Volume 3: List of Multi-run Quadratizations

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DECOMPOSITION OF A MONOMIAL

$$b_1b_2b_3\dots b_k = \min\left(b_1b_2\dots b_{k_1},b_{k_1+1}b_{k_1+2}\dots b_{k_2},b_{k_2+1}b_{k_2+2}\dots b_{k_3},...,b_{k_n+1}b_{k_n+2}...b_k\right) \quad (1)$$

$$b_1b_2b_3\dots b_k = \min\left(b_1,b_2,b_3,\dots,b_k\right) \quad \text{(Example of Eq. 1: Linearization of a degree-}k\text{ monomial).} \quad (2)$$

$$b_1b_2b_3b_4 = \min\left(b_1b_2,b_3b_4\right) \quad \text{(Example of Eq. 1: Quadratization of a degree-4 monomial).} \quad (3)$$

$$b_1b_2b_3b_4b_5b_6b_7b_8: \quad (4)$$
$$\longrightarrow 3b_a + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \quad (5)$$
$$\longrightarrow 3b_a + b_5b_6 + b_5b_7 + b_5b_8 + b_6b_7 + b_6b_8 + b_7b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \quad (6)$$

Quantum envelopes:
Cubic:

$$A_1B_2C_3: \quad (7)$$
$$\longrightarrow 1 + (A_1 - B_2C_3) \quad 6/8\ (75\%) \quad (8)$$
$$\longrightarrow 1 - (A_1 - B_2C_3) \quad 8/8(100\%) \quad (9)$$

where A, B and C can be any of the Pauli matrices X, Y or Z. For example:

$$Z_1Y_2Y_3: \quad (10)$$
$$\longrightarrow 1 + (Z_1 - Y_2Y_3) \quad 6/8\ (75\%) \quad (11)$$
$$\longrightarrow 1 - (Z_1 - Y_2Y_3) \quad 8/8(100\%) \quad (12)$$

or

$$X_1Y_2Z_3: \quad (13)$$
$$\longrightarrow 1 + (X_1 - Y_2Z_3) \quad 6/8\ (75\%) \quad (14)$$
$$\longrightarrow 1 - (X_1 - Y_2Z_3) \quad 8/8(100\%) \quad (15)$$

Quartic:

$$A_1B_2C_3D_4: \quad (16)$$
$$\longrightarrow 1 + (A_1B_2 - C_3D_4) \quad 12/16\ (75\%) \quad (17)$$
$$\longrightarrow 1 - (A_1B_2 - C_3D_4) \quad 16/16(100\%) \quad (18)$$

where A, B, C and D can be any of the Pauli matrices X, Y or Z. For example:

$Z_1X_2Y_3X_4$: (19)
$\longrightarrow$ $1+(Z_1X_2-Y_3X_4)$ 12/16 (75%) (20)
$\longrightarrow$ $1-(Z_1X_2-Y_3X_4)$ 16/16(100%) (21)

Degree-k:

$A_1B_2C_3...D_k$: (22)
$\longrightarrow$ $1+(A_1B_2-C_3...D_k)$ (75%) (23)
$\longrightarrow$ $1-(A_1B_2-C_3...D_k)$ (100%) (24)

where $A, B, C, ..., D$ can be any of the Pauli matrices X, Y or Z. For example:

$X_1Y_2X_3Z_4Y_5Z_6$: (25)
$\longrightarrow$ $1+(X_1Y_2-X_3Z_4Y_5Z_6)$ 48/64 (75%) (26)
$\longrightarrow$ $1-(X_1Y_2-X_3Z_4Y_5Z_6)$ 64/64(100%) (27)

DECOMPOSITION OF BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4+b_3b_4b_5b_6=\min\left(2b_3b_4, b_1b_2+b_5b_6\right) \qquad (k,n)=(4,6). \quad (28)$$

$$b_1b_2b_3b_4+b_3b_4b_5b_6=\min_{b_a}\left(b_2b_3+b_a(1-b_2-b_3+2b_4)+b_3b_4, b_1b_2+b_5b_6+b_5b_a\right) \qquad (k,n)=(4,6). \quad (29)$$

