Volume 3: List of Multi-run Quadratizations

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PRODUCT OF POLYNOMIALS

$$f_1 f_2 \dots f_\kappa = \min\left(f_1,f_2,\dots,f_\kappa\right), \qquad f_i\left(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}\right) \geq 0 \quad (1)$$

$$f_1 f_2 \dots f_\kappa = \min\left(f_1 f_2 \dots f_{\kappa-1} \max f_\kappa, f_\kappa - \min f_\kappa + f_1 f_2 \dots f_{\kappa-1}\right), \min f_\kappa < 0, \; f_{i<\kappa}\left(b_{k_i}, b_{k_i+1}, \dots, b_{k_{i+1}-1}\right) \geq 0 \quad (2)$$

$b_1b_2b_3b_4 + b_2b_3b_4 - b_3b_4b_5$: (Example of Eq. 2). (3)

$\longrightarrow 2b_3b_4$ 25/32 (78%) (4)

$\longrightarrow b_1b_2 + b_2 - b_5 - b_3b_4 + 1$ 32/32(100%) (5)

MONOMIALS

$$b_1b_2b_3\dots b_k = \min\left(b_1b_2\dots b_{k_1}, b_{k_1+1}b_{k_1+2}\dots b_{k_2}, b_{k_2+1}b_{k_2+2}\dots b_{k_3}, \dots, b_{k_n+1}b_{k_n+2}\dots b_k\right) \quad \text{(Example of Eq. 1).} \quad (6)$$

$$b_1b_2b_3\dots b_k = \min\left(b_1, b_2, b_3, \dots, b_k\right) \quad \text{(Example of Eq. 6: Linearization of a degree-}k\text{ monomial).} \quad (7)$$

$$b_1b_2b_3b_4 = \min\left(b_1b_2, b_3b_4\right) \quad \text{(Example of Eq. 6: Quadratization of a degree-4 monomial).} \quad (8)$$

$b_1b_2b_3b_4b_5b_6b_7b_8$: (9)

$\longrightarrow 3b_a + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - 2b_a(b_1 + b_2 + b_3 + b_4)$ (10)

$\longrightarrow 3b_a + b_5b_6 + b_5b_7 + b_5b_8 + b_6b_7 + b_6b_8 + b_7b_8 - 2b_a(b_5 + b_6 + b_7 + b_8)$ (11)

$$s_1s_2\dots s_k = \min\left(1 + s_1s_2 - s_3s_4\dots s_k, 1 - s_1s_2 + s_3s_4\dots s_k\right), s_i \in \{x, y, z\} \quad (12)$$

$x_1z_2x_3z_4y_5x_6$ (Example of Eq. 12). (13)

$\longrightarrow 1 + x_1z_2 - x_3z_4y_5x_6$ 48/64 (75%) (14)

$\longrightarrow 1 - x_1z_2 + x_3z_4y_5x_6$ 64/64(100%) (15)

HEURISTIC GADGETS

$z_1z_2x_3$ (16)

$\longrightarrow z_1z_2 + z_1x_3 + z_2x_3 - z_1 - z_2 - x_3 + 1$ 7/8 (88%) (17)

BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min\left(2b_3b_4, b_1b_2 + b_5b_6\right) \qquad (k,n) = (4,6). \quad (18)$$

