Volume 3: List of Multi-run Quadratizations

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(Dated: 20th October 2019)

DECOMPOSITION OF A MONOMIAL

$$b_1b_2b_3\dots b_k = \min\left(b_1b_2\dots b_{k_1}, b_{k_1+1}b_{k_1+2}\dots b_{k_2}, b_{k_2+1}b_{k_2+2}\dots b_{k_3}, ..., b_{k_n+1}b_{k_n+2}...b_k\right) \tag{1}$$

$$b_1b_2b_3\dots b_k = \min\left(b_1, b_2, b_3, \dots, b_k\right) \quad \text{(Example of Eq. 1: Linearization of a degree-}k\text{ monomial).} \tag{2}$$

$$b_1b_2b_3b_4 = \min\left(b_1b_2, b_3b_4\right) \quad \text{(Example of Eq. 1: Quadratization of a degree-4 monomial).} \tag{3}$$

$$b_1b_2b_3b_4b_5b_6b_7b_8 : \tag{4}$$
$$\longrightarrow\ 3b_a + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \tag{5}$$
$$\longrightarrow\ 3b_a + b_5b_6 + b_5b_7 + b_5b_8 + b_6b_7 + b_6b_8 + b_7b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \tag{6}$$

DECOMPOSITION OF BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min\left(b_2b_3 + b_3b_6, b_1b_4 + b_4b_5, b_1b_2 + b_5b_6 - b_3 - b_4 + 2\right) \quad (k, n) = (4, 6). \tag{7}$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min_{b_a}\left(b_2b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3b_4, b_1b_2 + b_5b_6 + b_5b_a\right) \quad (k, n) = (4, 6). \tag{8}$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \quad (k, n) = (4, 7). \tag{9}$$
$$\longrightarrow\ b_2b_3 + b_5b_6 + b_a(1 - b_5 - b_6 + b_7) \tag{10}$$
$$\longrightarrow\ b_1b_4 + b_4 + b_a \tag{11}$$
$$\longrightarrow\ b_5b_6 + b_1 + b_a(1 - b_5 - b_6 + b_7) \tag{12}$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \quad (k, n) = (5, 7). \tag{13}$$
$$\longrightarrow\ b_2b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(b_5 + b_6 + b_7 - 1) - b_5 - b_6 - b_7 + 1 \tag{14}$$
$$\longrightarrow\ b_1b_3 + b_3b_7 + b_a(1 + b_5 - b_7) - b_5 + 1 \tag{15}$$
$$\longrightarrow\ b_1b_4 + b_4b_6 - b_5b_6 + b_5b_a - b_5 + b_6 + 1 \tag{16}$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \quad (k, n) = (6, 7). \tag{17}$$
$$\longrightarrow\ 2b_3b_6 \tag{18}$$
$$\longrightarrow\ 2b_4b_5 - b_5b_6 + b_5 \tag{19}$$
$$\longrightarrow\ b_1b_4 - b_2b_5 + b_2b_6 + b_2b_7 + b_5b_7 - b_6b_7 - b_5 - b_6 + 2 \tag{20}$$
$$\longrightarrow\ b_1b_2 - b_1b_5 + b_1b_7 + b_2b_3 + b_3b_6 - b_3b_7 - b_4b_5 - b_5b_6 - b_3 + b_5 + 2 \tag{21}$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad (k,n) = (6,7). \tag{22}$$
$$\longrightarrow\ b_5b_6 + b_5b_7 - b_5b_8 + b_6b_8 - b_7b_8 + b_8 \tag{23}$$
$$\longrightarrow\ b_1b_4 + b_a(b_4 - b_7) + b_7 \tag{24}$$
$$\longrightarrow\ b_2b_3 + b_2b_7 - b_5b_6 - b_7b_a + b_5 + b_7 \tag{25}$$
$$\longrightarrow\ b_2b_3 + b_7b_a + b_3 \tag{26}$$

