2, b_3 b_4\right) \qquad \text{(Example of Eq. 1: Quadratization of a degree-4 monomial).} \tag{3}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 : \tag{4}$$
$$\longrightarrow 3b_a + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \tag{5}$$
$$\longrightarrow 3b_a + b_5 b_6 + b_5 b_7 + b_5 b_8 + b_6 b_7 + b_6 b_8 + b_7 b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \tag{6}$$

Quantum envelopes:
Cubic:

$$A_1 B_2 C_3 : \tag{7}$$
$$\longrightarrow 1 + (A_1 - B_2 C_3) \tag{8}$$
$$\longrightarrow 1 - (A_1 - B_2 C_3) \tag{9}$$

where A, B and C can be any of the Pauli matrices X, Y or Z. For example:

$$Z_1 Y_2 Y_3 : \tag{10}$$
$$\longrightarrow 1 + (Z_1 - Y_2 Y_3) \tag{11}$$
$$\longrightarrow 1 - (Z_1 - Y_2 Y_3) \tag{12}$$

or

$$X_1 Y_2 Z_3 : \tag{13}$$
$$\longrightarrow 1 + (X_1 - Y_2 Z_3) \tag{14}$$
$$\longrightarrow 1 - (X_1 - Y_2 Z_3) \tag{15}$$

Quartic:

$$Z_1 Z_2 Z_3 X_4 : \tag{16}$$
$$\longrightarrow 1 + (Z_1 Z_2 - Z_3 X_4) \tag{17}$$
$$\longrightarrow 1 - (Z_1 Z_2 - Z_3 X_4) \tag{18}$$

DECOMPOSITION OF BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min\left(2b_3b_4, b_1b_2 + b_5b_6\right) \qquad\qquad (k,n) = (4,6). \quad (19)$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min_{b_a}\left(b_2b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3b_4, b_1b_2 + b_5b_6 + b_5b_a\right) \qquad\qquad (k,n) = (4,6). \quad (20)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad\qquad (k,n) = (4,7). \quad (21)$$
$$\longrightarrow b_2b_3 + b_5b_6 + b_a(1 - b_5 - b_6 + b_7) \qquad\qquad (22)$$
$$\longrightarrow b_1b_4 + b_4 + b_a \qquad\qquad (23)$$
$$\longrightarrow b_5b_6 + b_1 + b_a(1 - b_5 - b_6 + b_7) \qquad\qquad (24)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad\qquad (k,n) = (4,7). \quad (25)$$
$$\longrightarrow b_3b_4 + b_4b_6 \qquad\qquad 89/128 \ (70\%) \quad (26)$$
$$\longrightarrow b_1b_2 + b_6b_7 \qquad\qquad 118/128 \ (92\%) \quad (27)$$
$$\longrightarrow b_2b_3 - b_5b_6 + b_5b_7 + b_5 \qquad\qquad 127/128 \ (99\%) \quad (28)$$
$$\longrightarrow b_1b_4 + 2b_5 - b_7 + 1 \qquad\qquad 128/128(100\%) \quad (29)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (5,7). \quad (30)$$
$$\longrightarrow b_2b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(b_5 + b_6 + b_7 - 1) - b_5 - b_6 - b_7 + 1 \qquad\qquad (31)$$
$$\longrightarrow b_1b_3 + b_3b_7 + b_a(1 + b_5 - b_7) - b_5 + 1 \qquad\qquad (32)$$
$$\longrightarrow b_1b_4 + b_4b_6 - b_5b_6 + b_5b_a - b_5 + b_6 + 1 \qquad\qquad (33)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (5,7). \quad (34)$$
$$\longrightarrow b_2b_3 + b_3b_7 \qquad\qquad 85/128 \ (66\%) \quad (35)$$
$$\longrightarrow 2b_4b_5 \qquad\qquad 121/128 \ (95\%) \quad (36)$$
$$\longrightarrow b_1b_2 + b_6b_7 - b_5 + 1 \qquad\qquad 128/128(100\%) \quad (37)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (6,7). \quad (38)$$
$$\longrightarrow 2b_3b_6 \qquad\qquad (39)$$
$$\longrightarrow 2b_4b_5 - b_5b_6 + b_5 \qquad\qquad (40)$$
$$\longrightarrow b_1b_4 - b_2b_5 + b_2b_6 + b_2b_7 + b_5b_7 - b_6b_7 - b_5 - b_6 + 2 \qquad\qquad (41)$$
$$\longrightarrow b_1b_2 - b_1b_5 + b_1b_7 + b_2b_3 + b_3b_6 - b_3b_7 - b_4b_5 - b_5b_6 - b_3 + b_5 + 2 \qquad\qquad (42)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (6,7). \quad (43)$$
$$\longrightarrow 2b_5b_6 \qquad\qquad 97/128 \ (76\%) \quad (44)$$
$$\longrightarrow b_1b_4 + b_4b_7 \qquad\qquad 119/128 \ (93\%) \quad (45)$$
$$\longrightarrow b_1b_3 + b_1b_7 + b_2b_3 - b_3b_6 + b_3b_7 - b_4b_5 - b_1 - b_7 + 2 \qquad\qquad 127/128 \ (99\%) \quad (46)$$
$$\longrightarrow b_1b_2 + b_2b_6 \qquad\qquad 128/128(100\%) \quad (47)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (6,7). \quad (48)$$

