# 1 Terminology

**Directed tree:** A directed graph that would be a tree if we ingored the directions of the edges.

**Arborescence:** A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

**Spanning subgraph** of $D(V, A)$: A subgraph $D(V, B)$ such that $B \subseteq A$.

**Branching system** of $D(V, A)$ with $k$ specified root nodes: A spanning subgraph of $D$ created by edge-disjoint arborescences rooted at the $k$ root nodes. Edge-disjoint means none of these arborescences share any edges. Every node has at most $k$ incoming edges (at most one from each arborescence).

**Optimal branching system** of a weighted digraph with $k$ specified root nodes: the branching system with the lowest total weight.

# 2 Algorithm

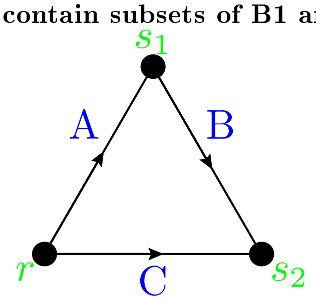
To find an OBS we need to find a min-cost set of edges which is both a basis of matroid $M_1$ and of $M_2$:

- $M_1$=$(E, J_1)$ in which $j \in J_1$ has no edges entering any root nodes, and at most $k$ edges entering each other node of the graph.
- $M_2$=$(E, J_2)$ in which $j \in J_2$ can be partitioned into at most $k$ **arborescences**.

# 3 Examples:

## 3.1 3-nodes

### 3.1.1 1 root (it seems that J1 also should contain subsets of B1 and B2)

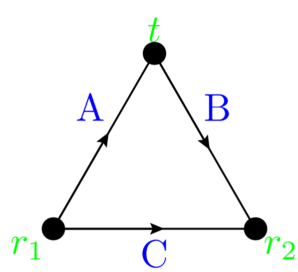



The branchings systems are:

$$B_1 = \{A, C\} \tag{3.1}$$
$$B_2 = \{A, B\} \tag{3.2}$$

**We also have that** $J_1 = J_2 = \{B_1, B_2\}$.

### 3.1.2 2 roots

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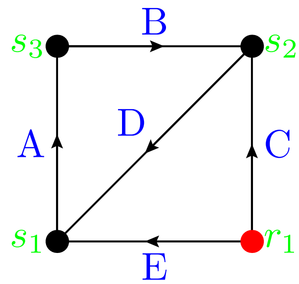
The only branching system is:

$$B = \{A\}. \tag{3.3}$$

**We also have that** $J_1 = J_2 = B$.

## 3.2 4-nodes:

### 3.2.1 1 root




The branching systems are:

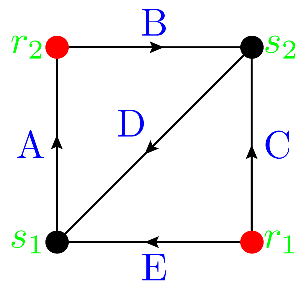
$$B_1 = \{E, B, A\}, \tag{3.4}$$
$$B_2 = \{E, B, D\}, \tag{3.5}$$
$$B_3 = \{D, C, B\}. \tag{3.6}$$

**We also have that:**

$$J_1 = \{B_1, B_2, B_3, \{A, C, B\}\} \tag{3.7}$$
$$J_2 = \{B_1, B_2, B_3\}. \tag{3.8}$$

Since $J_2 \subset J_1$,we know that $J_1 \cap J_2 = J_2$, **so the OBS will be the min-cost branching system in $J_2$.**

### 3.2.2 2 roots



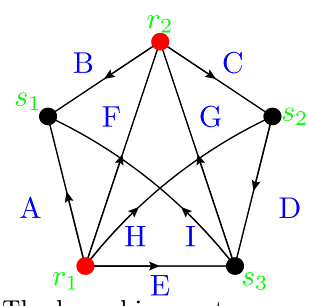

There is only one branching system:

$$B = \{E, B, D, C\}, \tag{3.9}$$

**We also have that:** $J_1 = J_2 = B$.

## 3.3 5-nodes:

### 3.3.1 2 roots




The branching systems are:

$$B_1 = \{B, C, A, H, D, E\}, \tag{3.10}$$
$$B_2 = \{B, C, D, E, H, I\}, \tag{3.11}$$
$$B_3 = \{C, D, I, A, H, E\}. \tag{3.12}$$

**We also have that:** $J_1 = J_2 = \{B_1, B_2, B_3\}$.