$$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : \qquad (k,n) = (5,8). \tag{27}$$
$$\longrightarrow\ b_1b_2 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8) \tag{28}$$
$$\longrightarrow\ b_3b_5 + b_8b_a + b_5 \tag{29}$$
$$\longrightarrow\ b_4b_7 + b_4 \tag{30}$$
$$\longrightarrow\ b_7b_8 + b_a(1 + b_6 - b_7 - b_8) + b_3 \tag{31}$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \qquad (k,n) = (6,8). \tag{32}$$
$$\longrightarrow\ b_1b_6 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8) \tag{33}$$
$$\longrightarrow\ b_2b_5 + b_4b_5 + b_4b_a \tag{34}$$
$$\longrightarrow\ b_3b_4 + b_3b_7 - b_a + 1 \tag{35}$$
$$\longrightarrow\ b_2b_4 + b_7b_8 \tag{36}$$
$$\longrightarrow\ b_3b_4 + b_4 \tag{37}$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 : \qquad (k,n) = (7,8). \tag{38}$$
$$\longrightarrow\ b_6b_7 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8) \tag{39}$$
$$\longrightarrow\ b_2b_3 + b_3b_4 \tag{40}$$
$$\longrightarrow\ b_1b_4 + b_4b_8 - b_6b_a + b_6 \tag{41}$$
$$\longrightarrow\ b_2b_3 + b_2 \tag{42}$$
$$\longrightarrow\ b_1b_5 + b_3b_5 + b_6b_a \tag{43}$$

$$b_1b_2b_3b_4 + b_5b_6b_7b_8 : \qquad (k,n) = (4,8). \tag{44}$$
$$\longrightarrow\ b_1b_2 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8) \tag{45}$$
$$\longrightarrow\ b_3b_4 + b_6b_8 + 2b_8b_a \tag{46}$$
$$\longrightarrow\ b_2b_3 + b_5b_7 + b_a\,(1 - b_6 + b_7) \tag{47}$$
$$\longrightarrow\ b_1b_4 + b_5b_7 - b_6b_8 + b_7b_a + b_6 \tag{48}$$

$$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10} : \qquad (k,n) = (5,10). \tag{49}$$
$$\longrightarrow\ b_2b_3 + b_6b_9 + b_9b_a \tag{50}$$
$$\longrightarrow\ b_1b_4 + b_8b_{10} + b_9b_a \tag{51}$$
$$\longrightarrow\ b_3b_5 + b_7b_{10} + b_1b_a + b_9b_a \tag{52}$$
$$\longrightarrow\ b_4b_5 + b_6b_9 + b_9b_a \tag{53}$$
$$\longrightarrow\ b_1b_2 + b_7b_9 + b_9b_a \tag{54}$$
$$\longrightarrow\ b_2b_5 + b_6b_8 + b_9b_a \tag{55}$$
$$\longrightarrow\ b_2b_3 + b_8b_{10} + b_9b_a \tag{56}$$
$$\longrightarrow\ b_1b_4 + b_6b_7 + b_9b_a \tag{57}$$

$$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (6,10). \tag{58}$$
$$\longrightarrow\ b_5b_6 + b_5b_7 + b_a(1 - b_{10}) \tag{59}$$
$$\longrightarrow\ b_1b_2 + b_8b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{60}$$
$$\longrightarrow\ b_3b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10} \tag{61}$$
$$\longrightarrow\ b_3b_6 - b_5b_{10} + b_6b_7 + b_a(1 - b_9) + b_{10} \tag{62}$$
$$\longrightarrow\ b_1b_2 + b_7b_{10} + b_a(1 - b_9 - b_{10}) + b_{10} \tag{63}$$
$$\longrightarrow\ b_3b_4 + b_8b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1 \tag{64}$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (7,10). \tag{65}$$
$$\longrightarrow\ b_4b_7 + b_6b_7 + b_a(1 - b_4 - b_7 + b_{10}) \tag{66}$$
$$\longrightarrow\ b_2b_5 + b_5b_9 + b_a \tag{67}$$
$$\longrightarrow\ b_1b_4 + b_4b_8 + b_a \tag{68}$$
$$\longrightarrow\ b_1b_3 + b_6b_{10} \tag{69}$$
$$\longrightarrow\ b_3b_6 + b_6b_9 + b_a \tag{70}$$
$$\longrightarrow\ b_2b_3 + b_8b_{10} \tag{71}$$
$$\longrightarrow\ b_1b_4 + b_9 \tag{72}$$

