We would like to quadratise

$$f = xb_1b_2b_3 + yb_2b_3b_4 + zb_3b_4b_1 + wb_4b_1b_2 + kb_1b_2b_3b_4.$$

Also assume $k \geq 0$ for now and write $\boldsymbol{b} = b_1b_2b_3b_4$.

Lemma 0.1

If $x \leq -k$ and $-\frac{k}{2} \leq y \leq z \leq 0 \leq w$, then $g = b_a(x(b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + zb_1b_4 + wb_1b_4 + kb_1b_4$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $w \geq 0$ and $k + z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_ay - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_ax - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z + b_a(-k - w + x + y - z)$. Since $-k \leq 0$ and $-w \leq 0$ and $y \leq z$ and $-x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z + b_a(-k - w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k+w+z-b_a(k+z)$. Since $-(k+z) \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w + z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x) \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. ∎

Lemma 0.2

If $x \leq -k$ and $-k/2 \leq y \leq z \leq w \leq 0$, then $b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1b_2 + b_1b_3) + y(-b_2 - b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 + b_3 - b_4 + b_1b_4) + w(+1 - b_1 + b_2 - b_4 + b_1b_4) + k(+1 - b_1 - b_4 + b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = k+w+z-b_a(k+w+z)$. Since $-\frac{k}{2} - w \leq 0$ and $-\frac{k}{2} - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_ay$. Since $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - w \leq 0$ and $y \leq z$ and $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \le 0$ and $x \le 0$ and $y \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \le w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k - 2w \le 0$ and $x \le 0$ and $y \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \le z$ and $y \le w$ and $x \le 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $\frac{k}{2} + w \ge 0$ and $\frac{k}{2} + z \ge 0$ and $x \ge 0$ and $y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \ge 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \le w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \ge 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \le w$ and $y \le z$ and $y \le 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x \le 0$ and $y \le 0$, $b_a^* = 1$, so $\min g = x = f$.

□

Lemma 0.3

If ???, then $b_a(x(-1+b_1+b_2+b_3)+y(-1+b_2+b_3+b_4)+z(-1+b_1+b_3+b_4)+w(-1+b_1+b_2+b_4)+k(-3+2b_1+2b_2+2b_3+2b_4))+x(+1-b_1-b_2-b_3+b_1b_2+b_1b_3+b_2b_3)+y(+1-b_2-b_3-b_4+b_2b_3+b_2b_4+b_3b_4)+z(+1-b_1-b_3-b_4+b_1b_3+b_1b_4+b_3b_4)+w(+1-b_1-b_2-b_4+b_1b_2+b_1b_4+b_2b_4)+k(+3-2b_1-2b_2-2b_3-2b_4+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$.

If $\boldsymbol{b} = 0001$, then $g = k + x - b_a(k + x)$.

If $\boldsymbol{b} = 0010$, then $g = k + w - b_a(k + w)$.

If $\boldsymbol{b} = 0011$, then $g = b_a(k + y + z)$.

If $\boldsymbol{b} = 0100$, then $g = k + z - b_a(k + z)$.

If $\boldsymbol{b} = 0101$, then $g = b_a(k + w + y)$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + x + y)$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$.

If $\boldsymbol{b} = 1000$, then $g = k + y - b_a(k + y)$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w + z)$.

If $\boldsymbol{b} = 1010$, then $g = b_a(k + x + z)$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(3k + w + x + y + 2z)$.

If $\boldsymbol{b} = 1100$, then $g = b_a(k + w + x)$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(3k + 2w + x + y + z)$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(3k + w + 2x + y + z)$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z + b_a(5k + 2w + 2x + 2y + 2z)$.