$b_1b_2b_3b_4+b_4b_5b_6b_7$: $(k,n)=(4,7)$. (30)
$\longrightarrow$ $b_2b_3+b_5b_6+b_a(1-b_5-b_6+b_7)$ (31)
$\longrightarrow$ $b_1b_4+b_4+b_a$ (32)
$\longrightarrow$ $b_5b_6+b_1+b_a(1-b_5-b_6+b_7)$ (33)

$b_1b_2b_3b_4+b_4b_5b_6b_7$: $(k,n)=(4,7)$. (34)
$\longrightarrow$ $b_3b_4+b_4b_6$ 89/128 (70%) (35)
$\longrightarrow$ $b_1b_2+b_6b_7$ 118/128 (92%) (36)
$\longrightarrow$ $b_2b_3-b_5b_6+b_5b_7+b_5$ 127/128 (99%) (37)
$\longrightarrow$ $b_1b_4+2b_5-b_7+1$ 128/128(100%) (38)

$b_1b_2b_3b_4b_5+b_3b_4b_5b_6b_7$: $(k,n)=(5,7)$. (39)
$\longrightarrow$ $b_2b_5+b_5b_6+b_5b_7+b_6b_7+b_a(b_5+b_6+b_7-1)-b_5-b_6-b_7+1$ (40)
$\longrightarrow$ $b_1b_3+b_3b_7+b_a(1+b_5-b_7)-b_5+1$ (41)
$\longrightarrow$ $b_1b_4+b_4b_6-b_5b_6+b_5b_a-b_5+b_6+1$ (42)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (43)
$\longrightarrow\ b_2b_3 + b_3b_7$ 85/128 (66%) (44)
$\longrightarrow\ 2b_4b_5$ 121/128 (95%) (45)
$\longrightarrow\ b_1b_2 + b_6b_7 - b_5 + 1$ 128/128(100%) (46)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7$: $(k,n) = (6,7)$. (47)
$\longrightarrow\ 2b_3b_6$ (48)
$\longrightarrow\ 2b_4b_5 - b_5b_6 + b_5$ (49)
$\longrightarrow\ b_1b_4 - b_2b_5 + b_2b_6 + b_2b_7 + b_5b_7 - b_6b_7 - b_5 - b_6 + 2$ (50)
$\longrightarrow\ b_1b_2 - b_1b_5 + b_1b_7 + b_2b_3 + b_3b_6 - b_3b_7 - b_4b_5 - b_5b_6 - b_3 + b_5 + 2$ (51)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7$: $(k,n) = (6,7)$. (52)
$\longrightarrow\ 2b_5b_6$ 97/128 (76%) (53)
$\longrightarrow\ b_1b_4 + b_4b_7$ 119/128 (93%) (54)
$\longrightarrow\ b_1b_3 + b_1b_7 + b_2b_3 - b_3b_6 + b_3b_7 - b_4b_5 - b_1 - b_7 + 2$ 127/128 (99%) (55)
$\longrightarrow\ b_1b_2 + b_2b_6$ 128/128(100%) (56)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7$: $(k,n) = (6,7)$. (57)
$\longrightarrow\ b_5b_6 + b_5b_7 - b_5b_8 + b_6b_8 - b_7b_8 + b_8$ (58)
$\longrightarrow\ b_1b_4 + b_a(b_4 - b_7) + b_7$ (59)
$\longrightarrow\ b_2b_3 + b_2b_7 - b_5b_6 - b_7b_a + b_5 + b_7$ (60)
$\longrightarrow\ b_2b_3 + b_7b_a + b_3$ (61)

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (62)
$\longrightarrow\ b_1b_2 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8)$ (63)
$\longrightarrow\ b_3b_5 + b_8b_a + b_5$ (64)
$\longrightarrow\ b_4b_7 + b_4$ (65)
$\longrightarrow\ b_7b_8 + b_a(1 + b_6 - b_7 - b_8) + b_3$ (66)