$$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (8,10). \tag{73}$$
$$\longrightarrow\ b_2b_4 + b_4b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{74}$$
$$\longrightarrow\ b_1b_7 + b_7b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{75}$$
$$\longrightarrow\ b_5b_8 + b_6b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{76}$$
$$\longrightarrow\ b_3b_6 + b_a(b_3 - b_{10}) + b_{10} \tag{77}$$
$$\longrightarrow\ b_1b_5 + b_5b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{78}$$
$$\longrightarrow\ b_6b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1 \tag{79}$$
$$\longrightarrow\ b_1b_2 - b_9b_a + b_{10} + 1 \tag{80}$$

$$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (9,10). \tag{81}$$
$$\longrightarrow\ b_1b_6 + b_6b_{10} + b_9b_a \tag{82}$$
$$\longrightarrow\ b_4b_7 + b_7b_8 \tag{83}$$
$$\longrightarrow\ b_4b_9 + b_a(b_9 - b_4) + b_4 \tag{84}$$
$$\longrightarrow\ b_2b_3 + b_3b_8 + b_9b_a \tag{85}$$
$$\longrightarrow\ b_1b_5 + b_2b_5 + b_9b_a \tag{86}$$
$$\longrightarrow\ b_2b_8 - b_6b_7 + b_8b_{10} + b_9b_a + 1 \tag{87}$$
$$\longrightarrow\ b_2b_{10} + b_2 \tag{88}$$

$$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 : \qquad (k,n) = (4,6). \tag{89}$$
$$\longrightarrow\ 3b_3b_4 + b_3b_5 + b_4b_5 - b_3 - b_4 - b_5 + 1 \qquad 41/64\ (64\%) \tag{90}$$
$$\longrightarrow\ b_1b_4 + b_3b_5 + b_4b_5 \qquad 56/64\ (88\%) \tag{91}$$
$$\longrightarrow\ b_1b_2 + b_2b_6 + b_3b_5 + b_5b_6 + b_2 - b_3 - b_4 - b_5 + 2 \qquad 64/64\ (100\%) \tag{92}$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 : \qquad (k,n) = (4,8). \tag{93}$$
$$\longrightarrow\ b_1b_4 + 2b_5b_6 \qquad 159/256\ (62\%) \tag{94}$$
$$\longrightarrow\ b_2b_3 + b_3b_5 + b_7b_8 \qquad 225/256\ (88\%) \tag{95}$$
$$\longrightarrow\ b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1 \qquad 244/256\ (95.3\%) \tag{96}$$
$$\longrightarrow\ b_2b_3 + b_6b_8 + b_6 \qquad 253/256\ (98.8\%) \tag{97}$$
$$\longrightarrow\ b_2b_3 + b_5b_7 + b_5 \qquad 256/256\ (100\%) \tag{98}$$

$$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \tag{99}$$
$$\longrightarrow\ b_1b_5 + 2b_5b_6 \tag{100}$$
$$\longrightarrow\ b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1 \tag{101}$$
$$\longrightarrow\ b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1) \tag{102}$$
$$+ b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5 \tag{103}$$
$$\longrightarrow\ b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1 \tag{104}$$
$$\longrightarrow\ b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2 \tag{105}$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 : \qquad (k,n) = (6,8). \tag{106}$$
$$\longrightarrow\ b_1b_3 + b_3b_5 + b_3b_8 + b_a\,(1 + b_6 - b_7) \tag{107}$$
$$\longrightarrow\ b_2b_6 + b_6b_7 + b_a\,(-b_6 + b_7) + b_6 \tag{108}$$
$$\longrightarrow\ b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4 \tag{109}$$
$$\longrightarrow\ -b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8 \tag{110}$$
$$+ b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3 \tag{111}$$
$$\longrightarrow\ b_1b_4 + b_6b_7 + b_7b_8 \tag{112}$$
$$\longrightarrow\ b_2b_7 + b_7b_8 + b_8b_a + b_2 \tag{113}$$

