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Jon Lee, Jennifer Ryan,

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Matroid Applications and Algorithms

JON LEE Department of Operations Research, Yale University, New Haven, CT 06520, BITNET: leejona@yalevm

JENNIFER RYAN Department of Mathematics, University of Colorado at Denver, P.O. Box 173364, Denver, CO 80217-3364, BITNET: jryan@cudenver

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Matroid theory provides a set of modeling tools with which many combinatorial and algebraic problems may be treated. Generic algorithms for the resulting matroid problems can be used to solve problems from a variety of application areas including engineering, scheduling, mathematics, and mathematical programming. In this paper, we give an introduction to matroid theory and algorithms, and a survey of algorithmic applications.

Matroids provide a setting in which combinatorial properties of vector spaces, projective planes, graphs, and many other structures can be studied. Much of matroid theory is combinatorial mathematics with no known algorithmic significance. Readers who are interested in the mathematics of matroid theory are referred to Welsh[157] and Oxley.[125] Algorithmic matroid theory is the focus of Lawler[101] and Recski.[134] While matroids were first introduced in the mid-1930s by Whitney[160] and Van der Waerden,[152] algorithmic matroid theory did not begin to flourish until the late 1960s. Since then, algorithms have been developed to solve several combinatorial problems defined with respect to one or several matroids. These combinatorial problems serve as mathematical models in such diverse areas as the optimal sequencing of jobs, the analysis of electrical networks and mechanical frameworks, network design, game theory, and preprocessing mathematical programs.

Section 1 summarizes the standard definitions, basic facts, and constructions of matroid theory that bear on applications. The aim of Section 1 is to acquaint the reader with elements of matroid theory that may serve as modeling tools, as well as enough of the theory and notation so that the reader may understand the algorithms and their efficient implementation.

Section 2 contains rudimentary algorithms for many of the combinatorial problems on matroids. All of the algorithms require the availability of a subroutine that is specific to each matroid in the application. Fortunately, most of the applications only involve matroids of a few different types, so the incremental work required to produce software to solve an additional application is often quite small. In many cases we do not present the most efficient algorithm for a matroid problem. However, we provide references to more efficient algorithms.

Many of the problems can be more efficiently solved with specialized algorithms that blur the line between the matroid problem and the application-specific subroutines that define the relevant matroids. This paper is not about such algorithms. Our viewpoint is that the availability of fast computers and well-coded, efficient, general purpose algorithms for matroid problems is sufficient for the solution of a wide variety of large problems in many application areas, with little application-specific code required. Of course, the solution of the largest problems, or the use of slower computers, may necessitate the development of special software.

Section 3 contains an assortment of applications of algorithmic matroid theory. The next two sections cover application areas that are more developed and involve special terminology. Section 4 covers applications in the analysis of electrical systems. Section 5 is devoted to statics. It is hoped that by reading Sections 3, 4, and 5, the reader will gain the insight necessary to recognize a matroid problem, and the applicability of an algorithm of Section 2. Finally, Section 6 contains references to a variety of applications of matroid theory in which the algorithmic consequences are less clear than the applications of Sections 3, 4, and 5. In many of these cases, the insight obtained by the application of matroid theory should not be underestimated.

Throughout, we omit proofs. In some cases, the application of matroid theory is simply a modeling exercise, in which case there is little to prove. For situations in which there is something to prove, we refer the reader to the appropriate literature.

We assume some familiarity with graph theory (see Bondy and Murty[15], especially its algorithmic aspects (see Lawler[101]). We briefly set our notation. If A is a matrix with columns labeled from the set E, and X is a subset of E, then A_X is the submatrix of A comprised of the columns labeled from X. The order of the columns

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is usually unimportant. The order r identity matrix is denoted I_r; the order is omitted if it is clear from context. In matrices, a block of zeros is denoted by $\mathbf{0}$. The rational (respectively, real, complex, binary) field is denoted by $\mathbf{Q}$ ($\mathbf{R}$, $\mathbf{C}$, $\mathbf{GF(2)}$). Disjoint set union is indicated by $\uplus$. A set $\{e\}$ consisting of a single element is often abbreviated by e.

1. Definitions, Basic Facts and Constructions

There are many equivalent axiomatizations of matroids. We give one such axiomatization that is convenient in its relationship to the algorithms of matroid theory. A *matroid* $\mathcal{M}$ is specified by an ordered pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I}$ is a set of subsets of E, called the *independence sets*, that satisfy the *independence axioms*:

(I0) $\emptyset \in \mathcal{I}$.
(I1) $X \in \mathcal{I},\ X' \subseteq X \Rightarrow X' \in \mathcal{I}$.
(I2) $X_1 \in \mathcal{I},\ X_2 \in \mathcal{I},\ |X_1| > |X_2| \Rightarrow \exists\ e \in X_1 \setminus X_2$ such that $X_2 \uplus e \in \mathcal{I}$.

In computation, we do not explicitly list all elements of $\mathcal{I}$ as their number may be exponential in $|E|$. We could just list the maximal independent sets, known as the *bases*, but there might be an exponential number of them as well. Instead, we provide a subroutine that determines whether an arbitrary subset of the ground set is independent or not. In practice, we require that the subroutine operate in a number of steps that is bounded by a polynomial in some parsimonious encoding of the problem. The common cardinality of all maximal independent subsets of $X \subseteq E$, denoted $r_{\mathcal{M}}(X)$, is the *rank* of X. The rank of the entire ground set, which is equal to the common size of all bases of $\mathcal{M}$, is the *rank* of $\mathcal{M}$. For a set $X \subseteq E$, the *span* of X, denoted $\mathrm{sp}_{\mathcal{M}}(X)$, is the maximal set containing X, whose rank does not exceed that of X.

Associated with the matroid $\mathcal{M} = (E, \mathcal{I})$ is its set of *circuits* $\mathcal{C}$ defined by

$$\mathcal{C} \equiv \{X \subseteq E\colon X \notin \mathcal{I},\ X \setminus e \in \mathcal{I}\ \forall\ e \in X\}.$$

That is, the circuits of $\mathcal{M}$ are the minimal sets that are not independent. If X is independent and $X \uplus e$ is not independent, then $X \uplus e$ contains a unique circuit $C(X, e)$. Most of the matroid algorithms require the ability to be able to easily determine the circuit $C(X, e)$. We will refer to $C(X, e)$ as a *fundamental circuit* with respect to X.

The *direct sum* of $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$, with $E_1 \cap E_2 = \emptyset$ is $\mathcal{M}_1 \oplus \mathcal{M}_2 = (E_1 \uplus E_2, \mathcal{I})$, where

$$\mathcal{I} = \{X \uplus Y \subseteq E_1 \uplus E_2\colon X \in \mathcal{I}_1,\ Y \in \mathcal{I}_2\}.$$

The *dual* $\mathcal{M}^*$ of $\mathcal{M} = (E, \mathcal{I})$, is the ordered pair $\mathcal{M}^* = (E, \mathcal{I}^*)$, where $\mathcal{I}^*$ is the set of all subsets of E whose complements contain bases of $\mathcal{M}$. The system $\mathcal{M}^*$ is also a matroid, and its set of bases, termed the set of *cobases* of $\mathcal{M}$, is the set of complements of bases of $\mathcal{M}$. Hence the dual of $\mathcal{M}^*$ is just $\mathcal{M}$ again. The circuits of $\mathcal{M}^*$, termed the *cocircuits* of $\mathcal{M}$, turn out to be the minimal sets among all non-empty subsets of E that fail to have single-element intersections with all circuits of $\mathcal{M}$. The rank function $r^*(\cdot)$ of $\mathcal{M}^*$ is easily obtained from the rank function $r(\cdot)$ of M:

$$r^*(X) = r(E \setminus X) + |X| - r(E) \qquad \forall\ X \subseteq E.$$

Because of the close connection between a matroid and its dual, we may sometimes work with the dual of the matroid that one "naturally" considers. We will do so when it is warranted by expected computational savings.

Many of the algorithms work with "minors" of matroids. Let F be a proper subset of E. The system $\mathcal{M} \setminus F$ ($\mathcal{M}$ *delete* F) is specified by $\mathcal{M} \setminus F = (E \setminus F, \mathcal{I} \setminus F)$, where

$$\mathcal{I} \setminus F \equiv \{X \subseteq E \setminus F\colon X \in \mathcal{I}\}.$$

For convenience, we also denote $\mathcal{M} \setminus F$ by $\mathcal{M} \cdot (E \setminus F)$, the *restriction* of $\mathcal{M}$ *to* $E \setminus F$. The system $\mathcal{M}/F$ ($\mathcal{M}$ *contract* F) is specified by $\mathcal{M}/F = (E \setminus F, \mathcal{I}/F)$, where

$$\mathcal{I}/F \equiv \{X \subseteq E \setminus F\colon X \uplus J \in \mathcal{I}\},$$

and J is any fixed maximal independent subset of F (in $\mathcal{M}$). For convenience, we also denote $\mathcal{M}/F$ by $\mathcal{M} \times (E \setminus F)$, the *contraction* of $\mathcal{M}$ *to* $E \setminus F$. It is an elementary fact that minors of matroids are also matroids. The following easily verified facts will be exploited without further mention:

$$\begin{aligned}
\mathcal{M}/F/H &= \mathcal{M}/H/F,\\
\mathcal{M} \setminus F \setminus H &= \mathcal{M} \setminus H \setminus F,\\
\mathcal{M} \setminus F/H &= \mathcal{M}/H \setminus F,\\
(\mathcal{M} \setminus F/H)^* &= \mathcal{M}^*/F \setminus H,\\
r_{M \setminus F}(X) &= r_M(X) \qquad \forall X \subseteq E \setminus F,\\
r_{M/F}(X) &= r_M(X \uplus F) - r_M(F) \forall X \subseteq E \setminus F.
\end{aligned}$$

One of the most useful matroids is defined on the edge-set of a finite undirected graph $G = (V, E)$, and is called the *graphic* matroid of G. The matroid $\mathcal{M}(G)$ has as its independent sets the forests of G, and the circuits of $\mathcal{M}(G)$ are the polygons of G. The rank of a set of edges $F \subseteq E$, is simply $|V|$ minus the number of connected components of the subgraph $G' = (V, F)$ (note that we count isolated vertices as components). The dual matroid, $\mathcal{M}^*(G)$, is the *cographic* matroid of G. The independent sets of $\mathcal{M}^*(G)$ are the sets of edges whose removal does not increase the number of connected components of the graph. The circuits of $\mathcal{M}^*(G)$ are the minimal subsets of edges whose removal increases the number of connected components of the graph, i.e., the "cut-sets" of G.

Minors of $\mathcal{M}(G)$ are also graphic, and graphical representations are easily obtained from G. The matroid $\mathcal{M}(G) \setminus F/H$ is the graphic matroid of the graph obtained from G by deleting the edges in F and contracting, to single vertices, the edges in H. Moreover, the cographic matroid of the resulting graph is precisely $\mathcal{M}^*(G)/F \setminus H$.

The *linear* (or *matric*) matroids form another important class of matroids. A linear matroid is specified by a matrix A over some field $\mathbf{F}$. The ground set of $\mathcal{M}_{\mathbf{F}}(A)$ is the set of column labels E of A, and the independent sets are the subsets of E whose corresponding column vectors are linearly independent over $\mathbf{F}$. The field $\mathbf{F}$ will be omitted if it is clear from context or irrelevant. If the field $\mathbf{F}$ is a finite field such as **GF(2)**, we need not concern ourselves with roundoff errors, and we may compute the rank of a set of elements $X \subseteq E$ by applying rudimentary Gaussian Elimination, or some variant, to the submatrix A_X. In many practical situations, the field is the rationals $\mathbf{Q}$ and rounding errors in computation with fixed precision arithmetic may be significant. We may avail ourselves of some of the algorithms of numerical linear algebra such as elimination methods with partial or complete pivoting and algorithms to compute the singular value decomposition of a matrix. Fortunately, many of these algorithms have been implemented and are widely available as FORTRAN-callable subroutines in packages such as IMSL and LINPACK.

A linear representation of $\mathcal{M}_{\mathbf{F}}(A_1) \oplus \mathcal{M}_{\mathbf{F}}(A_2)$ over $\mathbf{F}$ is given by

$$\begin{array}{cc} E_1 & E_2 \\ \end{array}$$
$$\left[\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right].$$

The dual and minors of $\mathcal{M}_{\mathbf{F}}(A)$ are linear over the same field $\mathbf{F}$, and it is easy to obtain such representations. Without loss of generality, we may assume that the rank of the matrix A is equal to the rank of the matroid $\mathcal{M}(A)$. Let β be a base of $\mathcal{M}(A)$. The *standard representative matrix* (with respect to β) $A_\beta^{-1}A$ is also a linear representation of $\mathcal{M}(A)$, which, after permutations (being sure to permute column labels with the columns), has the form

$$\begin{array}{ccc} & \beta & E\setminus\beta \\ A' = & \Big[\, I_r \;\Big| & N \;\Big], \end{array}$$

where r is the rank of $\mathcal{M}(A)$, and $N = A_\beta^{-1}A_{E\setminus\beta}$. It is easy to check that a standard representative matrix for $\mathcal{M}^*(A)$ is

$$\begin{array}{cc} \beta & E\setminus\beta \\ \Big[\, -N^t \;\Big| & I_{|E|-r} \Big]. \end{array}$$

A standard representative matrix for $\mathcal{M}(A) \setminus F/H$ is obtained in the following manner. First, we find a base β of $\mathcal{M}(A)$ that contains as few elements of F and as many elements of H as is possible. Form the standard representative matrix A' with respect to β. Let $F' = (E \setminus \beta) \cap (F \cup H)$ and $H' = \beta \cap (F \cup H)$. Remove all columns of A' labeled by elements of F'. Each column labeled from H' has a unique entry of 1 in some (unique) row. Remove all such rows and the associated columns labeled from H'. The resulting matrix is a standard representative matrix for $\mathcal{M}(A) \setminus F/H$.

A special class of matroids is linearly representable over every field. Such matroids are called *regular*. One canonical representation of a regular matroid is as the matroid of a totally unimodular matrix (a matrix in which every square non-singular submatrix has determinant ± 1 over the field $\mathbf{Q}$). One particularly nice feature of a totally unimodular standard representative matrix is that after an arbitrary sequence of pivots, it remains $\{0, \pm 1\}$-valued, hence rounding errors cannot occur. In fact, if A is a totally unimodular matrix, then $\mathcal{M}_{\mathbf{F}}(A) = \mathcal{M}_{\mathbf{Q}}(A)$ for every field $\mathbf{F}$; in particular, we may treat A as a matrix over the binary field **GF(2)**. A vertex-edge incidence matrix A of a directed graph G is totally unimodular, and the matroid $\mathcal{M}(A)$ is precisely the graphic matroid $\mathcal{M}(G)$.

A very useful matroid that allows a simple compound alternative structure as its set of independent sets is the *partition* matroid. Such a matroid $\mathcal{M}(E, \mathcal{I})$ is specified by a partition $\biguplus_{i=1}^{t} B_i$ of the ground set E, and integers d_i, $0 \leqslant d_i \leqslant |B_i|(1 \leqslant i \leqslant t)$. The set of independent sets is

$$\mathcal{I} \equiv \{X \subseteq E\colon |X \cap B_i| \leqslant d_i(1 \leqslant i \leqslant t)\},$$

and the rank of a subset X of E is precisely $\sum_{i=1}^{t} \min\{|X \cap B_i|, d_i\}$.