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (67)
$\longrightarrow\ b_2b_5 + b_5b_8$ 169/256 (66%) (68)
$\longrightarrow\ b_1b_4 + b_4b_7 - b_5b_8 + b_8$ 233/256 (91%) (69)
$\longrightarrow\ b_1b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 252/256 (98%) (70)
$\longrightarrow\ b_2b_3 + b_6b_7$ 256/256(100%) (71)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (72)
$\longrightarrow\ b_1b_6 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8)$ (73)
$\longrightarrow\ b_2b_5 + b_4b_5 + b_4b_a$ (74)
$\longrightarrow\ b_3b_4 + b_3b_7 - b_a + 1$ (75)
$\longrightarrow\ b_2b_4 + b_7b_8$ (76)
$\longrightarrow\ b_3b_4 + b_4$ (77)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (78)
$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (79)
$\longrightarrow\ b_1b_4 + b_4b_8$ 237/256 (93%) (80)
$\longrightarrow\ b_2b_3 + b_3b_7 - b_4b_6 + b_4b_8 - b_5b_7 - b_5b_8 + b_6b_8 - b_6 + b_7 - b_8 + 2$ 254/256 (99%) (81)
$\longrightarrow\ b_1b_2 + b_7b_8$ 256/256(100%) (82)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8$: $(k,n) = (7,8)$. (83)
$\longrightarrow\ b_6b_7 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8)$ (84)
$\longrightarrow\ b_2b_3 + b_3b_4$ (85)
$\longrightarrow\ b_1b_4 + b_4b_8 - b_6b_a + b_6$ (86)
$\longrightarrow\ b_2b_3 + b_2$ (87)
$\longrightarrow\ b_1b_5 + b_3b_5 + b_6b_a$ (88)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8$: $(k,n) = (7,8)$. (89)
$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (90)
$\longrightarrow\ b_1b_4 + b_4b_8$ 235/256 (92%) (91)
$\longrightarrow\ b_2b_3 + b_2b_7 - b_5b_6 + b_6b_8 + b_5 - b_6 - b_8 + 1$ 250/256 (98%) (92)
$\longrightarrow\ b_3b_7 + b_7b_8$ 254/256 (99%) (93)
$\longrightarrow\ b_3b_8 + b_3$ 256/256(100%) (94)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (95)
$\longrightarrow\ b_1b_2 + b_6b_8 + b_a\,(1 - b_6 + b_7 - b_8)$ (96)
$\longrightarrow\ b_3b_4 + b_6b_8 + 2b_8b_a$ (97)
$\longrightarrow\ b_2b_3 + b_5b_7 + b_a\,(1 - b_6 + b_7)$ (98)
$\longrightarrow\ b_1b_4 + b_5b_7 - b_6b_8 + b_7b_a + b_6$ (99)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (100)
$\longrightarrow\ b_1b_2 + b_6b_7$ 169/256 (66%) (101)
$\longrightarrow\ b_3b_4 + b_5b_8$ 238/256 (93%) (102)
$\longrightarrow\ b_1b_4 + b_5b_6 + b_5b_7 + b_6b_7 - b_5 - b_6 - b_7 + 1$ 248/256 (97%) (103)
$\longrightarrow\ b_2b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 254/256 (99%) (104)
$\longrightarrow\ b_1b_2 + b_5b_8$ 256/256(100%) (105)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (106)
$\longrightarrow\ b_2b_3 + b_6b_9 + b_9b_a$ (107)
$\longrightarrow\ b_1b_4 + b_8b_{10} + b_9b_a$ (108)
$\longrightarrow\ b_3b_5 + b_7b_{10} + b_1b_a + b_9b_a$ (109)
$\longrightarrow\ b_4b_5 + b_6b_9 + b_9b_a$ (110)
$\longrightarrow\ b_1b_2 + b_7b_9 + b_9b_a$ (111)
$\longrightarrow\ b_2b_5 + b_6b_8 + b_9b_a$ (112)
$\longrightarrow\ b_2b_3 + b_8b_{10} + b_9b_a$ (113)
$\longrightarrow\ b_1b_4 + b_6b_7 + b_9b_a$ (114)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (115)

$\longrightarrow\ b_1b_3 + b_9b_{10}$ 625/1024 (61%) (116)

$\longrightarrow\ b_2b_4 + b_7b_{10}$ 851/1024 (83%) (117)

$\longrightarrow\ b_3b_5 + b_5b_{10} + b_8b_9$ 924/1024 (90%) (118)

$\longrightarrow\ b_1b_2 + b_6$ 972/1024 (95%) (119)

$\longrightarrow\ b_3b_4 + b_8b_9$ 997/1024 (97%) (120)

