Quadratic Envelopes: A method that can quadratize with a negative number of auxiliary variables

Nike Dattani[1] and Andreas Soteriou[2]

[1]*Harvard-Smithsonian Center for Astrophysics, USA*
[2]*Surrey University, Department of Mathematics, UK*

We present a quadratization method that, for all of the many cases considered, provides quadratizations that require fewer auxiliary variables, less connectivity between the variables, and *much* smaller coefficients than any previously known quadratization method. The quadratizations presented in this work, are also obtainable much more quickly (in terms of computation runtime) than quadratizatons for functions of similar complexity have traditionally been in the past. The idea is that many *very* simple quadratic functions might *almost* reproduce a degree-k function perfectly, but each of these *almost* perfect quadratic functions can compensate for each others imperfections, hence leading to a perfect quadratization when all such quadratic functions are considered collectively. This collection is called a *quadratic envelope.* We are able to quadratize positive monomials with quadratic functions that have exponentially smaller coefficients than in [Boros *et al.* (2018)], while also having far less connectivity between the variables of the quadratics, in addition to requiring fewer auxiliary variables. We will also show examples in which an envelope of quadratic functions, each with *fewer* total variables than the original degree-k function, collectively can be made to *exactly* reproduce the degree-k function.

I. INTRODUCTION

Any real-valued function of binary variables can be turned into a quadratic one that maintains all the desired properties of the original degree-k function. We call this a 'quadratization' of the degree-k function. The first quadratization technique of which we are aware, is Rosenberg's substitution method from 1975 [1], which for quadratizing a single positive monomial of degree-k, would need $k-2$ auxiliary variables (meaning that the quadratic function obtained after the quadratization, would involve $k-2$ more auxiliary variables than the original degree-k monomial). This was cut by roughly one half in 2011 by Ishikawa, who introduced a quadratization for a positive degree-k monomial, which needed only $\frac{k-1}{2}$ auxiliary variables (rounded down if k were even). In 2018 several new quadratizations were introduced for positive degree-k monomials: one required only $k/4$ auxiliary variables, one required only $\log k$, and while it might be surprising that an improvement can be made over $\log k$, a quadratization was finally shown which required only $\log{(k/2)}$ (all three of these were presented in [2, 3], and the numbers are rounded *up* to the nearest integer). While it was astonishing that any positive monomial can be quadratized with only $\log{(k/2)}$ auxiliary variables, the resulting quadratic function would have very large coefficients and all possible quadratic terms over all original and auxiliary variables; and since large coefficients and a large number of quadratic terms, are two of the factors which most severely increase the cost when optimizing a quadratic discrete function, one may hesitate to use such quadratizations in practice.

Quadratization is also possible without the addition of any 'auxiliary variables' [4–8], and while these techniques should always be attempted, they all have limitations [8].

II. RESULTS

2-runs/0-aux case:

$$b_1b_2b_3b_4 = \min{(b_1b_2, b_3b_4)} \tag{1}$$

3-runs/0-aux case to be applied in Computer Vision and LHZ lattice:

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min{(b_2b_3 + b_3b_6, b_1b_4 + b_4b_5, b_1b_2 + b_5b_6 - b_3 - b_4 + 2)} \tag{2}$$

Linearization:

$$b_1b_2b_3 \dots b_k = \min{(b_1, b_2, b_3, \dots, b_k)} \tag{3}$$

3-runs/0-aux case:

$$
\begin{aligned}
& b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 : && (k,n) = (4,6). \quad (4) \\
\longrightarrow\ & 3b_3b_4 + b_3b_5 + b_4b_5 - b_3 - b_4 - b_5 + 1 && (5) \\
\longrightarrow\ & b_1b_4 + b_3b_5 + b_4b_5 && (6) \\
\longrightarrow\ & b_1b_2 + b_2b_6 + b_3b_5 + b_5b_6 + b_2 - b_3 - b_4 - b_5 + 2 && (7)
\end{aligned}
$$

III. EXAMPLES

IV. DISCUSSION

V. ACKNOWLEDGMENTS

[1] I. G. Rosenberg, Cahiers du Centre d'Etudes de Recherche Operationnelle **17**, 71 (1975).
[2] E. Boros, Y. Crama, and E. Rodríguez-Heck, *Quadratizations of symmetric pseudo-Boolean functions: sub-linear bounds on the number of auxiliary variables*, Tech. Rep. (2018).
[3] E. Boros, Y. Crama, and E. Rodriguez Heck, *Compact quadratizations for pseudo-Boolean functions* (2018).
[4] H. Ishikawa, in *2014 IEEE Conference on Computer Vision and Pattern Recognition* (IEEE, 2014) pp. 1362–1369.
[5] R. Tanburn, E. Okada, and N. Dattani, (2015), arXiv:1508.04816.
[6] E. Okada, R. Tanburn, and N. S. Dattani, (2015), arXiv:1508.07190.
[7] R. Dridi and H. Alghassi, Scientific Reports **7**, 43048 (2017).
[8] N. Dattani, *Quadratization in discrete optimization and quantum mechanics* (2019) arXiv:1901.04405.