A partition matroid with exactly one block (i.e., $t = 1$) is called a *uniform* matroid. Every partition matroid is the direct sum of uniform matroids. Let $\mathcal{U}_r^n$ denote a rank r uniform matroid on ground set $\{1, 2, \ldots, n\}$. The dual of $\mathcal{U}_r^n$ is $\mathcal{U}_{n-r}^n$. Let F and H be disjoint subsets of $\{1, 2, \ldots, n\}$ such that $|F| + |H| < n$. Let $f = |F|$ and $h = |H|$. Then

$$\mathcal{U}_r^n/F \setminus H \cong \mathcal{U}^{n-f-h}_{\min\{n-f-h,\, \max\{0,\, r-f\}\}}.$$

A very useful operator for joining matroids that may have overlapping ground sets is the union operator. Let $\mathcal{M}_i = (E_i, \mathcal{I}_i)(1 \leqslant i \leqslant t)$ be a multi-set of matroids. The *union* $\mathcal{M} = \bigvee_{i=1}^{t} \mathcal{M}_i$ of the matroids $\mathcal{M}_i$ $(1 \leqslant i \leqslant t)$ has ground set $E = \biguplus_{i=1}^{t} E_i$ and independent sets

$$\mathcal{I} = \left\{X = \bigcup_{i=1}^{t} X_i \subseteq E\colon X_i \in \mathcal{I}_i(1 \leqslant i \leqslant t)\right\}.$$

If the ground sets of the individual matroids are disjoint then the union is simply the direct sum. Independence and determination of fundamental circuits, in $\mathscr{M}$, of a subset of E can be determined by applying a "matroid partitioning" algorithm which we will discuss in Section 2.2. We mention that there is a formula for the rank function $r(\cdot)$ of $\mathscr{M} = \bigvee_{i=1}^{t} \mathscr{M}_i$ in terms of the rank functions r_i of $\mathscr{M}_i$ $(1 \leqslant i \leqslant t)$.

$$r(X) = \min_{S \subseteq X} \left\{ \sum_{i=1}^{t} r_i(S \cap E_i) + |X \setminus S| \right\} \quad \forall X \subseteq E.$$

It is not difficult to check that for $F \subseteq E$, $\mathscr{M} \setminus F = \bigvee_{i=1}^{t} (\mathscr{M}_i \setminus F)$. Contractions of $\mathscr{M}$ are not so easily realized, hence it is best to work with $\mathscr{M}$ directly. For example, $X \subseteq E \setminus F$ is independent in $\mathscr{M}/F$ if and only if $X \uplus J$ is independent in $\mathscr{M}$, where J is a base of $M.F$. The dual of $\mathscr{M}$ does not have a computationally useful description, hence rank in the dual is best determined by appealing to the general formula for the rank in the dual of a matroid.

The matchings in an arbitrary undirected graph $G = (V, E)$ yield a matroid in an interesting manner. Let W be an arbitrary non-empty subset of the vertex set V. Let $\mathscr{I}$ be the set of subsets of W that can be covered by some matching (i.e., a set of vertex-disjoint edges) in G. Then, $\mathscr{M} = (W, \mathscr{I})$ is a *matching* matroid. Independence in a matching matroid can be determined by solving a graph matching problem on G (see Lawler[101]). The matching matroid on the bipartite graph $G = (V_1 \uplus V_2, E)$, with $W = V_2$, is called a *transversal* matroid. Independence of $X \subseteq V_2$ in the transversal matroid can be determined by applying a maximum-cardinality bipartite matching algorithm to the subgraph of G induced by $V_1 \uplus X$ (see Lawler[101] or Edmonds[38,39]). It is interesting to note that every matching (respectively, transversal) matroid has a representation as a transversal (linear) matroid, but realizing such a representation is not a simple matter (see Edmonds and Fulkerson[47]).

Transversal matroids are related to matrices with algebraically independent entries. Let A be a matrix with columns (respectively, rows) labeled from a finite set V_1 (V_2). The *term rank* of A is the rank of A, computed over the rationals, where the non-zero elements of A are regarded as algebraically independent indeterminates over the rationals. We may associate, with A, a bipartite graph $G = (V_1 \uplus V_2, E)$, where $(i, j) \in E$ if and only if a_{ij} is nonzero. The rank of the transversal matroid that we associate with G is the term rank of A. This illustrates the fact that the term rank of a matrix may also be defined as the maximum number of nonzeros of A, no two of which are in the same row or the same column.

There is an interesting class of matroids defined on the vertex set of a directed graph $D = (N, \mathscr{A})$. Let M be a fixed subset of N. A subset X of N can be *linked into* M if there is a collection of vertex-disjoint directed paths from all vertices of X to a subset of M. The set $\mathscr{I}$ of subsets of N that can be linked into M is the set of independent sets of a matroid on N. Any such matroid $\mathscr{M} = (N, \mathscr{I})$ is called a *strict gammoid*. The dual of a strict gammoid is a transversal matroid, and conversely. We describe the duality constructively:

Let $\mathscr{M}_1$ be a strict gammoid defined on the vertex set of $D = (N, \mathscr{A})$ by distinguishing $M \subseteq N$. Let $V_1 = N$ and let V_2 be a copy of $N \setminus M$ (for $i \in N \setminus M$, let i' be the copy of i in V_2). Let $G = (V_1 \uplus V_2, E)$ be the bipartite graph having

$$E = \{(i, i') : i \in N \setminus M\} \uplus \{(i, j') : (i, j) \in \mathscr{A}, j \notin M\}.$$

The transversal matroid that we canonically associate with G is precisely the dual of $\mathscr{M}_1$. We can use this construction to determine whether a subset X of N is independent in $\mathscr{M}_1$. We simply check whether $N \setminus X$ contains a base of $\mathscr{M}_1^*$.

There is a reverse construction, but its utility is not apparent since we know of no direct way to determine independence in a strict gammoid. Let $\mathscr{M}_2$ be the transversal matroid that we associate with an arbitrary bipartite graph $G = (V_1 \uplus V_2, E)$. Let F be a maximum-cardinality matching in G, and let M (respectively, M') be the subset of V_1 (V_2) that is *not* covered by F. Note that M is the complement of a base of $\mathscr{M}_2$. For each $i \in V_1 \setminus M$, let i' be the vertex of $V_2 \setminus M'$ that is matched to i. Let $N = V_1$ and let

$$\mathscr{A} = \{(i, j) : i \in V_1 \setminus M, (j, i') \in E, j \neq i\}.$$

The strict gammoid associated with the directed graph $D = (N, \mathscr{A})$ and the distinguished vertex set M is precisely the dual of $\mathscr{M}_2$.

The class of transversal matroids (respectively, strict gammoids) is closed under deletion (contraction) but not under contraction (deletion). Hence, in some cases, it is easier to work with the full matroid when we seek to determine independence in arbitrary minors. An arbitrary minor of a strict gammoid is called a *gammoid*. It turns out that the class of gammoids coincides with the class of transversal matroids and their contractions.

Transversal matroids are a special case of a more general construction involving bipartite graphs. Let $G = (V_1 \uplus V_2, E)$ be a bipartite graph, and let $\mathscr{M}$ be a matroid having ground set V_1. Let $\mathscr{I}$ be the set of subsets of V_2 that can be matched into independent sets of $\mathscr{M}$. Then $\mathscr{M}' = (V_2, \mathscr{I})$ is a matroid. We may refer to the "generalized transversal matroid" $\mathscr{M}'$ as the matroid *induced* from $\mathscr{M}$ by G. If $\mathscr{M}$ is the free matroid on V_1, then $\mathscr{M}'$ is the usual transversal matroid that we associate with G.

There is a similar generalization of strict gammoids. Let $D = (N, \mathscr{A})$ be a directed graph, and let $\mathscr{M}$ be a matroid on N. Let $\mathscr{I}$ be the set of subsets of N that can be linked into independent subsets of $\mathscr{M}$. Then $\mathscr{M}' = (N, \mathscr{I})$ is a matroid. We may refer to the "generalized strict gammoid" $\mathscr{M}'$ as the matroid *induced* from $\mathscr{M}$ by D. If M is a fixed subset of N, and the independent subsets of $\mathscr{M}$ (on N) are precisely the subsets of M, then $\mathscr{M}'$ is the usual strict gammoid that we associate with D and the distinguished set of vertices M.

There is a very general way in which one matroid may induce another one through a "linking system." Linking systems, introduced by Schrijver,[137] provide a common generalization of the generalized strict gammoids and the generalized transversal matroids. Although we know of no application that makes use of this greater generality, the form of the generalization seems to be quite appealing as a potential modeling tool. A *linking system* $\mathscr{L}$ is specified by an ordered triple $\mathscr{L} = (E, F, \Lambda)$, where E and F are finite sets and Λ is a subset of $2^E \times 2^F$ satisfying:

(L0) $(\varnothing, \varnothing) \in \Lambda$.
(L1) $(X, Y) \in \Lambda \Rightarrow |X| = |Y|$.
(L2) $(X, Y) \in \Lambda,\ X' \subseteq X \Rightarrow \exists\ Y' \subseteq Y$ such that $(X', Y') \in \Lambda$.
(L3) $(X, Y) \in \Lambda,\ Y' \subseteq Y \Rightarrow \exists\ X' \subseteq X$ such that $(X', Y') \in \Lambda$.
(L4) $(X_1, Y_1) \in \Lambda,\ (X_2, Y_2) \in \Lambda \Rightarrow \exists\ X' \subseteq X_2 \setminus X_1$, $Y' \subseteq Y_1 \setminus Y_2$ such that $(X_1 \cup X', Y_2 \cup Y') \in \Lambda$.

The bipartite graph $G = (V_1 \cup V_2, E)$ naturally gives rise to the linking system $\mathscr{L} = (V_1, V_2, \Lambda)$, where

$$\Lambda = \{(X, Y): X \subseteq V_1,\ Y \subseteq V_2, \text{ and } \exists \text{ a perfect matching in the subgraph of } G \text{ induced by } X \cup Y\}.$$

The directed graph $D = (N, \mathscr{A})$, with some distinguished set of vertices M, gives rise to the linking system $\mathscr{L} = (N, M, \Lambda)$, where

$$\Lambda = \{(X, Y): X \subseteq M,\ Y \subseteq N, \text{ and } \exists \text{ a set of vertex-disjoint paths linking the elements of } X \text{ to the elements of } Y\}.$$

Let A be a matrix over $\mathbf{F}$, the rows (respectively, columns) of which are indexed from E (F). The matrix A yields the linking system $\mathscr{L} = (E, F, \Lambda)$, where

$$\Lambda = \{(X, Y): X \subseteq E,\ Y \subseteq F, \text{ and } A_{X,Y} \text{ is square and nonsingular}\}.$$

Now let $\mathscr{M} = (E, \mathscr{I})$ be a matroid and $\mathscr{L} = (e, F, \Lambda)$ be an arbitrary linking system. Let

$$\mathscr{I}' = \{Y \subseteq F: \exists\ X \in \mathscr{I} \text{ such that } (X, Y) \in \Lambda\}.$$

Schrijver demonstrates that $\mathscr{M}' = (F, \mathscr{I}')$ is a matroid, termed the matroid *induced* from $\mathscr{M}$ by $\mathscr{L}$. A formula for the rank function of $\mathscr{M}'$ gives us a clue as to how we might determine independence in $\mathscr{M}'$. Associated with the linking system $\mathscr{L}$ is its linking function $\lambda(\cdot, \cdot)$ defined by

$$\lambda(X, Y) = \max\{|X'|: X' \subseteq X,\ Y' \subseteq Y, \text{ and } (X', Y') \in \Lambda\}.$$

Let E and F be disjoint sets and let $\mathscr{L} = (E, F, \Lambda)$ be a linking system. The linking system $\mathscr{L}$ determines a matroid $\mathscr{N}$ on $E \cup F$ having rank function ρ specified by

$$\rho(X \cup Y) = \lambda(E \setminus X, Y) + |X| \quad \text{for } X \subseteq E,\ Y \subseteq F.$$

Let $r(\cdot)$ (respectively, $r'(\cdot)$) be the rank function of $\mathscr{M}$ ($\mathscr{M}'$). It can be shown that

$$r'(Y) = \min_{X \subseteq E} \{r(E \setminus X) + \lambda(X, Y)\} \qquad \forall\ Y \subseteq F.$$

The formulas for $r'(\cdot)$, $r(\cdot)$, the rank of the union of two matroids, and the rank of a contracted matroid can be used to give a computationally useful form for $\mathscr{M}'$. Namely, $\mathscr{M}' = (\mathscr{M} \vee \mathscr{N})/E$. We see the fundamental role of matroid union in many of the previous examples of matroids, and how an efficient method for determining independence in the union of matroids can be an extraordinarily useful tool. Such a method is discussed in Section 2.3.

2. Basic Algorithms

In this section we introduce a collection of generic matroid problems and associated algorithms that will serve as a tool-box available for applications described in the later sections. Many of the algorithms that we present are not the fastest, but they usually contain the important ideas of more complicated algorithms. References are given to the best algorithms for the various problems. A reader determined to implement one of the many algorithms for a particular problem should carefully consider the value of her time versus computer time.

In Sections 3, 4 and 5, those applications that require the solution of a particular basic matroid problem will have that problem indicated in the subsection heading, with the abbreviations introduced below.

2.1. Optimal-Weight Base

Let $\mathscr{M} = (E, \mathscr{I})$ be a matroid and let β be the set of bases of $\mathscr{M}$. A *weight function* on E is a function $w: 2^E \to \mathbf{R}$ such that $w(X) = \sum_{e \in X} w(e)$. The most fundamental matroid optimization problems are the *minimum-weight independent set problem* ([OPT-WT IND-SET]) $\min\{w(X): X \in \mathscr{I}\}$ and the closely related *minimum-weight base problem* ([OPT-WT BASE]) $\min\{w(X): X \in \beta\}$.

GREEDY ALGORITHM FOR THE MINIMUM-WEIGHT BASE PROBLEM

(0) Let $X = \emptyset$. Let $U = E$.

(1) If $U = \emptyset$ then stop, X is a minimum-weight base.

(2) Let e be a minimum-weight element of U.
$U \leftarrow U \setminus e$.
If $X \cup e \in \mathscr{I}$ then let $X \leftarrow X \cup e$.
Go to (1).

HYBRID ALGORITHM FOR THE MINIMUM-WEIGHT BASE PROBLEM

(0) Let $X = \emptyset$, $X' = \emptyset$ and $U = E$.

(1) Perform either of the following steps:

(1.1) If $X \cup U$ contains no circuit then let $X \leftarrow X \cup U$, and stop, X is a minimum-weight base. Otherwise, let C be a circuit contained in $X \cup U$. Let e be a maximum-weight element of C. Let $U \leftarrow U \setminus e$, $X' \leftarrow X' \cup e$. Go to (1).

(1.2) If $X' \cup U$ contains no cocircuit then let $X' \leftarrow X' \cup U$, and stop, X is a minimum-weight base. Otherwise, let C' be a cocircuit contained in $X' \cup U$. Let e be a minimum-weight element of C'. Let $U \leftarrow U \setminus e$, $X \leftarrow X \cup e$. Go to (1).

At each step of the algorithm, X is a minimum-weight base of $\mathscr{M}.\mathrm{sp}(X)$ (i.e., $\mathscr{M}$ restricted to the span of X). If the rank of $\mathscr{M}$ is known in advance, the algorithm may be terminated once the independent set X is a base of $\mathscr{M}$. At the conclusion of the algorithm $E \setminus X$ is a maximum-weight base of $\mathscr{M}^*$. If a minimum-weight *independent set* of $\mathscr{M}$ is desired, then the algorithm may be halted in step (1) when no negative-weight elements remain in U.

The proof of correctness of the greedy algorithm for these matroid problems was first given by Rado.[129] It was later rediscovered, independently, by Edmonds,[45] Gale[61] and Welsh.[156] Prior to the inception of matroid theory, Borůvka[16] motivated by the graphic case, proposed and proved the correctness of the greedy algorithm, without having the benefit of the definitions and tools of matroid theory. Confining himself to the graphic case, Kruskal,[96] aware of Borůvka's relatively obscure work, gave a simplified proof of the correctness of the greedy algorithm, pointed out the correctness of the dual greedy algorithm, and gave a variation that maintained a connectivity property. This last variation was also studied independently by Prim[127] and Dijkstra.[35] Edmonds coined the term "greedy algorithm" at least as early as 1967, and described the important viewpoint of regarding weighted matroid optimization problems as integer-linear (and sometimes even just linear!) programming problems (see Edmonds[44, 45]).

Rosentiehl[135] observed that many of the algorithms for the graphic case were hybrid "primal-dual" greedy algorithms. Lawler[101] noticed that this observation extends to general matroids:

At the conclusion of the algorithm, $X' = E \setminus X$ is a maximum-weight base of $\mathscr{M}^*$. Note that if only the step (1.2) is used, the hybrid algorithm reduces to the "primal" greedy algorithm above. Alternatively, when only the step (1.1) is used, the algorithm is the "dual" greedy algorithm (essentially, the greedy algorithm to determine a *maximum*-weight base of $\mathscr{M}^*$).

"Greedy-type" algorithms are widely used as tools in optimization. Most often, they are used as a heuristic only, and do not give optimal solutions. Korte and Lovász,[94, 95] Faigle,[50–52] Brylawski,[22] Dechter and Dechter,[34] and Goecke[65] discuss additional settings in which the greedy algorithm produces optimal solutions. Bagotskaya, Levit and Losev[5] give a setting for which an extension of the greedy algorithm applies. In Baumgarten,[9] Jenkyns,[90] Korte and Hausmann,[92] and Hausmann, Jenkyns and Korte[73] performance bounds are given for greedy algorithms on set systems that satisfy the first two independence axioms (I0) and (I1), but not (I2).