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min_{b_a}\left(b_2b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3b_4, b_1b_2 + b_5b_6 + b_5b_a\right) \qquad (k,n) = (4,6). \quad (19)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad (k,n) = (4,7). \quad (20)$$
$$\longrightarrow\ b_3b_4 + b_4b_6 + b_a(b_5 + b_7) \qquad 89/128\ \ (70\%) \quad (21)$$
$$\longrightarrow\ b_1b_2 + b_5b_7 + b_a(1 - b_5 + b_6 - b_7) \qquad 125/128\ \ (98\%) \quad (22)$$
$$\longrightarrow\ b_5b_7 + b_3 \qquad 128/128(100\%) \quad (23)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad (k,n) = (4,7). \quad (24)$$
$$\longrightarrow\ b_3b_4 + b_4b_6 \qquad 89/128\ \ (70\%) \quad (25)$$
$$\longrightarrow\ b_1b_2 + b_6b_7 \qquad 118/128\ \ (92\%) \quad (26)$$
$$\longrightarrow\ b_2b_3 - b_5b_6 + b_5b_7 + b_5 \qquad 127/128\ \ (99\%) \quad (27)$$
$$\longrightarrow\ b_1b_4 + 2b_5 - b_7 + 1 \qquad 128/128(100\%) \quad (28)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \quad (29)$$
$$\longrightarrow\ b_1b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(1 - b_5 - 2b_6 - b_7) + b_6 \qquad 188/256\ \ (73\%) \quad (30)$$
$$\longrightarrow\ b_3b_4 + b_a(b_4 - b_6) + b_6 \qquad 236/256\ \ (92\%) \quad (31)$$
$$\longrightarrow\ b_2b_3 + b_3b_6 - b_4b_6 + b_6b_a + b_6 \qquad 254/256\ \ (99\%) \quad (32)$$
$$\longrightarrow\ b_2b_5 + b_5b_7 \qquad 256/256(100\%) \quad (33)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \quad (34)$$
$$\longrightarrow\ b_2b_3 + b_3b_7 \qquad 85/128\ \ (66\%) \quad (35)$$
$$\longrightarrow\ 2b_4b_5 \qquad 121/128\ \ (95\%) \quad (36)$$
$$\longrightarrow\ b_1b_2 + b_6b_7 - b_5 + 1 \qquad 128/128(100\%) \quad (37)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad (k,n) = (6,7). \quad (38)$$
$$\longrightarrow\ b_5b_6 + b_6b_7 + b_a(1 + b_5 - b_6 - b_7) \qquad 196/256\ \ (77\%) \quad (39)$$
$$\longrightarrow\ b_1b_4 + b_2b_4 + b_7b_a \qquad 238/256\ \ (93\%) \quad (40)$$
$$\longrightarrow\ b_1b_3 + b_3b_7 - b_4b_6 + 2b_5b_a - b_6b_7 - b_5 + b_6 + b_7 + b_a + 1 \qquad 252/256\ \ (98\%) \quad (41)$$
$$\longrightarrow\ b_2b_6 + b_2 - b_6 + 1 \qquad 256/256(100\%) \quad (42)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad (k,n) = (6,7). \quad (43)$$
$$\longrightarrow\ 2b_5b_6 \qquad 97/128\ \ (76\%) \quad (44)$$
$$\longrightarrow\ b_1b_4 + b_4b_7 \qquad 119/128\ \ (93\%) \quad (45)$$
$$\longrightarrow\ b_1b_3 + b_1b_7 + b_2b_3 - b_3b_6 + b_3b_7 - b_4b_5 - b_1 - b_7 + 2 \qquad 127/128\ \ (99\%) \quad (46)$$
$$\longrightarrow\ b_1b_2 + b_2b_6 \qquad 128/128(100\%) \quad (47)$$

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (48)
$\longrightarrow\ b_3b_5 + b_7b_8 + b_a(-1 - b_6 + b_7 + b_8) + b_6 - b_7 - b_8 + 1$ 360/512 (70%) (49)
$\longrightarrow\ b_1b_4 + b_4b_8 + b_a(b_4 + b_6)$ 468/512 (91%) (50)
$\longrightarrow\ b_1b_2 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8)$ 496/512 (97%) (51)
$\longrightarrow\ b_3b_5 + b_5$ 512/512(100%) (52)

