

Volume 3: List of Multi-run Quadratizations

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(Dated: 22nd March 2020)

PRODUCT OF POLYNOMIALS

$$f_1 f_2 \dots f_\kappa = \min(f_1, f_2, \dots, f_\kappa), \quad f_i(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}) \geq 0 \quad (1)$$

$$f_1 f_2 \dots f_\kappa = \min(\max f_\kappa f_1 f_2 \dots f_{\kappa-1}, f_\kappa - \min f_\kappa + f_1 f_2 \dots f_{\kappa-1}), \quad \min f_\kappa < 0, \quad f_{i < \kappa}(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}) \geq 0 \quad (2)$$

$$\begin{aligned} b_1 b_2 b_3 b_4 + b_2 b_3 b_4 - b_3 b_4 b_5 : & \quad \text{(Example of Eq. 2).} \quad (3) \\ \longrightarrow 2b_3 b_4 & \quad 25/32 \text{ (78\%)} \quad (4) \\ \longrightarrow b_1 b_2 + b_2 - b_5 - b_3 b_4 + 1 & \quad 32/32(100\%) \quad (5) \end{aligned}$$

MONOMIALS

$$b_1 b_2 b_3 \dots b_k = \min(b_1 b_2 \dots b_{k_1}, b_{k_1+1} b_{k_1+2} \dots b_{k_2}, b_{k_2+1} b_{k_2+2} \dots b_{k_3}, \dots, b_{k_n+1} b_{k_n+2} \dots b_k) \quad \text{(Example of Eq. 1).} \quad (6)$$

$$b_1 b_2 b_3 \dots b_k = \min(b_1, b_2, b_3, \dots, b_k) \quad \text{(Example of Eq. 6: Linearization of a degree-} k \text{ monomial).} \quad (7)$$

$$b_1 b_2 b_3 b_4 = \min(b_1 b_2, b_3 b_4) \quad \text{(Example of Eq. 6: Quadratization of a degree-4 monomial).} \quad (8)$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 : \quad (9)$$

$$\longrightarrow 3b_a + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \quad (10)$$

$$\longrightarrow 3b_a + b_5 b_6 + b_5 b_7 + b_5 b_8 + b_6 b_7 + b_6 b_8 + b_7 b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \quad (11)$$

$$s_1 s_2 \dots s_k = \min(1 + s_1 s_2 - s_3 s_4 \dots s_k, 1 - s_1 s_2 - s_3 s_4 \dots s_k), s_i \in \{x, y, z\} \quad (12)$$

$$x_1 z_2 x_3 z_4 y_5 x_6 \quad \text{(Example of Eq. 12).} \quad (13)$$

$$\longrightarrow 1 + x_1 z_2 - x_3 z_4 y_5 x_6 \quad 48/64 \text{ (75\%)} \quad (14)$$

$$\longrightarrow 1 - x_1 z_2 - x_3 z_4 y_5 x_6 \quad 64/64(100\%) \quad (15)$$

BINOMIALS OF DEGREE- k TERMS

$$b_1 b_2 b_3 b_4 + b_3 b_4 b_5 b_6 = \min(2b_3 b_4, b_1 b_2 + b_5 b_6) \quad (k, n) = (4, 6). \quad (16)$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min_{b_a} (b_2b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3b_4, b_1b_2 + b_5b_6 + b_5b_a) \quad (k, n) = (4, 6). \quad (17)$$