2.2. Variations on the Optimal Weight Base Problem

Several variations on the problem of finding a minimum-weight base of a matroid have been studied. Fenner and Frieze[53] gave an algorithm that finds a base that maximizes the *product* of the weights of its elements ([OPT-PROD BASE]) (also see Gusfield[69]). The algorithm of Fenner and Frieze runs in polynomial time, provided, as usual, that independence can be tested in polynomial time. If the weights of the elements are all positive, the greedy algorithm will find a base maximizing the product of its weights.

If the weight on element e of E is given by $w(e)$, let $w'(e) = |w(e)|$. For any base B, let $w(B) = \Pi_{e\in B} w(e)$ and $w'(B) = \Pi_{e\in B} w'(e)$. If B is a base with an even number of negative weight elements, then $w(B) = w'(B)$. If every base has an odd number of negative weight elements, then an optimal solution can be found by minimizing $w'(B)$ over all bases B. Thus the base maximizing $w(B)$ can be found as follows:

Hassin and Tamir[72] address the problem of finding a base that maximizes a convex function of two weight functions ([OPT-CONV BASE]). Let $g(x, y)$ be a convex function, and let $A(\cdot)$ and $B(\cdot)$ be two weight functions on 2^E, defined by $A(S) = \sum_{i\in S} a_i$ and $B(S) = \sum_{i\in S} b_i$ for $S \subseteq E$. We assume that for every pair of distinct elements $i, j \in S$, (a_i, b_i) does not equal (a_j, b_j). The authors show how to perturb the system if this assumption is not valid. Every $S \subseteq E$ gives rise to a point in $\mathbf{R}^2$, $(A, B) = (A(S), B(S))$. To each point (A, B) in $\mathbf{R}^2$, there corresponds a graph $G(A, B)$ with nodes E, defined as follows:

1) For each pair of elements (i, j) of E there is an arc from i to j if and only if $g(A - a_i + a_j, B - b_i + b_j) > g(A, B)$.
2) For each pair of elements (i, j) of E there is an arc from j to i if and only if $g(A + a_i - a_j, B + b_i - b_j) > g(A, B)$.

The authors show that at least one of (1) and (2) always hold, and call the graph $G(A, B)$ a *tournament*. (Note that this differs from the usual definition of a tournament since both (1) and (2) may hold here.) Since there are only a finite number of tournaments on E, this assignment of points to tournaments defines an equivalence class on $\mathbf{R}^2$. Each tournament has an associated "optimal base" that

ALGORITHM FOR MAXIMIZING PRODUCT OF WEIGHTS OVER ALL BASES

1) Find the base with an even number of negative weight elements maximizing $w'(B)$, or determine that no such B exists. Fenner and Frieze present the following algorithm to accomplish this.

   Greedy-Exchange Procedure:

   Step a. Use the greedy algorithm to find B_1. If B_1 has an even number of negative elements then it is in the optimal solution over all bases with an even number of negative weight elements. Otherwise go to Step b.

   Step b. Among all oppositely signed pairs (e, f) with $e \notin B_1$ and $f \in C(B_1, e)$, let (e', f') be the pair maximizing $w(B_1 \cup e \setminus f)$. (If there is no such pair, stop, there is no base with an even number of negative weight elements.) $B_1 \cup e' \setminus f'$ is the optimal solution over all bases with an even number of negative weight elements.

2) If a base was found in Step a, STOP, the solution is optimal. Otherwise use the greedy algorithm to minimize $w'(B)$.

maximizes the objective function over all points assigned to the same tournament. Hassin and Tamir show that an optimal base of some tournament will be an overall optimal base. They give an $O(rn)$ algorithm for finding an optimal base of a given tournament (where r is the rank of the matroid and n is the number of elements). A polynomial algorithm for the original problem results when the number of distinct tournaments is polynomial in the size of the problem. In particular, Hassin and Tamir use these results to present a polynomial time algorithm when g is a polynomial, and a strongly polynomial algorithm when the degree of g is at most three.

MATROID PARTITIONING ALGORITHM

(0) Start with disjoint X_i satisfying $X \supseteq X_i \in \mathscr{I}_i$ $(1 \leqslant i \leqslant t)$.

(1) Let $X_0 = X \setminus \dot{\bigcup}_{i=1}^{t} X_i$.
If $X_0 = \varnothing$, then stop—the current partitioning partitions X.

(2) Build a directed graph $D = (N, \mathscr{A})$ with $N = X \cup \{\mathscr{O}_1, \ldots, \mathscr{O}_t\}$.
For $y \in X_i$ and $x \notin X_i$, the pair (x, y) is in $\mathscr{A}$ if $X_i \cup x \setminus y \in \mathscr{I}_i$ $(1 \leqslant i \leqslant t)$.
For $x \notin X_i$, the pair $(x, \mathscr{O}_i)$ is in $\mathscr{A}$ if $X_i \cup x \in \mathscr{I}_i$ $(1 \leqslant i \leqslant t)$.

(3) Find a "short-cut free" dipath in D, terminating at $\mathscr{O}_s$ for the *least s* possible, and beginning at some element $x_0 \in X_0$.

(4) If dipath $x_0 \to x_1 \to \cdots \to x_n \to \mathscr{O}_s$ is found in step (2),
then:
  For $j = 1, \ldots, n$:
    Let i be the index such that $x_j \in X_i$,
    and let $X_i \leftarrow X_i \cup x_{j-1} \setminus x_j$.
  Let $X_s = X_s \cup x_n$.
  Go to (1).
else:
  Stop, the current partitioning is optimal;
  $\dot{\bigcup}_{i=1}^{t} X_i$ is a basis of $\mathscr{M}.X$.

2.3. Matroid Partitioning

Let $\mathscr{M}_i = (E_i, \mathscr{I}_i)(1 \leqslant i \leqslant t)$ be a multi-set of matroids having rank functions $r_i(\cdot)(1 \leqslant i \leqslant t)$. We consider the *matroid partitioning problem* for a set $X \subseteq E \equiv \dot{\bigcup}_{i=1}^{t} E_i$: Find pair-wise disjoint subsets X_i of X, with $X_i \in \mathscr{I}_i$, so as to maximize $|\dot{\bigcup}_{i=1}^{t} X_i|$ ([PARTITION]). We noted in the previous section that the set of subsets X of E that are "partitionable" is the set of independent sets of the matroid $\mathscr{M} \equiv \bigvee_{i=1}^{t} \mathscr{M}_i$. Therefore, our present problem is that of finding a base of $\mathscr{M}.X$.

A related problem is the *lexicographic matroid partitioning problem*. Associated with any partition of a subset of X into independent sets X_i of the individual matroids is a t-vector x with $x_i = |X_i|$. The lexicographic ordering of these vectors induces a lexicographic ordering on the valid partitions. The lexicographic matroid partitioning problem is to find a valid partitioning of X that is lexicographically maximum. It may be shown that a lexicographically maximum partitioning is also a maximum-cardinality partitioning. Appropriately initialized, the algorithm that we give will actually solve the lexicographic problem. The algorithm is due to Edmonds,[40, 43] but our presentation is closer to that of Welsh[157] (see Knuth[91] and Greene and Magnanti[66] also).

By a "short-cut free" dipath in step (3), we mean one in which no pair of vertices x_j, x_k, with $j < k$, has the property that (x_j, x_k) is an edge of D that is not in the path. Such a dipath may be found by using breadth-first search to find a *shortest* dipath which, a fortiori, is short-cut free. Finding such a path to the *least s* only serves to satisfy the more stringent requirement of the lexicographic problem.

If the algorithm stops in step (4), evidence that X is not partitionable is at hand. Relative to the current digraph D, let S be the subset of elements of X that are not the initial vertex of any directed path terminating at an element of $\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}$. The set S satisfies

$$\sum_{i=1}^{t} r_i(S \cap E_i) + |X \setminus S| < |X|,$$

hence the rank of X in $\mathcal{M}$ (see Section 1) is less than its cardinality.

To solve the lexicographic problem, we should initialize the algorithm in step (0) with disjoint $X_i \in \mathcal{I}_i$ $(1 \leqslant i \leqslant t)$ such that the sets constitute a lexicographically maximum partition of their union. For example, we may let X_1 be a base of $\mathcal{M}_1$, X_2 be a base of $\mathcal{M}_2 \setminus X_1, \ldots,$ and X_t be a base of $\mathcal{M}_t \setminus (\uplus_{i=1}^{t} X_i)$. Alternatively, we may let $X_i = \emptyset$ $(1 \leqslant i \leqslant t)$.

The algorithm may be used to determine a fundamental circuit in $\mathcal{M}$. If the algorithm is initialized in step (0) with X_i $(1 \leqslant i \leqslant t)$ such that $X \setminus (\uplus_{i=1}^{t} X_i) = x$ and $\uplus_{i=1}^{t} X_i \uplus x$ is dependent then $C(\uplus_{i=1}^{t} X_i, x)$ can be determined by forming the digraph D and searching for a directed path from x to some element of $\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}$. The elements of $C(\uplus_{i=1}^{t} X_i, x)$ are precisely the vertices that can be reached by a directed path originating at x.

2.4. Maximum-Cardinality (2-)Matroid Intersection

Let $\mathcal{M}_i = (E, \mathcal{I}_i)$ be a pair of matroids. We consider the *maximum-cardinality (2-)matroid intersection problem* ([MAX-CARD 2-INT])

$$\max\{|X| : X \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

This problem was first studied by Edmonds,[44, 46] and solution methods were further developed by Lawler[99-101] and by Aigner and Dowling.[1] We give a direct algorithm for this problem although we note that the problem can be solved by recasting it as a lexicographic matroid partitioning problem: Let $\mathcal{M}_3$ be the free matroid on E. Find a lexicographically maximum partitioning of E with respect to $\mathcal{M}_1^* \vee \mathcal{M}_2 \vee \mathcal{M}_3$. A lexicographically maximum partitioning $\uplus_{i=1}^{3} X_i$ has the property that X_1 is a base of $\mathcal{M}_1^*$ and X_2 is a maximum-cardinality set that is independent both in $\mathcal{M}_1$ and $\mathcal{M}_2$. We note that interchanging the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$ may yield computational savings depending on their individual structure.

MAXIMUM-CARDINALITY (2-)MATROID INTERSECTION ALGORITHM

(0) Start with $X \in \mathcal{I}_1 \cap \mathcal{I}_2$.

(1) Build a directed bipartite graph $D = (X \uplus (E \setminus X), \mathcal{A})$ where $\mathcal{A}$ is defined by:
For $x \in X$ and $y \in E \setminus X$, the pair (x, y) is in $\mathcal{A}$ if $X \uplus y \notin \mathcal{I}_1$ and $X \uplus y \setminus x \in \mathcal{I}_1$.
For $x \in X$ and $y \in E \setminus X$, the pair (y, x) is in $\mathcal{A}$ if $X \uplus y \notin \mathcal{I}_2$ and $X \uplus y \setminus x \in \mathcal{I}_2$.

(2) A source (respectively, sink) of D is an element of $E \setminus X$ that has in-degree (out-degree) zero. Find a short-cut free dipath in D, beginning at some source and terminating at some sink.

(3) If dipath $y_1 \to x_1 \to y_2 \to x_2 \to \cdots \to x_{n-1} \to y_n$ is found in step (2),
then:
  For $j = n$ down to 2:
    Let $X \leftarrow X \setminus x_{j-1} \uplus y_j$.
  Let $X \leftarrow X \uplus y_1$.
  Go to (1).
else:
  Stop, X is a maximum-cardinality common independent set.

The algorithm has the property that as the "for-loop" within step (3) is iterated, X remains independent in $\mathcal{M}_1$. This may be computationally advantageous in applications (see Anstreicher, Lee and Rutherford,[2] for example).

Edmonds'[44] duality theorem for the maximum-cardinality (2-)matroid intersection problem asserts that

$$\max\{|X| : X \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(S_1) + r_2(S_2) : S_1 \cup S_2 = E\}.$$

The algorithm also essentially solves the minimization problem of Edmonds' theorem, which has its own applications. Let S be the subset of elements of E that are the final vertex of some directed path beginning at some source. Let $\mathrm{sp}_i(\cdot)$ be the span operator for $\mathcal{M}_i$ $(i = 1, 2)$. Let $S_1 = \mathrm{sp}_1((E \setminus S) \cap X)$, and let $S_2 = \mathrm{sp}_2(S \cap X) \setminus S_1$. Then S_1, S_2 solve the minimization problem.

2.5. Optimal-Weight (2-)Matroid Intersection

Now let $\mathcal{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ be a pair of matroids, and let $\mathcal{B}_i$ be the set of bases of $\mathcal{M}_i$. Let $w(\cdot)$ be a weight function on 2^E. We consider the *minimum-weight (2-)matroid intersection problem* ([OPT-WT 2-INT]) $\min\{w(X)\colon\ X \in \mathcal{I}_1 \cap \mathcal{I}_2\}$ and the closely related *minimum-weight (2-)matroid common-base problem* ([OPT-WT 2-BASE]) $\min\{w(X)\colon\ X \in \mathcal{B}_1 \cap \mathcal{B}_2\}$.

This problem was first studied by Edmonds[44, 46] and solution methods were further developed by Lawler.[99-101] Alternative approaches and algorithms have been developed by Iri and Tomizawa,[89] Frank,[54] Orlin and Vande Vate, [120] Brezovec, Cornuejols and Glover,[17] and by Camerini and Hamacher.[23]

We present a "primal" algorithm for the problem (see Lawler[100]). The set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ is *k-minimal* if X is a minimum-weight element of $\mathcal{I}_1 \cap \mathcal{I}_2$ among all those having cardinality k. Thus the algorithm can be initiated with the 0-minimal set $X = \emptyset$. Alternatively, the algorithm can be given an advanced start by using the "extended greedy algorithm" of Glover (see Brezovec, Cornuejols, and Glover[17]). The present algorithm uses similar ideas to those used in Section 2.4. The nodes of the digraph D are now weighted, with node $y \in E \setminus X$ receiving a weight of $w(y)$ and each node $x \in X$ is given a weight of $-w(x)$. At each step we find a minimum node-weight source-sink dipath such that every shortcut induces a source-sink dipath that has strictly greater weight. We will refer to such a dipath as an *augmenting dipath*.

We will first give a statement of the algorithm to find a minimum-weight set among all maximum-cardinality elements of $\mathcal{I}_1 \cap \mathcal{I}_2$. If $\mathcal{I}_1 \cap \mathcal{I}_2$ contains any set that is a base of $\mathcal{M}_1$ and $\mathcal{M}_2$, then the algorithm finds a minimum-weight common base of $\mathcal{M}_1$ and $\mathcal{M}_2$. We will indicate how the algorithm can be adapted to find the minimum-weight common independent set (of arbitrary cardinality).

MINIMUM-WEIGHT COMMON BASE ALGORITHM

(0) Start with k-minimal X, for some k.

(1) Build a directed bipartite graph $D = (X \cup (E \setminus X), \mathcal{A})$ where $\mathcal{A}$ is defined by:
For $x \in X$ and $y \in E \setminus X$, the pair (x, y) is in $\mathcal{A}$ if $X \cup y \notin \mathcal{I}_1$ and $X \cup y \setminus x \in \mathcal{I}_1$.
For $x \in X$ and $y \in E \setminus X$, the pair (y, x) is in $\mathcal{A}$ if $X \cup y \notin \mathcal{I}_2$ and $X \cup y \setminus x \in \mathcal{I}_2$.
Weight the nodes of D as follows:
  For $x \in X$, give x a weight of $-w(x)$.
  For $y \in E \setminus X$, give y a weight of $w(y)$.

(2) Find an augmenting dipath in D.

(3) If dipath $y_1 \to x_1 \to y_2 \to x_2 \to \cdots \to x_{n-1} \to y_n$ is found in step (2),
then:
  For $j = n$ down to 2:
    Let $X \leftarrow X \setminus x_{j-1} \cup y_j$.
  Let $X \leftarrow X \cup y_1$.
  Let $k \leftarrow k + 1$ and go to (1).
else:
  Stop, X is a minimum-weight set among all maximum-cardinality elements of $\mathcal{I}_1 \cap \mathcal{I}_2$. If $k = r_{\mathcal{M}_1}(E) = r_{\mathcal{M}_2}(E)$, then X is a minimum-weight common base.

It can be shown that a minimum-weight common independent set can be found by terminating the algorithm the first time the dipath found in step (3) of the algorithm has non-negative weight.

