# All 4-variable functions can be perfectly quadratized with only 1 auxiliary variable

Nike Dattani[1] and Hou Tin Chau[2]

[1]*Harvard-Smithsonian Center for Astrophysics, USA*
[2]*Cambridge University, Department of Mathematics, UK*

We prove that any function whose input is 4 binary variables and whose output is a real number, is perfectly equivalent to a function whose input is 5 binary variables and is minimized over the new variable. Our proof is constructive, so we provide quadratic functions that quadratize any 4-variable function, but there exists 7 different classes of 4-variable functions that each have their own 5-variable quadratization formula. Since we provide 'perfect' quadratizations, we can apply these formulas to any 4-variable subset of an $n$-variable function even if $n \gg 4$.

## I. INTRODUCTION

Many problems can be solved by minimizing a real-valued degree-$k$ function of binary variables with $k > 2$. Some examples include image de-blurring (where typically $k = 4$ but in general we can have $k = m^2$ with $m \geq 2$ being the length in pixels of the square-shaped mask) $> 0$[1, 2], integer factoring (where typically $k = 4$) [3–10], and determining whether or not a number $N$ is an $m$-color Ramsey number (where $k = \frac{mN(N-1)}{2}$) [11–13].

Solving such discrete optimization problems with $k > 2$ can be very difficult, and more algorithms have been developed for the $k = 2$ case (such as the algorithm known as "QPBO" and extensions of it [14]) than for the $k > 2$ case. Fortunately it is possible to turn any $k$-degree binary optimization problem into a 2-degree binary optimization problem, by a transformation called "quadratization" [15].

Quadratization methods exist which can turn an $n$-variable degree-$k$ problem into an $n$-variable quadratic problem (i.e. the number of variables does not change) [8, 13, 16, 17], but not every case can be quadratized without adding some auxiliary variables (so the number of variables in the quadratic problem is usually much more than in the original degree-$k$ problem). Coming up with better quadratizations (for example with fewer auxiliary variables) has been a very active area of research recently: The first quadratization method was published in 1975 [18], and some subsequent quadratization methods were published in 2004 [19], 2005 [20], and 2011 [1, 2, 21], but the rest of the methods were published in the last 5 years (from 2014-2018) [8, 13, 22–30].

In the most recent of these papers [29, 30], a remarkable discovery was made, that some entire functions (no matter how many terms and how many variables they contain) can be quadratized with only $\log_2(k) - 1$ auxiliary variables. For many functions this can still be very costly though, because for example, quadratizing each monomial term with $\log_2(k) - 1$ auxiliary variables can mean the new optimization problem has far more variables than the original one.

This motivated us to look for quadratizations that are compact, but also applicable to many real-world problems. The result of this study is the theorem described in the title of this paper, and explained in more detail in the section below.

## II. RESULTS

**Theorem 1: All 4-variable functions can be quadratized perfectly with only 1-auxiliary variable.**

By "perfect" quadratization we mean all $2^4$ outcomes of the 4-variable function are exactly preserved when minimizing over the auxiliary variable in the 5-variable quadratic function. Therefore any 4-variable subset of an $n$-variable problem can be quadratized with only 1-auxiliary.

We prove the theorem by providing an explicit quadratization for various different cases, of the following function of binary variables $b_i \in \{0, 1\}$:

$$\alpha_{1234}b_1b_2b_3b_4 + \alpha_{123}b_1b_2b_3 + \alpha_{124}b_1b_2b_4 + \alpha_{134}b_1b_3b_4 + \alpha_{234}b_2b_3b_4. \tag{1}$$

First we reduce to the case where $\alpha_{1234} \geq 0$, and $\alpha_{123} \leq \alpha_{124} \leq \alpha_{134} \leq \alpha_{234}$ and $-\frac{\alpha_{1234}}{2} \leq \alpha_{124} \leq \alpha_{134} \leq \alpha_{234}$. Then we will prove 7 Lemmas which cover all such cases.