$$\begin{aligned} & b_1b_2b_3b_4 + b_4b_5b_6b_7 : & (k, n) &= (4, 7). \quad (18) \\ \longrightarrow & b_3b_4 + b_4b_6 + b_a(b_5 + b_7) & 89/128 \quad (70\%) & (19) \\ \longrightarrow & b_1b_2 + b_5b_7 + b_a(1 - b_5 + b_6 - b_7) & 125/128 \quad (98\%) & (20) \\ \longrightarrow & b_5b_7 + b_3 & 128/128(100\%) & (21) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4 + b_4b_5b_6b_7 : & (k, n) &= (4, 7). \quad (22) \\ \longrightarrow & b_3b_4 + b_4b_6 & 89/128 \quad (70\%) & (23) \\ \longrightarrow & b_1b_2 + b_6b_7 & 118/128 \quad (92\%) & (24) \\ \longrightarrow & b_2b_3 - b_5b_6 + b_5b_7 + b_5 & 127/128 \quad (99\%) & (25) \\ \longrightarrow & b_1b_4 + 2b_5 - b_7 + 1 & 128/128(100\%) & (26) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : & (k, n) &= (5, 7). \quad (27) \\ \longrightarrow & b_1b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(1 - b_5 - 2b_6 - b_7) + b_6 & 188/256 \quad (73\%) & (28) \\ \longrightarrow & b_3b_4 + b_a(b_4 - b_6) + b_6 & 236/256 \quad (92\%) & (29) \\ \longrightarrow & b_2b_3 + b_3b_6 - b_4b_6 + b_6b_a + b_6 & 254/256 \quad (99\%) & (30) \\ \longrightarrow & b_2b_5 + b_5b_7 & 256/256(100\%) & (31) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : & (k, n) &= (5, 7). \quad (32) \\ \longrightarrow & b_2b_3 + b_3b_7 & 85/128 \quad (66\%) & (33) \\ \longrightarrow & 2b_4b_5 & 121/128 \quad (95\%) & (34) \\ \longrightarrow & b_1b_2 + b_6b_7 - b_5 + 1 & 128/128(100\%) & (35) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : & (k, n) &= (6, 7). \quad (36) \\ \longrightarrow & b_5b_6 + b_6b_7 + b_a(1 + b_5 - b_6 - b_7) & 196/256 \quad (77\%) & (37) \\ \longrightarrow & b_1b_4 + b_2b_4 + b_7b_a & 238/256 \quad (93\%) & (38) \\ \longrightarrow & b_1b_3 + b_3b_7 - b_4b_6 + 2b_5b_a - b_6b_7 - b_5 + b_6 + b_7 + b_a + 1 & 252/256 \quad (98\%) & (39) \\ \longrightarrow & b_2b_6 + b_2 - b_6 + 1 & 256/256(100\%) & (40) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : & (k, n) &= (6, 7). \quad (41) \\ \longrightarrow & 2b_5b_6 & 97/128 \quad (76\%) & (42) \\ \longrightarrow & b_1b_4 + b_4b_7 & 119/128 \quad (93\%) & (43) \\ \longrightarrow & b_1b_3 + b_1b_7 + b_2b_3 - b_3b_6 + b_3b_7 - b_4b_5 - b_1 - b_7 + 2 & 127/128 \quad (99\%) & (44) \\ \longrightarrow & b_1b_2 + b_2b_6 & 128/128(100\%) & (45) \end{aligned}$$

$$\begin{aligned} & b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : & (k, n) &= (5, 8). \quad (46) \\ \longrightarrow & b_3b_5 + b_7b_8 + b_a(-1 - b_6 + b_7 + b_8) + b_6 - b_7 - b_8 + 1 & 360/512 \quad (70\%) & (47) \\ \longrightarrow & b_1b_4 + b_4b_8 + b_a(b_4 + b_6) & 468/512 \quad (91\%) & (48) \\ \longrightarrow & b_1b_2 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8) & 496/512 \quad (97\%) & (49) \\ \longrightarrow & b_3b_5 + b_5 & 512/512(100\%) & (50) \end{aligned}$$

$$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : \quad (k, n) = (5, 8). \quad (51)$$

$$\longrightarrow b_2b_5 + b_5b_8 \quad 169/256 \quad (66\%) \quad (52)$$

$$\longrightarrow b_1b_4 + b_4b_7 - b_5b_8 + b_8 \quad 233/256 \quad (91\%) \quad (53)$$

$$\longrightarrow b_1b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1 \quad 252/256 \quad (98\%) \quad (54)$$

$$\longrightarrow b_2b_3 + b_6b_7 \quad 256/256(100\%) \quad (55)$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (6, 8). \quad (56)$$

$$\longrightarrow b_1b_6 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8) \quad 364/512 \quad (71\%) \quad (57)$$

$$\longrightarrow b_2b_3 + b_5b_8 - b_6b_8 + b_7b_a - b_7 + b_8 - b_a + 1 \quad 450/512 \quad (88\%) \quad (58)$$

$$\longrightarrow b_1b_4 + b_4 \quad 488/512 \quad (95\%) \quad (59)$$

$$\longrightarrow b_2b_3 + b_3b_7 - b_6b_8 + b_8 - b_a + 1 \quad 502/512 \quad (98\%) \quad (60)$$

$$\longrightarrow b_2b_5 + b_5 \quad 512/512(100\%) \quad (61)$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (6, 8). \quad (62)$$

$$\longrightarrow 2b_5b_6 \quad 193/256 \quad (75\%) \quad (63)$$

$$\longrightarrow b_1b_4 + b_4b_8 \quad 237/256 \quad (93\%) \quad (64)$$

$$\longrightarrow b_2b_3 + b_3b_7 - b_4b_6 + b_4b_8 - b_5b_7 - b_5b_8 + b_6b_8 - b_6 + b_7 - b_8 + 2 \quad 254/256 \quad (99\%) \quad (65)$$

$$\longrightarrow b_1b_2 + b_7b_8 \quad 256/256(100\%) \quad (66)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (7, 8). \quad (67)$$

$$\longrightarrow b_6b_7 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad 388/512 \quad (76\%) \quad (68)$$

$$\longrightarrow b_1b_3 + b_3b_8 + b_a(1 + b_8) \quad 470/512 \quad (92\%) \quad (69)$$