The digraph D formed in step (1) has no dicycles with negative weight. Hence, a variety of methods may be employed to find a minimum-weight source-sink dipath. Not every such method will yield an augmenting dipath. For example, the Bellman-Ford dynamic programming method will yield an augmenting path, but the Floyd-Warshall method may not (see Lawler[101] for a description of these methods).

In general, the problem of finding a minimum-weight base of matroid $\mathcal{M}$ on E that contains d_i elements from disjoint sets $B_i \subseteq E$ $(1 \leqslant i \leqslant t-1)$ can be recast as a minimum-weight common base problem. We simply take $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2$ to be the partition matroid determined by B_i $(1 \leqslant i \leqslant t)$, where $B_t \equiv E \setminus (\uplus_{1 \leqslant i \leqslant t-1} B_i)$, d_i $(1 \leqslant i \leqslant t-1)$, and $d_t = r_{\mathcal{M}}(E) - \sum_{i=1}^{t-1} d_i$. Several authors have developed refined algorithms for the special cases of $t = 2$ and 3. Glover and Klingman[63] consider the case $t = 2$ in developing what they call a "quasi-greedy" algorithm. Although Glover and Klingman presented their algorithm in the graphical setting, they observed that it could be generalized to matroids. Gabow and Tarjan[59] have studied the quasi-greedy algorithm for special matroid problems. Glover and Novick[64] present a "2-quasi greedy" algorithm for the case of $t = 3$. Brezovec, Cornuejols and Glover[18] address the case of general t. See Frederickson and Srinivas[56] for the case of small values of t.

2.6. Bounds on Optimal-Weight (k-)Matroid Intersection

Unfortunately, as the set of common independent sets of two matroids is not generally the set of independent sets of a matroid, it does not appear to be easy to find a minimum-weight common base of $k \geqslant 3$ matroids $\mathcal{M}_i$ $(1 \leqslant i \leqslant k)$ [OPT-WT k-BASE]. In fact the NP-hard problem of finding a minimum-weight Hamiltonian path can be formulated as the problem of finding a minimum-weight set that is a base of three matroids, so we are unlikely to be able to find a good algorithm for the problem. However, the $\binom{k}{2}$ related 2-matroid relaxations do provide lower bounds for the k-matroid problem.

Camerini and Maffioli[24] address the problem of finding much better bounds for the minimum-weight base of 3 matroids for use in branch-and-bound or heuristic search procedures. We present the straightforward extension to k matroids. We restrict ourselves to the case in which $k-2$ of the matroids, say $\mathcal{M}_i$ $(3 \leqslant i \leqslant k)$ are partition matroids. Independence in the partition matroids is relaxed and the objective function is altered as follows. If the partition matroid $\mathcal{M}_i$ has as its independent sets $\mathcal{I}_i \equiv \{X \subseteq E \mid |X \cap B_j^i| \leqslant d_j^i,\ 1 \leqslant j \leqslant t^i\}$, where E is the disjoint union $\uplus_{j=1}^{t} B_j^i$, for each i $(3 \leqslant i \leqslant k)$, then the function $\mu(X, \pi) \equiv \sum_{i=3}^{k} \sum_{j=1}^{t^i} \pi_j^i(|B_j^i \cap X| - d_j^i)$, for non-negative π_j^i $(3 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant t^i)$, is a weighted measure of the deviation of X from independence in the partition matroid. The original objective function is augmented by $\mu(X, \pi)$, and a set that is a minimum-weight common base of $\mathcal{M}_1$ and $\mathcal{M}_2$ is found, using the augmented objective function. Then iterative subgradient methods, such as those developed by Held and Karp[74, 75] and Held, Wolfe and Crowder[76] can be used to find the Lagrange multipliers π_j^i $(3 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant t^i)$ giving the best possible lower bound of this type. We mention that the optimal multipliers can be determined in polynomial-time using the Ellipsoid Method (see Grötschel, Lovász and Schrijver[68]). If all k of the matroids are partition matroids, this method leads to $\binom{k}{2}$ lower bounds.

2.7. Maximum-Cardinality Matroid Parity Set

The maximum-cardinality matroid parity (or matroid "matching") problem, generalizes both the problems of 2-matroid intersection and of matching in (not necessarily bipartite) graphs. The problem can be formulated as follows. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, suppose that the elements of E have been grouped into distinct pairs. Each element $e \in E$ has a *mate* $\bar{e} \in E$. Clearly, $\bar{\bar{e}} = e$. The *maximum-cardinality matroid parity problem* ([PARITY]) is to find a maximum-cardinality set X in $\mathcal{I}$ having the property that e is an element of X if and only if $\bar{e}$ is an element of X.

Consider the case where $\mathcal{M}$ is the union $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2$ of two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\bar{E}, \mathcal{I}_2)$ as in Section 1, where $\bar{E}$ is a copy of E. Then if an element e of E is paired with its copy $\bar{e}$ in $\bar{E}$, a maximum-cardinality parity set in $\mathcal{M}$ is a set X such that $X \cap E$ is independent in $\mathcal{M}_1$ and $X \cap \bar{E}$ is independent in $\mathcal{M}_2$. Identifying E and $\bar{E}$, we have that X is a maximum-cardinality independent set in $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus the cardinality intersection problem is a special case of the matroid parity problem.

A matching in a graph is a set of edges M so that no two edges in M share an endpoint. The problem of finding a maximum-cardinality matching in a graph can be recast as a matroid parity problem. Given a graph G with vertex set V and edge set E, construct a new graph G' by inserting a vertex on every edge of E. Denote the sets of vertices and edges of G' by V' and E' respectively. Now let $\mathcal{I}$ be the set of edges in E' that do not share an endpoint in V. It is easy to check that $\mathcal{M} = (E', \mathcal{I})$ is a partition matroid. If an edge e in E was split into the edges e' and e'' in E' then e' and e'' are paired (that is $\bar{e}' = e''$). The parity sets in $\mathscr{M}$ correspond exactly to the graph theoretic matchings in G.

Lovász[106] has shown that the maximum-cardinality matroid parity problem is NP-hard for general matroids. Moreover, Korte and Jensen[93] established an exponential-time lower bound for the problem. However for linear matroids the problem is solvable in polynomial time. Polynomial-time algorithms for determining a maximum-cardinality parity set for a linear matroid have given by Lovász,[107] Orlin, Vande Vate, Gugenheim and Hammond,[124] Orlin and Vande Vate,[121, 123], Vande Vate[153] and Gabow and Stallman.[58] The algorithm of Gabow and Stallman uses augmenting paths and generalizes the standard algorithms for matroid intersection and matching in graphs. As one would expect, given the more general nature of the problem, the augmenting path algorithm for matroid parity is more complicated than the algorithms for matroid intersection and graph matching. We will give the idea of the algorithm here; for details the reader is referred to Gabow and Stallman.[58]

The algorithm of Gabow and Stallman works on a slightly altered version of the parity problem, where we search for bases, rather than arbitrary independent sets. Given a matroid $\mathscr{M}$ with base B and an associated parity problem, we augment the ground set of $\mathscr{M}$ to get a matroid $\mathscr{M}'$ by duplicating each element e of B with a parallel element e' (i.e., an element e' such that $\{e, e'\}$ is a circuit of $\mathscr{M}'$). Elements e' are called *singletons* since they have no mates. When X is a maximum-cardinality parity set in $\mathscr{M}$, X together with the appropriate singletons forms a base X' of $\mathscr{M}'$. Thus the problem of finding a maximum-cardinality parity set in $\mathscr{M}$ is equivalent to that of finding a base of $\mathscr{M}'$ with as many matched pairs as possible.

The algorithm proceeds from one base of $\mathscr{M}'$ to another, increasing the number of matched pairs by one at each iteration. Given a base B we build a directed bipartite graph $D = (B \cup (E \setminus B), \mathscr{A})$. For $x \in B$ and $y \in E \setminus B$, the pair $(y, x) \in \mathscr{A}$ if $B \cup y \setminus x$ is a base of $\mathscr{M}'$. An *augmenting sequence for a base B of $\mathscr{M}'$* is composed of a set of distinct pairs $b_i, \bar{b}_i$ in B, singletons $\bar{b}_0$, b_{k+1} in B, and pairs $a_i, \bar{a}_i$ not in B, $0 \leqslant i \leqslant k$, such that the following sets are bases:

i) $B(a_0) = B \setminus \bar{b}_0 \cup a_0$,
ii) $B(b_i) = B(a_{i-1}) \cup \bar{a}_{i-1} \setminus b_i$, for $0 < i \leqslant k + 1$, and
iii) $B(a_i) = B(b_i) \setminus \bar{b}_i \cup a_i$.

The base $B(b_{k+1})$ will have two more matched pairs than B. Gabow and Stallman have shown that the base B has maximum-cardinality if and only if there is no augmenting sequence. Finding an augmenting sequence is quite complicated. At this point we refer the reader to Gabow and Stallman.[58]

2.8. Constrained Bottleneck Problems

Let w be a weight function on E, and let $\mathscr{F}$ be a subset of 2^E. Let f be a real-valued function on $\mathscr{F}$ that we can efficiently minimize, and let F be a real constant. For example, see Sections 2.1–2.7. The constrained bottleneck minimization problem ([BOTTLE]) is

$$\min_{X \in \mathscr{F}} \max_{e \in X} \{ w(e) \colon f(X) \leqslant F \}.$$

This problem can be efficiently solved using the "threshold method" of Edmonds and Fulkerson[48] (also see Lee[103]). The basic method accumulates elements in order of increasing weight in a set S, until S contains an element of $\mathscr{F}$ that has value less than or equal to F.

2.9. Graph Realization

If a matrix B is obtained from A by performing elementary row operations, appending or deleting zero-rows, and nonzero column scaling, we say that A and B are *projectively equivalent*. The *graph realization problem* ([GRAPH REALIZATION]) is to determine whether a real matrix A is projectively equivalent to the node-arc incidence matrix of a digraph, and, if so, to determine a projective transformation that realizes the equivalence. Bixby and Cunningham[72] address this problem and the related problem of determining whether a matroid is graphic (also see Tutte,[149–151] Lofgren,[105] Auslander and Trent,[3, 4] Iri,[80, 82, 83, 85] Tomizawa,[147] Fujishige,[57] Bixby,[10, 11] Seymour,[140] Wagner,[154] Truemper[148] and Bixby and Wagner[13]).

Let the columns of A be labeled from E. The method requires an algorithm to determine an undirected graph $G = (V, E)$, such that $\mathscr{M}(G)$ is identical to $\mathscr{M}_{\mathrm{GF(2)}}(A)$, if such a graph exists. We then check whether the node-arc incidence matrix of an arbitrary orientation of G is projectively equivalent to the given matrix A.

Suppose that A and A' are matrices, over the same field and having the same column set E, and we wish to determine if they are projectively equivalent. There is no loss of generality in assuming that both matrices are standard representative matrices, with respect to the same base β, and that the supports of the two matrices are identical. That is,

$$A = [I \mid N],$$
$$A' = [I \mid N'],$$

and $n_{ij} = 0$ if and only if $n'_{ij} = 0$. We consider the rows of the matrices to be labeled from β. We use the scaling procedure below to compute scalars r_i $(i \in \beta)$ and c_j $(j \in E \setminus \beta)$, so that $r_i n_{ij} c_j = n'_{ij}$ for all $i \in \beta$, $j \in E \setminus \beta$.

SCALING PROCEDURE

(0) Let H be a bipartite graph with vertex set $\beta \cup (E \setminus \beta)$. The pair (i, j) is an edge of H if and only if n_{ij} is nonzero. Designate all vertices of H unlabeled and unscanned.

(1) If all vertices of H are labeled and scanned, then go to Step 3. If every labeled vertex has been scanned, but there is an unlabeled vertex, let x be such a vertex, and designate x labeled with $l(x) = 1$.

(2) Let x be any labeled and unscanned vertex. For each vertex y joined to x by the edge $(i, j) = (y, x)$ or (x, y), give y the label $l(y) = n'_{ij}/(l(x)n_{ij})$, and designate y as labeled. After labeling all such vertices y, designate x as scanned

(3) Let $r_i = l(i)$ for all $i \in \beta$, and let $c_j = l(j)$ for all $j \in E \setminus \beta$. For each $i \in \beta$, scale row i of A by r_i, and column i of A by $1/r_i$. Then for each $j \in E \setminus \beta$, scale column j of A by c_j. The matrices A and A' are projectively equivalent precisely when the scaled version of A is equal to A'.

Let $\underline{A}$ be the matrix obtained from A by replacing nonzeros by 1's. To complete the description of the algorithm, we indicate how to determine if $\mathcal{M} = \mathcal{M}_{\mathbf{GF(2)}}(\underline{A})$ is graphic. Again, we assume that $\underline{A}$ is a standard representative matrix. If each column of $\underline{A}$ has at most two 1's, then $\mathcal{M}$ is obviously graphic; by appending a row to $\underline{A}$ that is the **GF(2)** sum of the rows of $\underline{A}$, we obtain the vertex-edge incidence matrix of a graph G having $\mathcal{M}(G) = \mathcal{M}$. A subset S of E is a *separator* of $\mathcal{M}$ if $r_{\mathcal{M}}(E \setminus S) + r_{\mathcal{M}}(S) = r_{\mathcal{M}}(E)$. That is, $(S, E \setminus S)$ is a 1-separation of $\mathcal{M}$. The set S is an *elementary separator* if it is a minimal non-empty separator. Let Y be a cocircuit of $\mathcal{M}$. A *bridge* of Y in $\mathcal{M}$ is an elementary separator of $\mathcal{M}/Y$. The cocircuit Y is a *separating cocircuit* if it has more than one bridge. Let B be a bridge of Y in $\mathcal{M}$. The matroid $\mathcal{M}/(E \setminus (B \cup Y))$ is called a *Y-component*. The *B-segments* of Y are the parallel classes of $\mathcal{M}/(E \setminus (B \cup Y)) \setminus B$. The *bridge graph* of the cocircuit Y has a vertex for every bridge of Y. If B and B' are bridges of Y, then $\{B, B'\}$ is an edge of the bridge graph of Y if B and B' fail to have S- and S'-segments, respectively, such that $S \cup S' = Y$.

Choose a column of $\underline{A}$ that has a 1 in at least three rows. The support of each row is a cocircuit of $\mathcal{M}$. If each such cocircuit has one bridge, then $\mathcal{M}$ is not graphic. Otherwise choose such a cocircuit Y that has more than one bridge. If the bridge graph of Y is not bipartite, then $\mathcal{M}$ is not graphic. Otherwise, apply this entire procedure, recursively, to the Y-components of $\mathcal{M}$. In this case, $\mathcal{M}$ is graphic if and only if each Y-component is graphic. It remains only to discuss how a graph G can be found such that $\mathcal{M}(G) = \mathcal{M}$, provided that we have the bipartite bridge graph of Y and a graph G', for each Y-component $\mathcal{M}'$, such that $\mathcal{M}(G') = \mathcal{M}'$.

First, we define a certain composition of two graphs. Let $H = (W, F)$ and $H' = (W', F')$ be two graphs with $w \in W$, $w' \in W'$, and $W \cap W' = \emptyset$. Let $\delta(w) = \{w, v_1), (w, v_2), \ldots, (w, v_d)\}$ be the set of edges incident to w, and let $\delta(w') = \{w', v'_1), (w', v'_2), \ldots, (w', v'_d)\}$ be the set of edges incident to w'. The *Y-composition* $H \circ H'$ is the graph having vertex-set $(W \setminus w) \cup (W' \setminus w')$ and edge-set $(F \setminus \delta(w)) \cup (F' \setminus \delta(w')) \cup \{(v_1, v'_1), (v_2, v'_2), \ldots, (v_d, v'_d)\}$.

To assemble G, we separately Y-compose the graphs corresponding to the Y-components determined by the bridges on each side of the bipartite bridge graph. Finally, we Y-compose the two resulting graphs to form G. For this procedure to make sense, we must order the component graphs so that the new graph obtained after each Y-composition has $Y = \delta(v)$ for some vertex v. If the graphs are ordered $G_1, G_2, G_3, \ldots, G_k$, then we Y-compose from left to right; that is, $(\cdots((G_1 \circ G_2) \circ G_3) \circ \cdots) \circ G_k$. Let $B_1, B_2, B_3, \ldots, B_k$ be the bridges on one side of the bridge graph of Y. For each i $(1 \leqslant i \leqslant k)$, let S_i be a maximum-cardinality B_i-segment of Y. We assume that B_1 has been chosen so that $|S_1| = \max\{|S_i|: 1 \leqslant i \leqslant k\}$, and so that there are more than two B_1-segments if this is possible. Next, for each other i $(2 \leqslant i \leqslant k)$, let T_i be a B_i-segment of Y such that $S_1 \cup T_i = Y$. We assume that $B_2, B_3, \ldots, B_k$ are ordered so that for $2 \leqslant i < j \leqslant k$, we have $|T_i| \leqslant |T_j|$, and whenever $|T_i| = |T_j|$, there are two B_j-

segments only if there are two B_i-segments. The same procedure is applied to each side of the bridge graph. Bixby and Cunningham[12] demonstrate that this ordering guarantees that all of the required Y-compositions can be performed.

