All 4-variable functions can be perfectly quadratized with only 1 auxiliary variable

Nike Dattani[1] and Hou Tin Chau[2]

[1] *Harvard-Smithsonian Center for Astrophysics, USA*
[2] *Cambridge University, Department of Mathematics, UK*

We prove that any function whose input is 4 binary variables and whose output is a real number, is perfectly equivalent to a function whose input is 5 binary variables and is minimized over the new variable. Our proof is constructive, so we provide quadratic functions that quadratize any 4-variable function, but there exists 8 different classes of 4-variable functions that each have their own 5-variable quadratization formula. Since we provide 'perfect' quadratizations, we can apply these formulas to any 4-variable subset of an n-variable function even if $n \gg 4$.

I. INTRODUCTION

Many problems can be solved by minimizing a real-valued degree-k function of binary variables with $k > 2$. Some examples include image de-blurring (where typically $k = 4$ but in general we can have $k = m^2$ with $m \geq 2$ being the length in pixels of the square-shaped mask) [1, 2], integer factoring (where typically $k = 4$) [3–10], and determining whether or not a number N is an m-color Ramsey number (where $k = \frac{mN(N-1)}{2}$) [11–13].

Solving such discrete optimization problems with $k > 2$ can be very difficult, and more algorithms have been developed for the $k = 2$ case (such as the algorithm known as "QPBO" and extensions of it [14]) than for the $k > 2$ case. Fortunately it is possible to turn any k-degree binary optimization problem into a 2-degree binary optimization problem, by a transformation called "quadratization" [15].

Quadratization methods exist which can turn an n-variable degree-k problem into an n-variable quadratic problem (i.e. the number of variables does not change) [8, 13, 16, 17], but not every case can be quadratized without adding some auxiliary variables (so the number of variables in the quadratic problem is usually much more than in the original degree-k problem). Coming up with better quadratizations (for example with fewer auxiliary variables) has been a very active area of research recently: The first quadratization method was published in 1975 [18], and some subsequent quadratization methods were published in 2004 [19], 2005 [20], and 2011 [1, 2, 21], but the rest of the methods were published in the last 5 years (from 2014-2018) [8, 13, 22–30].

In the most recent of these papers [29, 30], a remarkable discovery was made, that some entire functions (no matter how many terms and how many variables they contain) can be quadratized with only $\log_2(k) - 1$ auxiliary variables. Unfortunately those functions are either extremely obscure functions that would rarely come up in a real-world problem, or they are just monomials (meaning that for a real-world problem containing many terms, each of these terms would be quadratized separately, with $\log_2(k) - 1$ new auxiliary variables being added each time, and adding up to a potentially untractable number of total variables in the quadratic optimization problem).

This motivated us to look for quadratizations that are compact, but also applicable to many real-world problems. The result of this study is the theorem described in the title of this paper, and explained in more detail in the section below.

II. RESULTS

Theorem I: All 4-variable functions can be quadratized perfectly with only 1-auxiliary variable.

By "perfect" quadratization we mean all 2^4 outcomes of the 4-variable function are exactly preserved when minimizing over the auxiliary variable in the 5-variable quadratic function. Therefore any 4-variable subset of an n-variable problem can be quadratized with only 1-auxiliary.

We prove the theorem by providing an explicit quadratization for various different cases, of the following function of binary variables $b_i \in \{0,1\}$:

$$\alpha_{1234}b_1b_2b_3b_4 + \alpha_{123}b_1b_2b_3 + \alpha_{234}b_2b_3b_4 + \alpha_{134}b_1b_3b_4 + \alpha_{124}b_1b_2b_4. \tag{1}$$

Case 1: $\alpha_i \le 0$, $\forall i$

$$b_a\left(\alpha_{1234}\left(\sum_i b_i - 3\right) + \sum_i \alpha_i\left(\sum_j b_j - 2\right)\right) \tag{2}$$