□

Lemma 0.4

If ???, then $b_a(x(+b_1-b_2-b_3)+y(-b_2-b_3+b_4)+z(1-b_1+b_3-b_4)+w(1-b_1+b_2-b_4)+k(1-b_1-b_4))+x(-b_1+b_1b_2+b_1b_3)+y(-b_4+b_2b_4+b_3b_4)+z(+b_1b_4)+w(+b_1b_4)+k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=b_a(k+w+z)$.
If $\boldsymbol{b}=0001$, then $g=b_ay-y$.
If $\boldsymbol{b}=0010$, then $g=b_a(k+w-x-y+2z)$.
If $\boldsymbol{b}=0011$, then $g=-b_a(x-z)$.
If $\boldsymbol{b}=0100$, then $g=b_a(k+2w-x-y+z)$.
If $\boldsymbol{b}=0101$, then $g=b_a(w-x)$.
If $\boldsymbol{b}=0110$, then $g=b_a(k+2w-2x-2y+2z)$.
If $\boldsymbol{b}=0111$, then $g=y+b_a(w-2x-y+z)$.
If $\boldsymbol{b}=1000$, then $g=b_ax-x$.
If $\boldsymbol{b}=1001$, then $g=k+w-x-y+z-b_a(k+w-x-y+z)$.
If $\boldsymbol{b}=1010$, then $g=-b_a(y-z)$.
If $\boldsymbol{b}=1011$, then $g=k+w+z-b_a(k+w)$.
If $\boldsymbol{b}=1100$, then $g=b_a(w-y)$.
If $\boldsymbol{b}=1101$, then $g=k+w+z-b_a(k+z)$.
If $\boldsymbol{b}=1110$, then $g=x+b_a(w-x-2y+z)$.
If $\boldsymbol{b}=1111$, then $g=k+w+x+y+z-b_a(k+x+y)$.

□

Lemma 0.5

If ???, then $b_a(x(+b_1-b_2-b_3)+y(-b_2-b_3+b_4)+z(1-b_1+b_3-b_4)+w(1-b_1+b_2-b_4)+k(1-b_1-b_4))+x(-b_1+b_1b_2+b_1b_3)+y(-b_4+b_2b_4+b_3b_4)+z(+b_1b_4)+w(+b_1b_4)+k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=b_a(k+w+z)$.
If $\boldsymbol{b}=0001$, then $g=b_ay-y$.
If $\boldsymbol{b}=0010$, then $g=b_a(k+w-x-y+2z)$.
If $\boldsymbol{b}=0011$, then $g=-b_a(x-z)$.
If $\boldsymbol{b}=0100$, then $g=b_a(k+2w-x-y+z)$.
If $\boldsymbol{b}=0101$, then $g=b_a(w-x)$.
If $\boldsymbol{b}=0110$, then $g=b_a(k+2w-2x-2y+2z)$.
If $\boldsymbol{b}=0111$, then $g=y+b_a(w-2x-y+z)$.
If $\boldsymbol{b}=1000$, then $g=b_ax-x$.
If $\boldsymbol{b}=1001$, then $g=k+w-x-y+z-b_a(k+w-x-y+z)$.
If $\boldsymbol{b}=1010$, then $g=-b_a(y-z)$.
If $\boldsymbol{b}=1011$, then $g=k+w+z-b_a(k+w)$.
If $\boldsymbol{b}=1100$, then $g=b_a(w-y)$.
If $\boldsymbol{b}=1101$, then $g=k+w+z-b_a(k+z)$.
If $\boldsymbol{b}=1110$, then $g=x+b_a(w-x-2y+z)$.
If $\boldsymbol{b}=1111$, then $g=k+w+x+y+z-b_a(k+x+y)$.

□

Lemma 0.6

If ???, then $b_a(x(-1+b_1+b_2+b_3)+y(-1+b_2+b_3+b_4)+z(-1+b_1+b_3+b_4)+w(-1+b_1+b_2+b_4)+k(-3+2b_1+2b_2+2b_3+2b_4))+x(+1-b_1-b_2-b_3+b_1b_2+b_1b_3+b_2b_3)+y(+1-b_2-b_3-b_4+b_2b_3+b_2b_4+b_3b_4)+z(+1-b_1-b_3-b_4+b_1b_3+b_1b_4+b_3b_4)+w(+1-b_1-b_2-b_4+b_1b_2+b_1b_4+b_2b_4)+k(+3-2b_1-2b_2-2b_3-2b_4+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=3k+w+x+y+z-b_a(3k+w+x+y+z)$.
If $\boldsymbol{b}=0001$, then $g=k+x-b_a(k+x)$.
If $\boldsymbol{b}=0010$, then $g=k+w-b_a(k+w)$.
If $\boldsymbol{b}=0011$, then $g=b_a(k+y+z)$.
If $\boldsymbol{b}=0100$, then $g=k+z-b_a(k+z)$.
If $\boldsymbol{b}=0101$, then $g=b_a(k+w+y)$.
If $\boldsymbol{b}=0110$, then $g=b_a(k+x+y)$.
If $\boldsymbol{b}=0111$, then $g=y+b_a(3k+w+x+2y+z)$.
If $\boldsymbol{b}=1000$, then $g=k+y-b_a(k+y)$.
If $\boldsymbol{b}=1001$, then $g=b_a(k+w+z)$.
If $\boldsymbol{b}=1010$, then $g=b_a(k+x+z)$.
If $\boldsymbol{b}=1011$, then $g=z+b_a(3k+w+x+y+2z)$.
If $\boldsymbol{b}=1100$, then $g=b_a(k+w+x)$.
If $\boldsymbol{b}=1101$, then $g=w+b_a(3k+2w+x+y+z)$.
If $\boldsymbol{b}=1110$, then $g=x+b_a(3k+w+2x+y+z)$.
If $\boldsymbol{b}=1111$, then $g=k+w+x+y+z+b_a(5k+2w+2x+2y+2z)$.