$$\longrightarrow b_2b_4 - b_3b_8 + b_4b_5 + b_a(1 - b_7) + b_8 \quad 500/512 \quad (98\%) \quad (70)$$

$$\longrightarrow b_2b_5 + b_2b_8 - b_4b_8 - b_6b_7 + b_6b_8 + b_a(-1 - b_4 - b_7 + b_8) - b_3 + b_7 - b_8 + 4 \quad 508/512 \quad (99\%) \quad (71)$$

$$\longrightarrow b_2b_5 - b_7b_8 + b_5 + 1 \quad 512/512(100\%) \quad (72)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (7, 8). \quad (73)$$

$$\longrightarrow 2b_5b_6 \quad 193/256 \quad (75\%) \quad (74)$$

$$\longrightarrow b_1b_4 + b_4b_8 \quad 235/256 \quad (92\%) \quad (75)$$

$$\longrightarrow b_2b_3 + b_2b_7 - b_5b_6 + b_6b_8 + b_5 - b_6 - b_8 + 1 \quad 250/256 \quad (98\%) \quad (76)$$

$$\longrightarrow b_3b_7 + b_7b_8 \quad 254/256 \quad (99\%) \quad (77)$$

$$\longrightarrow b_3b_8 + b_3 \quad 256/256(100\%) \quad (78)$$

$$b_1b_2b_3b_4 + b_5b_6b_7b_8 : \quad (k, n) = (4, 8). \quad (79)$$

$$\longrightarrow b_2b_3 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad 390/512 \quad (76\%) \quad (80)$$

$$\longrightarrow b_1b_4 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad 480/512 \quad (94\%) \quad (81)$$

$$\longrightarrow b_2b_4 + b_5 - b_a + 1 \quad 506/512 \quad (99\%) \quad (82)$$

$$\longrightarrow b_1b_3 - b_6b_a + b_5 + 1 \quad 512/512(100\%) \quad (83)$$

$$b_1b_2b_3b_4 + b_5b_6b_7b_8 : \quad (k, n) = (4, 8). \quad (84)$$

$$\longrightarrow b_1b_2 + b_6b_7 \quad 169/256 \quad (66\%) \quad (85)$$

$$\longrightarrow b_3b_4 + b_5b_8 \quad 238/256 \quad (93\%) \quad (86)$$

$$\longrightarrow b_1b_4 + b_5b_6 + b_5b_7 + b_6b_7 - b_5 - b_6 - b_7 + 1 \quad 248/256 \quad (97\%) \quad (87)$$

$$\longrightarrow b_2b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1 \quad 254/256 \quad (99\%) \quad (88)$$