$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (53)
$\longrightarrow\ b_2b_5 + b_5b_8$ 169/256 (66%) (54)
$\longrightarrow\ b_1b_4 + b_4b_7 - b_5b_8 + b_8$ 233/256 (91%) (55)
$\longrightarrow\ b_1b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 252/256 (98%) (56)
$\longrightarrow\ b_2b_3 + b_6b_7$ 256/256(100%) (57)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (58)
$\longrightarrow\ b_1b_6 + b_7b_8 + b_a(1 + b_6 - b_7 - b_8)$ 364/512 (71%) (59)
$\longrightarrow\ b_2b_3 + b_5b_8 - b_6b_8 + b_7b_a - b_7 + b_8 - b_a + 1$ 450/512 (88%) (60)
$\longrightarrow\ b_1b_4 + b_4$ 488/512 (95%) (61)
$\longrightarrow\ b_2b_3 + b_3b_7 - b_6b_8 + b_8 - b_a + 1$ 502/512 (98%) (62)
$\longrightarrow\ b_2b_5 + b_5$ 512/512(100%) (63)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (64)
$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (65)
$\longrightarrow\ b_1b_4 + b_4b_8$ 237/256 (93%) (66)
$\longrightarrow\ b_2b_3 + b_3b_7 - b_4b_6 + b_4b_8 - b_5b_7 - b_5b_8 + b_6b_8 - b_6 + b_7 - b_8 + 2$ 254/256 (99%) (67)
$\longrightarrow\ b_1b_2 + b_7b_8$ 256/256(100%) (68)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8$: $(k,n) = (7,8)$. (69)
$\longrightarrow\ b_6b_7 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 388/512 (76%) (70)
$\longrightarrow\ b_1b_3 + b_3b_8 + b_a(1 + b_8)$ 470/512 (92%) (71)
$\longrightarrow\ b_2b_4 - b_3b_8 + b_4b_5 + b_a(1 - b_7) + b_8$ 500/512 (98%) (72)
$\longrightarrow\ b_2b_5 + b_2b_8 - b_4b_8 - b_6b_7 + b_6b_8 + b_a(-1 - b_4 - b_7 + b_8) - b_3 + b_7 - b_8 + 4$ 508/512 (99%) (73)
$\longrightarrow\ b_2b_5 - b_7b_8 + b_5 + 1$ 512/512(100%) (74)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8$: $(k,n) = (7,8)$. (75)
$\longrightarrow\ 2b_5b_6$ 193/256 (75%) (76)
$\longrightarrow\ b_1b_4 + b_4b_8$ 235/256 (92%) (77)
$\longrightarrow\ b_2b_3 + b_2b_7 - b_5b_6 + b_6b_8 + b_5 - b_6 - b_8 + 1$ 250/256 (98%) (78)
$\longrightarrow\ b_3b_7 + b_7b_8$ 254/256 (99%) (79)
$\longrightarrow\ b_3b_8 + b_3$ 256/256(100%) (80)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (81)
$\longrightarrow\ b_2b_3 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 390/512 (76%) (82)
$\longrightarrow\ b_1b_4 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8)$ 480/512 (94%) (83)
$\longrightarrow\ b_2b_4 + b_5 - b_a + 1$ 506/512 (99%) (84)
$\longrightarrow\ b_1b_3 - b_6b_a + b_5 + 1$ 512/512(100%) (85)

$b_1b_2b_3b_4 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (86)
$\longrightarrow\ b_1b_2 + b_6b_7$ 169/256 (66%) (87)
$\longrightarrow\ b_3b_4 + b_5b_8$ 238/256 (93%) (88)
$\longrightarrow\ b_1b_4 + b_5b_6 + b_5b_7 + b_6b_7 - b_5 - b_6 - b_7 + 1$ 248/256 (97%) (89)
$\longrightarrow\ b_2b_3 + b_6b_7 + b_6b_8 + b_7b_8 - b_6 - b_7 - b_8 + 1$ 254/256 (99%) (90)
$\longrightarrow\ b_1b_2 + b_5b_8$ 256/256(100%) (91)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (92)
$\longrightarrow\ b_1b_4 + b_7b_9$ 625/1024 (61%) (93)
$\longrightarrow\ b_3b_5 + b_6b_8$ 889/1024 (87%) (94)
$\longrightarrow\ b_2b_5 + b_7b_{10}$ 972/1024 (95%) (95)
$\longrightarrow\ b_2b_4 + b_6b_8$ 999/1024 (98%) (96)
$\longrightarrow\ b_1b_3 + b_9b_{10} + b_9b_a$ 1016/1024 (99%) (97)
$\longrightarrow\ b_1b_5 + b_6b_9$ 1020/1024 (99%) (98)
$\longrightarrow\ b_1b_4 + b_8b_{10}$ 1022/1024 (99%) (99)
$\longrightarrow\ b_2b_3 - b_4b_{10} + b_7b_9 + b_9b_a + 1$ 1024/1024(100%) (100)