$$\longrightarrow b_5b_6 + b_5b_7 - b_5b_8 + b_6b_8 - b_7b_8 + b_8 \qquad\qquad (49)$$

$$\longrightarrow b_1b_4 + b_a(b_4 - b_7) + b_7 \qquad\qquad (50)$$

$$\longrightarrow b_2b_3 + b_2b_7 - b_5b_6 - b_7b_a + b_5 + b_7 \qquad\qquad (51)$$

$$\longrightarrow b_2b_3 + b_7b_a + b_3 \qquad\qquad (52)$$

$$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (5,8). \quad (53)$$

$$\longrightarrow b_1b_2 + b_6b_8 + b_a\left(1 - b_6 + b_7 - b_8\right) \qquad\qquad (54)$$

$$\longrightarrow b_3b_5 + b_8b_a + b_5 \qquad\qquad (55)$$

$$\longrightarrow b_4b_7 + b_4 \qquad\qquad (56)$$

$$\longrightarrow b_7b_8 + b_a(1 + b_6 - b_7 - b_8) + b_3 \qquad\qquad (57)$$

$$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (5,8). \quad (58)$$

$$\longrightarrow b_2b_5 + b_5b_8 \qquad\qquad 169/256 \ (66\%) \quad (59)$$

$$\longrightarrow b_1b_4 + b_4b_7 - b_5b_8 + b_8 \qquad\qquad 233/256 \ (91\%) \quad (60)$$

$$\longrightarrow b_1b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1 \qquad\qquad 252/256 \ (98\%) \quad (61)$$

$$\longrightarrow b_2b_3 + b_6b_7 \qquad\qquad 256/256(100\%) \quad (62)$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (6,8). \quad (63)$$

$$\longrightarrow b_1b_6 + b_6b_8 + b_a\left(1 - b_6 + b_7 - b_8\right) \qquad\qquad (64)$$

$$\longrightarrow b_2b_5 + b_4b_5 + b_4b_a \qquad\qquad (65)$$

$$\longrightarrow b_3b_4 + b_3b_7 - b_a + 1 \qquad\qquad (66)$$

$$\longrightarrow b_2b_4 + b_7b_8 \qquad\qquad (67)$$

$$\longrightarrow b_3b_4 + b_4 \qquad\qquad (68)$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (6,8). \quad (69)$$

$$\longrightarrow 2b_5b_6 \qquad\qquad 193/256 \ (75\%) \quad (70)$$

$$\longrightarrow b_1b_4 + b_4b_8 \qquad\qquad 237/256 \ (93\%) \quad (71)$$

$$\longrightarrow b_2b_3 + b_3b_7 - b_4b_6 + b_4b_8 - b_5b_7 - b_5b_8 + b_6b_8 - b_6 + b_7 - b_8 + 2 \qquad\qquad 254/256 \ (99\%) \quad (72)$$

$$\longrightarrow b_1b_2 + b_7b_8 \qquad\qquad 256/256(100\%) \quad (73)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (7,8). \quad (74)$$

$$\longrightarrow b_6b_7 + b_6b_8 + b_a\left(1 - b_6 + b_7 - b_8\right) \qquad\qquad (75)$$

$$\longrightarrow b_2b_3 + b_3b_4 \qquad\qquad (76)$$

$$\longrightarrow b_1b_4 + b_4b_8 - b_6b_a + b_6 \qquad\qquad (77)$$

$$\longrightarrow b_2b_3 + b_2 \qquad\qquad (78)$$

$$\longrightarrow b_1b_5 + b_3b_5 + b_6b_a \qquad\qquad (79)$$

$b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_2 b_3 b_4 b_5 b_6 b_7 b_8$: $\qquad\qquad (k, n) = (7, 8). \quad (80)$

