1 Terminology

Directed tree: A directed graph (digraph) that would be a tree if we ignored the directions of the edges.

Arborescence: A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

Spanning subgraph of $G = (V, E)$: Just a subgraph (V, B) of G such that $B \subseteq E$.

Branching system of $G = (V, E)$ with k specified root nodes: A collection of edge-disjoint arborescences rooted at the k root nodes, with each arborescence being a spanning subgraph of $(V\backslash R, E)$ with R containing all roots that are not part of the arborescence. By "edge-disjoint" we mean that none of these arborescences share any edges. Also, every node has at most k incoming edges (at most one from each arborescence).

Optimal branching system of a weighted digraph with k specified root nodes: the branching system with the lowest total weight.

2 Algorithm overview

To find an OBS we need to find a min-cost set of edges which is both a basis of matroid M_1 and of M_2:

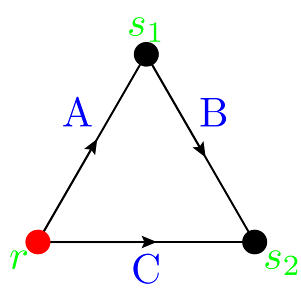
- M_1=(E, J_1) in which $j \in J_1$ has no edges entering any root nodes, and at most k edges entering each other node of the graph.
- M_2=(E, J_2) in which $j \in J_2$ can be partitioned into k or fewer **forests**.

If $\mathcal{B}(M_1)$ is the set of all bases of M_1 and $\mathcal{B}(M_2)$ is the set of all bases of M_2, then the optimal branching system will be the least-cost element in $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$.

3 Examples:

3.1 3-nodes

3.1.1 1 root, 3 edges



The branching systems are:

$$\mathscr{B}_1 = \{A, C\} \tag{3.1}$$
$$\mathscr{B}_2 = \{A, B\} \tag{3.2}$$

We also have:

$$\begin{aligned} J_1 &= P(\mathscr{B}_1) \cup P(\mathscr{B}_2) && (3.3)\\ &= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}\} && (3.4)\\ & && (3.5)\\ J_2 &= P(\mathscr{B}_1) \cup P(\mathscr{B}_2) \cup P\{B, C\} && (3.6)\\ &= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}, \{B, C\}\}. && (3.7) \end{aligned}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 respectively) are:

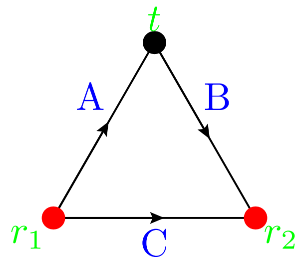
$$\mathcal{B}(M_1) = \{\mathscr{B}_1, \mathscr{B}_2\} \tag{3.8}$$
$$\mathcal{B}(M_2) = \{\mathscr{B}_1, \mathscr{B}_2, \{B, C\}\} \tag{3.9}$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\mathscr{B}_1, \mathscr{B}_2\}, \tag{3.10}$$

which are precisely the branching systems that we found by brute force!

3.1.2 2 roots, 3 edges



There's no branching systems because it's impossible for an arborescence rooted at r_2 to be a spanning subgraph of the original graph (no arborescence rooted at r_2 can contain all of the original graph's vertices).

We also have:

$$J_1 = \{\emptyset, \{A\}\} \tag{3.11}$$
$$J_2 = P\{A, B; C\} \tag{3.12}$$
$$= \{\emptyset, A, B, C, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}. \tag{3.13}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 respectively) are:

$$\mathcal{B}(M_1) = \{\{A\}\} \tag{3.14}$$
$$\mathcal{B}(M_2) = \{\{A, B, C\}\}. \tag{3.15}$$

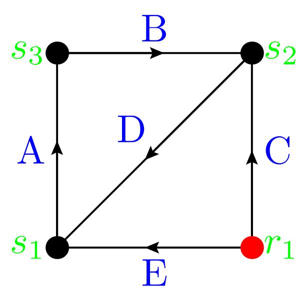
So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \tag{3.16}$$

meaning that no branching systems exist, which is exactly what we said at the beginning!

