1 Terminology

Directed tree: A directed graph that would be a tree if we ingored the directions of the edges.

Arborescence: A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

Spanning subgraph of $D(V, A)$: A subgraph $D(V, B)$ such that $B \subseteq A$.

Branching system of $D(V, A)$ with k specified root nodes: A collection of edge-disjoint arborescences rooted at the k root nodes, with each arborescence being a spanning subgraph of $D'(V \backslash R, A)$ with R containing all roots that are not part of the arborescence. By "edge-disjoint" we mean that none of these arborescences share any edges. Also, every node has at most k incoming edges (at most one from each arborescence).

Optimal branching system of a weighted digraph with k specified root nodes: the branching system with the lowest total weight.

2 Algorithm

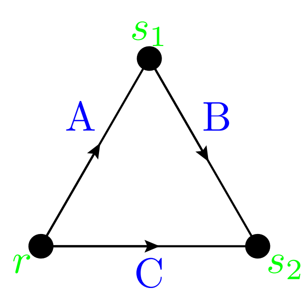
To find an OBS we need to find a min-cost set of edges which is both a basis of matroid M_1 and of M_2:

- M_1=(E, J_1) in which $j \in J_1$ has no edges entering any root nodes, and at most k edges entering each other node of the graph.
- M_2=(E, J_2) in which $j \in J_2$ can be partitioned into at most k **arborescences**.

3 Examples:

3.1 3-nodes

3.1.1 1 root



The branchings systems are:

$$B_1 = \{A, C\} \tag{3.1}$$
$$B_2 = \{A, B\} \tag{3.2}$$

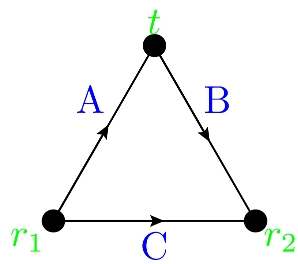
We also have:

$$J_1 = \{P(B_1) \cup P(B_2)\} \tag{3.3}$$
$$= \{\emptyset, A, C, \{A, C\}, B, \{A, B\}\} \tag{3.4}$$

$$J_2 = J_1 \tag{3.5}$$

Therefore, we have $J_1 \cap J_2 = J_1$ and if $c(A) < 0$ and $c(B) = c(D) > 0$,we have that the OBS obtained from minimizing over $J_1 \cap J_2$ is $\{A\}$, even though it is a non-spanning arborescence! Also if $c(A) = c(B) = c(C) > 0$, the OBS obtained from minimizing over $J_1 \cap J_2$ would be $\{\emptyset\}$ for which if we consider the root to be r, the arborescence would also not be spanning the graph.

3.1.2 2 roots



The only branching system is:

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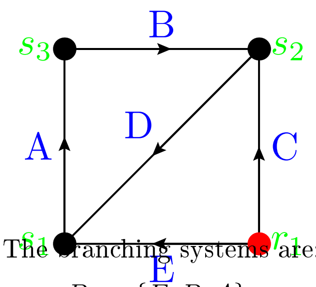
$$B = \{A\}. \tag{3.6}$$

We also have that $J_1 = J_2 = B$.

3.2 4-nodes:

3.2.1 1 root



The branching systems are:

$$B_1 = \{E, B, A\}, \tag{3.7}$$

$$B_2 = \{E, B, D\}, \tag{3.8}$$

$$B_3 = \{D, C, B\}. \tag{3.9}$$

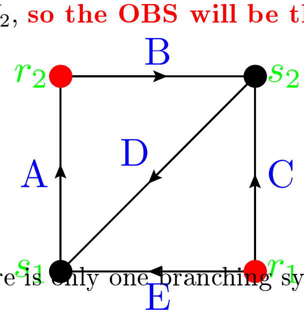
We also have that:

$$J_1 = \{B_1, B_2, B_3, \{A, C, B\}\} \tag{3.10}$$

$$J_2 = \{B_1, B_2, B_3\}. \tag{3.11}$$

Since $J_2 \subset J_1$,we know that $J_1 \cap J_2 = J_2$, **so the OBS will be the min-cost branching system in** J_2**.**

3.2.2 2 roots



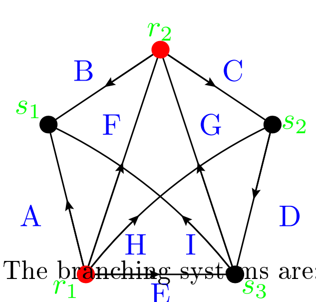
There is only one branching system:

$$B = \{E, B, D, C\}, \tag{3.12}$$

We also have that: $J_1 = J_2 = B$.

3.3 5-nodes:

3.3.1 2 roots



The branching systems are:

$$B_1 = \{B, C, A, H, D, E\}, \tag{3.13}$$

$$B_2 = \{B, C, D, E, H, I\}, \tag{3.14}$$

$$B_3 = \{C, D, I, A, H, E\}. \tag{3.15}$$

We also have that: $J_1 = J_2 = \{B_1, B_2, B_3\}$.