$$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (8,10). \tag{114}$$
$$\longrightarrow\ b_3b_5 + b_5b_9 + b_5b_{10} \tag{115}$$
$$\longrightarrow\ b_1b_4 + b_4b_7 + b_4b_9 \tag{116}$$
$$\longrightarrow\ b_1b_6 + b_2b_6 - b_5b_6 + b_6b_{10} + b_6 \tag{117}$$
$$\longrightarrow\ b_4b_8 + 2b_8 \tag{118}$$
$$\longrightarrow\ b_1b_7 + b_2b_7 - b_a(b_4 + b_6) + b_7b_9 + b_4 + 1 \tag{119}$$
$$\longrightarrow\ b_2b_3 + b_3b_5 + b_3 \tag{120}$$
$$\longrightarrow\ b_1b_2 + b_2b_9 - b_5b_6 - b_5b_a + b_6b_9 + 2 \tag{121}$$
$$\longrightarrow\ b_2b_7 + b_2 + b_{10} \tag{122}$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}: \qquad (k,n) = (6,10). \tag{123}$$
$$\longrightarrow\ b_2b_5 + b_5b_6 + b_5b_9 - b_9b_{11} + b_9 \tag{124}$$
$$\longrightarrow\ b_1b_4 + b_4b_6 + b_6b_{10} - b_9b_{11} + b_9 - 2b_{11} + 2 \tag{125}$$
$$\longrightarrow\ b_3b_4 + b_3b_7 + b_9b_{10} - b_9b_{11} + b_9 \tag{126}$$
$$\longrightarrow\ b_2b_3 + b_8b_{11} - b_9b_{11} + b_8 + b_9 \tag{127}$$
$$\longrightarrow\ b_1b_6 + b_6b_7 - b_9b_{11} + b_{10}b_{11} + b_9 \tag{128}$$
$$\longrightarrow\ b_1b_3 + b_7b_{10} - b_9b_{11} + b_7 + b_9 \tag{129}$$
$$\longrightarrow\ b_2b_4 + b_4b_7 - b_9b_{11} + 2b_{10}b_{11} + b_9 \tag{130}$$
$$\longrightarrow\ b_2b_6 + b_6b_7 + b_6b_8 - b_9b_{11} + b_9 \tag{131}$$
$$\longrightarrow\ b_1b_4 - b_2b_5 - b_3b_{10} + b_5b_8 - b_7b_{11} + b_8b_{10} - b_9b_{11} + b_{10}b_{11} + b_9 + 2 \qquad 1020/1024\ (99.6\%) \tag{132}$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}: \qquad (k,n) = (4,10). \tag{133}$$
$$\longrightarrow\ b_3b_4 + b_4b_5 + b_8b_{10} + b_a(1 - b_9) \tag{134}$$
$$\longrightarrow\ b_1b_2 + b_a(1 + b_3 - b_9) + b_5b_6 + b_7b_9 \tag{135}$$
$$\longrightarrow\ b_1b_3 + b_6b_7 + b_7b_{10} + b_a \tag{136}$$
$$\longrightarrow\ b_4b_6 + b_7b_9 + b_a(1 - b_9) + b_4 \tag{137}$$
$$\longrightarrow\ b_2b_3 + b_5b_7 + b_8b_9 + b_a(1 - b_9) \tag{138}$$
$$\longrightarrow\ b_2b_3 + b_8b_{10} + b_a(1 - b_9) + b_6 - b_9 + 1 \tag{139}$$
$$\longrightarrow\ b_1b_2 + b_5b_7 + b_8b_{10} + b_a(1 - b_9) - b_3 + 1 \tag{140}$$
$$\longrightarrow\ b_5b_6 + b_a(1 - b_9) + b_3 + b_9 \tag{141}$$
$$\longrightarrow\ b_7b_8 + b_2 + b_7 \qquad 1023/1024\ (99.9\%) \tag{142}$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8: \qquad (k,n) = (5,8). \tag{143}$$
$$\longrightarrow\ b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8) \tag{144}$$
$$\longrightarrow\ b_1b_3 + b_3b_6 + b_6b_7 \tag{145}$$
$$\longrightarrow\ b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7) \tag{146}$$
$$\longrightarrow\ b_2b_5 + b_6b_8 + b_6 \tag{147}$$
$$\longrightarrow\ b_1b_3 + b_3 + b_8 \tag{148}$$