□

Lemma 0.7

If ???, then $b_a(x(1-b_1-b_2-b_3)+y(1-b_2-b_3-b_4)+z(1-b_1-b_3-b_4)+w(1-b_1-b_2-b_4)+k(3-2b_1-2b_2-2b_3-2b_4))+x(+b_1b_2+b_1b_3+b_2b_3)+y(+b_2b_3+b_2b_4+b_3b_4)+z(+b_1b_3+b_1b_4+b_3b_4)+w(+b_1b_2+b_1b_4+b_2b_4)+k(+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=b_a(3k+w+x+y+z)$.
If $\boldsymbol{b}=0001$, then $g=b_a(k+x)$.
If $\boldsymbol{b}=0010$, then $g=b_a(k+w)$.
If $\boldsymbol{b}=0011$, then $g=k+y+z-b_a(k+y+z)$.
If $\boldsymbol{b}=0100$, then $g=b_a(k+z)$.
If $\boldsymbol{b}=0101$, then $g=k+w+y-b_a(k+w+y)$.
If $\boldsymbol{b}=0110$, then $g=k+x+y-b_a(k+x+y)$.
If $\boldsymbol{b}=0111$, then $g=3k+w+x+3y+z-b_a(3k+w+x+2y+z)$.
If $\boldsymbol{b}=1000$, then $g=b_a(k+y)$.
If $\boldsymbol{b}=1001$, then $g=k+w+z-b_a(k+w+z)$.
If $\boldsymbol{b}=1010$, then $g=k+x+z-b_a(k+x+z)$.
If $\boldsymbol{b}=1011$, then $g=3k+w+x+y+3z-b_a(3k+w+x+y+2z)$.
If $\boldsymbol{b}=1100$, then $g=k+w+x-b_a(k+w+x)$.
If $\boldsymbol{b}=1101$, then $g=3k+3w+x+y+z-b_a(3k+2w+x+y+z)$.
If $\boldsymbol{b}=1110$, then $g=3k+w+3x+y+z-b_a(3k+w+2x+y+z)$.

If $\boldsymbol{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$.

□

Lemma 0.8

If ???, then $b_a(x(-b_1+b_2+b_3)+y(+b_2+b_3-b_4)+z(-1+b_1-b_3+b_4)+w(-1+b_1-b_2+b_4)+k(-1+b_1+b_4))+x(-b_2-b_3+b_1b_2+b_1b_3)+y(-b_2-b_3+b_2b_4+b_3b_4)+z(+1-b_1+b_3-b_4+b_1b_4)+w(+1-b_1+b_2-b_4+b_1b_4)+k(+1-b_1-b_4+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = k + w + z - b_a(k + w + z)$.
If $\boldsymbol{b} = 0001$, then $g = -b_a y$.
If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z - b_a(k + w - x - y + 2z)$.
If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$.
If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z - b_a(k + 2w - x - y + z)$.
If $\boldsymbol{b} = 0101$, then $g = w - x - b_a(w - x)$.
If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z - b_a(k + 2w - 2x - 2y + 2z)$.
If $\boldsymbol{b} = 0111$, then $g = w - 2x + z - b_a(w - 2x - y + z)$.
If $\boldsymbol{b} = 1000$, then $g = -b_a x$.
If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$.
If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$.
If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$.
If $\boldsymbol{b} = 1100$, then $g = w - y - b_a(w - y)$.
If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$.
If $\boldsymbol{b} = 1110$, then $g = w - 2y + z - b_a(w - x - 2y + z)$.
If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$.

□