Table I. All possible cases of 4-variable functions.

| $\alpha_{ijk} \le -\alpha_{1234}$ | $-\alpha_{1234} \le \alpha_{ijk} \le -\frac{\alpha_{1234}}{2}$ | $-\frac{\alpha_{1234}}{2} \le \alpha_{ijk} \le 0$ | $0 \le \alpha_{ijk}$ | |
|---|---|---|---|---|
| | | | $\alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234}$ | Lemma 1 |
| | | $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | Lemma 1 |
| | $\alpha_{123}$ | | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | Lemma 8 |
| $\alpha_{123}$ | | | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | Lemma 2 |
| | | $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | Lemma 1 |
| | $\alpha_{123}$ | $\alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | Lemma 3, X (flip $b_2, b_4$) |
| $\alpha_{123}$ | | $\alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | Lemma 8, X |
| | $\alpha_{123}, \alpha_{124}$ | | $\alpha_{134}, \alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}$ | $\alpha_{124}$ | | $\alpha_{134}, \alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}, \alpha_{124}$ | | | $\alpha_{134}, \alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| | | $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | Lemma 1 |
| | $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | Lemma 7, X (flip $b_1, b_4$) |
| $\alpha_{123}$ | | $\alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | Lemma 3 |
| | $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}$ | $\alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}$ | $\alpha_{124}$ | $\alpha_{134}$ | $\alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}, \alpha_{124}$ | | $\alpha_{134}$ | $\alpha_{234}$ | Lemma 1, X (flip $b_3, b_4$) |
| | $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | | $\alpha_{234}$ | Lemma 7, X (flip all) |
| $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}$ | | $\alpha_{234}$ | Lemma 3, X (flip $b_2, b_3$) |
| $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}$ | | $\alpha_{234}$ | Lemma 8, X (flip $b_3, b_4$) |
| $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | | | $\alpha_{234}$ | Lemma 2, (flip $b_2, b_3, b_4$) |
| | | $\alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234}$ | | Lemma 1 |
| | $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | | Lemma 6 |
| $\alpha_{123}$ | | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | | Lemma 7 |
| | $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}$ | $\alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | | Lemma 1, X (flip $b_3, b_4$) |
| $\alpha_{123}, \alpha_{124}$ | | $\alpha_{134}, \alpha_{234}$ | | Lemma 1, X (flip $b_3, b_4$) |
| | $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | | Lemma 6, X (flip $b_3, b_4$) |
| $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | | Lemma 7, X (flip $b_2, b_3$) |
| $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}$ | $\alpha_{234}$ | | Lemma 3, X (flip $b_2, b_3$) |
| $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | | $\alpha_{234}$ | | Lemma 8, X (flip all) |
| | $\alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234}$ | | | Lemma 1, X (flip all) |
| $\alpha_{123}$ | $\alpha_{124}, \alpha_{134}, \alpha_{234}$ | | | Lemma 1, X (flip all) |
| $\alpha_{123}, \alpha_{124}$ | $\alpha_{134}, \alpha_{234}$ | | | Lemma 1, X (flip all) |
| $\alpha_{123}, \alpha_{124}, \alpha_{134}$ | $\alpha_{234}$ | | | Lemma 1, X (flip all) |
| $\alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{234}$ | | | | Lemma 1, X (flip all) |

**Lemma X: If a function of the form in Eq. 1 can be perfectly quadratized, then the function obtained by flipping two bits in the original function, can also be perfectly quadratized. Here is how:**

$$\alpha_{1234}\bar{b}_1\bar{b}_2b_3b_4+\alpha_{123}\bar{b}_1\bar{b}_2b_3+\alpha_{124}\bar{b}_1\bar{b}_2b_4+\alpha_{134}\bar{b}_1b_3b_4+\alpha_{234}\bar{b}_2b_3b_4 \tag{2}$$

$$\begin{aligned}
&=\alpha_{1234}\left(b_3b_4-b_2b_3b_4-b_1b_3b_4+b_1b_2b_3b_4\right)+\alpha_{123}\left(b_3-b_1b_3-b_2b_3+b_1b_2b_3\right)+\alpha_{124}\left(b_4-b_1b_4-b_2b_4+b_1b_2b_4\right) && (3)\\
&+\alpha_{134}\left(b_3b_4-b_1b_3b_4\right)+\alpha_{234}\left(b_3b_4-b_2b_3b_4\right) && (4)\\
&=\alpha_{1234}b_1b_2b_3b_4+\alpha_{123}b_1b_2b_3+\alpha_{124}b_1b_2b_4+\left(-\alpha_{134}-\alpha_{1234}\right)b_1b_3b_4+\left(-\alpha_{234}-\alpha_{1234}\right)b_2b_3b_4 && (5)\\
&+\alpha_{1234}b_3b_4+\alpha_{123}\left(b_3-b_1b_3-b_2b_3\right)+\alpha_{124}\left(b_4-b_1b_4-b_2b_4\right)+\left(\alpha_{134}+\alpha_{234}\right)b_3b_4 && (6)\\
&=\alpha_{1234}b_1b_2b_3b_4+\alpha_{123}b_1b_2b_3+\alpha_{124}b_1b_2b_4+\bar{\alpha}_{134}b_1b_3b_4+\bar{\alpha}_{234}b_2b_3b_4+f_{\text{quadratic}}\left(b_1,b_2,b_3,b_4\right) && (7)
\end{aligned}$$

