1 Terminology

Directed tree: A directed graph that would be a tree if we ingored the directions of the edges.

Arborescence: A directed tree with a "root" such that every node of it has a unique path to it from the root. There's always exactly one root.

Spanning subgraph of $G = (V, E)$: Just a subgraph (V, B) of G such that $B \subseteq E$.

Branching system of $G = (V, E)$ with k specified root nodes: A collection of edge-disjoint arborescences rooted at the k root nodes, with each arborescence being a spanning subgraph of $(V \backslash R, E)$ with R containing all roots that are not part of the arborescence. By "edge-disjoint" we mean that none of these arborescences share any edges. Also, every node has at most k incoming edges (at most one from each arborescence).

Optimal branching system of a weighted digraph with k specified root nodes: the branching system with the lowest total weight.

2 Algorithm

To find an OBS we need to find a min-cost set of edges which is both a basis of matroid M_1 and of M_2:

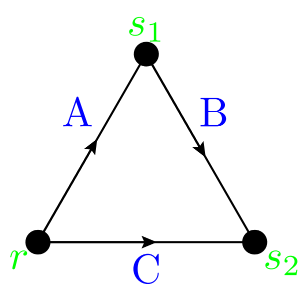
- M_1=(E, J_1) in which $j \in J_1$ has no edges entering any root nodes, and at most k edges entering each other node of the graph.
- M_2=(E, J_2) in which $j \in J_2$ can be partitioned into at most k **forests**.

If $\mathcal{B}(M_1)$ is the set of all bases of M_1 and $\mathcal{B}(M_2)$ is the set of all bases of M_2, then the optimal branching system will be the least-cost element in $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$.

3 Examples:

3.1 3-nodes

3.1.1 1 root



The branching systems are:

$$\mathscr{B}_1 = \{A, C\} \tag{3.1}$$
$$\mathscr{B}_2 = \{A, B\} \tag{3.2}$$

We also have:

$$J_1 = \{P(\mathscr{B}_1) \cup P(\mathscr{B}_2)\} \tag{3.3}$$
$$= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}\} \tag{3.4}$$
$$\tag{3.5}$$
$$J_2 = \{P(\mathscr{B}_1) \cup P(\mathscr{B}_2) \cup P(\{B, C\})\} \tag{3.6}$$
$$= \{\emptyset, \{A\}, \{C\}, \{A, C\}, \{B\}, \{A, B\}, \{B, C\}\}. \tag{3.7}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 resepctively) are:

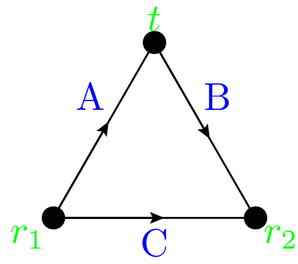
$$\mathcal{B}(M_1) = \{\mathscr{B}_1, \mathscr{B}_2\} \tag{3.8}$$
$$\mathcal{B}(M_2) = \{\mathscr{B}_1, \mathscr{B}_2, \{B, C\}\} \tag{3.9}$$

So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\mathscr{B}_1, \mathscr{B}_2\}, \tag{3.10}$$

which are precisely the branching systems that we found by brute force!

3.1.2 2 roots



There's no branching systems because it's impossible for an arborescence rooted at r_2 to be a spanning subgraph of the original graph (no arborescence rooted at r_2 can contain all of the original graph's vertices).

We also have:

$$J_1 = \{\emptyset, \{A\}\} \tag{3.11}$$
$$J_2 = P(\{A, B; C\}) \tag{3.12}$$
$$= \{A, B, C, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}. \tag{3.13}$$

The bases of M_1 and M_2 (which are associated with J_1 and J_2 resepctively) are:

$$\mathcal{B}(M_1) = \{\{A\}\} \tag{3.14}$$
$$\mathcal{B}(M_2) = \{\{A, B, C\}\}. \tag{3.15}$$

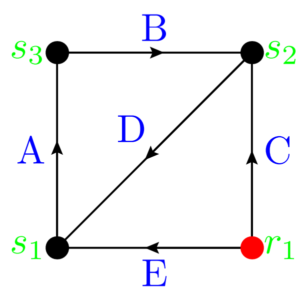
So we have that:

$$\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \emptyset. \tag{3.16}$$

meaning that no branching systems exist, which is exactly what we said at the beginning!

3.2 4-nodes:

3.2.1 1 root



The branching systems are:

$$\mathscr{B}_1 = \{E, A, B\}, \tag{3.17}$$
$$\mathscr{B}_2 = \{C, E, A\}, \tag{3.18}$$
$$\mathscr{B}_3 = \{C, D, A\}. \tag{3.19}$$

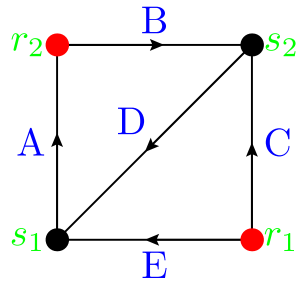
We also have that:

$$b(J_1) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \{A, B, D\}\} \tag{3.20}$$
$$b(J_2) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \{A, B, C\}, \{B, C, E\}, \{B, D, C\}, \{B, D, E\}, \{A, D, E\}\}. \tag{3.21}$$

So we find that:

$$b(J_1) \cap b(J_2) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}. \tag{3.22}$$

3.2.2 2 roots



There is only one branching system:

$$\mathscr{B} = \{E, B, D, C\}. \tag{3.23}$$

We also have that:

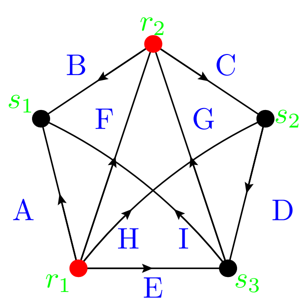
$$b(J_1) = \{\mathscr{B}\} \tag{3.24}$$
$$b(J_2) = \{A, B, C, D, E\}, \tag{3.25}$$
$$b(J_1) \cap b(J_2) = \emptyset \tag{3.26}$$

So we need to remove A (which is going into a root). By doing that we have:

$$b(J_2) = b(J_1), \tag{3.27}$$
$$b(J_2) \cap b(J_2) = \mathscr{B}. \tag{3.28}$$

3.3 5-nodes:

3.3.1 2 roots



The branching systems are:

$$\mathscr{B}_1 = \{A, H, E; B, C, D\}, \tag{3.29}$$
$$\mathscr{B}_2 = \{E, I, H; B, C, D\}, \tag{3.30}$$
$$\mathscr{B}_3 = \{A, H, E; C, D, I\}. \tag{3.31}$$

$$b(J_1) = \{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_2\} \tag{3.32}$$
$$b(J_2) = \{B, A, I, D; C, H, E\} \tag{3.33}$$

No overlap between b(J_1) and b(J_2)