$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10}$: $(k,n) = (5,10)$. (101)
$\longrightarrow\ b_1b_3 + b_9b_{10}$ 625/1024 (61%) (102)
$\longrightarrow\ b_2b_4 + b_7b_{10}$ 851/1024 (83%) (103)
$\longrightarrow\ b_3b_5 + b_5b_{10} + b_8b_9$ 924/1024 (90%) (104)
$\longrightarrow\ b_1b_2 + b_6$ 972/1024 (95%) (105)
$\longrightarrow\ b_3b_4 + b_8b_9$ 997/1024 (97%) (106)
$\longrightarrow\ b_1b_5 + b_7b_{10}$ 1010/1024 (99%) (107)
$\longrightarrow\ b_2b_3 - b_1b_7 - b_1b_{10} - b_2b_8 - b_2b_{10} + b_3b_5 + b_6b_9 + b_7b_{10} - b_8b_9 + b_9b_{10} - b_3 - b_7 + b_8 + 3$ 1016/1024 (99%) (108)
$\longrightarrow\ b_1b_3 + b_7b_8$ 1020/1024 (99%) (109)
$\longrightarrow\ b_2b_4 + b_2b_6 - b_2b_9 - b_3b_{10} - b_5b_7 + b_7b_{10} + b_9b_{10} - b_{10} + 2$ 1023/1024 (99%) (110)
$\longrightarrow\ b_2b_5 + b_2b_9 + b_6b_8$ 1024/1024(100%) (111)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (112)
$\longrightarrow\ b_4b_5 + b_5b_9$ 657/1024 (64%) (113)
$\longrightarrow\ b_2b_6 + b_6b_8$ 905/1024 (88%) (114)
$\longrightarrow\ b_1b_3 + b_7b_8$ 982/1024 (96%) (115)
$\longrightarrow\ b_2b_3 + b_a(b_{10} - b_9) + b_9$ 1011/1024 (99%) (116)
$\longrightarrow\ b_2b_4 + b_7b_{10}$ 1020/1024 (99%) (117)
$\longrightarrow\ b_9b_{10} + b_1$ 1023/1024 (99%) (118)
$\longrightarrow\ b_7b_8 + b_4$ 1024/1024(100%) (119)

$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (120)
$\longrightarrow\ 2b_5b_6$ 769/1024 (75%) (121)
$\longrightarrow\ b_1b_3 + b_8b_9$ 934/1024 (92%) (122)
$\longrightarrow\ b_2b_4 + b_7b_{10} + b_8b_9 - b_8 - b_9 + 1$ 997/1024 (97%) (123)
$\longrightarrow\ -b_1b_3 + b_1b_9 + b_2b_4 + b_4b_9 + b_5b_8 + b_8b_9 - b_5 - b_8 - b_9 + 2$ 769/1024 (99%) (124)
$\longrightarrow\ b_1b_3 + b_7b_{10} - b_8 - b_9 + 2$ 1014/1024 (99%) (125)
$\longrightarrow\ b_2b_3 + b_8b_9$ 1024/1024(100%) (126)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (127)
$\longrightarrow\ b_3b_5 + b_5b_8$ 649/1024 (63%) (128)
$\longrightarrow\ b_2b_4 + b_4b_9$ 893/1024 (87%) (129)
$\longrightarrow\ b_1b_7 + b_7b_{10}$ 985/1024 (96%) (130)
$\longrightarrow\ b_1b_6 + b_6b_9 + b_a$ 1015/1024 (99%) (131)
$\longrightarrow\ b_2b_3 + b_8b_{10} + b_a$ 1022/1024 (99%) (132)
$\longrightarrow\ b_1b_3 + b_8b_9$ 1024/1024(100%) (133)

$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (134)
$\longrightarrow\ b_3b_7 + b_7b_{10}$ 649/1024 (63%) (135)
$\longrightarrow\ 2b_4b_6$ 937/1024 (92%) (136)
$\longrightarrow\ b_1b_5 + b_5b_8$ 1001/1024 (98%) (137)
$\longrightarrow\ b_1b_2 + b_9b_{10}$ 1019/1024 (99%) (138)
$\longrightarrow\ b_2b_3 + b_8$ 1023/1024 (99%) (139)
$\longrightarrow\ b_3b_7 + b_9b_{10}$ 1024/1024(100%) (140)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (141)
$\longrightarrow\ b_2b_8 + b_8b_9$ 645/1024 (63%) (142)
$\longrightarrow\ b_1b_3 + b_3b_{10} + b_9b_a$ 887/1024 (87%) (143)
$\longrightarrow\ b_4b_6 + b_5b_6$ 977/1024 (95%) (144)
$\longrightarrow\ b_2b_7 + b_7b_{10}$ 1007/1024 (98%) (145)
$\longrightarrow\ b_1b_4 + b_4b_5 + 2b_9b_a$ 1018/1024 (99%) (146)
$\longrightarrow\ b_1b_5 + b_5b_9$ 1024/1024(100%) (147)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (148)
$\longrightarrow\ 4b_3b_7$ 768/1024 (75%) (149)
$\longrightarrow\ b_2b_8 + b_8b_9$ 933/1024 (91%) (150)
$\longrightarrow\ 2b_4b_6 + b_8b_9 - b_8 - b_9 + 1$ 1005/1024 (98%) (151)
$\longrightarrow\ b_1b_5 + b_5b_{10} + b_8b_9 - b_8 - b_9 + 1$ 1022/1024 (99%) (152)
$\longrightarrow\ b_1b_2 + b_8b_9 + b_9b_{10} - b_8 - b_9 + 1$ 1024/1024(100%) (153)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (154)
$\longrightarrow\ b_1b_9 + b_9b_{10} + b_{10}b_a$ 643/1024 (63%) (155)
$\longrightarrow\ b_2b_4 + b_4b_5$ 883/1024 (86%) (156)
$\longrightarrow\ b_3b_7 + b_3b_8$ 973/1024 (95%) (157)
$\longrightarrow\ b_2b_6 + b_6b_8$ 1003/1024 (98%) (158)
$\longrightarrow\ b_2b_5 + b_5b_7 - b_{10}b_a + b_{10}$ 1015/1024 (99%) (159)
$\longrightarrow\ b_1b_8 + b_7b_8$ 1019/1024 (99%) (160)
$\longrightarrow\ b_2b_7 + b_2b_{10} - b_4b_5 - b_{10}b_a + b_{10} + 1$ 1023/1024 (99%) (161)
$\longrightarrow\ b_4b_7 + b_7$ 1024/1024(100%) (162)