$\longrightarrow 2 b_5 b_6 \qquad\qquad 193/256 \ (75\%) \quad (81)$

$\longrightarrow b_1 b_4 + b_4 b_8 \qquad\qquad 235/256 \ (92\%) \quad (82)$

$\longrightarrow b_2 b_3 + b_2 b_7 - b_5 b_6 + b_6 b_8 + b_5 - b_6 - b_8 + 1 \qquad\qquad 250/256 \ (98\%) \quad (83)$

$\longrightarrow b_3 b_7 + b_7 b_8 \qquad\qquad 254/256 \ (99\%) \quad (84)$

$\longrightarrow b_3 b_8 + b_3 \qquad\qquad 256/256 (100\%) \quad (85)$

$b_1 b_2 b_3 b_4 + b_5 b_6 b_7 b_8$: $\qquad\qquad (k, n) = (4, 8). \quad (86)$

$\longrightarrow b_1 b_2 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) \qquad\qquad (87)$

$\longrightarrow b_3 b_4 + b_6 b_8 + 2 b_8 b_a \qquad\qquad (88)$

$\longrightarrow b_2 b_3 + b_5 b_7 + b_a (1 - b_6 + b_7) \qquad\qquad (89)$

$\longrightarrow b_1 b_4 + b_5 b_7 - b_6 b_8 + b_7 b_a + b_6 \qquad\qquad (90)$

$b_1 b_2 b_3 b_4 + b_5 b_6 b_7 b_8$: $\qquad\qquad (k, n) = (4, 8). \quad (91)$

$\longrightarrow b_1 b_2 + b_6 b_7 \qquad\qquad 169/256 \ (66\%) \quad (92)$

$\longrightarrow b_3 b_4 + b_5 b_8 \qquad\qquad 238/256 \ (93\%) \quad (93)$

$\longrightarrow b_1 b_4 + b_5 b_6 + b_5 b_7 + b_6 b_7 - b_5 - b_6 - b_7 + 1 \qquad\qquad 248/256 \ (97\%) \quad (94)$

$\longrightarrow b_2 b_3 + b_6 b_7 + b_6 b_8 + b_7 b_8 - b_6 - b_7 - b_8 + 1 \qquad\qquad 254/256 \ (99\%) \quad (95)$

$\longrightarrow b_1 b_2 + b_5 b_8 \qquad\qquad 256/256 (100\%) \quad (96)$

$b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10}$: $\qquad\qquad (k, n) = (5, 10). \quad (97)$

$\longrightarrow b_2 b_3 + b_6 b_9 + b_9 b_a \qquad\qquad (98)$

$\longrightarrow b_1 b_4 + b_8 b_{10} + b_9 b_a \qquad\qquad (99)$

$\longrightarrow b_3 b_5 + b_7 b_{10} + b_1 b_a + b_9 b_a \qquad\qquad (100)$

$\longrightarrow b_4 b_5 + b_6 b_9 + b_9 b_a \qquad\qquad (101)$

$\longrightarrow b_1 b_2 + b_7 b_9 + b_9 b_a \qquad\qquad (102)$

$\longrightarrow b_2 b_5 + b_6 b_8 + b_9 b_a \qquad\qquad (103)$

$\longrightarrow b_2 b_3 + b_8 b_{10} + b_9 b_a \qquad\qquad (104)$

$\longrightarrow b_1 b_4 + b_6 b_7 + b_9 b_a \qquad\qquad (105)$

$b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10}$: $\qquad\qquad (k, n) = (5, 10). \quad (106)$

$\longrightarrow b_1 b_3 + b_9 b_{10} \qquad\qquad 625/1024 \ (61\%) \quad (107)$