A real matrix is a *generalized-network incidence matrix* if it has at most two nonzero entries in each column. The associated generalized network is *bicircular* if it has no square singular submatrix with exactly two nonzeros in every row and column. A problem related to the graph realization problem is the problem of determining a projective equivalence transformation from a real matrix to a generalized-network incidence matrix if such a transformation exists. This problem is solved by Shull, Orlin, Shuchat and Gardner[143] and by Coullard, del Greco and Wagner[32] for the case in which such a transformation to the generalized-network incidence matrix of a bicircular generalized network is known to exist.

3. Applications

Most of the applications documented in Sections 3, 4 and 5 require an algorithm for the solution of one of the matroid problems presented in Section 2. This is indicated in the subsection heading. Some applications simply require matroid rank evaluations; no reference to an algorithm will be made for these. We have only included detailed descriptions of those applications where a polynomial-time algorithm results. This excludes applications where an exponential amount of time (in terms of the natural data encoding of the application) is required to execute a fundamental step of the matroid algorithm.

The most prolific application of algorithmic matroid theory has occurred in the areas of electrical network theory and statics. The applications within each of these two fields rely on common notation and structure. For this reason, separate sections have been devoted to each of these areas. The applications to electrical networks are discussed in Section 4 and to statics in Section 5. This section is devoted to those applications outside of the areas of electrical networks and statics.

The matroid structure of the minimum-weight spanning tree problem is evident. It is not always the case that the matroid structure is so obvious. The following three examples illustrate settings in which problems can be recast as that of finding an optimal-weight independent set, or base, of an appropriate matroid.

The Job Sequencing Problem [OPT-WT BASE]. Lawler[101] presented a job scheduling problem which can be cast as a matroid problem. We are given n jobs to be processed on a single machine. For each job j, $1 \leqslant j \leqslant n$, not completed by a deadline d_j, we pay a penalty p_j. All jobs require one unit of processing time. The problem is to determine the order that minimizes the total penalty to be paid. Given a sequence, the order in which the late jobs are completed is irrelevant; the same penalty is paid no matter how late the job is. Moreover, without loss of generality, we can assume that the on-time jobs are processed in order of their deadlines. Thus the problem is reduced to selecting the jobs that will be done on time. Finally, minimizing the penalties paid is equivalent to maximizing the penalties not paid.

Let E be the set of jobs, and let $\mathscr{I}$ be the set of subsets of jobs that will all be completed on time if ordered by increasing deadlines. It is easy to check that $M = (E, \mathscr{I})$ is a transversal matroid. Lawler demonstrates that the matroid greedy algorithm can be applied to derive the following optimal sequencing rule.

> Choose the jobs in order of decreasing penalties, ensuring that any job chosen, along with the jobs previously chosen, can be completed on time. ∎

A Problem in Network Design [OPT-WT BASE]. Topological network design problems are often difficult. The problem of finding a minimum-weight spanning tree is an exception. Another exception appears in Du and Miller,[36] where a topology problem that has application in the design of vacuum systems is shown to be solvable in polynomial time by the greedy algorithm.

A vacuum system is simply a system designed to create a vacuum. Chambers are interconnected with valves. Du and Miller simplify the problem of designing a vacuum system satisfying certain interconnection requirements and having the minimum number of valves as follows.

Let X be a set with subsets $X_1, X_2, \ldots, X_n$ such that $X = \bigcup_{i=1}^{n} X_i$. Find a graph G with vertex set X having the minimum number of edges such that the following property holds:

(P) For any subset I of $\{1, \ldots, n\}$, the subgraph G_I induced by $\bigcap_{i \in I} X_i$ is connected.

Let $\mathscr{I}$ be the set of *complements* of edge-sets of graphs satisfying property (P), and let E be the edges of the complete graph on n vertices. Du and Miller show that $(E, \mathscr{I})$ is a matroid, and hence a largest cardinality edge-set in $\mathscr{I}$ (equivalently a minimum-cardinality edge-set satisfying (P)) can be found by the dual matroid greedy algorithm. Note that a largest cardinality set in $\mathscr{I}$ is a basis of M and hence its complement is a basis of M^*. Note that the weights here are all equal to 1. ∎

Finding a Shortest Cycle Basis of a Graph [OPT-WT BASE]. Let A be the vertex-edge incidence matrix of an undirected graph $G = (V, E)$. The *cycle space* of G is the null space of A over **GF[2]**. Let w be a nonnegative weight function on E. The *weight* of a vector in the cycle space is the sum of the weights of the elements indexed by the vector. The *weight* of a set of vectors in the cycle space is the sum of the weights of its elements. Horton[78] demonstrates that the following algorithm can be used to find a minimum-weight basis of the cycle space.

MINIMUM-WEIGHT CYCLE BASIS ALGORITHM

(1) Find a minimum-weight path $P(x, y)$ between each pair of vertices x, y.

(2) For each vertex v and edge (x, y), let C^v_{xy} be the mod 2 sum of the incidence vectors of $P(v, x)$, $P(v, y)$ and (x, y).

(3) Let D be a matrix over **GF[2]**, having the C^v_{xy} as columns. Solve the minimum-weight matroid base problem on $\mathcal{M}(D)$.

Commutativity of diagrams. A *diagram* is a directed, acyclic graph $D = (N, \mathcal{A})$ with a set of spaces associated with N and a set of maps associated with $\mathcal{A}$. It is assumed that the set of maps $f(e)$ $(e \in \mathcal{A})$ forms a semi-group with the associative multiplication denoted by $\circ$. Diagrams arise in many areas of mathematics, and it is important to know which diagrams commute. For a dipath $p = (e_1, e_2, \ldots, e_m)$ $(e_i \in \mathcal{A}, 1 \leqslant i \leqslant m)$, let $f(p) \equiv f(e_1) \circ f(e_2) \circ \cdots \circ f(e_m)$. A diagram *commutes* if $f(p) = f(q)$ for each pair of dipaths p and q that have the same initial and terminal vertices. A *bilinking* is an unordered pair of dipaths that have no internal vertices in common and have the same initial and terminal vertices. The diagram commutes if and only if $f(p) = f(q)$ for every bilinking (p, q).

An algorithm to determine whether a diagram commutes is given by Murota.[115] The algorithm involves the concept of a *preordered matroid*. This is a matroid $\mathcal{M}$ together with a transitive, reflexive, binary relation $>$ on the ground set E. The preorder and the matroid induce a "closure" function $\sigma(\cdot)$ on E. Namely,

$$\sigma(B) = \{b \in E\colon b \in \mathrm{sp}_{\mathcal{M}}(\{a \in B\colon b > a\})\} \quad \forall\, B \subseteq E.$$

A set $B \subseteq E$ is a *spanning set* in the preordered matroid if $\sigma(B) = E$. Murota shows that all minimal spanning sets have the same cardinality, hence a minimal spanning set can be determined by starting with an arbitrary spanning set and considering each element for removal once—removing elements so that the set remains spanning. This is similar to the dual greedy algorithm of Section 2, but we emphasize that $\sigma(\cdot)$ is *not* the span function of a matroid; in particular, the primal greedy algorithm does not correctly solve the problem.

For the particular problem at hand, we shall specify the appropriate preordered matroid. A matroid $\mathcal{M}$ is defined on the set of bilinkings E, by considering linear dependence of their $\{0, 1\}$-valued incidence vectors over **GF(2)**. A preorder $>$ is defined on E by $(p_1, q_1) > (p_2, q_2)$ if there is a dipath from the initial vertex of (p_1, q_1) to the initial vertex of (p_2, q_2) and a dipath from the terminal vertex of (p_2, q_2) to the terminal vertex of (p_1, q_1). A minimal spanning set of the preordered matroid $(\mathcal{M}, >)$ specifies a minimum-cardinality set of equations to be verified in determining whether the diagram commutes.

To show that a good algorithm results, we must provide a method for determining an initial spanning set of the preordered matroid that is not too large. For each vertex $v \in N$, let $N(v)$ be the set of vertices of N that may be reached from v by a dipath of D. Also, let $G(v)$ be the subgraph of D induced by $N(v)$, and let $T(v)$ be a directed spanning tree of $G(v)$, rooted at v. The set of edges $\{e\} \uplus T(v)$ contains a unique bilinking, for each arc e of $G(v)$ that is not in $T(v)$. Let $R(v)$ be the set of bilinkings determined in the above manner, with respect to $T(v)$. Murota demonstrates that $R \equiv \bigcup_{v \in N} R(v)$ is a spanning set of the preordered matroid, and $|R| \leqslant |V|\,|E|$. ∎

Maximizing the Reliability/Cost Ratio Over All Spanning Trees [OPT-WT BASE, OPT-CONV BASE]. Let G be a connected graph on edge-set E. For edge e, let p_e denote the probability that e will not fail and let $c_e > 0$ be its cost. The *reliability/cost ratio* of an edge-set S is given by $\Pi_{e \in S}\, p_e / \Sigma_{e \in S}\, c_e$. The problem is to maximize the ratio over all spanning trees S of G. Maximizing the ratio is equivalent to maximizing the logarithm of the ratio which is given by $g(S) = \Sigma_{e \in S} \log(p_e) - \log(\Sigma_{e \in S}\, c_e)$. If all of the edge-costs are identical, the problem is essentially a maximum-weight base problem. In general, $g(S)$ is a convex function of two weight functions on 2^S. The method of Hassin and Tamir[72] described in Section 2 can be used to find the optimal spanning tree. Hassin and Tamir show that only $O(n^2)$ spanning trees need to be considered, and that their algorithm can be refined to solve the maximum reliability/cost ratio problem in $O(n^4)$ time.

Let t_e be a transit time associated with each edge e. The variation of the problem for which we wish to minimize, over spanning trees with bounded reliability/cost ratio, the maximum of t_e (over e in the tree) can be solved using [BOTTLE] together with [OPT-CONV BASE]. ∎

Applications of the matroid intersection algorithm abound in the literature. Many of these fall under the categories of electrical network theory or statics, and are included in the later sections. Some additional applications are presented here. In many of the applications, one of the matroids is a partition matroid; hence, the presence of the simple compound alternative structure of a partition matroid should lead a modeller to consider whether a problem at hand may be cast as a matroid intersection problem.

0–1 Matrices with Prescribed Row and Column Sums [MAX-CARD 2-INT]. We wish to find a $\{0, 1\}$-valued $m \times n$ matrix A having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$, or prove that no such matrix exists. We may as well assume that the r_i $(1 \leqslant i \leqslant m)$ and c_j $(1 \leqslant j \leqslant n)$ are nonnegative integers satisfying $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. This problem was solved by Gale[60] and by Ryser.[136]

Let $R_i = \{e_{ij}\colon 1 \leqslant j \leqslant n\}$ $(1 \leqslant i \leqslant m)$, and let $C_j = \{e_{ij}\colon 1 \leqslant i \leqslant m\}$ $(1 \leqslant j \leqslant n)$. Let $E = \bigcup_{i=1}^{m} R_i = \bigcup_{j=1}^{n} C_j$. Let $\mathscr{M}$ (respectively, $\mathscr{N}$) be the partition matroid on E with independent sets $\{X \subseteq E\colon |X \cap R_i| \leqslant r_i \ (1 \leqslant i \leqslant m)\}$ ($\{X \subseteq E\colon |X \cap C_j| \leqslant c_j \ (1 \leqslant j \leqslant n)\}$). If the size of a maximum-cardinality set X that is independent in $\mathscr{M}$ and $\mathscr{N}$ is $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$, then the incidence matrix of X has the prescribed row and column sums. Otherwise, no such matrix exists.

The special case of this problem for which $c_j = 2$ $(1 \leqslant j \leqslant n)$ is equivalent to the problem of realizing a graph with a prescribed degree sequence. ■

The Assignment Problems [OPT-WT 2-BASE, OPT-WT k-BASE]. For $k \geqslant 2$, let V_i $(1 \leqslant i \leqslant k)$ be finite sets of equal cardinality n. Let E be a set of subsets of $\biguplus_{i=1}^{k} V_i$ such that every element of E has exactly one element in each set V_i $(1 \leqslant i \leqslant k)$. The pair $H = (\biguplus_{i=1}^{k} V_i, E)$ is a "k-uniform, k-partite hypergraph." A *perfect matching* is an n-element subset of E such that every element of $\biguplus_{i=1}^{k} V_i$ appears (once). Let c be a weight function on 2^E. The *k-dimensional assignment problem* is the problem of finding a minimum-weight perfect matching in H. When $k = 2$, H is a bipartite graph, and the problem is known as the *assignment problem*. The general problem can easily be recast as that of finding a minimum-weight common base of k matroids on ground set E. Let $\mathscr{M}_i$ be a partition matroid on E whose independent sets consist of all sets meeting no element of V_i more than once $(1 \leqslant i \leqslant k)$. A subset of E that is a base of all $\mathscr{M}_i$ $(1 \leqslant i \leqslant k)$ is a perfect matching of G. Kuhn[97] first introduced an augmenting path algorithm to solve the 2-dimensional assignment problem. This algorithm motivated the development of 2-matroid intersection algorithms. The problem is NP-Hard for $k \geqslant 3$ (see Garey and Johnson[62]), but lower-bounds on the optimal solution can be calculated by applying methods for [OPT-WT k-BASE]. ■

Optimal Branchings [OPT-WT 2-INT, MAX-CARD 2-INT]. The problem of finding an "optimal branching" or "arborescence" of a network was addressed by Edmonds[42] and by Chu and Liu.[25] A *branching*, or *arborescence*, in a connected, digraph is a collection of arcs forming a spanning tree in the underlying undirected graph, such that no node has more than one arc leading into it. If $\mathscr{M}_1$ is the graphic matroid on the underlying undirected graph, and if $\mathscr{M}_2$ is the partition matroid whose independent sets are those collections of arcs having the property that no node has more than one arc leading into it, then arc-sets that are bases in $\mathscr{M}_1$ and independent in $\mathscr{M}_2$ are exactly the branchings. Thus a branching in a digraph can be found by a maximum-cardinality 2-matroid intersection algorithm. Let n be the number of nodes. If the edges of the digraph have weights, then a minimum-weight branching can be found by determining a minimum-weight set of cardinality $n - 1$ that is independent in $\mathscr{M}_1$ and $\mathscr{M}_2$. ■

Lower Bounds on the Asymmetric Traveling Salesman Problem [OPT-WT 3-BASE]. The asymmetric travelling salesman problem is to find a minimum-weight directed Hamiltonian tour in a connected digraph with arc weights. Let $\mathscr{M}_1$ be the *1-tree* (or *bicycle*) matroid on the underlying undirected graph. That is, the independent sets of $\mathscr{M}$ are the edge-sets that contain no more than 1 polygon of the graph. Let $\mathscr{M}_2$ (respectively, $\mathscr{M}_3$) be the matroid on the set of arcs such that $\mathscr{I}_2$ $(\mathscr{I}_3)$ is the set of subsets of arcs that have no more than one element leading into (out of) it. The directed Hamiltonian tours of the digraph are precisely the sets that are bases of all three matroids. The "assignment problem lower bound" arises by ignoring independence in $\mathscr{M}_1$. Two other lower bounds can be calculated by applying bounding methods for [OPT-WT 3-BASE], since $\mathscr{M}_2$ and $\mathscr{M}_3$ are partition matroids. ■

Crashing a Maximum-Weight Complementary Basis [OPT-WT 2-INT]. A novel application of the matroid intersection algorithm is given by Anstreicher, Lee and Rutherford.[2] Let A be the $m \times 2m$ matrix $[I, -M]$, and let $E = \{1, 2, \ldots, 2m\}$. Let $\bar{i} = i + m$ for $1 \leqslant i \leqslant m$. A subset γ of E is called *complementary* if $|\gamma \cap \{i, \bar{i}\}| \leqslant 1$ *for* $1 \leqslant i \leqslant m$. A subset β of E that is a base of $\mathscr{M}(A)$ and is complementary is called a *complementary basis* of A. Anstreicher, et al. consider the following problem. Given a complementary subset of E, β, find a complementary basis of A, β', that shares as many elements as possible with β. This problem arises naturally when a complementary set of columns is proposed as an initial basis for a "warm start" of Lemke's algorithm for the linear complementarity problem (see e.g. Murty[119]), but the set of columns is rank-deficient.