$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (9,10)$. (163)
$\longrightarrow\ 2b_2b_3 - b_8b_9 + b_9$ 577/1024 (56%) (164)
$\longrightarrow\ 3b_8b_9$ 961/1024 (94%) (165)
$\longrightarrow\ 2b_4b_6 - b_8b_9 - b_8b_{10} + b_{10} + 1$ 1009/1024 (99%) (166)
$\longrightarrow\ 2b_5b_7 - b_8b_{10} + b_{10}$ 1021/1024 (99%) (167)
$\longrightarrow\ b_1b_6 + b_{10}$ 1024/1024(100%) (168)

DEGREE-k, EXACT-k-OF-n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $(k,n) = (4,6)$. (169)
$\longrightarrow\ b_1b_4 + 2b_4b_5 + b_7$ 44/64 (69%) (170)
$\longrightarrow\ b_1b_3 + b_2b_3 + b_3b_6 + b_6b_7$ 60/64 (94%) (171)
$\longrightarrow\ b_2b_4 + b_5b_6 + b_2$ 64/64(100%) (172)

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $(k,n) = (4,6)$. (173)
$\longrightarrow\ b_2b_4 + 2b_4b_5$ 43/64 (67%) (174)
$\longrightarrow\ b_1b_3 + b_2b_3 + b_2b_5 + b_3b_6 - b_4b_5 - b_2 + 1$ 60/64 (94%) (175)
$\longrightarrow\ b_1b_2 + b_2b_5 + b_5b_6$ 64/64(100%) (176)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (177)
$\longrightarrow\ b_1b_4 + 2b_5b_6$ 159/256 (62%) (178)
$\longrightarrow\ b_2b_3 + b_3b_5 + b_7b_8$ 225/256 (88%) (179)
$\longrightarrow\ b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1$ 244/256 (95.3%) (180)
$\longrightarrow\ b_2b_3 + b_6b_8 + b_6$ 253/256 (98.8%) (181)
$\longrightarrow\ b_2b_3 + b_5b_7 + b_5$ 256/256 (100%) (182)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $(k,n) = (4,8)$. (183)
$\longrightarrow\ b_2b_4 + 2b_5b_6$ 159/256 (62%) (184)
$\longrightarrow\ b_3b_6 + b_7b_8 + b_3$ 212/256 (83%) (185)
$\longrightarrow\ b_2b_4 - b_5b_7 + b_7b_8 + b_4 + b_7$ 234/256 (91%) (186)
$\longrightarrow\ b_1b_3 + 2b_5b_6$ 253/256 (99%) (187)
$\longrightarrow\ b_7b_8 + b_1 + b_6$ 256/256(100%) (188)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (189)
$\longrightarrow\ b_1b_5 + b_5b_6 + b_6b_7 + b_a(-2 - b_5 + 2b_6 + b_7) + b_5 - 2b_6 - b_7 + 2$ 86/128 (67%) (190)
$\longrightarrow\ b_1b_3 + b_3b_4 - b_3b_6 + b_3b_7 + b_a(b_5 + b_7) + b_3$ 112/128 (88%) (191)
$\longrightarrow\ b_1b_4 + b_2b_4 + b_4b_7 + b_5b_7 + b_a(-1 - b_6 - b_7) - b_5 + b_6 + 2$ 124/128 (97%) (192)
$\longrightarrow\ b_2b_4 - 2b_5b_a + b_6b_7 + b_2 + b_5 + 1$ 128/128(100%) (193)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $(k,n) = (5,7)$. (194)