$$\longrightarrow b_1b_2 + b_5b_8 \quad 256/256(100\%) \quad (89)$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (5, 10). \quad (90) \\
\longrightarrow & b_1 b_4 + b_7 b_9 & 625/1024 \quad (61\%) \quad (91) \\
\longrightarrow & b_3 b_5 + b_6 b_8 & 889/1024 \quad (87\%) \quad (92) \\
\longrightarrow & b_2 b_5 + b_7 b_{10} & 972/1024 \quad (95\%) \quad (93) \\
\longrightarrow & b_2 b_4 + b_6 b_8 & 999/1024 \quad (98\%) \quad (94) \\
\longrightarrow & b_1 b_3 + b_9 b_{10} + b_9 b_a & 1016/1024 \quad (99\%) \quad (95) \\
\longrightarrow & b_1 b_5 + b_6 b_9 & 1020/1024 \quad (99\%) \quad (96) \\
\longrightarrow & b_1 b_4 + b_8 b_{10} & 1022/1024 \quad (99\%) \quad (97) \\
\longrightarrow & b_2 b_3 - b_4 b_{10} + b_7 b_9 + b_9 b_a + 1 & 1024/1024(100\%) \quad (98)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (5, 10). \quad (99) \\
\longrightarrow & b_1 b_3 + b_9 b_{10} & 625/1024 \quad (61\%) \quad (100) \\
\longrightarrow & b_2 b_4 + b_7 b_{10} & 851/1024 \quad (83\%) \quad (101) \\
\longrightarrow & b_3 b_5 + b_5 b_{10} + b_8 b_9 & 924/1024 \quad (90\%) \quad (102) \\
\longrightarrow & b_1 b_2 + b_6 & 972/1024 \quad (95\%) \quad (103) \\
\longrightarrow & b_3 b_4 + b_8 b_9 & 997/1024 \quad (97\%) \quad (104) \\
\longrightarrow & b_1 b_5 + b_7 b_{10} & 1010/1024 \quad (99\%) \quad (105) \\
\longrightarrow & b_2 b_3 - b_1 b_7 - b_1 b_{10} - b_2 b_8 - b_2 b_{10} + b_3 b_5 + b_6 b_9 + b_7 b_{10} - b_8 b_9 + b_9 b_{10} - b_3 - b_7 + b_8 + 3 & 1016/1024 \quad (99\%) \quad (106) \\
\longrightarrow & b_1 b_3 + b_7 b_8 & 1020/1024 \quad (99\%) \quad (107) \\
\longrightarrow & b_2 b_4 + b_2 b_6 - b_2 b_9 - b_3 b_{10} - b_5 b_7 + b_7 b_{10} + b_9 b_{10} - b_{10} + 2 & 1023/1024 \quad (99\%) \quad (108) \\
\longrightarrow & b_2 b_5 + b_2 b_9 + b_6 b_8 & 1024/1024(100\%) \quad (109)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). \quad (110) \\
\longrightarrow & b_4 b_5 + b_5 b_9 & 657/1024 \quad (64\%) \quad (111) \\
\longrightarrow & b_2 b_6 + b_6 b_8 & 905/1024 \quad (88\%) \quad (112) \\
\longrightarrow & b_1 b_3 + b_7 b_8 & 982/1024 \quad (96\%) \quad (113) \\
\longrightarrow & b_2 b_3 + b_a(b_{10} - b_9) + b_9 & 1011/1024 \quad (99\%) \quad (114) \\
\longrightarrow & b_2 b_4 + b_7 b_{10} & 1020/1024 \quad (99\%) \quad (115) \\
\longrightarrow & b_9 b_{10} + b_1 & 1023/1024 \quad (99\%) \quad (116) \\
\longrightarrow & b_7 b_8 + b_4 & 1024/1024(100\%) \quad (117)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). \quad (118) \\
\longrightarrow & 2b_5 b_6 & 769/1024 \quad (75\%) \quad (119) \\
\longrightarrow & b_1 b_3 + b_8 b_9 & 934/1024 \quad (92\%) \quad (120) \\
\longrightarrow & b_2 b_4 + b_7 b_{10} + b_8 b_9 - b_8 - b_9 + 1 & 997/1024 \quad (97\%) \quad (121) \\
\longrightarrow & -b_1 b_3 + b_1 b_9 + b_2 b_4 + b_4 b_9 + b_5 b_8 + b_8 b_9 - b_5 - b_8 - b_9 + 2 & 769/1024 \quad (99\%) \quad (122) \\
\longrightarrow & b_1 b_3 + b_7 b_{10} - b_8 - b_9 + 2 & 1014/1024 \quad (99\%) \quad (123) \\
\longrightarrow & b_2 b_3 + b_8 b_9 & 1024/1024(100\%) \quad (124)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (7, 10). \quad (125) \\
\longrightarrow & b_3 b_5 + b_5 b_8 & 649/1024 \quad (63\%) \quad (126) \\
