

# 1 Terminology

**Directed tree:** A directed graph (digraph) that would be a tree if we ignored the directions of the edges.

**Arborescence:** A directed tree with a “root” such that every node of it has a unique path to it from the root. There’s always exactly one root.

**Spanning subgraph** of  $G = (V, E)$ : Just a subgraph  $(V, B)$  of  $G$  such that  $B \subseteq E$ .

**Branching system** of  $G = (V, E)$  with  $k$  specified root nodes: A collection of edge-disjoint arborescences rooted at the  $k$  root nodes, with each arborescence being a spanning subgraph of  $(V \setminus R, E)$  with  $R$  containing all roots that are not part of the arborescence. By “edge-disjoint” we mean that none of these arborescences share any edges. Also, every node has at most  $k$  incoming edges (at most one from each arborescence).

**Optimal branching system** of a weighted digraph with  $k$  specified root nodes: the branching system with the lowest total weight.

## 2 Algorithm

To find an OBS we need to find a min-cost set of edges which is both a basis of matroid  $M_1$  and of  $M_2$ :

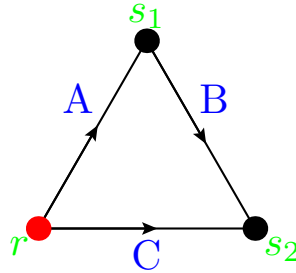
- $M_1 = (E, J_1)$  in which  $j \in J_1$  has no edges entering any root nodes, and at most  $k$  edges entering each other node of the graph.
- $M_2 = (E, J_2)$  in which  $j \in J_2$  can be partitioned into  $k$  or fewer **forests**.

If  $\mathcal{B}(M_1)$  is the set of all bases of  $M_1$  and  $\mathcal{B}(M_2)$  is the set of all bases of  $M_2$ , then the optimal branching system will be the least-cost element in  $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ .

## 3 Examples:

### 3.1 3-nodes

#### 3.1.1 1 root, 3 edges



The branching systems are:

$$\mathcal{B}_1 = \{A, C\} \tag{3.1}$$

$$\mathcal{B}_2 = \{A, B\} \tag{3.2}$$

We also have:

$$J_1 = P(\mathcal{B}_1) \cup P(\mathcal{B}_2) \tag{3.3}$$

$$= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}\} \tag{3.4}$$

$$\tag{3.5}$$

$$J_2 = P(\mathcal{B}_1) \cup P(\mathcal{B}_2) \cup P\{B, C\} \tag{3.6}$$

$$= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}, \{B, C\}\}. \tag{3.7}$$

The bases of  $M_1$  and  $M_2$  (which are associated with  $J_1$  and  $J_2$  respectively) are:

$$\mathcal{B}(M_1) = \{\mathcal{B}_1, \mathcal{B}_2\} \tag{3.8}$$

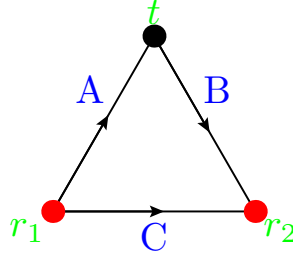
$$\mathcal{B}(M_2) = \{\mathcal{B}_1, \mathcal{B}_2, \{B, C\}\} \tag{3.9}$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\mathcal{B}_1, \mathcal{B}_2\}, \tag{3.10}$$

which are precisely the branching systems that we found by brute force!

### 3.1.2 2 roots, 3 edges



There's no branching systems because it's impossible for an arborescence rooted at  $r_2$  to be a spanning subgraph of the original graph (no arborescence rooted at  $r_2$  can contain all of the original graph's vertices).

We also have:

$$J_1 = \{\emptyset, \{A\}\} \quad (3.11)$$

$$J_2 = P\{A, B, C\} \quad (3.12)$$

$$= \{\emptyset, A, B, C, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}. \quad (3.13)$$

The bases of  $M_1$  and  $M_2$  (which are associated with  $J_1$  and  $J_2$  respectively) are:

$$\mathcal{B}(M_1) = \{\{A\}\} \quad (3.14)$$

$$\mathcal{B}(M_2) = \{\{A, B, C\}\}. \quad (3.15)$$

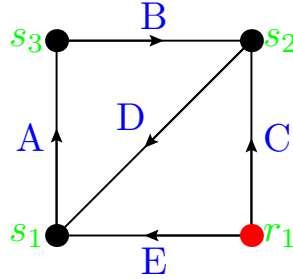
So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \quad (3.16)$$

meaning that no branching systems exist, which is exactly what we said at the beginning!