Let $\mathscr{I}_1$ be the set of all subsets of E corresponding to linearly independent subsets of columns of A, i.e., $\mathscr{I}_1$ is the set of independent sets of $\mathscr{M}_1 \equiv \mathscr{M}(A)$. Let $\mathscr{I}_2$ be the set of all complementary subsets of E. Then $\mathscr{M}_2 = (E, \mathscr{I}_2)$ is a partition matroid on E. Define the weight function $w(\cdot)$ by $w(e) = 1$ for $e \in \beta$ and $w(e) = 0$ for $e \in E \setminus \beta$. The problem can be recast as that of finding a maximum-weight set that is a base of both $\mathscr{M}_1$ and $\mathscr{M}_2$. Anstreicher et al. treat the problem in more generality, where the elements of E may have arbitrary weights. The authors show how the algorithm of Section 2.5 can be specialized and implemented to solve their problem. They provide computational evidence that their approach may be valuable as a preprocessor for some applications of Lemke's algorithm. ■

Dynamic Matroid Intersection and Dynamic Spanning Forests ([OPT-WT 2-INT]). Hamacher[70] considers the construction over time of a hierarchical information system that is represented as a directed tree. The root of the tree is the head of the hierarchy. If the possible branches of the tree are weighted by a preference number, then the "static" problem of finding an optimal hierarchy is simply the optimal branching problem discussed above. Hamacher has generalized this problem to the case where the hierarchical system may vary over time. We now want to maximize the sum of the preference numbers over all connections and over all time periods in consideration.

The above problem may be modeled as a *dynamic matroid intersection problem*. The dynamic matroid intersection problem for $\mathscr{M}_1 = (E, \mathscr{I}_1)$ and $\mathscr{M}_2 = (E, \mathscr{I}_2)$ arises when each element e of the ground set E requires $\tau(e)$ (a positive integer) units of time to install. For each time period t $(1 \leqslant t \leqslant T)$, we seek a set $X(t) \subseteq E$. For each t the set $X'(t)$ is defined by $X'(t) \equiv \{e \in E\colon e \in X(t - \tau(e)\}$. The set $\{X(t)\colon 1 \leqslant t \leqslant T\}$ is a dynamic matroid intersection if the following three conditions are satisfied for every time period t.

i) $X(t)$ is independent in $\mathscr{M}_1$,
ii) $X'(t)$ is independent in $\mathscr{M}_2$, and
iii) if $e \in X(t)$, then $t + \tau(e) \leqslant T$.

Hamacher[70] presents a polynomial time algorithm to find a maximum-weight (or maximum-cardinality) dynamic matroid intersection. The solution is achieved by reducing the problem to static problems in larger matroids called the *time expanded matroids*.

Let $\mathscr{I}_i$ be the set of independent sets of matroid $\mathscr{M}_i$ $(i = 1, 2)$. The ground set of the time expanded matroids is $E^T = \{e_t\colon e \in E, 1 \leqslant t \leqslant T, t + \tau(e) \leqslant T\}$. For $I \subseteq E^T$, let $I_1(t) = \{e\colon e_t \in I\}$ and $I_2(t) = \{e\colon e_{t-\tau(e)} \in I\}$. The time expanded matroids $\mathscr{M}_1^T$ and $\mathscr{M}_2^T$ are defined by $\mathscr{M}_1^T = (E^T, \mathscr{I}_1^T)$ and $\mathscr{M}_2^T = (E^T, \mathscr{I}_2^T)$ where

1) $\mathscr{I}_1^T = \{I \subseteq E^T\colon I_1(t) \in \mathscr{I}_1, \text{ for } 1 \leqslant t \leqslant T\}$, and
2) $\mathscr{I}_2^T = \{I \subseteq E^T\colon I_2(t) \in \mathscr{I}_2, \text{ for } 1 \leqslant t \leqslant T\}$.

Note that condition (1) states that $I_1(t)$ is independent in $\mathscr{M}_1$ during every time period t. Condition (2) states that $I_2(t)$ is independent in $\mathscr{M}_2$ for every t. Hamacher shows that $\{X(t) \subseteq E\colon 1 \leqslant t \leqslant T\}$ is a dynamic matroid intersection of $\mathscr{M}_1$ and $\mathscr{M}_2$ if and only if $\bigcup_{t=1}^{T}\{e_t\colon e \in X(t)\}$ is independent in $\mathscr{M}_1^T$ and $\mathscr{M}_2^T$. Thus the dynamic matroid intersection problem (and hence the optimal time-varying hierarchy) can be solved as a "static" matroid intersection problem. ■

Structural Controllability of Descriptor Systems [PARTITION, OPT-WT 2-BASE]. A linear dynamical system may be described in the "descriptor form":

$$F\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{u},$$

where the n-vector $\bar{x}$ is the "descriptor vector" and the m-vector $\bar{u}$ is the "initial conditions vector." The matrices F, A and B are real and constant. Murota[116] addresses a type of controllability of such systems. In order that the system be uniquely solvable for arbitrary initial conditions u, we must have $\det(A - sF) \neq 0$, where the indeterminate s stands for differentiation with respect to time. We assume that the matrix F (respectively, A, B) is the sum of the rational matrix Q_F (Q_A, Q_B) and the real matrix T_F (T_A, T_B), where the set of non-zero entries of T_F, T_A and T_B are algebraically independent over $\mathbf{Q}$. Murota notes that the system is uniquely solvable if the rank of

$$\mathscr{M}([I \mid Q_A - Q_F]) \vee \mathscr{M}([I \mid T_A - T_F])$$

is exactly n.

There are many definitions of controllable. We say that the system is *controllable* if

$$\operatorname{rank}([A - zF \mid B]) = n \quad \forall\, z \in \mathbf{C}.$$

Controllability does not appear to be an easy condition to check. Murota rephrases the problem in terms of matroids. First, the controllability condition can be checked for $z = 0$, where it is equivalent to the rank of

$$\mathscr{M}([I \mid Q_A \mid Q_F]) \vee \mathscr{M}([I \mid T_A \mid T_F])$$

being exactly $2n$. To check the condition for $z \neq 0$, we define

$$\begin{array}{rl} & \quad\;\; V \quad\;\; Y \qquad\quad X \qquad\quad U \\ Q_D = & [\,\mathbf{0} \mid I \mid Q_A - Q_F \mid Q_B\,] \\ T_{D1} = & [\,I \mid \mathbf{0} \mid T_F \mid \mathbf{0}\,] \\ T_{D0} = & [\,I \mid I \mid T_A \mid T_B\,]. \end{array}$$

The column labels X and U are naturally associated with the vectors x and u, respectively. The sets V and Y may be viewed as disjoint indexes into the rows of $[A - sF \mid B]$. Let $E = V \uplus Y \uplus X \uplus U$, and let

$$\mathcal{M}_D = \mathcal{M}(Q_D) \vee \mathcal{M}(T_{D1}) \vee \mathcal{M}(T_{D0}).$$

Let r_i $(1 \leqslant i \leqslant n)$ and c_j $(1 \leqslant j \leqslant m + n)$ be such that

$$[Q_A - sQ_F | Q_B] = \mathrm{diag}(s^{r_1}, \ldots, s^{r_n})[Q_A - Q_F | Q_B] \, \mathrm{diag}(s^{-c_1}, \ldots, s^{-c_{n+m}}).$$

Such constants can be found by the scaling procedure of Section 2.8. Define two weight functions w_1 and w_2 on E by

$$w_1(v) = 0 \quad \text{and} \quad w_2(v) = 0 \quad \text{for } v \in V,$$
$$w_1(y) = -r_i \quad \text{and} \quad w_2(y) = 0 \quad \text{for } y \in Y,$$
$$w_1(x) = -c_j \quad \text{and} \quad w_2(x) = 1 \quad \text{for } x \in X,$$
$$w_1(u) = -c_{n+j} \quad \text{and} \quad w_2(u) = 0 \quad \text{for } u \in U.$$

Let $\hat{E}$ be the set of loops of $\mathcal{M}_D^*$. Note that $e \in \hat{E}$ if and only if the rank of $E \setminus e$ is exactly one less than the rank of E in $\mathcal{M}_D$.

Suppose that the system is uniquely solvable and is controllable for $z = 0$. Then the system is controllable (for all complex z) if and only if $w_1(S_1) + w_2(S_2)$ is constant for all disjoint sets S_1, $S_2 \subseteq \hat{E}$, such that S_1 is independent in $\mathcal{M}(Q_D).\hat{E}$, S_2 is independent in $\mathcal{M}(T_{D1}).\hat{E}$, and $\hat{E} \setminus (S_1 \uplus S_2)$ is independent in $\mathcal{M}(T_{D0}).\hat{E}$. We check this condition by maximizing and then minimizing $w_1(S_1) + w_2(S_2)$ over appropriate S_1 and S_2. We do this in a slightly different manner than Murota, so that our basic algorithms need not be modified. We make copies of some of the elements in E to obtain E', and define a single weight function w' on E':

$$\text{Let } F^i = \{f^i \colon f \in F\} \text{ for } F \in \{V, Y, X, U\} \ (i \in \mathbf{Z}).$$

$$E' \equiv V^1 \uplus Y^1 \uplus Y^2 \uplus X^1 \uplus X^2 \uplus X^3 \; U^1 \uplus U^2.$$

For $v \in V$: $w'(v) = w_1(v) = w_2(v) = 0.$

For $y \in Y$: $w'(y^1) = w_1(y) = -r_i,$

$w'(y^2) = w_2(y) = 0.$

For $x \in X$: $w'(x^1) = w_1(x) = c_j,$

$w'(x^2) = w_2(x) = 1,$

$w'(x^3) = 0.$

For $u \in U$: $w'(u^1) = w_1(u) = -c_{n+j},$

$w'(u^2) = w_2(u) = 0.$

Now we modify the matroids at hand. Let $\mathcal{M}'(Q_D)$ (respectively, $\mathcal{M}'(Q_{D1})$, $\mathcal{M}'(Q_{D0})$) be identical to $\mathcal{M}(Q_D)(\mathcal{M}(Q_{D1}), \mathcal{M}'(Q_{D0}))$, but defined on $V^1 \uplus Y^1 \uplus X^1 \uplus U^1(V^1 \uplus Y^2 \uplus X^2 \uplus U^2,\ V^1 \uplus Y^2 \uplus X^3 \uplus U^2)$. Let $\mathcal{M}_P$ be a partition matroid on E', that has as independent sets those subsets of E' that have single copies of elements from E. Lastly, let $\hat{E}'$ be the image of $\hat{E}$ in E'. We solve our problem by finding a maximum-weight (and then minimum-weight) common base of $\mathcal{M}_{\mathrm{P}}.\hat{E}'$ and $\mathcal{M}'(Q_D).\hat{E}' \vee \mathcal{M}'(T_{D1}).\hat{E}' \vee \mathcal{M}(T_{D0}).\hat{E}'$. ■

The Shannon Switching Game [PARTITION]. The Shannon Switching Game was introduced by Shannon.[141] The game is played by two players on the edges of an undirected graph $G = (V, E)$ with a distinguished (nonloop) edge e. The players are called "short" and "cut." They play alternately, with the short player claiming edges, and the cut player deleting edges. No player is allowed to play the distinguished edge e. The short player attempts to claim enough edges to complete a path joining the two endpoints of e. The cut player attempts to prevent short from completing the path through her own choice of edges. The cut player goes first.

Lehman[104] gave a matroidal interpretation of the game and characterized those games for which the short player has a winning strategy. We can rephrase the short player's goal as that of claiming a set $T \subseteq E \setminus e$ so that e is contained in $\mathrm{sp}_{\mathcal{M}(G)}(T)$. The cut player's goal is to prevent the short player from spanning e. Lehman showed that there is a winning strategy for the short player if and only if there are a pair of disjoint sets T_1 and T_2 in $E \setminus e$ such that $e \in \mathrm{sp}_{\mathcal{M}(G)}(T_1) = \mathrm{sp}_{\mathcal{M}(G)}(T_2)$. The sets T_1 and T_2 may be taken to be independent in $\mathcal{M}(G)$.

The existence of the sets T_1 and T_2 can be used to formulate a winning strategy for the short player. First, we point out that the game is equivalent to one in which the short player contracts edges of G, her goal being to reduce e to a loop. The short player need only contract edges from $T_1 \uplus T_2$, so T_1 and T_2 can be taken to be connected (i.e., trees). Each time the cut player deletes an edge, she disconnects at most one of the trees. If neither tree is severed, the short player may contract any unplayed edge. If a tree is severed, the short player contracts an edge of the other tree that reconnects the severed tree. Eventually, one of the trees is empty, winning the game for short.

Edmonds[41] showed that the existence of such sets T_1 and T_2 can be determined by applying a matroid partitioning algorithm. We apply a partition algorithm to determine a base $J = T_1' \uplus T_2'$ of $(\mathcal{M}(G) \setminus e) \vee (\mathcal{M}(G) \setminus e)$ with T_1' and T_2' independent in $\mathcal{M}(G)$. If J is also a base of $\mathcal{M}(G) \vee \mathcal{M}(G)$, then the short player has a winning strategy. In this case, the matroid partition algorithm of Section 2.3 can be used to determine a set S such that $|J| = 2r_{\mathcal{M}(G)}(S) + |E \setminus S|$. The sets $T_i = T_i' \cap S$ $(i = 1, 2)$ can be shown to satisfy the requirements of the short player.

Edmonds[41] describes a winning strategy for the cut player if J is not a base of $\mathcal{M}(G) \vee \mathcal{M}(G)$. (See also Bruno and Weinberg[19]). Note that the event that the cut player wins is equivalent (see Section 1) to the event that she has contracted a set of elements T in $\mathcal{M}^*(G)$ so that e is a loop of $\mathcal{M}^*(G)$.

If J is not a base of $\mathcal{M}(G) \vee \mathcal{M}(G)$, then (without loss of generality) $e \notin \text{sp}(T_1')$, and so $J \uplus e$ is a base of $\mathcal{M}(G) \vee \mathcal{M}(G)$. Also, again using the matroid partition algorithm, there is a set S' in $E \setminus e$ such that $|J| = 2\, r_{\mathcal{M}(G)}(S') + |(E \setminus e) \setminus S'|$. Again letting $T_i = T_i' \cap S'$ $(i = 1, 2)$, it can be shown that $\text{sp}(T_1) = \text{sp}(T_2) \supseteq S' \supseteq (E \setminus e) \setminus J$. We now consider the game on the matroid $\mathcal{M}' = \mathcal{M}(G)/\text{sp}(T_1)$. If the cut player can win the game with the edges of $\text{sp}(T_1)$ already contracted then she can clearly win the game on $\mathcal{M}(G)$. Note that the ground set of $\mathcal{M}'$, E', is contained in $J \uplus e$, and contains e. Thus there exists F_1 and F_2 independent in $\mathcal{M}'$ such that $E' = F_1 \uplus F_2$. Suppose that $e \in F_2$. Since $F_2 = E' \setminus F_1$, and similarly $F_1 = E' \setminus F_2$, each F_i contains a base of $\mathcal{M}'^*$. F_2 contains e, but there is an element $f \in C(F_2, f) \cap F_1$ such that $F_2 \setminus e \uplus f$ still contains a base of $\mathcal{M}'^*$. Now f is the only element of $F_1 \cap F_2$.