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$$b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4: \qquad (k,n) = (3,4). \tag{149}$$
$$\longrightarrow\ 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3 \tag{150}$$
$$\longrightarrow\ 3b_1b_3 + b_2b_4 \tag{151}$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (7,10). \tag{152}$$

$$\longrightarrow\ b_1b_2 + b_2b_5 + b_5b_7 + b_5b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{153}$$

$$\longrightarrow\ b_5b_6 + b_6b_8 + b_6b_9 + b_6b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{154}$$

$$\longrightarrow\ b_1b_7 + b_5b_7 + b_6b_7 + b_7b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{155}$$

$$\longrightarrow\ b_4b_5 + b_4b_8 + b_4b_{10} + b_a(b_4 - b_9 - b_{10}) + b_9 + b_{10} \tag{156}$$

$$\longrightarrow\ b_1b_3 + b_3b_4 + b_3b_9 - b_6b_7 + b_6b_{10} + b_a(1 - b_9 - b_{10}) - b_6 + b_{10} + 2 \tag{157}$$

$$\longrightarrow\ b_2b_3 + b_2b_8 + b_3b_9 - b_5b_7 + b_6b_9 - b_a(1 + b_{10}) - b_7 + b_9 + b_{10} + 3 \tag{158}$$

$$\longrightarrow\ b_2b_3 - b_6b_7 + b_6b_8 + b_7b_8 - b_a(b_9 + b_{10}) + b_8 + b_9 + b_{10} + 1 \tag{159}$$

$$\longrightarrow\ -b_1b_2 + b_1b_4 + b_2b_5 - b_4b_9 + b_5b_8 + b_5b_9 - b_a(b_9 + b_{10}) - b_4 + b_5 - b_7 + 2b_9 + b_{10} + 2 \tag{160}$$

$$\longrightarrow\ b_2b_8 + b_8b_{10} + b_a(b_8 - b_9 - b_{10}) + b_1 + b_9 + b_{10} \qquad 1021/1024\ (99.7\%) \tag{161}$$

DECOMPOSITION OF DEGREE-k, NOT EXACT-k-OF-n QUADRINOMIALS

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 : \qquad (k,n) = (4,4). \tag{162}$$

$$\longrightarrow\ b_1b_2 + b_1b_3 + b_1b_4 + 4b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1 \tag{163}$$

$$\longrightarrow\ b_1b_2 + b_1b_3 + 4b_1b_4 + b_2b_4 \tag{164}$$

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + 3b_1b_3b_4 + b_2b_3b_4 : \qquad (k,n) = (4,4). \tag{165}$$

$$\longrightarrow\ 2b_1b_2 + 5b_1b_4 + b_3b_4 \tag{166}$$

$$\longrightarrow\ -b_1b_2 + 3b_1b_3 + 4b_2b_3 + 2b_2b_4 - 4b_3b_4 + 4b_3 - b_4 + 1 \tag{167}$$

$$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 : \qquad (k,n) = (4,4). \tag{168}$$

$$\longrightarrow\ 4b_1b_3 \tag{169}$$

$$\longrightarrow\ 2b_1b_2 + b_1b_4 + b_2b_4 \tag{170}$$