$\longrightarrow\ b_1b_5 + b_7b_{10}$ 1010/1024 (99%) (121)

$\longrightarrow\ -b_1b_7 - b_1b_{10} + b_2b_3 - b_2b_8 - b_2b_{10} + b_3b_5$ (122)

$\quad + b_6b_9 + b_7b_{10} - b_8b_9 + b_9b_{10} - b_3 - b_7 + b_8 + 3$ 1016/1024 (99%) (123)

$\longrightarrow\ b_1b_3 + b_7b_8$ 1020/1024 (99%) (124)

$\longrightarrow\ b_2b_4 + b_2b_6 - b_2b_9 - b_3b_{10} - b_5b_7 + b_7b_{10} + b_9b_{10} - b_{10} + 2$ 1023/1024 (99%) (125)

$\longrightarrow\ b_2b_5 + b_2b_9 + b_6b_8$ 1024/1024(100%) (126)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (127)

$\longrightarrow\ b_5b_6 + b_5b_7 + b_a(1 - b_{10})$ (128)

$\longrightarrow\ b_1b_2 + b_8b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ (129)

$\longrightarrow\ b_3b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10}$ (130)

$\longrightarrow\ b_3b_6 - b_5b_{10} + b_6b_7 + b_a(1 - b_9) + b_{10}$ (131)

$\longrightarrow\ b_1b_2 + b_7b_{10} + b_a(1 - b_9 - b_{10}) + b_{10}$ (132)

$\longrightarrow\ b_3b_4 + b_8b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1$ (133)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (134)

$\longrightarrow\ 2b_5b_6$ 769/1024 (75%) (135)

$\longrightarrow\ b_1b_3 + b_8b_9$ 934/1024 (92%) (136)

$\longrightarrow\ b_2b_4 + b_7b_{10} + b_8b_9 - b_8 - b_9 + 1$ 997/1024 (97%) (137)

$\longrightarrow\ -b_1b_3 + b_1b_9 + b_2b_4 + b_4b_9 + b_5b_8 + b_8b_9 - b_5 - b_8 - b_9 + 2$ 769/1024 (99%) (138)

$\longrightarrow\ b_1b_3 + b_7b_{10} - b_8 - b_9 + 2$ 1014/1024 (99%) (139)

$\longrightarrow\ b_2b_3 + b_8b_9$ 1024/1024(100%) (140)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (141)

$\longrightarrow\ b_4b_7 + b_6b_7 + b_a(1 - b_4 - b_7 + b_{10})$ (142)

$\longrightarrow\ b_2b_5 + b_5b_9 + b_a$ (143)

$\longrightarrow\ b_1b_4 + b_4b_8 + b_a$ (144)

$\longrightarrow\ b_1b_3 + b_6b_{10}$ (145)

$\longrightarrow\ b_3b_6 + b_6b_9 + b_a$ (146)

$\longrightarrow\ b_2b_3 + b_8b_{10}$ (147)