The cut players strategy is as follows. She must first delete the element f. Now F_1 and F_2 are disjoint sets spanning the remaining elements of $\mathcal{M}'^*$, including e. Thus F_1 and F_2 satisfy the property that for any edge g *not* in F_i, $i = 1, 2$, (in particular for $g = e$), $F_i \uplus g$ contains a circuit of $\mathcal{M}'^*$, or equivalently, a cutset in the graph corresponding to $\mathcal{M}'$. The cut player should maintain this property to ensure a win. When the short player contracts an edge, the cut player should check whether F_1 and F_2 both still contain a base of $\mathcal{M}'^*$. If so, the cut player may delete any edge. Otherwise, the short player has (without loss of generality) contracted an edge of F_1, and F_1 no longer contains a base of $\mathcal{M}'^*$. That is, there is an element g of F_2 such that g can be added to F_1 without creating a cut. The largest independent set in F_1 can be augmented with g to form a base of $\mathcal{M}'^*$. The cut player should delete this edge g. Thus after every move the cut player makes, e forms a cut with elements of F_1 and also with elements of F_2. Thus no matter which edge the short player chooses to contract, she can never reduce e to a loop. ■

Matrix Separation [MAX-CARD 2-INT]. Let A be a full row-rank matrix over field $\mathbf{F}$, the columns of which are indexed from E. Let $r(\cdot)$ be the rank function of $\mathcal{M} = \mathcal{M}_{\mathbf{F}}(A)$. Let F_1 and F_2 be disjoint subsets of E having equal cardinality k $(\geqslant 1)$. An (F_1, F_2)*-separation* is a partition $E_1 \uplus E_2 = E$ with $F_1 \subseteq E_1$, $F_2 \subseteq E_2$ and $r(E_1) + r(E_2) \leqslant r(E) + k - 1$. If $E_1 \uplus E_2$ is an (F_1, F_2)-separation then elementary row operations can be performed on A so that it has the form

$$\begin{array}{cc} E_1 & E_2 \end{array}$$
$$\left[\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right],$$

where A_1 and A_2 overlap on $r(E_1) + r(E_2 - r(E)$ (hence, no more than $k - 1$) rows. A *k-separation* is an (F_1, F_2)-separation for some choice of disjoint subsets F_1 and F_2 of E, having cardinality k. A method for separating A can be used by a preprocessor for solving the linear program $\max\{cx\colon\ Ax = b,\ x \geqslant 0\}$ by decomposition methods. The method is also used by an algorithm for determining whether a matroid is regular. As a biproduct, one has an algorithm for determining whether a $\{0, \pm 1\}$-valued matrix is totally unimodular. (See the section below on recognizing totally unimodular matrices.)

Let $E' \equiv E \setminus (F_1 \uplus F_2)$, and let $\mathcal{M}_1 = \mathcal{M}/F_1 \setminus F_2$ and $\mathcal{M}_2 = \mathcal{M}/F_2 \setminus F_1$. Let $r_i(\cdot)$ be the rank function of $\mathcal{M}_i$ on E' $(i = 1, 2)$. Find S_1, S_2 that solve

$$\min\{r_1(S_1) + r_2(S_2)\colon S_1 \uplus S_2 = E\}.$$

By Edmonds' duality theorem, this is essentially a maximum-cardinality 2-matroid intersection problem (see the final paragraph of Section 2.4). Using properties of the rank function under minors (see Section 1), it can be shown that $E_1 \uplus E_2$ is an (F_1, F_2)-separation, where $E_i \equiv S_i \uplus F_i$ $(i = 1, 2)$, precisely when the minimum, above, is no more than $r(E) + r(F_1) + r(F_2) + k - 1$. A complete discussion can be found in Cunningham and Edmonds[33] and in Bixby.[10] ■

Recognizing Totally Unimodular Matrices [GRAPH REALIZATION, MAX-CARD 2-INT]. Totally unimodular matrices (see Section 1) are very important in linear programming (see, e.g., Schrijver[138]). If A is totally unimodular, the linear programming problem $\max\{cx\colon Ax = b,\ x \geqslant 0\}$, has an integer-valued optimal solution, for all vectors c and integer-valued vectors b for which an optimal solution exists (see Hoffman and Kruskal[77]).

Seymour[139] showed how every regular matroid can be constructed by "joining" graphic matroids, cographic matroids and isomorphs of R_{10}, where R_{10} is the binary matroid arising from the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We exploit the relationship between totally unimodular matrices and regular matroids (see Section 1) to determine whether the $\{0, \pm 1\}$-valued matrix A is totally unimodular. We may as well assume that A is a standard representative matrix, since if it is not we may append an identity matrix to it. We first determine whether $\mathcal{M} = \mathcal{M}_{\mathbf{GF(2)}}(A)$ is regular. If it is then $|A|$ (replace each -1 with $+1$) has a unique signing, up to projective equivalence, so that the resulting matrix is totally unimodular. We then check whether this signed matrix is projectively equivalent to A.

Matroid sums are intimately related to separations. This relationship is used to determine whether a matrix is totally unimodular. 1-sums are simply direct sums (see Section 1). If $E_1 \cup E_2$ is a 1-separation of $\mathcal{M}$, then $\mathcal{M}$ is the 1-sum of $\mathcal{M} \setminus E_2$ and $\mathcal{M} \setminus E_1$. Note that the set of circuits of a 1-sum is the union of the sets of circuits of the summands.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two matroids whose ground sets E_1 and E_2 overlap on one element e (that is not a loop of either of the matroids or their duals), then their 2-sum is a matroid on $E = (E_1 \cup E_2) \setminus e$, the circuits of which are the circuits of $\mathcal{M}_1 \setminus e$ and $\mathcal{M}_2 \setminus e$, and those minimal subsets of E of the form $C = (C_1 \setminus e) \cup (C_2 \setminus e)$ where C_i $(i = 1, 2)$ is a circuit of $\mathcal{M}_i$ containing e. Suppose that $\mathcal{M}$ has no 1-separation, and $E_1 \cup E_2$ is a 2-separation. Then we can perform elementary row operations so that A has the form:

$$\begin{array}{cc} E_1 & E_2 \\ \left[\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right] \end{array},$$

where A_1 and A_2 overlap on 1 row. The matroid $\mathcal{M}$ is the 2-sum of the matroids having binary representations

$$\begin{array}{cc} E_1 & e \\ \left[\begin{array}{c|c} A_1 & \begin{array}{c} \mathbf{0} \\ \hline 1 \end{array} \end{array}\right] \end{array}, \qquad \begin{array}{cc} e & E_2 \\ \left[\begin{array}{c|c} \begin{array}{c} 1 \\ \hline \mathbf{0} \end{array} & A_2 \end{array}\right] \end{array}.$$

Finally, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be binary matroids on E_1 and E_2, where E_1 and E_2 overlap on three elements, $\{e_1, e_2, e_3\}$, which form a circuit of both matroids and a cocircuit of neither. Their 3-sum is a matroid on $E = E_1 \cup E_2 \setminus \{e_1, e_2, e_3\}$ whose circuits are subsets of E that are circuits in $\mathcal{M}_1 \setminus \{e_1, e_2, e_3\}$ or $\mathcal{M}_2 \setminus \{e_1, e_2, e_3\}$, together with those minimal subsets of E that are the symmetric differences of circuits of $\mathcal{M}_1$ and $\mathcal{M}_2$. Suppose that $\mathcal{M}$ has no 1- or 2-separation, and that $E_1 \cup E_2$ is a 3-separation. Then we can perform elementary row operations so that A has the form:

$$A = \begin{array}{cc} E_1 & E_2 \\ \left[\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right] \end{array},$$

where A_1 and A_2 overlap on 2 rows. The matroid $\mathcal{M}$ is the 3-sum of the matroids having binary representations

$$\begin{array}{cccc} E_1 & e_1 & e_2 & e_3 \end{array} \qquad \begin{array}{cccc} e_1 & e_2 & e_3 & E_2 \end{array}$$
$$\left[\begin{array}{c|ccc} & & \mathbf{0} & \\ \cline{2-4} A_1 & 1 & 0 & 1 \\ & 0 & 1 & 1 \end{array}\right], \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & \\ 0 & 1 & 1 & A_2 \\ \hline & \mathbf{0} & & \end{array}\right].$$

Seymour's theorem asserts that if $\mathcal{M}$ is regular then it is either graphic, cographic, isomorphic to R_{10}, has a 1- or 2-separation, or has a 3-separation (E_1, E_2) with $|E_i| \geqslant 4$ $(i = 1, 2)$. Thus in order to test if $\mathcal{M}$ is regular, we carry out the following:

TEST FOR REGULARITY

i) Test if $\mathcal{M}$ is graphic using an algorithm to test whether a binary matroid is graphic. If $\mathcal{M}$ is graphic then stop, since $\mathcal{M}$ is regular.

ii) Test if $\mathcal{M}$ is cographic by testing whether $\mathcal{M}^*$ is graphic. If $\mathcal{M}$ is cographic then stop, since $\mathcal{M}$ is regular.

iii) Test if M is isomorphic to R_{10}. This is easily checked by enumerating the possible isomorphisms. If $\mathcal{M}$ is isomorphic to R_{10} then stop, since $\mathcal{M}$ is regular.

iv) If $\mathcal{M}$ has a 1-separation, determine matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ so that $\mathcal{M}$ is the 1-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, and, recursively, test $\mathcal{M}_1$ and $\mathcal{M}_2$ for regularity. If $\mathcal{M}$ has no 1-separation, but $\mathcal{M}$ has 2-separation, determine matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ so that $\mathcal{M}$ is the 2-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, and, recursively, test $\mathcal{M}_1$ and $\mathcal{M}_2$ for regularity. If $\mathcal{M}$ has no 1- or 2-separation, but $\mathcal{M}$ has a 3-separation (E_1, E_2) with $|E_i| \geqslant 4$, determine matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ so that $\mathcal{M}$ is the 3-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, and, recursively, test $\mathcal{M}_1$ and $\mathcal{M}_2$ for regularity. If $\mathcal{M}$ has no 1-, 2-, or 3-separation, then stop, since $\mathcal{M}$ is not regular.

We remark that k-separations can be found by checking for (F_1, F_2)-separations for each of the $|E|!/k!k!(|E| - 2k)!$ choices of F_1 and F_2 satisfying $|F_i| = k$ $(i = 1, 2)$. The 3-separations of the required type may be found by checking for 4-separations.

Bland and Edmonds[14] showed that if a matrix A is totally unimodular, its decomposition (as above) can be used to develop a polynomial-time algorithm for solving linear programs having the constraint matrix A. The algorithm generalizes the scaling method of Edmonds and Karp.[49]

Testing for regularity also has an application to geometry. Let A be a real $m \times n$ matrix with distinct non-null columns. The *zonotope* associated with A is the set:

$$Z = \{z \in \mathbf{R}^m\colon z = Ax, \text{ where } |x_j| \leqslant 1 \ (1 \leqslant j \leqslant n)\}.$$

The zonotope Z *packs* $\mathbf{R}^m$ if translates of Z can be used to fill $\mathbf{R}^m$ in such a way that the copies of Z only intersect on their boundaries. Shepard[142] and McMullen[113] showed that Z packs $\mathbf{R}^m$ if and only if $\mathscr{M}(A)$ is binary. Since A is a real matrix, the condition is equiv alent to $\mathscr{M}$ being regular. ■

4. Electrical Systems

The topology of an electrical system is naturally modeled by a graph, and its physical properties can often be modeled by a system of linear equations. Consequently, it is not surprising that techniques from linear algebra and graph theory have long been in use in the study of electrical networks. This together with the natural voltage/current duality makes the study of electrical systems a natural application area for maroid theory. The potential for matroid theory as a tool in the analysis of electrical networks was realized in the mid 1960s (see Iri[81] and Minty[114]).

The full potential of matroid theory in the study of electrical systems was not realized until the 1970s, when *algorithmic* matroid theory was developed. There have been several approaches to the study of electrical systems via matroid theory. In Iri,[86] Petersen,[126] and Recski,[132] some of the key uses of matroid theory in electrical network theory are summarized. In addition, the book by Recski[134] provides a more detailed account of much of the methodology.

Most of the work involves variations of the following model. We consider systems with a set $\mathscr{E}$ of n "branches." The state of the system is completely described by real-valued voltages and currents for each branch. It is convenient to let $\mathscr{E}_V$ and $\mathscr{E}_I$ denote a pair of disjoint copies of the set of branches $\mathscr{E}$. Then for $e \in E$, e_v (respectively, e_i) denotes the copy of e in $\mathscr{E}_V$ $(\mathscr{E}_I)$, and $x(e_v)$ $(x(e_i))$ denotes the voltage (current) of element e. A cardinality p element subset $\mathscr{P}$ of the set of branches is the set of "ports." Let $\mathscr{P}_V$ (respectively, $\mathscr{P}_I$) denote the port voltage (current) elements. A *port* is a branch that has no physical characteristics, but the voltage or current may be exogenously specified. That is, a port can be specified to be either a voltage or a current source. We denote the voltage (respectively, current) elements of the voltage sources by $\mathscr{S}_V$ $(\mathscr{S}_I)$. We denote the voltage (respectively, current) elements of the current sources by $\mathscr{T}_V$ $(\mathscr{T}_I)$. We note that $|\mathscr{S}_V \cup \mathscr{T}_I| = p$.

Topological Solvability of a p-port. Relationships that must be satisfied based on Kirchoff's voltage and current laws are specified by a system of n linear (homogeneous) real equations, where the variables are the n voltages and n currents. The equations that must be satisfied are determined by the topology of the system. We associate with the system a directed graph G, the edges of which are in one-to-one correspondence with the branches of the system. Kirchoff's voltage (respectively, current) law asserts that the net voltage (current) change around any elementary circuit (cocircuit) of the network must be zero. If the nullity of the (directed) vertex-edge incidence matrix of the network is r, then Kirchoff's voltage law imposes r independent equations and the current law imposes an additional $n - r$ independent equations. Thus the "topological" solution space of a system having n branches is an n-dimensional subspace of $\mathbf{R}^{2n}$. Let A_V (respectively, A_I) be the coefficient matrix of the system of equations associated with Kirchoff's voltage (current) law. The matroid $\mathscr{M}(A_V)$ is isomorphic to the graphic matroid $\mathscr{M}(G)$ of the graph G, and $\mathscr{M}(A_I)$ is isomorphic to its dual $\mathscr{M}^*(G)$—the cographic matroid of G. For an arbitrary fixing of p of the voltages and currents at p distinct ports (i.e., at each port either the voltage or the current is fixed), the topological system has a solution if the remaining set of $2n - p$ columns is rank n. That is, if $(\mathscr{M}(G) \setminus \mathscr{S}_V) \oplus (\mathscr{M}^*(G) \setminus \mathscr{T}_I)$ is rank n. ■

Solvability of a p-port [PARTITION]. We may also consider the "physical" characteristics of the system determined by the set $\mathscr{I}$ of $n - p$ *internal* (i.e., non-port) branches. Typically, one assumes that the physical characteristics can be specified by a system of $n - p$ independent linear equations in the $2(n - p)$ variables corresponding to the voltages and currents of the internal branches. For example, an internal branch may be a resistor satisfying Ohm's Law; that is the voltage and current of the resistor are arbitrary except they must satisfy a proportionality restriction. The *sign* of the resistor is the sign of the proportionality constant. The homogeneity assumption is not very restrictive since the ports afford us the flexibility to specify outside electric sources. Let F be the coefficient matrix of the system of equations.

The matrix F has all zero entries in the $2p$ columns corresponding to ports. In studying the solvability of the entire system of physical and topological equations, we naturally consider the matroid $\mathscr{M}$ of the matrix

$$M = \begin{array}{cc} \overbrace{\qquad}^{\mathscr{E}_V} & \overbrace{\qquad}^{\mathscr{E}_I} \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} A_V & \mathbf{0} \\ \hline \mathbf{0} & A_I \\ \hline \multicolumn{2}{c}{F} \end{array}\right]} \end{array}.$$

There are two combinatorial approaches that one may take with respect to the submatrix F. The most natural approach is to consider the entries of F to be rational numbers—exact measurements of the physical properties of the system, and then $\mathscr{M}$ is the matroid of the rational matrix M. An alternative approach views the nonzero elements of F as arbitrary algebraically independent real numbers. The latter approach recognizes that manufacturing processes cannot produce a device to exact specifications and that no two devices will behave in exactly the same manner. Under this "generality" assumption, $\mathscr{M}$ is the matroid $(\mathscr{M}(G) \oplus \mathscr{M}^*(G)) \vee \mathscr{M}(F)$, where $\mathscr{M}(F)$ is a transversal matroid on $\mathscr{E}_V \cup \mathscr{E}_I$. In either case, independence in $\mathscr{M}$ can be checked efficiently. We note that a mixed approach can be taken (see Iri[86]). We may suppose that some of the nonzero entries of F are rational and some are algebraically independent real numbers. The matrix F may be expressed in the form $Q + T$ where Q is a rational matrix, and T is a matrix of algebraically independent real numbers. Let $\mathscr{R}$ be a set of elements corresponding to the rows of M. The rank of M, over the reals, may be expressed as

$$\max\{r(X) + \tau(E \setminus X) \colon X \subseteq \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{R}\} - |\mathscr{R}|,$$

where $r(\cdot)$ is the rank function of the matroid of the matrix

$$\begin{array}{ccc} \overbrace{\qquad}^{\mathscr{E}_V} & \overbrace{\qquad}^{\mathscr{E}_I} & \overbrace{\qquad}^{\mathscr{R}} \\ \multicolumn{3}{c}{\left[\begin{array}{c|c|c} A_V & \mathbf{0} & \\ \cline{1-2} \mathbf{0} & A_I & I \\ \cline{1-2} \multicolumn{2}{c|}{Q} & \end{array}\right]} \end{array}.$$

and $\tau(\cdot)$ is the rank function of the matroid of the matrix

$$\begin{array}{ccc} \overbrace{\quad}^{\mathscr{E}_V} & \overbrace{\quad}^{\mathscr{E}_I} & \overbrace{\quad}^{\mathscr{R}} \\ \multicolumn{3}{c}{\left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \\ \cline{1-2} \mathbf{0} & \mathbf{0} & I \\ \cline{1-2} \multicolumn{2}{c|}{T} & \end{array}\right]} \end{array}.$$

This latter matroid is a transversal matroid, hence its rank function may be evaluated in a purely combinatorial manner.