Case 2: $\alpha_{1234} \ge 0$

Case 2.1: $-\alpha_{1234} \ge \alpha_{123}$

Case 2.1.1: $-\alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le \alpha_{124} \le 0$

$$b_a\left(\alpha_{123}\left(-b_1+b_2+b_3\right)+\alpha_{234}\left(b_2+b_3-b_4\right)+\alpha_{134}\left(b_1-b_3+b_4-1\right)+\alpha_{124}\left(b_1-b_2+b_4-1\right)+\alpha_{1234}\left(b_1+b_4-1\right)\right) \tag{3}$$
$$+\alpha_{123}\left(-b_2-b_3+b_1b_2+b_1b_3\right)+\alpha_{234}\left(-b_2-b_3+b_2b_4+b_3b_4\right) \tag{4}$$
$$+\alpha_{134}\left(1-b_1+b_3-b_4+b_1b_4\right)+\alpha_{124}\left(1-b_1+b_2-b_4+b_1b_4\right)+\alpha_{1234}\left(1-b_1-b_4+b_1b_4\right) \tag{5}$$

Case 2.1.2: $-\alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le 0 \le \alpha_{124}$

$$b_a\left(\alpha_{123}\left(b_1-b_2-b_3\right)+\alpha_{234}\left(-b_2-b_3+b_4\right)+\alpha_{134}\left(1-b_1+b_3-b_4\right)+\alpha_{124}\left(1-b_1+b_2-b_4\right)+\alpha_{1234}\left(1-b_1-b_4\right)\right) \tag{6}$$
$$+\alpha_{123}\left(-b_1+b_1b_2+b_1b_3\right)+\alpha_{234}\left(-b_4+b_2b_4+b_3b_4\right)+\left(\alpha_{134}+\alpha_{124}+\alpha_{1234}\right)b_1b_4 \tag{7}$$

Case 2.2: $-\frac{\alpha_{1234}}{2} \ge \alpha_{123}$

Case 2.2.1: $\alpha_{1234}+\alpha_{123}+\alpha_{234} \le 0$

Case 2.2.1.1: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le \alpha_{124} \le 0$

$$b_a\left(\alpha_{123}\left(-b_1+b_2+b_3\right)+\alpha_{234}\left(b_2+b_3-b_4\right)+\alpha_{134}\left(b_1-b_3+b_4-1\right)+\alpha_{124}\left(b_1-b_2+b_4-1\right)+\alpha_{1234}\left(b_1+b_4-1\right)\right) \tag{8}$$
$$+\alpha_{123}\left(-b_2-b_3+b_1b_2+b_1b_3\right)+\alpha_{234}\left(-b_2-b_3+b_2b_4+b_3b_4\right) \tag{9}$$
$$+\alpha_{134}\left(1-b_1+b_3-b_4+b_1b_4\right)+\alpha_{124}\left(1-b_1+b_2-b_4+b_1b_4\right)+\alpha_{1234}\left(1-b_1-b_4+b_1b_4\right) \tag{10}$$

Case 2.2.1.2: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le 0 \le \alpha_{124}$

Case 2.2.1.3: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le 0 \le \alpha_{134} \le \alpha_{124}$

Case 2.2.2: $\alpha_{1234}+\alpha_{123}+\alpha_{234} \ge 0$

Case 2.2.2.1: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le \alpha_{124} \le 0$

Case 2.2.2.2: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le 0 \le \alpha_{124}$

Case 2.2.2.3: $-\alpha_{1234} \le \alpha_{123} \le \alpha_{1234}/2 \le \alpha_{234} \le 0 \le \alpha_{134} \le \alpha_{124}$

Case 2.3: $-\frac{\alpha_{1234}}{2} \le \alpha_{123}$

Case 2.3.1: $-\alpha_{1234}/2 \le \alpha_{234} \le \alpha_{134} \le \alpha_{124} \le 0$

Case 2.3.2: $-\alpha_{1234}/2 \leq \alpha_{234} \leq \alpha_{134} \leq 0 \leq \alpha_{124}$

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