$\longrightarrow b_2 b_4 + b_7 b_{10} \qquad\qquad 851/1024 \ (83\%) \quad (108)$

$\longrightarrow b_3 b_5 + b_5 b_{10} + b_8 b_9 \qquad\qquad 924/1024 \ (90\%) \quad (109)$

$\longrightarrow b_1 b_2 + b_6 \qquad\qquad 972/1024 \ (95\%) \quad (110)$

$\longrightarrow b_3 b_4 + b_8 b_9 \qquad\qquad 997/1024 \ (97\%) \quad (111)$

$\longrightarrow b_1 b_5 + b_7 b_{10} \qquad\qquad 1010/1024 \ (99\%) \quad (112)$

$\longrightarrow -b_1 b_7 - b_1 b_{10} + b_2 b_3 - b_2 b_8 - b_2 b_{10} + b_3 b_5 \qquad\qquad (113)$

$\qquad + b_6 b_9 + b_7 b_{10} - b_8 b_9 + b_9 b_{10} - b_3 - b_7 + b_8 + 3 \qquad\qquad 1016/1024 \ (99\%) \quad (114)$

$\longrightarrow b_1 b_3 + b_7 b_8 \qquad\qquad 1020/1024 \ (99\%) \quad (115)$

$\longrightarrow b_2 b_4 + b_2 b_6 - b_2 b_9 - b_3 b_{10} - b_5 b_7 + b_7 b_{10} + b_9 b_{10} - b_{10} + 2 \qquad\qquad 1023/1024 \ (99\%) \quad (116)$

$\longrightarrow b_2 b_5 + b_2 b_9 + b_6 b_8 \qquad\qquad 1024/1024 (100\%) \quad (117)$

$$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}: \qquad\qquad (k,n) = (6,10). \quad (118)$$
$$\longrightarrow b_5b_6 + b_5b_7 + b_a(1 - b_{10}) \qquad\qquad (119)$$
$$\longrightarrow b_1b_2 + b_8b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \qquad\qquad (120)$$
$$\longrightarrow b_3b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10} \qquad\qquad (121)$$
$$\longrightarrow b_3b_6 - b_5b_{10} + b_6b_7 + b_a(1 - b_9) + b_{10} \qquad\qquad (122)$$
$$\longrightarrow b_1b_2 + b_7b_{10} + b_a(1 - b_9 - b_{10}) + b_{10} \qquad\qquad (123)$$
$$\longrightarrow b_3b_4 + b_8b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1 \qquad\qquad (124)$$

$$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}: \qquad\qquad (k,n) = (6,10). \quad (125)$$
$$\longrightarrow 2b_5b_6 \qquad\qquad 769/1024 \ (75\%) \quad (126)$$
$$\longrightarrow b_1b_3 + b_8b_9 \qquad\qquad 934/1024 \ (92\%) \quad (127)$$
$$\longrightarrow b_2b_4 + b_7b_{10} + b_8b_9 - b_8 - b_9 + 1 \qquad\qquad 997/1024 \ (97\%) \quad (128)$$
$$\longrightarrow \ -b_1b_3 + b_1b_9 + b_2b_4 + b_4b_9 + b_5b_8 + b_8b_9 - b_5 - b_8 - b_9 + 2 \qquad\qquad 769/1024 \ (99\%) \quad (129)$$
$$\longrightarrow b_1b_3 + b_7b_{10} - b_8 - b_9 + 2 \qquad\qquad 1014/1024 \ (99\%) \quad (130)$$
$$\longrightarrow b_2b_3 + b_8b_9 \qquad\qquad 1024/1024(100\%) \quad (131)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}: \qquad\qquad (k,n) = (7,10). \quad (132)$$
$$\longrightarrow b_4b_7 + b_6b_7 + b_a(1 - b_4 - b_7 + b_{10}) \qquad\qquad (133)$$
$$\longrightarrow b_2b_5 + b_5b_9 + b_a \qquad\qquad (134)$$
$$\longrightarrow b_1b_4 + b_4b_8 + b_a \qquad\qquad (135)$$
$$\longrightarrow b_1b_3 + b_6b_{10} \qquad\qquad (136)$$
$$\longrightarrow b_3b_6 + b_6b_9 + b_a \qquad\qquad (137)$$
$$\longrightarrow b_2b_3 + b_8b_{10} \qquad\qquad (138)$$
$$\longrightarrow b_1b_4 + b_9 \qquad\qquad (139)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}: \qquad\qquad (k,n) = (7,10). \quad (140)$$
$$\longrightarrow b_3b_7 + b_7b_{10} \qquad\qquad 649/1024 \ (63\%) \quad (141)$$
$$\longrightarrow 2b_4b_6 \qquad\qquad 937/1024 \ (92\%) \quad (142)$$
$$\longrightarrow b_1b_5 + b_5b_8 \qquad\qquad 1001/1024 \ (98\%) \quad (143)$$
$$\longrightarrow b_1b_2 + b_9b_{10} \qquad\qquad 1019/1024 \ (99\%) \quad (144)$$
$$\longrightarrow b_2b_3 + b_8 \qquad\qquad 1023/1024 \ (99\%) \quad (145)$$
$$\longrightarrow b_3b_7 + b_9b_{10} \qquad\qquad 1024/1024(100\%) \quad (146)$$