\longrightarrow & b_2 b_4 + b_4 b_9 & 893/1024 \quad (87\%) \quad (127) \\
\longrightarrow & b_1 b_7 + b_7 b_{10} & 985/1024 \quad (96\%) \quad (128) \\
\longrightarrow & b_1 b_6 + b_6 b_9 + b_a & 1015/1024 \quad (99\%) \quad (129) \\
\longrightarrow & b_2 b_3 + b_8 b_{10} + b_a & 1022/1024 \quad (99\%) \quad (130) \\
\longrightarrow & b_1 b_3 + b_8 b_9 & 1024/1024(100\%) \quad (131)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (7, 10). \quad (132) \\
& \longrightarrow b_3 b_7 + b_7 b_{10} & 649/1024 \quad (63\%) \quad (133) \\
& \longrightarrow 2b_4 b_6 & 937/1024 \quad (92\%) \quad (134) \\
& \longrightarrow b_1 b_5 + b_5 b_8 & 1001/1024 \quad (98\%) \quad (135) \\
& \longrightarrow b_1 b_2 + b_9 b_{10} & 1019/1024 \quad (99\%) \quad (136) \\
& \longrightarrow b_2 b_3 + b_8 & 1023/1024 \quad (99\%) \quad (137) \\
& \longrightarrow b_3 b_7 + b_9 b_{10} & 1024/1024(100\%) \quad (138)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (139) \\
& \longrightarrow b_2 b_8 + b_8 b_9 & 645/1024 \quad (63\%) \quad (140) \\
& \longrightarrow b_1 b_3 + b_3 b_{10} + b_9 b_a & 887/1024 \quad (87\%) \quad (141) \\
& \longrightarrow b_4 b_6 + b_5 b_6 & 977/1024 \quad (95\%) \quad (142) \\
& \longrightarrow b_2 b_7 + b_7 b_{10} & 1007/1024 \quad (98\%) \quad (143) \\
& \longrightarrow b_1 b_4 + b_4 b_5 + 2b_9 b_a & 1018/1024 \quad (99\%) \quad (144) \\
& \longrightarrow b_1 b_5 + b_5 b_9 & 1024/1024(100\%) \quad (145)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (146) \\
& \longrightarrow 4b_3 b_7 & 768/1024 \quad (75\%) \quad (147) \\
& \longrightarrow b_2 b_8 + b_8 b_9 & 933/1024 \quad (91\%) \quad (148) \\
& \longrightarrow 2b_4 b_6 + b_8 b_9 - b_8 - b_9 + 1 & 1005/1024 \quad (98\%) \quad (149) \\
& \longrightarrow b_1 b_5 + b_5 b_{10} + b_8 b_9 - b_8 - b_9 + 1 & 1022/1024 \quad (99\%) \quad (150) \\
& \longrightarrow b_1 b_2 + b_8 b_9 + b_9 b_{10} - b_8 - b_9 + 1 & 1024/1024(100\%) \quad (151)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (9, 10). \quad (152) \\
& \longrightarrow b_1 b_9 + b_9 b_{10} + b_{10} b_a & 643/1024 \quad (63\%) \quad (153) \\
& \longrightarrow b_2 b_4 + b_4 b_5 & 883/1024 \quad (86\%) \quad (154) \\
& \longrightarrow b_3 b_7 + b_3 b_8 & 973/1024 \quad (95\%) \quad (155) \\
& \longrightarrow b_2 b_6 + b_6 b_8 & 1003/1024 \quad (98\%) \quad (156) \\
& \longrightarrow b_2 b_5 + b_5 b_7 - b_{10} b_a + b_{10} & 1015/1024 \quad (99\%) \quad (157) \\
& \longrightarrow b_1 b_8 + b_7 b_8 & 1019/1024 \quad (99\%) \quad (158) \\
& \longrightarrow b_2 b_7 + b_2 b_{10} - b_4 b_5 - b_{10} b_a + b_{10} + 1 & 1023/1024 \quad (99\%) \quad (159) \\
& \longrightarrow b_4 b_7 + b_7 & 1024/1024(100\%) \quad (160)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (9, 10). \quad (161) \\
& \longrightarrow 2b_2 b_3 - b_8 b_9 + b_9 & 577/1024 \quad (56\%) \quad (162) \\
& \longrightarrow 3b_8 b_9 & 961/1024 \quad (94\%) \quad (163) \\
& \longrightarrow 2b_4 b_6 - b_8 b_9 - b_8 b_{10} + b_{10} + 1 & 1009/1024 \quad (99\%) \quad (164) \\
& \longrightarrow 2b_5 b_7 - b_8 b_{10} + b_{10} & 1021/1024 \quad (99\%) \quad (165) \\
& \longrightarrow b_1 b_6 + b_{10} & 1024/1024(100\%) \quad (166)
\end{aligned}$$

