

Quadratic Envelopes: A method that can quadratize with a negative number of auxiliary variables

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We present a quadratization method that, for all of the many cases considered, provides quadratizations that require fewer auxiliary variables, less connectivity between the variables, and *much* smaller coefficients than any previously known quadratization method. The quadratizations presented in this work, are also obtainable much more quickly (in terms of computation runtime) than quadratizations for functions of similar complexity have traditionally been in the past. The idea is that many *very* simple quadratic functions might *almost* reproduce a degree- k function perfectly, but each of these *almost* perfect quadratic functions can compensate for each others imperfections, hence leading to a perfect quadratization when all such quadratic functions are considered collectively. This collection is called a *quadratic envelope*. We are able to quadratize positive monomials with quadratic functions that have exponentially smaller coefficients than in [Boros *et al.* (2018)], while also having far less connectivity between the variables of the quadratics, in addition to requiring fewer auxiliary variables. We will also show examples in which an envelope of quadratic functions, each with *fewer* total variables than the original degree- k function, collectively can be made to *exactly* reproduce the degree- k function.

I. INTRODUCTION

Any real-valued function of binary variables can be turned into a quadratic one that maintains all the desired properties of the original degree- k function. We call this a ‘quadratization’ of the degree- k function. The first quadratization technique of which we are aware, is Rosenberg’s substitution method from 1975 [1], which for quadratizing a single positive monomial of degree- k , would need $k - 2$ auxiliary variables (meaning that the quadratic function obtained after the quadratization, would involve $k - 2$ more auxiliary variables than the original degree- k monomial). This was cut by roughly one half in 2011 by Ishikawa, who introduced a quadratization for a positive degree- k monomial, which needed only $\frac{k-1}{2}$ auxiliary variables (rounded down if k were even). In 2018 several new quadratizations were introduced for positive degree- k monomials: one required only $k/4$ auxiliary variables, one required only $\log k$, and while it might be surprising that an improvement can be made over $\log k$, a quadratization was finally shown which required only $\log(k/2)$ (all three of these were presented in [2, 3], and the numbers are rounded *up* to the nearest integer). While it was astonishing that any positive monomial can be quadratized with only $\log(k/2)$ auxiliary variables, the resulting quadratic function would have very large coefficients and all possible quadratic terms over all original and auxiliary variables; and since large coefficients and a large number of quadratic terms, are two of the factors which most severely increase the cost when optimizing a quadratic discrete function, one may hesitate to use such quadratizations in practice.

Quadratization is also possible without the addition of any ‘auxiliary variables’ [4–8], and while these techniques should always be attempted, they in many cases might only reproduce the ground state of the degree- k function rather than the whole spectrum [8]. One of

these zero-auxiliary-variable methods that does reproduce the full spectrum, the Split-Reduc method [9], can be very powerful when quadratizing multiple terms that have common variables, but for the degree- k positive monomial alone, its application can be far more costly than the methods described in the previous paragraph (this will be discussed more later in the paper when we compare and contrast Split-Reduc, methods involving auxiliaries, and our new concept of quadratic envelopes).

In this work we introduce the concept of quadratic envelopes, which seem to be capable of ‘perfectly’ quadratizing functions more efficiently than any of the above methods (by ‘perfectly’ we mean, reproducing the whole spectrum and not just the ground state). Even for just preserving the ground state, quadratic envelopes are often the most efficient way to quadratize a function, and in other cases should likely always be considered in combination with other methods in order to achieve the most efficient quadratization possible.