$\longrightarrow\ 2b_4b_5 + b_4b_6$ 81/128 (63%) (195)
$\longrightarrow\ b_1b_3 + b_3b_6 + b_3b_7 - b_4b_5 + b_5$ 111/128 (87%) (196)
$\longrightarrow\ b_1b_2 + b_2b_6 - b_4b_5 + b_6b_7 + b_5$ 122/128 (95%) (197)
$\longrightarrow\ 2b_4b_5 + b_5$ 128/128(100%) (198)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8$: $(k,n) = (6,8)$. (199)
$\longrightarrow\ b_1b_6 + 2b_6b_7$ 164/256 (64%) (200)
$\longrightarrow\ b_1b_5 + b_2b_5 - b_3b_6 + b_5b_8 + b_3$ 219/256 (86%) (201)
$\longrightarrow\ b_2b_4 + b_4b_7 + b_4b_8 - b_6 + 1$ 243/256 (95%) (202)
$\longrightarrow\ b_2b_3 + b_3b_8 - b_5b_6 + b_3 + b_6$ 253/256 (99%) (203)
$\longrightarrow\ b_1b_2 + b_2b_6 + b_5b_7 - b_6b_7 + b_7b_8 - b_5 + 1$ 256/256(100%) (204)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (8,10)$. (205)
$\longrightarrow\ 3b_5b_8$ 769/1024 (75%) (206)
$\longrightarrow\ 2b_2b_6 + b_4b_6$ 931/1024 (91%) (207)
$\longrightarrow\ b_1b_7 - b_5b_{10} + b_7b_9 + b_9b_{10} - b_6 + b_{10} + 1$ 984/1024 (96%) (208)
$\longrightarrow\ 3b_2b_3 + b_3b_{10} - b_6b_8 + 1$ 1011/1024 (99%) (209)
$\longrightarrow\ b_4b_7 + b_4b_8 - b_3 + b_4 - b_8 + 2$ 1019/1024 (99%) (210)
$\longrightarrow\ b_2b_3 - b_2b_4 - b_3b_4 - b_3b_8 - b_5b_{10} - b_6b_9 + b_7b_8 + b_7b_9 + b_8b_9 + b_7 + 3$ 1023/1024 (99%) (211)
$\longrightarrow\ b_2b_8 + 2b_8b_9$ 1024/1024(100%) (212)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (213)
$\longrightarrow\ b_5b_6 + b_5b_8 + b_5b_9 + b_8b_{11} + b_9b_{11} + b_{10}b_{11}$ 583/1024 (57%) (214)
$\longrightarrow\ b_1b_2 + b_4b_7 + b_7b_{10} + b_9b_{11} - b_9 - b_{11} + 1$ 815/1024 (80%) (215)
$\longrightarrow\ b_1b_6 + b_5b_6 + b_6 - b_{11} + 1$ 917/1024 (90%) (216)
$\longrightarrow\ b_3b_4 + b_3b_7 + b_8b_9 + b_9b_{11}$ 979/1024 (96%) (217)
$\longrightarrow\ b_2b_4 + b_4b_8 + b_8b_9 - b_9b_{11} + b_9 - b_{11} + 1$ 1007/1024 (98%) (218)
$\longrightarrow\ b_1b_3 + b_7b_{10} + b_{10}b_{11} + b_3$ 1016/1024 (99%) (219)
$\longrightarrow\ b_1b_4 + b_4b_8 + b_7b_{10} + b_9b_{11} + b_{10}b_{11} - b_9 - b_{11} + 1$ 1021/1024 (99%) (220)
$\longrightarrow\ b_1b_3 - b_2b_{11} + b_7b_8 + b_8b_9 - b_{10}b_{11} - b_{11} + 3$ 1024/1024(100%) (221)