DEGREE- k , EXACT- k -OF- n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 :$	$(k, n) = (4, 6). \quad (167)$
$\longrightarrow b_1b_4 + 2b_4b_5 + b_7$	$44/64 \quad (69\%) \quad (168)$
$\longrightarrow b_1b_3 + b_2b_3 + b_3b_6 + b_6b_7$	$60/64 \quad (94\%) \quad (169)$
$\longrightarrow b_2b_4 + b_5b_6 + b_2$	$64/64(100\%) \quad (170)$
$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 :$	$(k, n) = (4, 6). \quad (171)$
$\longrightarrow b_2b_4 + 2b_4b_5$	$43/64 \quad (67\%) \quad (172)$
$\longrightarrow b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$	$60/64 \quad (94\%) \quad (173)$
$\longrightarrow b_1b_2 + b_2b_5 + b_5b_6$	$64/64(100\%) \quad (174)$
$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 :$	$(k, n) = (4, 8). \quad (175)$
$\longrightarrow b_1b_4 + 2b_5b_6$	$159/256 \quad (62\%) \quad (176)$
$\longrightarrow b_2b_3 + b_3b_5 + b_7b_8$	$225/256 \quad (88\%) \quad (177)$
$\longrightarrow b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1$	$244/256 \quad (95.3\%) \quad (178)$
$\longrightarrow b_2b_3 + b_6b_8 + b_6$	$253/256 \quad (98.8\%) \quad (179)$
$\longrightarrow b_2b_3 + b_5b_7 + b_5$	$256/256 \quad (100\%) \quad (180)$
$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 :$	$(k, n) = (4, 8). \quad (181)$
$\longrightarrow b_2b_4 + 2b_5b_6$	$159/256 \quad (62\%) \quad (182)$
$\longrightarrow b_3b_6 + b_7b_8 + b_3$	$212/256 \quad (83\%) \quad (183)$
$\longrightarrow b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7$	$234/256 \quad (91\%) \quad (184)$
$\longrightarrow b_1b_3 + 2b_5b_6$	$253/256 \quad (99\%) \quad (185)$
$\longrightarrow b_7b_8 + b_1 + b_6$	$256/256(100\%) \quad (186)$
$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 :$	$(k, n) = (5, 7). \quad (187)$
$\longrightarrow b_1b_5 + b_5b_6 + b_6b_7 + b_a(-2 - b_5 + 2b_6 + b_7) + b_5 - 2b_6 - b_7 + 2$	$86/128 \quad (67\%) \quad (188)$
$\longrightarrow b_1b_3 + b_3b_4 - b_3b_6 + b_3b_7 + b_a(b_5 + b_7) + b_3$	$112/128 \quad (88\%) \quad (189)$
$\longrightarrow b_1b_4 + b_2b_4 + b_4b_7 + b_5b_7 + b_a(-1 - b_6 - b_7) - b_5 + b_6 + 2$	$124/128 \quad (97\%) \quad (190)$
$\longrightarrow b_2b_4 - 2b_5b_a + b_6b_7 + b_2 + b_5 + 1$	$128/128(100\%) \quad (191)$
$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 :$	$(k, n) = (5, 7). \quad (192)$
$\longrightarrow 2b_4b_5 + b_4b_6$	$81/128 \quad (63\%) \quad (193)$
$\longrightarrow b_1b_3 + b_3b_6 + b_3b_7 - b_4b_5 + b_5$	$111/128 \quad (87\%) \quad (194)$
$\longrightarrow b_1b_2 + b_2b_6 - b_4b_5 + b_6b_7 + b_5$	$122/128 \quad (95\%) \quad (195)$
$\longrightarrow 2b_4b_5 + b_5$	$128/128(100\%) \quad (196)$
$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 :$	$(k, n) = (6, 8). \quad (197)$
$\longrightarrow b_1b_6 + 2b_6b_7$	$164/256 \quad (64\%) \quad (198)$
$\longrightarrow b_1b_5 + b_2b_5 - b_3b_6 + b_5b_8 + b_3$	$219/256 \quad (86\%) \quad (199)$
$\longrightarrow b_2b_4 + b_4b_7 + b_4b_8 - b_6 + 1$	$243/256 \quad (95\%) \quad (200)$
$\longrightarrow b_2b_3 + b_3b_8 - b_5b_6 + b_3 + b_6$	$253/256 \quad (99\%) \quad (201)$
$\longrightarrow b_1b_2 + b_2b_6 + b_5b_7 - b_6b_7 + b_7b_8 - b_5 + 1$	$256/256(100\%) \quad (202)$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (8, 10). \quad (203) \\
& \longrightarrow 3b_5 b_8 & 769/1024 \quad (75\%) \quad (204) \\
& \longrightarrow 2b_2 b_6 + b_4 b_6 & 931/1024 \quad (91\%) \quad (205) \\
& \longrightarrow b_1 b_7 - b_5 b_{10} + b_7 b_9 + b_9 b_{10} - b_6 + b_{10} + 1 & 984/1024 \quad (96\%) \quad (206) \\
& \longrightarrow 3b_2 b_3 + b_3 b_{10} - b_6 b_8 + 1 & 1011/1024 \quad (99\%) \quad (207) \\
& \longrightarrow b_4 b_7 + b_4 b_8 - b_3 + b_4 - b_8 + 2 & 1019/1024 \quad (99\%) \quad (208) \\
& \longrightarrow b_2 b_3 - b_2 b_4 - b_3 b_4 - b_3 b_8 - b_5 b_{10} - b_6 b_9 + b_7 b_8 + b_7 b_9 + b_8 b_9 + b_7 + 3 & 1023/1024 \quad (99\%) \quad (209) \\
& \longrightarrow b_2 b_8 + 2b_8 b_9 & 1024/1024(100\%) \quad (210)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). \quad (211) \\
& \longrightarrow b_5 b_6 + b_5 b_8 + b_5 b_9 + b_8 b_{11} + b_9 b_{11} + b_{10} b_{11} & 583/1024 \quad (57\%) \quad (212) \\
& \longrightarrow b_1 b_2 + b_4 b_7 + b_7 b_{10} + b_9 b_{11} - b_9 - b_{11} + 1 & 815/1024 \quad (80\%) \quad (213) \\
& \longrightarrow b_1 b_6 + b_5 b_6 + b_6 - b_{11} + 1 & 917/1024 \quad (90\%) \quad (214) \\
& \longrightarrow b_3 b_4 + b_3 b_7 + b_8 b_9 + b_9 b_{11} & 979/1024 \quad (96\%) \quad (215) \\
& \longrightarrow b_2 b_4 + b_4 b_8 + b_8 b_9 - b_9 b_{11} + b_9 - b_{11} + 1 & 1007/1024 \quad (98\%) \quad (216) \\
& \longrightarrow b_1 b_3 + b_7 b_{10} + b_{10} b_{11} + b_3 & 1016/1024 \quad (99\%) \quad (217) \\