## 3.2 4-nodes:

### 3.2.1 1 root



The branching systems are:

$$\mathcal{B}_1 = \{E, A, B\}, \quad (3.17)$$

$$\mathcal{B}_2 = \{C, E, A\}, \quad (3.18)$$

$$\mathcal{B}_3 = \{C, D, A\}. \quad (3.19)$$

We also have that:

$$J_1 = P(\mathcal{B}_1) \cup P(\mathcal{B}_2) \cup P(\mathcal{B}_3) \cup P\{A, B, D\} \quad (3.20)$$

$$J_2 = P(\mathcal{B}_1) \cup P(\mathcal{B}_2) \cup P(\mathcal{B}_3) \cup P\{A, B, C\} \cup P\{B, C, E\} \cup P\{B, D, C\} \cup P\{B, D, E\} \cup P\{A, D, E\}. \quad (3.21)$$

The bases of  $M_1$  and  $M_2$  (which are associated with  $J_1$  and  $J_2$  respectively) are:

$$\mathcal{B}(M_1) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \{A, B, D\}\} \quad (3.22)$$

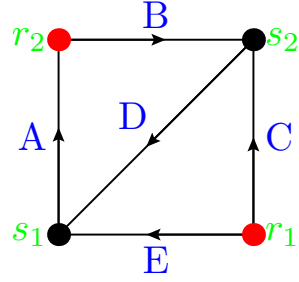
$$\mathcal{B}(M_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \{A, B, C\}, \{B, C, E\}, \{B, D, C\}, \{B, D, E\}, \{A, D, E\}\}. \quad (3.23)$$

So we have that:

$$\mathcal{B}(J_1) \cap \mathcal{B}(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}, \quad (3.24)$$

which are precisely the branching systems that we found by brute force!

### 3.2.2 2 roots



There is only one branching system:

$$\mathcal{B} = \{E, B, D, C\}. \quad (3.25)$$

We also have that:

$$J_1 = P(\mathcal{B}) \quad (3.26)$$

$$J_2 = P\{A, B, C, D, E\} \quad (3.27)$$

The bases of  $M_1$  and  $M_2$  (which are associated with  $J_1$  and  $J_2$  respectively) are:

$$\mathcal{B}(M_1) = \{\mathcal{B}\} \quad (3.28)$$

$$\mathcal{B}(M_2) = \{\{A, B, C, D, E\}\}. \quad (3.29)$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \quad (3.30)$$

This is not what we want, but if we remove  $A$  (which is going into a root), we get:

$$J_1 = \{P(\mathcal{B})\} \quad (3.31)$$

$$J_2 = J_1 \quad (3.32)$$

$$\mathcal{B}(M_1) = \mathcal{B} \quad (3.33)$$

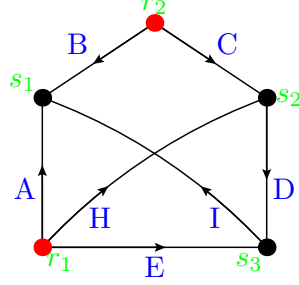
$$\mathcal{B}(M_2) = \mathcal{B}(M_1) \quad (3.34)$$

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \mathcal{B}, \quad (3.35)$$

which are precisely the branching systems that we listed at the beginning!

### 3.3 5-nodes:

#### 3.3.1 2 roots



The branching systems are:

$$\mathcal{B}_1 = \{A, H, E; B, C, D\}, \quad (3.36)$$

$$\mathcal{B}_2 = \{E, I, H; B, C, D\}, \quad (3.37)$$

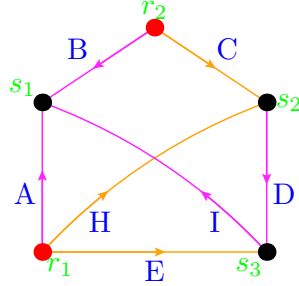
$$\mathcal{B}_3 = \{A, H, E; C, D, I\}. \quad (3.38)$$

We also have that:

$$J_1 = P(\mathcal{B}_1) \cup P(\mathcal{B}_2) \cup P(\mathcal{B}_3) \quad (3.39)$$

$$J_2 = P\{B, A, I, D; C, H, E\}. \quad (3.40)$$

$J_2$  contains all edges, and we know that this can be partitioned into  $k = 2$  forests because we have partitioned it into the pink and orange forests below:



We also know that a branching system can't include all of  $A, I$  and  $B$  at the same time since the arborescences aren't supposed to have cycles. Also  $J_1$  doesn't allow having  $3 > k = 2$  incoming edges, but if we include  $A, I$  and  $B$  at the same time, then vertex  $s_1$  would have 3 incoming edges.

The bases of  $M_1$  and  $M_2$  (which are associated with  $J_1$  and  $J_2$  respectively) are:

$$\mathcal{B}(M_1) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\} \quad (3.41)$$

$$\mathcal{B}(M_2) = \{\{B, A, I, D; C, H, E\}\} \quad (3.42)$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \quad (3.43)$$

This is not what we want, since we would like the intersection to contain the branching systems that we listed at the beginning.

This problem can be fixed by requiring that the members of  $J_2$  can be partitioned into  $k$  (or fewer) arborescences rooted at the specified roots, rather than just requiring them to be partitionable into  $k$  (or fewer) forests.