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\hspace{6cm}$ $(k, n) = (8, 10).$ $\quad$ (147)

$\longrightarrow b_2b_4 + b_4b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ $\hspace{9cm}$ (148)

$\longrightarrow b_1b_7 + b_7b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10}$ $\hspace{9cm}$ (149)

$\longrightarrow b_5b_8 + b_6b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ $\hspace{9cm}$ (150)

$\longrightarrow b_3b_6 + b_a(b_3 - b_{10}) + b_{10}$ $\hspace{11cm}$ (151)

$\longrightarrow b_1b_5 + b_5b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ $\hspace{9cm}$ (152)

$\longrightarrow b_6b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1$ $\hspace{10cm}$ (153)

$\longrightarrow b_1b_2 - b_9b_a + b_{10} + 1$ $\hspace{12cm}$ (154)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\hspace{6cm}$ $(k, n) = (8, 10).$ $\quad$ (155)

$\longrightarrow 4b_3b_7$ $\hspace{9cm}$ 768/1024 (75%) (156)

$\longrightarrow b_2b_8 + b_8b_9$ $\hspace{9cm}$ 933/1024 (91%) (157)

$\longrightarrow 2b_4b_6 + b_8b_9 - b_8 - b_9 + 1$ $\hspace{7cm}$ 1005/1024 (98%) (158)

$\longrightarrow b_1b_5 + b_5b_{10} + b_8b_9 - b_8 - b_9 + 1$ $\hspace{6cm}$ 1022/1024 (99%) (159)

$\longrightarrow b_1b_2 + b_8b_9 + b_9b_{10} - b_8 - b_9 + 1$ $\hspace{6cm}$ 1024/1024(100%) (160)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\hspace{5cm}$ $(k, n) = (9, 10).$ $\quad$ (161)

$\longrightarrow b_1b_6 + b_6b_{10} + b_9b_a$ $\hspace{11cm}$ (162)

$\longrightarrow b_4b_7 + b_7b_8$ $\hspace{12cm}$ (163)

$\longrightarrow b_4b_9 + b_a(b_9 - b_4) + b_4$ $\hspace{11cm}$ (164)

$\longrightarrow b_2b_3 + b_3b_8 + b_9b_a$ $\hspace{11cm}$ (165)

$\longrightarrow b_1b_5 + b_2b_5 + b_9b_a$ $\hspace{11cm}$ (166)

$\longrightarrow b_2b_8 - b_6b_7 + b_8b_{10} + b_9b_a + 1$ $\hspace{10cm}$ (167)

$\longrightarrow b_2b_{10} + b_2$ $\hspace{12cm}$ (168)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\hspace{5cm}$ $(k, n) = (9, 10).$ $\quad$ (169)

$\longrightarrow 2b_2b_3 - b_8b_9 + b_9$ $\hspace{8cm}$ 577/1024 (56%) (170)

$\longrightarrow 3b_8b_9$ $\hspace{9cm}$ 961/1024 (94%) (171)

$\longrightarrow 2b_4b_6 - b_8b_9 - b_8b_{10} + b_{10} + 1$ $\hspace{6cm}$ 1009/1024 (99%) (172)