& \longrightarrow b_1 b_4 + b_4 b_8 + b_7 b_{10} + b_9 b_{11} + b_{10} b_{11} - b_9 - b_{11} + 1 & 1021/1024 \quad (99\%) \quad (218) \\
& \longrightarrow b_1 b_3 - b_2 b_{11} + b_7 b_8 + b_8 b_9 - b_{10} b_{11} - b_{11} + 3 & 1024/1024(100\%) \quad (219)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 + b_5 b_6 b_7 b_8 b_9 b_{10} : & (k, n) = (6, 10). \quad (220) \\
& \longrightarrow 2b_3 b_4 + b_7 b_{10} & 591/1024 \quad (58\%) \quad (221) \\
& \longrightarrow 2b_3 b_5 + b_5 b_6 & 847/1024 \quad (83\%) \quad (222) \\
& \longrightarrow b_1 b_2 + b_7 b_8 + b_8 b_9 & 951/1024 \quad (93\%) \quad (223) \\
& \longrightarrow 3b_5 b_6 & 995/1024 \quad (97\%) \quad (224) \\
& \longrightarrow b_1 b_3 + b_3 b_4 + b_9 b_{10} & 1009/1024 \quad (99\%) \quad (225) \\
& \longrightarrow b_1 b_2 + b_5 b_7 + b_7 b_{10} & 1018/1024 \quad (99\%) \quad (226) \\
& \longrightarrow 2b_1 b_4 - b_1 b_{10} + b_2 b_4 + b_4 b_5 + b_4 b_{10} + b_5 b_8 - b_6 b_8 + b_8 b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3 & 1023/1024 \quad (99\%) \quad (227) \\
& \longrightarrow b_2 b_8 + b_3 b_6 + b_6 b_8 & 1024/1024(100\%) \quad (228)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + b_4 b_5 b_6 b_7 + b_7 b_8 b_9 b_{10} : & (k, n) = (4, 10). \quad (229) \\
& \longrightarrow b_2 b_3 + b_6 b_7 + b_7 b_9 + 2b_9 b_a & 581/1024 \quad (57\%) \quad (230) \\
& \longrightarrow b_2 b_4 + b_4 b_6 + b_a(b_9 - b_{10}) + b_{10} & 823/1024 \quad (80\%) \quad (231) \\
& \longrightarrow b_1 b_3 + b_5 b_6 + b_8 b_9 + b_a(b_9 - b_{10}) - b_9 + 1 & 930/1024 \quad (91\%) \quad (232) \\
& \longrightarrow b_1 b_4 + b_4 b_5 + b_4 b_{10} + b_8 b_{10} + b_a(1 - b_7 + b_9) & 978/1024 \quad (96\%) \quad (233) \\
& \longrightarrow b_1 b_4 + b_7 b_8 + b_a(1 + b_9) + b_7 & 1000/1024 \quad (98\%) \quad (234) \\
& \longrightarrow b_2 b_3 + b_a(b_9 - b_{10}) + b_5 + b_{10} & 1015/1024 \quad (99\%) \quad (235) \\
& \longrightarrow b_1 b_3 + b_6 + b_{10} & 1020/1024 \quad (99\%) \quad (236) \\
& \longrightarrow b_5 b_6 + b_2 + b_8 + b_a & 1024/1024(100\%) \quad (237)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10} : & (k, n) = (4, 10). \quad (238) \\
& \longrightarrow b_3b_4 + b_4b_6 + b_9b_{10} & 581/1024 \quad (57\%) \quad (239) \\
& \longrightarrow b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9 & 759/1024 \quad (74\%) \quad (240) \\
& \longrightarrow b_5b_6 + b_8b_9 + b_1 + b_8 & 842/1024 \quad (82\%) \quad (241) \\
& \longrightarrow b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8 & 935/1024 \quad (91\%) \quad (242) \\
& \longrightarrow b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1 & 969/1024 \quad (95\%) \quad (243) \\
& \longrightarrow b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9 & 992/1024 \quad (97\%) \quad (244) \\
& \longrightarrow b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1 & 1004/1024 \quad (98\%) \quad (245) \\
& \longrightarrow b_1b_3 + b_6b_7 + b_9b_{10} & 1013/1024 \quad (99\%) \quad (246) \\
& \longrightarrow b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9 & 1019/1024 \quad (99\%) \quad (247) \\
& \longrightarrow b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9 & 1022/1024 \quad (99\%) \quad (248) \\
& \longrightarrow -b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1 & 1023/1024 \quad (99\%) \quad (249) \\
& \longrightarrow b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2 & 1024/1024(100\%) \quad (250)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : & (k, n) = (5, 8). \quad (251) \\
& \longrightarrow b_1b_3 + b_3b_4 + b_6b_8 - b_6b_9 + b_7b_9 - b_8b_9 + b_9 & 156/256 \quad (61\%) \quad (252) \\
& \longrightarrow b_1b_5 + b_5b_7 + b_7b_9 + b_8b_9 + b_5 - b_7 - b_9 + 1 & 202/256 \quad (79\%) \quad (253) \\
& \longrightarrow b_2b_4 + b_6b_8 + b_6b_9 - b_7b_9 + b_8b_9 + b_7 - b_8 - b_9 + 1 & 230/256 \quad (90\%) \quad (254) \\
& \longrightarrow b_2b_4 + b_4b_8 + b_4 - b_9 + 1 & 246/256 \quad (96\%) \quad (255) \\
& \longrightarrow b_1b_5 + 2b_6 & 252/256 \quad (98\%) \quad (256) \\
& \longrightarrow b_2b_5 + b_7b_8 + b_5 & 256/256(100\%) \quad (257)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : & (k, n) = (5, 8). \quad (258) \\
& \longrightarrow b_4b_5 + 2b_5b_6 & 165/256 \quad (64\%) \quad (259) \\
& \longrightarrow b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7 & 215/256 \quad (84\%) \quad (260) \\
& \longrightarrow b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7 & 242/256 \quad (95\%) \quad (261) \\
& \longrightarrow b_1b_3 + b_5b_6 + b_6b_7 & 254/256 \quad (99\%) \quad (262) \\
& \longrightarrow b_1b_2 + b_5b_6 + b_6b_8 & 256/256(100\%) \quad (263)
\end{aligned}$$