Together, the topological and physical characteristics determine $2n - p$ homogeneous equations in $2n$ unknowns. For a particular specification of p voltage and current sources at the ports, the network is *solvable* if the remaining $2n - p$ voltages and currents are uniquely determined by the voltages of the voltage sources and the currents of the current sources. That is, if the rank of $\mathscr{M} \setminus (\mathscr{S}_V \cup \mathscr{T}_I)$ is $2n - p$. ■

Solvability of a p Port: Positive Resistors. If all of the internal branches are positive resistors, the system is solvable if and only if the graph of the system has a spanning forest containing every voltage source and no current source (see Recski[134]). That is, the system is solvable if and only if the rank of $\mathscr{M}(G)$ exceeds the rank of $\mathscr{M}(G)/\mathscr{S}_V \setminus \mathscr{T}_I$ by exactly $|\mathscr{S}_V|$. ■

Hybrid Immitance Description [PARTITION]. Among the complements of bases of $\mathscr{M}$, there might be at least one with the property that for any port exactly one of the voltage or current elements appear, and none of the non-port voltages and currents appear. Such a base determines a "hybrid immitance description" of the network. This simply means that there is a partition of the p ports into voltage and current sources so that the sources uniquely determine the remaining $2n - p$ voltages and currents. Let $\mathscr{B}$ be a partition matroid on $\mathscr{E}_V \cup \mathscr{E}_I$ having as blocks the pairs of voltages and currents for each branch. A set B is independent in $\mathscr{B}$ if $|B \cap \{e_v, e_i\}|$ is no more than 1 for each $e \in \mathscr{P}$ and exactly 0 for each $e \in \mathscr{I}$. A result of Iri and Tomizawa[88] is that the system has at least one hybrid immitance description if and only if $\mathscr{E}_V \cup \mathscr{E}_I$ is independent in $\mathscr{M} \vee \mathscr{B}$. A hybrid immitance description can be found by applying a matroid partitioning algorithm to find a partition $M \cup B$ of $\mathscr{E}_V \cup \mathscr{E}_I$ with M independent in $\mathscr{M}$ and B independent in $\mathscr{B}$. The voltage and current sources of the hybrid immitance description are precisely the set of elements B.

It is interesting to note that in the case where all branches are ports, and ignoring the special structure of $\mathscr{M}$, this pair of matroids is used by Anstreicher, Lee and Rutherford[2] in their study of the linear complementarity problem, and by Recski[133] in the consideration of planar frameworks that can be fixed by using horizontal and vertical "tracks." ■

Terminal Solvability [PARTITION]. The system is *terminal solvable* if, for some choice of voltage or current sources at each port, the remaining *port* voltages and currents are uniquely determined. The system is terminal solvable, if $\mathscr{P}_V \cup \mathscr{P}_I$ is independent in $(\mathscr{M} \times (\mathscr{P}_V \cup \mathscr{P}_I)) \vee (\mathscr{B} \setminus (\mathscr{P}_V \cup \mathscr{P}_I)$ (see Recski[130]).

Solvability with Nullators and Norators. A port of a system need not be terminated by exactly one source (i.e., a voltage or current source). A *nullator* is a device that acts as both a current and voltage source. A *norator* attached to a port indicates that neither the voltage nor the current is specified. The system is *solvable* if, for some choice of voltage or current sources at the ports, including the possibility of nullators and norators, all of the remaining voltages and currents are uniquely determined. That is, if $\mathcal{M}$ is rank $2n - p$ and there exists a base of $\mathcal{M}$ containing all $2(n - p)$ voltage and current elements of internal branches. ■

Terminal Solvability with Nullators and Norators [PARITY]. The system is terminal solvable, including the possibility of nullators and norators, if $\mathcal{M} \times (\mathcal{P}_V \cup \mathcal{P}_I)$ is rank p. Recski[131] showed that a matroid parity algorithm can determine a minimum-cardinality set of nullators that can be joined to the ports of a system so that the resulting electrical network has a unique solution, by pairing the current and the voltage of each port. ■

There is a related problem in the study of power systems that does not fit into the above framework.

Topological Observability of a Power System [MAX-CARD 2-INT]. An n-bus power system is described by an undirected *network graph* on n vertices. The vertices correspond to power system buses and the edges correspond to transmission lines. Measurements are associated with edges in the network graph. A measurement is associated with an edge if it is a line flow measurement associated with the transmission line corresponding to the edge, or if it is an injection measurement associated with one of the buses at the ends of the transmission line corresponding to the edge (see Quintana, Simoes-Costa and Mandel[128]).

The state of the power system is described by n voltage magnitudes (V) and $n - 1$ voltage angles (δ). One bus is chosen as the reference bus, and voltage angles are with respect to the voltage angle of the reference bus. The power system is *observable* with respect to the measurements if the measurements allow the determination of the bus voltage magnitude and angle at every bus of the network. The line flow and bus injection measurements are nonlinear functions of the bus voltage magnitudes and angles. By linearizing these nonlinear functions about a nominal operating point (normally, all voltages are set equal to 1 and all angles are set to 0), the observability question amounts to determining whether this approximating linear system has full rank (i.e., $2n - 1$). This is referred to as *topological observability*. The system must be observable in order to carry out basic monitoring and control.

There is a decoupling principle for power system networks which allows the topological observability question to be split into calculations involving the magnitudes and angles separately. *Active power* P (respectively, *reactive power* Q) in an element is the product of the current with the voltage component that is in (out of) phase with the current. The system is *$P - \delta$ topologically observable* with respect to a measurement set consisting of active line flow and bus injection measurements if the approximating linear system involving the voltage angles has full rank ($n - 1$). The system is *$Q - V$ topologically observable* with respect to a measurement set consisting of reactive line flow and bus injection measurements (as well as bus voltage magnitude measurements) if the approximating linear system involving the voltage magnitudes has full rank (n). The system is topologically observable precisely when it is observable in the above two senses.

To determine whether the network is topologically observable (in either of the above two senses) with respect to the set of measurements, we construct an (undirected) *measurement graph* G_m. The measurement graph also has vertex set V. The edges are determined as follows. If the line flow on transmission line $\{i, j\}$ is measured, then $\{i, j\}$ is an edge of G_m. If the line flow is measured at both ends of $\{i, j\}$, then two edges connect i and j in G_m. If a bus injection at bus i is monitored, then vertex i is connected to each of its adjacent vertices in G_m. The measurement graph depends on whether we are considering the $P - \delta$ problem (in which case we consider active measurements) or the $Q - V$ problem (in which case we consider reactive measurements). If the voltage magnitude of bus i is measured, a loop is created at vertex i in G_m (but only for the $Q - V$ problem).

Quintana, Simoes-Costa and Mandel[128] demonstrate that the system is topologically observable (in either sense) if and only if there is a spanning tree of G_m with no more than one edge associated with each measurement. Letting $\mathcal{M}_m$ be the partition matroid on the edge set of G_m for which the independent sets have no more than one edge per measurement. Then the system is topologically observable if and only if there is a base of $\mathcal{M}(G_m)$ that is independent in $\mathcal{M}_m$.

Clements, Davis and Krumpholz[26] point out that each critical element of the matroid intersection (i.e., an element that appears in every base of $\mathcal{M}(G_m)$ that is independent in $\mathcal{M}_m$) corresponds to a line whose loss would render the system unobservable. ■

5. Statics

Some problems in statics can also be modeled as matroidal problems. Recski[132] and Lovász and Recski[109] summarize the primary uses of matroid theory in the study of "bar and joint frameworks." Recski[134] gives a fuller treatment. A *framework* in $\mathbf{R}^k$ is an imbedding of a graph $G = (V, E)$ so that the edges of the graph, referred to as (rigid) *bars*, are straight lines in the space and the vertices of the graph, referred to as (flexible) *joints*, are points in the space. An (*infinitesimal*) *motion* of the framework is an assignment of a velocity k-vector m_i to each joint x_i so that the rigidity of every bar is respected. That is,

$$\langle m_i - m_j, x_i - x_j \rangle = 0 \qquad \forall (x_i, x_j) \in E.$$

Let ε be the number of bars and n be the number of joints of a framework. We see that motions of the framework are characterized by solutions of a system of ε linear homogeneous real equations in kn variables $M \equiv \{m_i^l l\ 1 \leqslant l \leqslant k,\ 1 \leqslant i \leqslant n\}$.

Rigidity. The framework is (*infinitesimally*) *rigid* if every infinitesimal motion is a (restriction to the vertices of a) Euclidean motion of the space. The Euclidean motions form a subspace of dimension $k(k+1)/2$. Let A be the coefficient matrix of the system of equations. The framework is rigid if and only if the rank of A is $kn - k(k+1)/2$. ■

Rigidity with Pins in the Plane [PARITY]. A matroid parity algorithm can be used to determine a minimum-cardinality set of joints of a planar structure to pin so that the structure is infinitesimally rigid (see Lovász[108]). Mansfield[111] showed that the corresponding three-dimensional problem is *NP*-Hard.

Rigidity with Tracks in the Plane [PARTITION]. We may exclude the use of pins but allow vertical and horizontal *tracks* each of which only allows motion in one of the k coordinate directions. Let $\mathscr{B}_{p,k}$ $(0 \leqslant p \leqslant k)$ denote a partition matroid on M, where the blocks are the sets $M_i \equiv \{m_i^l\colon 1 \leqslant l \leqslant k\}$ $(1 \leqslant i \leqslant n)$ and independent sets have no more than p elements from each block. Recski[133] showed that a planar framework can be fixed to the plane using tracks (but no pins) if and only if M is independent in $\mathscr{M}(A) \vee \mathscr{B}_{1,2}$. An appropriate set of tracks can be determined by the application of a matroid partitioning algorithm. ■

Rigidity with Tracks and Walls in 3-Space [PARTITION] . Recski extends this last idea to frameworks in 3-space. Tracks that allow motion in any one of the three coordinate directions are permitted. In addition, we may allow *walls* that allow motion in any two of the three coordinate directions. A framework in 3-space can be fixed using walls (but no pins or tracks) if and only if M is independent in $\mathscr{M}(A) \vee \mathscr{B}_{1,3}$. Also, we have that a framework in 3-space can be fixed using tracks and walls (but no pins) if and only if M is independent in $\mathscr{M}(A) \vee \mathscr{B}_{2,3}$. ■

Generic Rigidity in the Plane [PARTITION]. A graph is *generically rigid* (in k-space) if it corresponds to a rigid framework (in k-space) for suitably chosen lengths of the bars. Alternatively, a graph is considered to be generically rigid if it is rigid as a framework where the lengths of the edges are taken to be algebraically independent over the reals. Generic rigidity cannot be determined as easily as rigidity can. We cannot even resort to computing the term-rank of A since the matrix entries are not algebraically independent real numbers. For the present, we will restrict ourselves to the plane. A framework in the plane is not rigid if $\varepsilon < 2n - 3$. A rigid framework in the plane has unnecessary bars if $\varepsilon > 2n - 3$. Laman[98] showed that for graphs with $\varepsilon = 2n - 3$, generic rigidity is equivalent to $\varepsilon' \leqslant 2n' - 3$ for *every* subgraph having ε' edges and n' vertices. Laman's result does not demonstrate that generic rigidity in the plane can be determined in polynomial time since it appears that every vertex-induced subgraph must be examined. However, Lovász and Yemini[110] showed that a graph is generically rigid in the plane if each of the graphs obtained by doubling an edge contains 2 edge-disjoint spanning trees. Hence, with at most ε applications of a matroid partitioning algorithm, the planar generic rigidity of a graph satisfying $\varepsilon \geqslant 2n - 3$ can be determined. If $\varepsilon > 2n - 3$, and the graph is generically rigid, a set F of $\varepsilon - 2n + 3$ edges can be found so that the graph remains generically rigid after removing F. We simply consider edges for removal, one at a time, so that rigidity is preserved. ■

Generic Rigidity on Other Surfaces [PARTITION]. Laman's theorem does not generalize to 3-space. Furthermore, no good characterization of generic rigidity in 3-space is known. Although no generalization is known for frameworks in 3-space, there are efficient matroid partitioning procedures for determining the rigidity of frameworks on other surfaces than Euclidean 2-space. Whiteley[158] used matroid union to characterize generic rigidity for frameworks on the surfaces of the circular cylinder, right circular cones, the regular tetrahedron, and the "flat k-torus" in k-space (Euclidean k-space in which two points are identified if their difference is integer-valued). Bars on these surfaces are defined to be geodesics. Motions are also defined by wrapping a piece of a Euclidean space onto the surface. Motions on the surface are the images of motions in the Euclidean space. A characterization of rigidity for some of these surfaces requires the uses of the *bicycle* matroid $\mathscr{M}_1(G)$ of the graph $G = (V, E)$ in which the independent sets are the edge-sets for which no connected component contains more than one circuit of G. The simple graph $G = (V, E)$, on n vertices, is a minimal graph (with respect to E) that is generically rigid with respect to the surface S if and only if $|E| = \varepsilon(S)$ and E is the disjoint union of $k(S)$

bases of $\mathcal{M}(G)$ and $k_1(S)$ bases of $\mathcal{M}_1(G)$, where for each surface S, the constants $\varepsilon(S)$, $k(S)$ and $k_1(S)$ are specified below.

Surface S	$\varepsilon(S)$ Edges	$k(S)$ Graphic Bases	$k_1(S)$ Bicycle Bases
Circular cylinder	$2n - 2$	2	0
Right circular cone	$2n - 1$	1	1
Tetrahedron	$2n$	0	2
Flat k-torus	$kn - k$	k	0

Notice that $\varepsilon(S) = m(S)n - d(S)$, where $m(S)$ is the dimension of the surface S and $d(S)$ is its global degrees of freedom. The graph $G = (V, E)$ is generically rigid on S if and only if E contains the disjoint union of $k(S)$ bases of $\mathcal{M}(G)$ (that is, $k(S)$ spanning trees) and $k_1(S)$ bases of $\mathcal{M}_1(G)$. ■

6. Remarks

There are other applications of matroid theory and algorithms in which the computational significance is less clear-cut than in the applications of Sections 3, 4 and 5. Such applications are no less important or insightful, but they are not within the focus of the present paper. Below, we briefly mention some of these applications.

Marcotte and Trotter[112] studied an application of matroid theory to the scheduling of jobs with precedence constraints. Network reliability issues can also be addressed via matroid theory (see Ball and Nemhauser,[6] Ball and Provan,[7, 8] Colbourn,[27, 28] and Colbourn and Pulleyblank,[29] for example). Bruno and Weinberg[21] abstracted the theory of electrical networks so that the underlying graph is replaced by a regular matroid. This framework of "generalized networks" provides an interesting setting for examining questions of network synthesis using algorithms for graph realizability (see Weinberg[155]). Duffin and Morley[37] demonstrated how "Wang algebra" and the theory of regular matroids can be used to determine the joint resistance of a generalized electrical network. The problem of finding a sparse basis for a subspace of $\mathbf{R}^n$ was studied by Coleman and Pothen.[30, 31] Greene[67] showed how matroids can be used to capture approximate linear dependencies in multivariate data. Decomposition techniques for isolating the inconsistencies in systems of equations have been extensively studied (see Bruno and Weinberg,[20] Iri[84, 86, 87] Sugihara,[145] Murota,[117] and Murota, Iri and Nakamura,[118], for example). Matroid theory has also been used in scene analysis (see Sugihara,[144, 146] Imai, [79] and Whiteley,[159], for example); a central issue is the reconstruction of a three-dimensional polytope from the incidence structure of its vertices and faces.

There is an intimate connection between matroids and submodular functions. We have neglected to discuss applications of submodularity that do not easily fit within the framework of matroids. Available algorithmic tools are network flow algorithms that allow submodular capacity constraints on certain subsets of arcs (see Hassin,[71] and Lawler and Martel[102]), and an algorithm for minimizing a submodular function (see Grötschel, Lovász and Schrijver[68]), for example. Frank and Tardos[55] give a unified presentation of submodular functions in combinatorial optimization.

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