$\longrightarrow 2b_5b_7 - b_8b_{10} + b_{10}$ $\hspace{7cm}$ 1021/1024 (99%) (173)

$\longrightarrow b_1b_6 + b_{10}$ $\hspace{8cm}$ 1024/1024(100%) (174)

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $\hspace{6cm}$ $(k, n) = (4, 6).$ $\quad$ (175)

$\longrightarrow b_2b_4 + 2b_4b_5$ $\hspace{9cm}$ 43/64 (67%) (176)

$\longrightarrow b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$ $\hspace{5cm}$ 60/64 (94%) (177)

$\longrightarrow b_1b_2 + b_2b_5 + b_5b_6$ $\hspace{8cm}$ 64/64(100%) (178)

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 : \qquad\qquad (k,n) = (4,8). \quad (179)$$

$$\longrightarrow b_1b_4 + 2b_5b_6 \qquad\qquad 159/256 \ (62\%) \quad (180)$$

$$\longrightarrow b_2b_3 + b_3b_5 + b_7b_8 \qquad\qquad 225/256 \ (88\%) \quad (181)$$

$$\longrightarrow b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1 \qquad\qquad 244/256 \ (95.3\%) \quad (182)$$

$$\longrightarrow b_2b_3 + b_6b_8 + b_6 \qquad\qquad 253/256 \ (98.8\%) \quad (183)$$

$$\longrightarrow b_2b_3 + b_5b_7 + b_5 \qquad\qquad 256/256 \ (100\%) \quad (184)$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 : \qquad\qquad (k,n) = (4,8). \quad (185)$$

$$\longrightarrow b_2b_4 + 2b_5b_6 \qquad\qquad 159/256 \ (62\%) \quad (186)$$

$$\longrightarrow b_3b_6 + b_7b_8 + b_3 \qquad\qquad 212/256 \ (83\%) \quad (187)$$

$$\longrightarrow b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7 \qquad\qquad 234/256 \ (91\%) \quad (188)$$

$$\longrightarrow b_1b_3 + 2b_5b_6 \qquad\qquad 253/256 \ (99\%) \quad (189)$$

$$\longrightarrow b_7b_8 + b_1 + b_6 \qquad\qquad 256/256 (100\%) \quad (190)$$

$$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (5,7). \quad (191)$$

$$\longrightarrow b_1b_5 + 2b_5b_6 \qquad\qquad\qquad (192)$$

$$\longrightarrow b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1 \qquad\qquad (193)$$

$$\longrightarrow b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1) \qquad\qquad (194)$$

$$\qquad + b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5 \qquad\qquad (195)$$

$$\longrightarrow b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1 \qquad\qquad (196)$$

$$\longrightarrow b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2 \qquad\qquad (197)$$

$$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 : \qquad\qquad (k,n) = (5,7). \quad (198)$$

$$\longrightarrow 2b_4b_5 + b_4b_6 \qquad\qquad 81/128 \ (63\%) \quad (199)$$

$$\longrightarrow \qquad\qquad 111/128 \ (87\%) \quad (200)$$

$$\longrightarrow \qquad\qquad 122/128 \ (95\%) \quad (201)$$

$$\longrightarrow \qquad\qquad 128/128 (100\%) \quad (202)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (6,8). \quad (203)$$

$$\longrightarrow b_1b_3 + b_3b_5 + b_3b_8 + b_a \left(1 + b_6 - b_7\right) \qquad\qquad (204)$$

$$\longrightarrow b_2b_6 + b_6b_7 + b_a \left(-b_6 + b_7\right) + b_6 \qquad\qquad (205)$$

$$\longrightarrow b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4 \qquad\qquad (206)$$

$$\longrightarrow \ - b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8 \qquad\qquad (207)$$

$$\qquad + b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3 \qquad\qquad (208)$$

$$\longrightarrow b_1b_4 + b_6b_7 + b_7b_8 \qquad\qquad (209)$$

$$\longrightarrow b_2b_7 + b_7b_8 + b_8b_a + b_2 \qquad\qquad (210)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (6,8). \quad (211)$$

$$\longrightarrow b_1b_6 + 2b_6b_7 \qquad\qquad 164/256 \ (64\%) \quad (212)$$

$$\longrightarrow b_1b_5 + b_2b_5 - b_3b_6 + b_5b_8 + b_3 \qquad\qquad 219/256 \ (86\%) \quad (213)$$

$$\longrightarrow b_2b_4 + b_4b_7 + b_4b_8 - b_6 + 1 \qquad\qquad 243/256 \ (95\%) \quad (214)$$

$$\longrightarrow \qquad\qquad 253/256 \ (99\%) \quad (215)$$

$$\longrightarrow b_1b_2 + b_2b_6 + b_5b_7 - b_6b_7 + b_7b_8 - b_5 + 1 \qquad\qquad 256/256 (100\%) \quad (216)$$