$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (6,10)$. (222)
$\longrightarrow\ 2b_3b_4 + b_7b_{10}$ 591/1024 (58%) (223)
$\longrightarrow\ 2b_3b_5 + b_5b_6$ 847/1024 (83%) (224)
$\longrightarrow\ b_1b_2 + b_7b_8 + b_8b_9$ 951/1024 (93%) (225)
$\longrightarrow\ 3b_5b_6$ 995/1024 (97%) (226)
$\longrightarrow\ b_1b_3 + b_3b_4 + b_9b_{10}$ 1009/1024 (99%) (227)
$\longrightarrow\ b_1b_2 + b_5b_7 + b_7b_{10}$ 1018/1024 (99%) (228)
$\longrightarrow\ 2b_1b_4 - b_1b_{10} + b_2b_4 + b_4b_5 + b_4b_{10} + b_5b_8 - b_6b_8 + b_8b_9 + b_7(b_{10} - b_6 - b_5 - b_1) + 3$ 1023/1024 (99%) (229)
$\longrightarrow\ b_2b_8 + b_3b_6 + b_6b_8$ 1024/1024(100%) (230)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $(k,n) = (4,10)$. (231)
$\longrightarrow\ b_2b_3 + b_6b_7 + b_7b_9 + 2b_9b_a$ 581/1024 (57%) (232)
$\longrightarrow\ b_2b_4 + b_4b_6 + b_a(b_9 - b_{10}) + b_{10}$ 823/1024 (80%) (233)
$\longrightarrow\ b_1b_3 + b_5b_6 + b_8b_9 + b_a(b_9 - b_{10}) - b_9 + 1$ 930/1024 (91%) (234)
$\longrightarrow\ b_1b_4 + b_4b_5 + b_4b_{10} + b_8b_{10} + b_a(1 - b_7 + b_9)$ 978/1024 (96%) (235)
$\longrightarrow\ b_1b_4 + b_7b_8 + b_a(1 + b_9) + b_7$ 1000/1024 (98%) (236)
$\longrightarrow\ b_2b_3 + b_a(b_9 - b_{10}) + b_5 + b_{10}$ 1015/1024 (99%) (237)
$\longrightarrow\ b_1b_3 + b_6 + b_{10}$ 1020/1024 (99%) (238)
$\longrightarrow\ b_5b_6 + b_2 + b_8 + b_a$ 1024/1024(100%) (239)

$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10}$: $(k,n) = (4,10)$. (240)
$\longrightarrow\ b_3b_4 + b_4b_6 + b_9b_{10}$ 581/1024 (57%) (241)
$\longrightarrow\ b_1b_2 + b_5b_7 - b_8b_9 + b_9b_{10} + b_9$ 759/1024 (74%) (242)
$\longrightarrow\ b_5b_6 + b_8b_9 + b_1 + b_8$ 842/1024 (82%) (243)
$\longrightarrow\ b_2b_4 + b_7b_{10} - b_8b_9 + b_7 + b_8$ 935/1024 (91%) (244)
$\longrightarrow\ b_2b_4 + b_4b_6 - b_8b_9 + b_8b_{10} - b_7 + b_8 + 1$ 969/1024 (95%) (245)
$\longrightarrow\ b_1b_3 + b_3b_4 + b_5b_7 + b_7b_9 - b_8b_9 + b_9$ 992/1024 (97%) (246)
$\longrightarrow\ b_2b_3 + b_3b_5 + b_3b_{10} + b_4b_8 + b_5b_6 - b_4 + 1$ 1004/1024 (98%) (247)
$\longrightarrow\ b_1b_3 + b_6b_7 + b_9b_{10}$ 1013/1024 (99%) (248)
$\longrightarrow\ b_1b_9 + b_7b_8 - b_8b_9 - b_9b_{10} + b_1 + b_7 + b_8 + b_9$ 1019/1024 (99%) (249)
$\longrightarrow\ b_2b_3 + b_5b_6 - b_8b_9 + b_9b_{10} + b_9$ 1022/1024 (99%) (250)
$\longrightarrow\ -b_1b_5 + b_1b_8 + b_3b_7 + b_3 + b_7 + 1$ 1023/1024 (99%) (251)
$\longrightarrow\ b_1b_5 - b_1b_{10} + b_2 + b_8 - b_{10} + 2$ 1024/1024(100%) (252)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (253)
$\longrightarrow\ b_1b_3 + b_3b_4 + b_6b_8 - b_6b_9 + b_7b_9 - b_8b_9 + b_9$ 156/256 (61%) (254)
$\longrightarrow\ b_1b_5 + b_5b_7 + b_7b_9 + b_8b_9 + b_5 - b_7 - b_9 + 1$ 202/256 (79%) (255)
$\longrightarrow\ b_2b_4 + b_6b_8 + b_6b_9 - b_7b_9 + b_8b_9 + b_7 - b_8 - b_9 + 1$ 230/256 (90%) (256)
$\longrightarrow\ b_2b_4 + b_4b_8 + b_4 - b_9 + 1$ 246/256 (96%) (257)
$\longrightarrow\ b_1b_5 + 2b_6$ 252/256 (98%) (258)
$\longrightarrow\ b_2b_5 + b_7b_8 + b_5$ 256/256(100%) (259)

