1 Terminology

Directed tree: A directed graph that would be a tree if we ingored the directions of the edges.

Arborescence: A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

Spanning subgraph of $D(V, A)$: A subgraph $D(V, B)$ such that $B \subseteq A$.

Branching system of $D(V, A)$ with k specified root nodes: A collection of edge-disjoint arborescences rooted at the k root nodes, with each arborescence being a spanning subgraph of $D'(V \backslash R, A)$ with R containing all roots that are not part of the arborescence. By "edge-disjoint" we mean that none of these arborescences share any edges. Also, every node has at most k incoming edges (at most one from each arborescence).

Optimal branching system of a weighted digraph with k specified root nodes: the branching system with the lowest total weight.

2 Algorithm

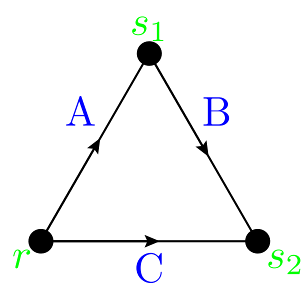
To find an OBS we need to find a min-cost set of edges which is both a basis of matroid M_1 and of M_2:

- M_1=(E, J_1) in which $j \in J_1$ has no edges entering any root nodes, and at most k edges entering each other node of the graph.
- M_2=(E, J_2) in which $j \in J_2$ can be partitioned into at most k **arborescences**.

3 Examples:

3.1 3-nodes

3.1.1 1 root



The branching systems are:

$$\mathcal{B}_1 = \{A, C\} \tag{3.1}$$
$$\mathcal{B}_2 = \{A, B\} \tag{3.2}$$

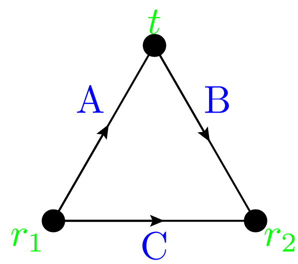
We also have:

$$J_1 = \{P(B_1) \cup P(B_2)\} \tag{3.3}$$
$$= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}\} \tag{3.4}$$

$$J_2 = J_1 \tag{3.5}$$

Therefore basis(J1) = basis(J2) and their intersection is trivial.

3.1.2 2 roots



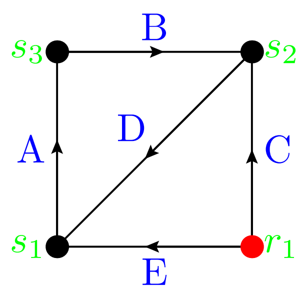
There's no branching systems because it's impossible for an arborescence rooted at r_2 to be spanning.

$$J_1 = \{\emptyset, \{A\}\} \tag{3.6}$$
$$J_2 = P(\{A, C, B\}) \tag{3.7}$$

Therefore $b(J_1) = \{\{A\}\}$ and $b(J_2) = \{\{A, B, C\}\}$ and $b(J_1) \cap b(J_2) = \emptyset$ meaning that there's no branching systems.

3.2 4-nodes:

3.2.1 1 root



The branching systems are:

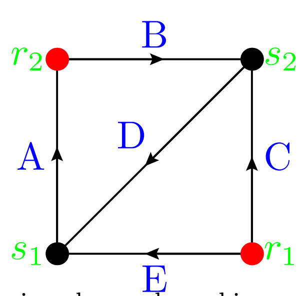
$$\mathcal{B}_1 = \{E, A, B\}, \tag{3.8}$$
$$\mathcal{B}_2 = \{C, E, A\}, \tag{3.9}$$
$$\mathcal{B}_3 = \{C, D, A\}. \tag{3.10}$$

We also have that:

$$b(J_1) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \{A, B, D\}\} \tag{3.11}$$
$$b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}. \tag{3.12}$$

Since $b(J_2) \subset b(J_1)$,we know that $b(J_1) \cap b(J_2) = b(J_2)$, **so the OBS will be the min-cost branching system in J_2.**

3.2.2 2 roots



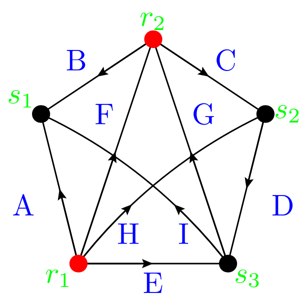
There is only one branching system:

$$\mathcal{B} = \{E, B, D, C\}, \tag{3.13}$$

We also have that: $b(J_1) = b(J_2) = \mathcal{B}$.

3.3 5-nodes:

3.3.1 2 roots



The branching systems are:

$$\mathcal{B}_1 = \{A, H, E; B, C, D\}, \tag{3.14}$$
$$\mathcal{B}_2 = \{E, I, H; B, C, D\}, \tag{3.15}$$
$$\mathcal{B}_3 = \{A, H, E; C, D, I\}. \tag{3.16}$$

We also have that: $b(J_1) = b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$.