3.2 4-nodes:

3.2.1 1 root



The branching systems are:

$$\mathscr{B}_1 = \{E, A, B\}, \tag{3.17}$$
$$\mathscr{B}_2 = \{C, E, A\}, \tag{3.18}$$
$$\mathscr{B}_3 = \{C, D, A\}. \tag{3.19}$$

We also have that:

$$J_1 = P(\mathscr{B}_1) \cup P(\mathscr{B}_2) \cup P(\mathscr{B}_3) \cup P\{A, B, D\} \tag{3.20}$$
$$J_2 = P(\mathscr{B}_1) \cup P(\mathscr{B}_2) \cup P(\mathscr{B}_3) \cup P\{A, B, C\} \cup P\{B, C, E\} \cup P\{B, D, C\} \cup P\{B, D, E\} \cup P\{A, D, E\}. \tag{3.21}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 respectively) are:

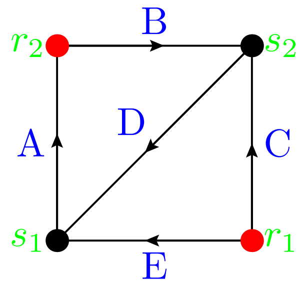
$$\mathcal{B}(M_1) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \{A, B, D\}\} \tag{3.22}$$
$$\mathcal{B}(M_2) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \{A, B, C\}, \{B, C, E\}, \{B, D, C\}, \{B, D, E\}, \{A, D, E\}\}. \tag{3.23}$$

So we have that:

$$\mathcal{B}(J_1) \cap \mathcal{B}(J_2) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}, \tag{3.24}$$

which are precisely the branching systems that we found by brute force!

3.2.2 2 roots



There is only one branching system:

$$\mathscr{B} = \{E, B, D, C\}. \tag{3.25}$$

We also have that:

$$J_1 = P(\mathscr{B}) \tag{3.26}$$
$$J_2 = P\{A, B, C, D, E\} \tag{3.27}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 respectively) are:

$$\mathcal{B}(M_1) = \{\mathscr{B}\} \tag{3.28}$$
$$B(M_2) = \{\{A, B, C, D, E\}\}. \tag{3.29}$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \tag{3.30}$$

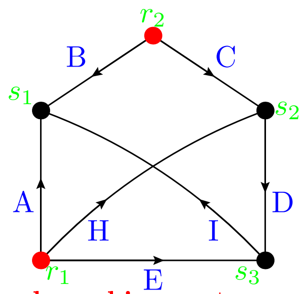
This is not what we want, but if we remove A (which is going into a root), we get:

$$J_1 = \{P(\mathscr{B})\} \tag{3.31}$$
$$J_2 = J_1 \tag{3.32}$$
$$\mathcal{B}(M_1) = \mathscr{B} \tag{3.33}$$
$$\mathcal{B}(M_2) = \mathcal{B}(M_1) \tag{3.34}$$
$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \mathscr{B}, \tag{3.35}$$

which are precisely the branching systems that we listed at the beginning!

3.3 5-nodes:

3.3.1 2 roots



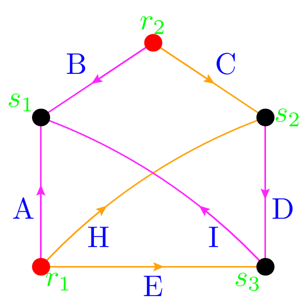
The branching systems are:

$$\mathscr{B}_1 = \{A, H, E; B, C, D\}, \tag{3.36}$$
$$\mathscr{B}_2 = \{E, I, H; B, C, D\}, \tag{3.37}$$
$$\mathscr{B}_3 = \{A, H, E; C, D, I\}. \tag{3.38}$$

We also have that:

$$J_1 = P(\mathscr{B}_1) \cup P(\mathscr{B}_2) \cup P(\mathscr{B}_3) \tag{3.39}$$
$$J_2 = P\{B, A, I, D; C, H, E\}. \tag{3.40}$$

J_2 contains all edges, and we know that this can be partitioned into $k = 2$ forests because we have partitioned it into the pink and orange forests below:



We also know that a branching system can't include all of A, I and B at the same time since the arborescences aren't supposed to have cycles. Also J_1 doesn't allow having $3 > k = 2$ incoming edges, but if we include A, I and B at the same time, then vertex s_1 would have 3 incoming edges.

The bases of M_1 and M_2 (which are associated with J_1 and J_2 respectively) are:

$$\mathcal{B}(M_1) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\} \tag{3.41}$$
$$\mathcal{B}(M_2) = \{\{B, A, I, D; C, H, E\}\} \tag{3.42}$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \tag{3.43}$$

This is not what we want, since we would like the intersection to contain the branching systems that we listed at the beginning.

This problem can be fixed by eliminating members of J_2 that have more than $k(n-k)$ edges, where $n = |V|$. If we simply remove the 7-edge member of J_2 in Eq. 3.40, we will get that Eq. 3.43 is the set of all desired branching systems. This also works for 7 more graphs that I'm working on adding to this document.