DEGREE- k , EXACT- k -OF- n QUADRINOMIALS

$$\begin{aligned}
& b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 : & (k, n) = (3, 4). \quad (264) \\
& \longrightarrow 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3 & 13/16 \quad (81\%) \quad (265) \\
& \longrightarrow 2b_1b_3 + b_2b_3 + b_2 & 16/16(100\%) \quad (266)
\end{aligned}$$

$$\begin{aligned}
& b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} : & (k, n) = (7, 10). \quad (267) \\
& \longrightarrow 4b_4b_5 & 769/1024 \quad (75\%) \quad (268) \\
& \longrightarrow b_2b_6 + 2b_3b_6 + b_6b_9 & 915/1024 \quad (89\%) \quad (269) \\
& \longrightarrow b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10} & 974/1024 \quad (95\%) \quad (270) \\
& \longrightarrow b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10} & 995/1024 \quad (97\%) \quad (271) \\
& \longrightarrow b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10} & 1008/1024 \quad (98\%) \quad (272) \\
& \longrightarrow b_1b_2 + b_2b_4 + b_9b_{10} + b_9 & 1016/1024 \quad (99\%) \quad (273) \\
& \longrightarrow b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8 & 1023/1024 \quad (99\%) \quad (274) \\
& \longrightarrow b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2 & 1024/1024(100\%) \quad (275)
\end{aligned}$$

DEGREE- k , NOT EXACT- k -OF- n MULTINOMIALS

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (276) \\
\longrightarrow & b_1 b_2 + 4b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - b_1 - b_2 - b_3 - b_4 + 1 & 12/16 \quad (75\%) \quad (277) \\
\longrightarrow & b_1 b_2 + b_1 b_3 + 4b_1 b_4 + b_2 b_4 & 16/16(100\%) \quad (278)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (279) \\
\longrightarrow & b_1 b_2 + 4b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - b_1 - b_2 - b_3 - b_4 + 1 & 12/16 \quad (75\%) \quad (280) \\
\longrightarrow & 2b_2 b_3 + 3b_2 b_4 + b_3 b_4 & 16/16(100\%) \quad (281)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_2 b_4 + 3b_1 b_3 b_4 + b_2 b_3 b_4 : & (k, n) = (4, 4). \quad (282) \\
\longrightarrow & 2b_1 b_2 + 5b_1 b_4 + b_3 b_4 & 11/16 \quad (69\%) \quad (283) \\
\longrightarrow & -b_1 b_2 + 3b_1 b_3 + 4b_2 b_3 + 2b_2 b_4 - 4b_3 b_4 + 4b_3 - b_4 + 1 & 16/16(100\%) \quad (284)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_3 b_4 : & (k, n) = (4, 4). \quad (285) \\
\longrightarrow & 4b_1 b_3 & 13/16 \quad (81\%) \quad (286) \\
\longrightarrow & 2b_1 b_2 + b_1 b_4 + b_2 b_4 & 16/16(100\%) \quad (287)
\end{aligned}$$

$$\begin{aligned}
& b_1 b_2 b_3 b_4 + 2b_1 b_2 b_3 + b_1 b_3 b_4 : & (k, n) = (4, 4). \quad (288) \\
\longrightarrow & 2b_1 b_3 + 2b_3 b_4 & 12/16 \quad (75\%) \quad (289) \\
\longrightarrow & 3b_1 b_2 + b_1 b_4 & 16/16(100\%) \quad (290)
\end{aligned}$$