$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8$: $(k,n) = (5,8)$. (260)
$\longrightarrow\ b_4b_5 + 2b_5b_6$ 165/256 (64%) (261)
$\longrightarrow\ b_2b_4 + b_3b_4 + b_4b_8 - b_5b_7 + b_7$ 215/256 (84%) (262)
$\longrightarrow\ b_2b_3 + b_3b_6 - b_4b_5 - b_5b_7 + b_7b_8 + b_5 + b_7$ 242/256 (95%) (263)
$\longrightarrow\ b_1b_3 + b_5b_6 + b_6b_7$ 254/256 (99%) (264)
$\longrightarrow\ b_1b_2 + b_5b_6 + b_6b_8$ 256/256(100%) (265)

DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4$: $(k,n) = (3,4)$. (266)
$\longrightarrow\ 2b_1b_2 + b_1b_3 + 2b_1b_4 + b_2b_3 + 2b_2b_4 + b_3b_4 - 2b_1 - 2b_2 - b_3 - 2b_4 + 3$ 13/16 (81%) (267)
$\longrightarrow\ 2b_1b_3 + b_2b_3 + b_2$ 16/16(100%) (268)

$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10}$: $(k,n) = (7,10)$. (269)
$\longrightarrow\ 4b_4b_5$ 769/1024 (75%) (270)
$\longrightarrow\ b_2b_6 + 2b_3b_6 + b_6b_9$ 915/1024 (89%) (271)
$\longrightarrow\ b_1b_7 + b_5b_7 + b_6b_7 + b_7b_{10}$ 974/1024 (95%) (272)
$\longrightarrow\ b_1b_2 + b_2b_8 + b_7b_8 + b_9b_{10}$ 995/1024 (97%) (273)
$\longrightarrow\ b_2b_3 + b_3b_4 + b_3b_6 + b_9b_{10}$ 1008/1024 (98%) (274)
$\longrightarrow\ b_1b_2 + b_2b_4 + b_9b_{10} + b_9$ 1016/1024 (99%) (275)
$\longrightarrow\ b_1b_3 - b_2b_8 + b_7b_8 + b_8b_9 + b_8b_{10} + 2b_8$ 1023/1024 (99%) (276)
$\longrightarrow\ b_1b_8 + b_2b_7 - b_5b_{10} + b_7b_8 + b_8b_9 - b_5 + 2$ 1024/1024(100%) (277)

DEGREE-k, NOT EXACT-k-OF-n MULTINOMIALS

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4$: $(k,n) = (4,4)$. (278)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (279)
$\longrightarrow\ b_1b_2 + b_1b_3 + 4b_1b_4 + b_2b_4$ 16/16(100%) (280)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4$: $(k,n) = (4,4)$. (281)
$\longrightarrow\ b_1b_2 + 4b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - b_1 - b_2 - b_3 - b_4 + 1$ 12/16 (75%) (282)
$\longrightarrow\ 2b_2b_3 + 3b_2b_4 + b_3b_4$ 16/16(100%) (283)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_2b_4 + 3b_1b_3b_4 + b_2b_3b_4$: $(k,n) = (4,4)$. (284)
$\longrightarrow\ 2b_1b_2 + 5b_1b_4 + b_3b_4$ 11/16 (69%) (285)
$\longrightarrow\ -b_1b_2 + 3b_1b_3 + 4b_2b_3 + 2b_2b_4 - 4b_3b_4 + 4b_3 - b_4 + 1$ 16/16(100%) (286)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4$: $(k,n) = (4,4)$. (287)
$\longrightarrow\ 4b_1b_3$ 13/16 (81%) (288)
$\longrightarrow\ 2b_1b_2 + b_1b_4 + b_2b_4$ 16/16(100%) (289)

$b_1b_2b_3b_4 + 2b_1b_2b_3 + b_1b_3b_4$: $(k,n) = (4,4)$. (290)
$\longrightarrow\ 2b_1b_3 + 2b_3b_4$ 12/16 (75%) (291)
$\longrightarrow\ 3b_1b_2 + b_1b_4$ 16/16(100%) (292)