1 Terminology

Directed tree: A directed graph that would be a tree if we ingored the directions of the edges.

Arborescence: A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

Spanning subgraph of $D(V, A)$: A subgraph $D(V, B)$ such that $B \subseteq A$.

Branching system of $D(V, A)$ with k specified root nodes: A collection of edge-disjoint arborescences rooted at the k root nodes, with each arborescence being a spanning subgraph of $D'(V \backslash R, A)$ with R containing all roots that are not part of the arborescence. By "edge-disjoint" we mean that none of these arborescences share any edges. Also, every node has at most k incoming edges (at most one from each arborescence).

Optimal branching system of a weighted digraph with k specified root nodes: the branching system with the lowest total weight.

2 Algorithm

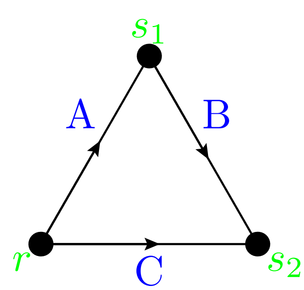
To find an OBS we need to find a min-cost set of edges which is both a basis of matroid M_1 and of M_2:

- M_1=(E, J_1) in which $j \in J_1$ has no edges entering any root nodes, and at most k edges entering each other node of the graph.
- M_2=(E, J_2) in which $j \in J_2$ can be partitioned into at most k **forests**.

3 Examples:

3.1 3-nodes

3.1.1 1 root



The branching systems are:

$$\mathcal{B}_1 = \{A, C\} \tag{3.1}$$
$$\mathcal{B}_2 = \{A, B\} \tag{3.2}$$

We also have:

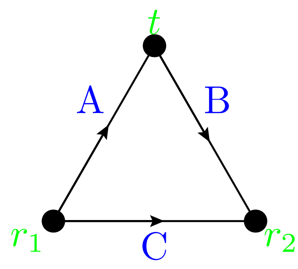
$$\begin{aligned} J_1 &= \{P(B_1) \cup P(B_2)\} && (3.3)\\ &= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}\} && (3.4) \end{aligned}$$

$$\begin{aligned} J_2 &= \{P(\{A, B\}) \cup P(A, C) \cup P(B, C)\} && (3.5)\\ b(J_1) &= \{\mathcal{B}_1, \mathcal{B}_2\} && (3.6)\\ b(J_2) &= \{\mathcal{B}_1, \mathcal{B}_2, \{B, C\}\}. && (3.7) \end{aligned}$$

So we have that:

$$b(J_1) \cap b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2\}. \tag{3.8}$$

3.1.2 2 roots



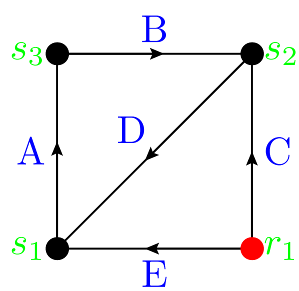
There's no branching systems because it's impossible for an arborescence rooted at r_2 to be spanning.

$$J_1 = \{\emptyset, \{A\}\} \tag{3.9}$$
$$J_2 = P(\{A, C, B\}) \tag{3.10}$$

Therefore $b(J_1) = \{\{A\}\}$ and $b(J_2) = \{\{A, B, C\}\}$ and $b(J_1) \cap b(J_2) = \emptyset$ meaning that there's no branching systems.

3.2 4-nodes:

3.2.1 1 root



The branching systems are:

$$\mathcal{B}_1 = \{E, A, B\}, \tag{3.11}$$
$$\mathcal{B}_2 = \{C, E, A\}, \tag{3.12}$$
$$\mathcal{B}_3 = \{C, D, A\}. \tag{3.13}$$

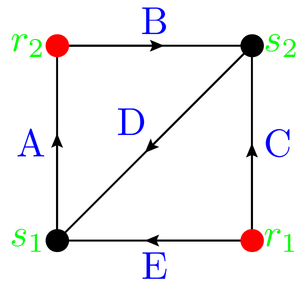
We also have that:

$$b(J_1) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \{A, B, D\}\} \tag{3.14}$$
$$b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \{A, B, C\}, \{B, C, E\}, \{B, D, C\}, \{B, D, E\}, \{A, D, E\}\}. \tag{3.15}$$

So we find that:

$$b(J_1) \cap b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}. \tag{3.16}$$

3.2.2 2 roots



There is only one branching system:

$$\mathcal{B} = \{E, B, D, C\}. \tag{3.17}$$

We also have that:

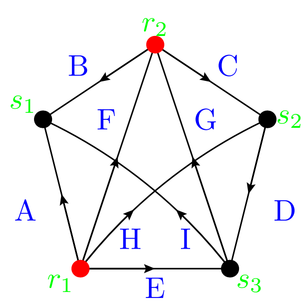
$$b(J_1) = \{\mathcal{B}\} \tag{3.18}$$
$$b(J_2) = \{A, B, C, D, E\}, \tag{3.19}$$
$$b(J_1) \cap b(J_2) = \emptyset \tag{3.20}$$

So we need to remove A (which is going into a root). By doing that we have:

$$b(J_2) = b(J_1), \tag{3.21}$$
$$b(J_2) \cap b(J_2) = \mathcal{B}. \tag{3.22}$$

3.3 5-nodes:

3.3.1 2 roots



The branching systems are:

$$\mathcal{B}_1 = \{A, H, E; B, C, D\}, \tag{3.23}$$
$$\mathcal{B}_2 = \{E, I, H; B, C, D\}, \tag{3.24}$$
$$\mathcal{B}_3 = \{A, H, E; C, D, I\}. \tag{3.25}$$

We also have that: $b(J_1) = b(J_2) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$.