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\qquad (k,n) = (8,10).$ (217)

$\longrightarrow b_3b_5 + b_5b_9 + b_5b_{10}$ (218)

$\longrightarrow b_1b_4 + b_4b_7 + b_4b_9$ (219)

$\longrightarrow b_1b_6 + b_2b_6 - b_5b_6 + b_6b_{10} + b_6$ (220)

$\longrightarrow b_4b_8 + 2b_8$ (221)

$\longrightarrow b_1b_7 + b_2b_7 - b_a(b_4 + b_6) + b_7b_9 + b_4 + 1$ (222)

$\longrightarrow b_2b_3 + b_3b_5 + b_3$ (223)

$\longrightarrow b_1b_2 + b_2b_9 - b_5b_6 - b_5b_a + b_6b_9 + 2$ (224)

$\longrightarrow b_2b_7 + b_2 + b_{10}$ (225)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\qquad (k,n) = (8,10).$ (226)

$\longrightarrow 3b_5b_8$ $\qquad$ 769/1024 (75%) (227)

$\longrightarrow 2b_2b_6 + b_4b_6$ $\qquad$ 931/1024 (91%) (228)

$\longrightarrow b_1b_7 - b_5b_{10} + b_7b_9 + b_9b_{10} - b_6 + b_{10} + 1$ $\qquad$ 984/1024 (96%) (229)

$\longrightarrow 3b_2b_3 + b_3b_{10} - b_6b_8 + 1$ $\qquad$ 1011/1024 (99%) (230)

$\longrightarrow b_4b_7 + b_4b_8 - b_3 + b_4 - b_8 + 2$ $\qquad$ 1019/1024 (99%) (231)

$\longrightarrow b_2b_3 - b_2b_4 - b_3b_4 - b_3b_8 - b_5b_{10} - b_6b_9 + b_7b_8 + b_7b_9 + b_8b_9 + b_7 + 3$ $\qquad$ 1023/1024 (99%) (232)

$\longrightarrow b_2b_8 + 2b_8b_9$ $\qquad$ 1024/1024(100%) (233)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $\qquad (k,n) = (6,10).$ (234)

$\longrightarrow 2b_3b_4 + b_7b_{10}$ $\qquad$ 591/1024 (58%) (235)

$\longrightarrow 2b_3b_5 + b_5b_6$ $\qquad$ 847/1024 (83%) (236)

$\longrightarrow b_1b_2 + b_7b_8 + b_8b_9$ $\qquad$ 951/1024 (93%) (237)

$\longrightarrow 3b_5b_6$ $\qquad$ 995/1024 (97%) (238)

$\longrightarrow b_1b_3 + b_3b_4 + b_9b_{10}$ $\qquad$ 1009/1024 (99%) (239)

$\longrightarrow b_1b_2 + b_5b_7 + b_7b_{10}$ $\qquad$ 1018/1024 (99%) (240)

$\longrightarrow 2b_1b_4 - b_1b_{10} + b_2b_4 + b_4b_5 + b_4b_{10} + b_5b_8 - b_6b_8 + b_8b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3$ $\qquad$ 1023/1024 (99%) (241)

$\longrightarrow b_2b_8 + b_3b_6 + b_6b_8$ $\qquad$ 1024/1024(100%) (242)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $\qquad (k,n) = (4,10).$ (243)

$\longrightarrow b_3b_4 + b_4b_6 + b_9b_{10}$ $\qquad$ 581/1024 (57%) (244)

$\longrightarrow b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9$ $\qquad$ 759/1024 (74%) (245)

$\longrightarrow b_5b_6 + b_8b_9 + b_1 + b_8$ $\qquad$ 842/1024 (82%) (246)