II. A QUICK EXAMPLE

Let us review the idea behind most known quadratizations. Consider the equality:

$$-b_1b_2b_3 = \min_{b_a} (2b_a - b_1b_a - b_2b_a - b_3b_a), \quad (1)$$

where all b_i are either 0 or 1 and the subscript a in b_a indicates that this is an ‘auxiliary’ variable because it does not appear in the original cubic function on the left side. This quadratization was first published by Kolmogorov and Zabih in 2004 [10] and generalized by Freedman and Drineas in 2005 beyond cubics and to negative monomials of any degree k . For clarity we explicitly show why Eq. 1 is true by constructing the truth tables for the left and right sides:

b_1	b_2	b_3	$-b_1b_2b_3$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	-1

b_1	b_2	b_3	b_a	$2b_a - b_1b_a - b_2b_a - b_3b_a$	\min_{b_a}
0	0	0	0	0	0
0	0	0	1	0	0
0	0	1	0		0
0	0	1	1		0
0	1	0	0		0
0	1	0	1		0
0	1	1	0		0
0	1	1	1		0
1	0	0	0		0
1	0	0	1		0
1	0	1	0		0
1	0	1	1		0
1	1	0	0		0
1	1	0	1		0
1	1	1	0		-1
1	1	1	1	-1	-1

It is impossible to construct a single quadratic function without any auxiliary variables, that reproduces the spectrum as seen in the tables above. However, minimizing over the quadratic function b_1b_2 and the linear function b_3 , perfectly yields the full spectrum of $b_1b_2b_3$ without introducing any auxiliary variables:

b_1	b_2	b_3	b_1b_2	b_3	$\min(b_1b_2, b_3)$
0	0	0		0	0
0	0	1	0	1	0
0	1	0		0	0
0	1	1	0	1	0
1	0	0		0	0
1	0	1	0	1	0
1	1	0	1	0	0
1	1	1	1	1	1

We call the set $\{b_1b_2, b_3\}$ a *quadratic envelope* for $b_1b_2b_3$. At each point (b_1, b_2, b_3) , the minimum over all functions in the quadratic envelope is equal to the degree- k function $b_1b_2b_3$.

III. GENERALIZATIONS OF THE QUICK EXAMPLE

The simple example from the previous section, of a quadratic envelope reproducing $b_1b_2b_3$, can be generalized:

$$b_1b_2b_3 \dots b_k = \min(b_1b_2 \dots b_{k_1}, b_{k_1+1}b_{k_1+2} \dots b_{k_2}, \dots, b_{k_{n-1}+1}b_{k_{n-1}+2} \dots b_k), \quad (2)$$

where k_1, k_2, \dots, k_n can be chosen arbitrarily between 1 and k (if $k_1 > 2$ or if any $k_i > k_{i-1} + 2$ then it is no longer a quadratic envelope but an envelope of a higher degree). One interesting quadratic envelope which is an example of Eq. 2 is:

$$b_1b_2b_3 \dots b_k = \min(b_1, b_2, b_3, \dots, b_k), \quad (3)$$

which is also an example of a *linear envelope*.

IV. USAGE

2-runs/0-aux case:

$$b_1b_2b_3b_4 = \min(b_1b_2, b_3b_4) \quad (4)$$

3-runs/0-aux case to be applied in Computer Vision and LHZ lattice:

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min(b_2b_3 + b_3b_6, b_1b_4 + b_4b_5, b_1b_2 + b_5b_6 - b_3 - b_4 - b_5 + 1) \quad (5)$$

Linearization:

$$b_1b_2b_3 \dots b_k = \min(b_1, b_2, b_3, \dots, b_k) \quad (6)$$

3-runs/0-aux case:

$$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 : \quad (k, n) = (4, 6) \quad (7)$$

$$\longrightarrow 3b_3b_4 + b_3b_5 + b_4b_5 - b_3 - b_4 - b_5 + 1 \quad (8)$$

$$\longrightarrow b_1b_4 + b_3b_5 + b_4b_5 \quad (9)$$

$$\longrightarrow b_1b_2 + b_2b_6 + b_3b_5 + b_5b_6 + b_2 - b_3 - b_4 - b_5 + 2 \quad (10)$$

V. EXAMPLES

VI. DISCUSSION

VII. ACKNOWLEDGMENTS

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