where we have defined:

$$\begin{aligned}
\bar{\alpha}_{134}&\equiv-\alpha_{134}-\alpha_{1234}, && (8)\\
\bar{\alpha}_{234}&\equiv-\alpha_{234}-\alpha_{1234}, && (9)\\
f_{\text{quadratic}}\left(b_1,b_2,b_3,b_4\right)&\equiv\alpha_{1234}b_3b_4+\alpha_{123}\left(b_3-b_1b_3-b_2b_3\right)+\alpha_{124}\left(b_4-b_1b_4-b_2b_4\right)+\left(\alpha_{134}+\alpha_{234}\right)b_3b_4. && (10)
\end{aligned}$$

---

**Lemma 1: If $\alpha_{1234}\geq 0$, $\alpha_{ijk}\geq-\frac{\alpha_{1234}}{2}$ , then Eq. 1 is perfectly quadratized by:**

$$\left(3\alpha_{1234}+\sum_{ijk}\alpha_{ijk}\right)b_a+\alpha_{1234}\sum_{ij}b_ib_j+\sum_{ij}\sum_{k\notin ij}\alpha_{ijk}b_ib_j-\sum_i\left(2\alpha_{1234}+\sum_{jk,i\neq jk}\alpha_{ijk}\right)b_ib_a. \tag{11}$$

---

**Lemma 2: If $\alpha_{1234}\leq 0$ and $\alpha_{ijk}\leq 0$, then Eq. 1 is perfectly quadratized by:**

$$\left(\alpha_{1234}\left(\sum_i b_i-3\right)+\sum_{ijk}\alpha_{ijk}\left(\sum_{l\in ijk}b_l-2\right)\right)b_a. \tag{12}$$

---

**Lemma 3: If $\alpha_{123}\leq-\alpha_{1234}$ and $-\frac{\alpha_{1234}}{2}\leq\alpha_{124}\leq\alpha_{134}\leq 0\leq\alpha_{234}$, then Eq. 1 is perfectly quadratized by:**

$$-\sum_i\alpha_{12i}b_i+\sum_{\substack{i=3,4\\j=1,2}}\alpha_{12i}b_ib_j+\left(\alpha_{134}+\alpha_{234}+\alpha_{1234}\right)b_3b_4+\sum_i\alpha_{12i}\left(b_i-b_1-b_2\right)b_a+\sum_i\alpha_{i34}\left(1-b_3+b_i-b_4\right)b_a+\alpha_{1234}(1-b_3-b_4). \tag{13}$$

---

**Lemma 6: If $-\alpha_{1234}\leq\alpha_{123}\leq-\frac{\alpha_{1234}}{2}\leq\alpha_{124}\leq\alpha_{134}\leq\alpha_{234}\leq 0$ and $\alpha_{123}+\alpha_{124}\geq-\alpha_{1234}$, then Eq. 1 is perfectly quadratized by:**

$$\sum_{ijk}\alpha_{ijk}\sum_{lm\subset ijk}b_lb_m+\alpha_{1234}\sum_{ij}b_ib_j+b_a\left(\sum_{ijk}\alpha_{ijk}\left(1-\sum_{l\in ijk}b_l\right)+\alpha_{1234}\left(3-2\sum_i b_i\right)\right). \tag{14}$$

---

**Lemma 7: If $\alpha_{123}\leq-\frac{\alpha_{1234}}{2}\leq\alpha_{124}\leq\alpha_{134}\leq\alpha_{234}\leq 0$ and $\alpha_{123}+\alpha_{124}\leq-\alpha_{1234}$, then Eq. 1 is perfectly quadratized by:**

$$\begin{aligned}
&\sum_{i=3,4}\alpha_{12i}\left(-b_1-b_2+b_1b_i+b_2b_i\right)+\sum_{i=1,2}\alpha_{i34}\left(1+b_i-b_3-b_4+b_3b_4\right)+\alpha_{1234}\left(1-b_3-b_4+b_3b_4\right) && (15)\\
&+b_a\left(\sum_{i=3,4}\alpha_{12i}\left(b_1+b_2-b_i\right)+\sum_{i=1,2}\alpha_{i34}\left(-b_i+b_3+b_4-1\right)+\alpha_{1234}\left(b_3+b_4-1\right)\right) && (16)
\end{aligned}$$