$\longrightarrow\ b_1b_4 + b_9$ (148)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (149)
$\longrightarrow\ b_3b_7 + b_7b_{10}$ 649/1024 (63%) (150)
$\longrightarrow\ 2b_4b_6$ 937/1024 (92%) (151)
$\longrightarrow\ b_1b_5 + b_5b_8$ 1001/1024 (98%) (152)
$\longrightarrow\ b_1b_2 + b_9b_{10}$ 1019/1024 (99%) (153)
$\longrightarrow\ b_2b_3 + b_8$ 1023/1024 (99%) (154)
$\longrightarrow\ b_3b_7 + b_9b_{10}$ 1024/1024(100%) (155)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (156)
$\longrightarrow\ b_2b_4 + b_4b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ (157)
$\longrightarrow\ b_1b_7 + b_7b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10}$ (158)
$\longrightarrow\ b_5b_8 + b_6b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ (159)
$\longrightarrow\ b_3b_6 + b_a(b_3 - b_{10}) + b_{10}$ (160)
$\longrightarrow\ b_1b_5 + b_5b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10}$ (161)
$\longrightarrow\ b_6b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1$ (162)
$\longrightarrow\ b_1b_2 - b_9b_a + b_{10} + 1$ (163)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (164)
$\longrightarrow\ 4b_3b_7$ 768/1024 (75%) (165)
$\longrightarrow\ b_2b_8 + b_8b_9$ 933/1024 (91%) (166)
$\longrightarrow\ 2b_4b_6 + b_8b_9 - b_8 - b_9 + 1$ 1005/1024 (98%) (167)
$\longrightarrow\ b_1b_5 + b_5b_{10} + b_8b_9 - b_8 - b_9 + 1$ 1022/1024 (99%) (168)
$\longrightarrow\ b_1b_2 + b_8b_9 + b_9b_{10} - b_8 - b_9 + 1$ 1024/1024(100%) (169)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (170)
$\longrightarrow\ b_1b_6 + b_6b_{10} + b_9b_a$ (171)
$\longrightarrow\ b_4b_7 + b_7b_8$ (172)
$\longrightarrow\ b_4b_9 + b_a(b_9 - b_4) + b_4$ (173)
$\longrightarrow\ b_2b_3 + b_3b_8 + b_9b_a$ (174)
$\longrightarrow\ b_1b_5 + b_2b_5 + b_9b_a$ (175)
$\longrightarrow\ b_2b_8 - b_6b_7 + b_8b_{10} + b_9b_a + 1$ (176)
$\longrightarrow\ b_2b_{10} + b_2$ (177)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (178)
$\longrightarrow\ 2b_2b_3 - b_8b_9 + b_9$ 577/1024 (56%) (179)
$\longrightarrow\ 3b_8b_9$ 961/1024 (94%) (180)
$\longrightarrow\ 2b_4b_6 - b_8b_9 - b_8b_{10} + b_{10} + 1$ 1009/1024 (99%) (181)
$\longrightarrow\ 2b_5b_7 - b_8b_{10} + b_{10}$ 1021/1024 (99%) (182)
$\longrightarrow\ b_1b_6 + b_{10}$ 1024/1024(100%) (183)

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $(k,n) = (4,6)$. (184)
$\longrightarrow\ b_2b_4 + 2b_4b_5$ 43/64 (67%) (185)
$\longrightarrow\ b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$ 60/64 (94%) (186)
$\longrightarrow\ b_1b_2 + b_2b_5 + b_5b_6$ 64/64(100%) (187)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (188)
$\longrightarrow\ b_1b_4 + 2b_5b_6$ 159/256 (62%) (189)
$\longrightarrow\ b_2b_3 + b_3b_5 + b_7b_8$ 225/256 (88%) (190)
$\longrightarrow\ b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1$ 244/256 (95.3%) (191)
$\longrightarrow\ b_2b_3 + b_6b_8 + b_6$ 253/256 (98.8%) (192)
$\longrightarrow\ b_2b_3 + b_5b_7 + b_5$ 256/256 (100%) (193)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (194)
$\longrightarrow\ b_2b_4 + 2b_5b_6$ 159/256 (62%) (195)
$\longrightarrow\ b_3b_6 + b_7b_8 + b_3$ 212/256 (83%) (196)
$\longrightarrow\ b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7$ 234/256 (91%) (197)
$\longrightarrow\ b_1b_3 + 2b_5b_6$ 253/256 (99%) (198)
$\longrightarrow\ b_7b_8 + b_1 + b_6$ 256/256(100%) (199)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (200)
$\longrightarrow\ b_1b_5 + 2b_5b_6$ (201)
$\longrightarrow\ b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1$ (202)
$\longrightarrow\ b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1)$ (203)
$\quad + b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5$ (204)
$\longrightarrow\ b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1$ (205)
$\longrightarrow\ b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2$ (206)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (207)
$\longrightarrow\ 2b_4b_5 + b_4b_6$ 81/128 (63%) (208)
$\longrightarrow\ b_1b_3 + b_3b_6 + b_3b_7 - b_4b_5 + b_5$ 111/128 (87%) (209)
$\longrightarrow\ b_1b_2 + b_2b_6 - b_4b_5 + b_6b_7 + b_5$ 122/128 (95%) (210)
$\longrightarrow\ 2b_4b_5 + b_5$ 128/128(100%) (211)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (212)
$\longrightarrow\ b_1b_3 + b_3b_5 + b_3b_8 + b_a\,(1 + b_6 - b_7)$ (213)
$\longrightarrow\ b_2b_6 + b_6b_7 + b_a\,(-b_6 + b_7) + b_6$ (214)
$\longrightarrow\ b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4$ (215)
$\longrightarrow\ -b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8$ (216)
$\quad + b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3$ (217)
$\longrightarrow\ b_1b_4 + b_6b_7 + b_7b_8$ (218)
$\longrightarrow\ b_2b_7 + b_7b_8 + b_8b_a + b_2$ (219)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (220)