$\longrightarrow b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8$ $\qquad$ 935/1024 (91%) (247)

$\longrightarrow b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1$ $\qquad$ 969/1024 (95%) (248)

$\longrightarrow b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9$ $\qquad$ 992/1024 (97%) (249)

$\longrightarrow b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1$ $\qquad$ 1004/1024 (98%) (250)

$\longrightarrow b_1b_3 + b_6b_7 + b_9b_{10}$ $\qquad$ 1013/1024 (99%) (251)

$\longrightarrow b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9$ $\qquad$ 1019/1024 (99%) (252)

$\longrightarrow b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9$ $\qquad$ 1022/1024 (99%) (253)

$\longrightarrow -b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1$ $\qquad$ 1023/1024 (99%) (254)

$\longrightarrow b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2$ $\qquad$ 1024/1024(100%) (255)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 :$ $(k, n) = (5, 8).$ (256)

$\longrightarrow b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8)$ (257)

$\longrightarrow b_1b_3 + b_3b_6 + b_6b_7$ (258)

$\longrightarrow b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7)$ (259)

$\longrightarrow b_2b_5 + b_6b_8 + b_6$ (260)

$\longrightarrow b_1b_3 + b_3 + b_8$ (261)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 :$ $(k, n) = (5, 8).$ (262)

$\longrightarrow b_4b_5 + 2b_5b_6$ 165/256 (64%) (263)

$\longrightarrow b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7$ 215/256 (84%) (264)

$\longrightarrow b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7$ 242/256 (95%) (265)

$\longrightarrow b_1b_3 + b_5b_6 + b_6b_7$ 254/256 (99%) (266)

$\longrightarrow b_1b_2 + b_5b_6 + b_6b_8$ 256/256(100%) (267)

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k, n) = (3, 4).$ (268)

$\longrightarrow 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3$ 13/16 (81%) (269)

$\longrightarrow 2b_1b_3 + b_2b_3 + b_2$ 16/16(100%) (270)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} :$ $(k, n) = (7, 10).$ (271)

$\longrightarrow 4b_4b_5$ 769/1024 (75%) (272)

$\longrightarrow b_2b_6 + 2b_3b_6 + b_6b_9$ 915/1024 (89%) (273)

$\longrightarrow b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10}$ 974/1024 (95%) (274)

$\longrightarrow b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10}$ 995/1024 (97%) (275)

$\longrightarrow b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10}$ 1008/1024 (98%) (276)

$\longrightarrow b_1b_2 + b_2b_4 + b_9b_{10} + b_9$ 1016/1024 (99%) (277)

$\longrightarrow b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8$ 1023/1024 (99%) (278)

$\longrightarrow b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2$ 1024/1024(100%) (279)

DECOMPOSITION OF DEGREE-k, NOT EXACT-k-OF-n QUADRINOMIALS

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k, n) = (4, 4).$ (280)

$\longrightarrow b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (281)

$\longrightarrow b_1b_2 + b_1b_3 + 4b_1b_4 + b_2b_4$ 16/16(100%) (282)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k, n) = (4, 4).$ (283)

$\longrightarrow b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (284)

$\longrightarrow 2b_2b_3 + 3b_2b_4 + b_3b_4$ 16/16(100%) (285)

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + 3b_1b_3b_4 + b_2b_3b_4 : \qquad (k,n) = (4,4). \quad (286)$$
$$\longrightarrow 2b_1b_2 + 5b_1b_4 + b_3b_4 \qquad (287)$$
$$\longrightarrow -b_1b_2 + 3b_1b_3 + 4b_2b_3 + 2b_2b_4 - 4b_3b_4 + 4b_3 - b_4 + 1 \qquad (288)$$

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 : \qquad (k,n) = (4,4). \quad (289)$$
$$\longrightarrow 4b_1b_3 \qquad (290)$$
$$\longrightarrow 2b_1b_2 + b_1b_4 + b_2b_4 \qquad (291)$$

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 : \qquad (k,n) = (4,4). \quad (292)$$
$$\longrightarrow 2b_1b_3 + 2b_3b_4 \qquad 12/16 \ (75\%) \quad (293)$$
$$\longrightarrow 3b_1b_2 + b_1b_4 \qquad 16/16(100\%) \quad (294)$$