---

**Lemma 8: If $-\alpha_{1234}\leq\alpha_{123}\leq-\frac{\alpha_{1234}}{2}$ and $0\leq\alpha_{234}\leq\alpha_{134}\leq\alpha_{124}$, then Eq. 1 is perfectly quadratized by 11**

---

[1] H. Ishikawa, IEEE Transactions on Pattern Analysis and Machine Intelligence **33**, 1234 (2011).
[2] A. Fix, A. Gruber, E. Boros, and R. Zabih, in *2011 International Conference on Computer Vision* (IEEE, 2011) pp. 1020–1027.
[3] N. S. Dattani and N. Bryans, (2014), arXiv:1411.6758.
[4] C. J. C. Burges, Microsoft Research **MSR-TR-200** (2002).
[5] X. Peng, Z. Liao, N. Xu, G. Qin, X. Zhou, D. Suter, and J. Du, Physical Review Letters **101**, 220405 (2008).
[6] G. Schaller and R. Schützhold, Quantum Information & Computation **10**, 109 (2010).
[7] N. Xu, J. Zhu, D. Lu, X. Zhou, X. Peng, and J. Du, Physical Review Letters **108**, 130501 (2012).
[8] R. Tanburn, E. Okada, and N. Dattani, *Reducing multi-qubit interactions in adiabatic quantum computation without adding auxiliary qubits. Part 1: The "deduc-reduc" method and its application to quantum factorization of numbers* (2015) arXiv:1508.04816.
[9] O. Lunt, R. Tanburn, E. Okada, and N. S. Dattani, Physical Review A (in preparation) (2015).
[10] Z. Li, N. S. Dattani, X. Chen, X. Liu, H. Wang, R. Tanburn, H. Chen, X. Peng, and J. Du, http://arxiv.org/abs/1706.08061 (2017), arXiv:1706.08061.
[11] F. Gaitan and L. Clark, (2012), arXiv:arXiv:1103.1345v3.
[12] Z. Bian, F. Chudak, W. G. Macready, L. Clark, and F. Gaitan, Physical Review Letters **111**, 130505 (2013).
[13] E. Okada, R. Tanburn, and N. S. Dattani, *Reducing multi-qubit interactions in adiabatic quantum computation without adding auxiliary qubits. Part 2: The "split-reduc" method and its application to quantum determination of Ramsey numbers* (2015) arXiv:1508.07190.
[14] C. Rother, V. Kolmogorov, V. Lempitsky, and M. Szummer, in *2007 IEEE Conference on Computer Vision and Pattern Recognition* (IEEE, 2007) pp. 1–8.
[15] N. Dattani, *Quadratization in discrete optimization and quantum mechanics* (2019) arXiv:1901.04405.
[16] H. Ishikawa, in *2014 IEEE Conference on Computer Vision and Pattern Recognition* (IEEE, 2014) pp. 1362–1369.
[17] R. Dridi and H. Alghassi, Scientific Reports **7**, 43048 (2017).
[18] I. G. Rosenberg, Cahiers du Centre d'Etudes de Recherche Operationnelle **17**, 71 (1975).
[19] V. Kolmogorov and R. Zabih, IEEE Transactions on Pattern Analysis and Machine Intelligence **26**, 147 (2004).
[20] D. Freedman and P. Drineas, in *2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05)*, Vol. 2 (IEEE, 2005) pp. 939–946.
[21] A. C. Gallagher, D. Batra, and D. Parikh, in *CVPR 2011* (IEEE, 2011) pp. 1857–1864.
[22] M. Anthony, E. Boros, Y. Crama, and A. Gruber, (2014), arXiv:1404.6535.
[23] M. Anthony, E. Boros, Y. Crama, and A. Gruber, (2015).
[24] M. Anthony, E. Boros, Y. Crama, and A. Gruber, Discrete Applied Mathematics **203**, 1 (2016).
[25] M. Leib, P. Zoller, and W. Lechner, Quantum Science and Technology **1**, 15008 (2016).
[26] A. Rocchetto, S. C. Benjamin, and Y. Li, Science Advances **2** (2016), 10.1126/sciadv.1601246.
[27] M. Anthony, E. Boros, Y. Crama, and A. Gruber, Mathematical Programming **162**, 115 (2017).
[28] N. Chancellor, S. Zohren, and P. A. Warburton, npj Quantum Information **3**, 21 (2017).
[29] E. Boros, Y. Crama, and E. Rodríguez-Heck, *Quadratizations of symmetric pseudo-Boolean functions: sub-linear bounds on the number of auxiliary variables*, Tech. Rep. (2018).
[30] E. Boros, Y. Crama, and E. Rodriguez Heck, *Compact quadratizations for pseudo-Boolean functions* (2018).