$\longrightarrow\ b_1b_6 + 2b_6b_7$ 164/256 (64%) (221)

$\longrightarrow\ b_1b_5 + b_2b_5 - b_3b_6 + b_5b_8 + b_3$ 219/256 (86%) (222)

$\longrightarrow\ b_2b_4 + b_4b_7 + b_4b_8 - b_6 + 1$ 243/256 (95%) (223)

$\longrightarrow\ b_2b_3 + b_3b_8 - b_5b_6 + b_3 + b_6$ 253/256 (99%) (224)

$\longrightarrow\ b_1b_2 + b_2b_6 + b_5b_7 - b_6b_7 + b_7b_8 - b_5 + 1$ 256/256(100%) (225)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (226)

$\longrightarrow\ b_3b_5 + b_5b_9 + b_5b_{10}$ (227)

$\longrightarrow\ b_1b_4 + b_4b_7 + b_4b_9$ (228)

$\longrightarrow\ b_1b_6 + b_2b_6 - b_5b_6 + b_6b_{10} + b_6$ (229)

$\longrightarrow\ b_4b_8 + 2b_8$ (230)

$\longrightarrow\ b_1b_7 + b_2b_7 - b_a(b_4 + b_6) + b_7b_9 + b_4 + 1$ (231)

$\longrightarrow\ b_2b_3 + b_3b_5 + b_3$ (232)

$\longrightarrow\ b_1b_2 + b_2b_9 - b_5b_6 - b_5b_a + b_6b_9 + 2$ (233)

$\longrightarrow\ b_2b_7 + b_2 + b_{10}$ (234)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (235)

$\longrightarrow\ 3b_5b_8$ 769/1024 (75%) (236)

$\longrightarrow\ 2b_2b_6 + b_4b_6$ 931/1024 (91%) (237)

$\longrightarrow\ b_1b_7 - b_5b_{10} + b_7b_9 + b_9b_{10} - b_6 + b_{10} + 1$ 984/1024 (96%) (238)

$\longrightarrow\ 3b_2b_3 + b_3b_{10} - b_6b_8 + 1$ 1011/1024 (99%) (239)

$\longrightarrow\ b_4b_7 + b_4b_8 - b_3 + b_4 - b_8 + 2$ 1019/1024 (99%) (240)

$\longrightarrow\ b_2b_3 - b_2b_4 - b_3b_4 - b_3b_8 - b_5b_{10} - b_6b_9 + b_7b_8 + b_7b_9 + b_8b_9 + b_7 + 3$ 1023/1024 (99%) (241)

$\longrightarrow\ b_2b_8 + 2b_8b_9$ 1024/1024(100%) (242)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (243)

$\longrightarrow\ 2b_3b_4 + b_7b_{10}$ 591/1024 (58%) (244)

$\longrightarrow\ 2b_3b_5 + b_5b_6$ 847/1024 (83%) (245)

$\longrightarrow\ b_1b_2 + b_7b_8 + b_8b_9$ 951/1024 (93%) (246)

$\longrightarrow\ 3b_5b_6$ 995/1024 (97%) (247)

$\longrightarrow\ b_1b_3 + b_3b_4 + b_9b_{10}$ 1009/1024 (99%) (248)

$\longrightarrow\ b_1b_2 + b_5b_7 + b_7b_{10}$ 1018/1024 (99%) (249)

$\longrightarrow\ 2b_1b_4 - b_1b_{10} + b_2b_4 + b_4b_5 + b_4b_{10} + b_5b_8 - b_6b_8 + b_8b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3$ 1023/1024 (99%) (250)

$\longrightarrow\ b_2b_8 + b_3b_6 + b_6b_8$ 1024/1024(100%) (251)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $(k,n) = (4,10)$. (252)
$\longrightarrow\ b_3b_4 + b_4b_6 + b_9b_{10}$ 581/1024 (57%) (253)
$\longrightarrow\ b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9$ 759/1024 (74%) (254)
$\longrightarrow\ b_5b_6 + b_8b_9 + b_1 + b_8$ 842/1024 (82%) (255)
$\longrightarrow\ b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8$ 935/1024 (91%) (256)
$\longrightarrow\ b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1$ 969/1024 (95%) (257)
$\longrightarrow\ b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9$ 992/1024 (97%) (258)
$\longrightarrow\ b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1$ 1004/1024 (98%) (259)
$\longrightarrow\ b_1b_3 + b_6b_7 + b_9b_{10}$ 1013/1024 (99%) (260)
$\longrightarrow\ b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9$ 1019/1024 (99%) (261)
$\longrightarrow\ b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9$ 1022/1024 (99%) (262)
$\longrightarrow\ -b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1$ 1023/1024 (99%) (263)
$\longrightarrow\ b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2$ 1024/1024(100%) (264)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (265)
$\longrightarrow\ b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8)$ (266)
$\longrightarrow\ b_1b_3 + b_3b_6 + b_6b_7$ (267)
$\longrightarrow\ b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7)$ (268)
$\longrightarrow\ b_2b_5 + b_6b_8 + b_6$ (269)
$\longrightarrow\ b_1b_3 + b_3 + b_8$ (270)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (271)
$\longrightarrow\ b_4b_5 + 2b_5b_6$ 165/256 (64%) (272)
$\longrightarrow\ b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7$ 215/256 (84%) (273)
$\longrightarrow\ b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7$ 242/256 (95%) (274)
$\longrightarrow\ b_1b_3 + b_5b_6 + b_6b_7$ 254/256 (99%) (275)
$\longrightarrow\ b_1b_2 + b_5b_6 + b_6b_8$ 256/256(100%) (276)

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4$: $(k,n) = (3,4)$. (277)
$\longrightarrow\ 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3$ 13/16 (81%) (278)
$\longrightarrow\ 2b_1b_3 + b_2b_3 + b_2$ 16/16(100%) (279)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (280)
$\longrightarrow\ 4b_4b_5$ 769/1024 (75%) (281)
$\longrightarrow\ b_2b_6 + 2b_3b_6 + b_6b_9$ 915/1024 (89%) (282)
$\longrightarrow\ b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10}$ 974/1024 (95%) (283)
$\longrightarrow\ b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10}$ 995/1024 (97%) (284)
$\longrightarrow\ b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10}$ 1008/1024 (98%) (285)
$\longrightarrow\ b_1b_2 + b_2b_4 + b_9b_{10} + b_9$ 1016/1024 (99%) (286)
$\longrightarrow\ b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8$ 1023/1024 (99%) (287)
$\longrightarrow\ b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2$ 1024/1024(100%) (288)

DECOMPOSITION OF DEGREE-k, NOT EXACT-k-OF-n MULTINOMIALS

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (289)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (290)
$\longrightarrow\ b_1b_2 + b_1b_3 + 4b_1b_4 + b_2b_4$ 16/16(100%) (291)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (292)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (293)
$\longrightarrow\ 2b_2b_3 + 3b_2b_4 + b_3b_4$ 16/16(100%) (294)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + 3b_1b_3b_4 + b_2b_3b_4 :$ $(k,n) = (4,4).$ (295)
$\longrightarrow\ 2b_1b_2 + 5b_1b_4 + b_3b_4$ (296)
$\longrightarrow\ -b_1b_2 + 3b_1b_3 + 4b_2b_3 + 2b_2b_4 - 4b_3b_4 + 4b_3 - b_4 + 1$ (297)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 :$ $(k,n) = (4,4).$ (298)
$\longrightarrow\ 4b_1b_3$ (299)
$\longrightarrow\ 2b_1b_2 + b_1b_4 + b_2b_4$ (300)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4 :$ $(k,n) = (4,4).$ (301)
$\longrightarrow\ 2b_1b_3 + 2b_3b_4$ 12/16 (75%) (302)
$\longrightarrow\ 3b_1b_2 + b_1b_4$ 16/16(100%) (303)