WLOG $x \le y \le z \le w$, and $k \ge 0$ unless otherwise stated (by flipping one bit if necessary). We would like to quadratise

$$f = xb_1b_2b_3 + yb_2b_3b_4 + zb_3b_4b_1 + wb_4b_1b_2 + kb_1b_2b_3b_4.$$

We shall write $\boldsymbol{b} = b_1b_2b_3b_4$.

List of Cases:

1. An even number of x, y, z, w lie on each side of $-\frac{k}{2}$, i.e. $-\frac{k}{2} \le x \le y \le z \le w$ or $x \le y \le -\frac{k}{2} \le z \le w$ or $x \le y \le z \le w \le -\frac{k}{2}$. Done by lemma 0.1 and flipping pairs of bits.
2. At least three of x, y, z, w lie in $(-\infty, -k] \cup [0, \infty)$. By flipping bits WLOG $y, z, w \ge 0$. Summarised in proposition 0.7.
   a) $x \ge -\frac{k}{2}$. Then we are in a previous case.
   b) $-k \le x \le -\frac{k}{2}$. Done by lemma 0.1.
   c) $x \le -k$. Done by lemma 0.2 and flipping one bit.
3. An odd number of x, y, z, w lie on each side of $-\frac{k}{2}$ and at least two of x, y, z, w lie in $[-k, 0]$. WLOG $x \le -\frac{k}{2}, y, z, w \ge -\frac{k}{2}$.
   a) $-\frac{k}{2} \le y \le 0 \le z \le w$. Note that $x \ge -k$ in this cases because at least two of x, y, z, w lie in $[-k, 0]$.
      i. $k + x + y \le 0$. Done by 0.10.
      ii. $k + x + y \ge 0$. Done by 0.9.
   b) $-\frac{k}{2} \le y \le z \le 0 \le w$.
      i. $x \le -k$. Done by 0.8.
      ii. $x \ge -k$ and $k + x + y \le 0$. Done by 0.10.
      iii. $x \ge -k$ and $k + x + y \ge 0$. Done by 0.9.
   c) $-\frac{k}{2} \le y \le z \le w \le 0$.
      i. $k + x + y \le 0$. Done by 0.12.
      ii. $k + x + y \ge 0$. Done by 0.11.

Lemma 0.1

Suppose $k \ge 0$. If either $x, y, z, w \ge -\frac{k}{2}$, or $-k \le x \le -\frac{k}{2}$ and $w, y, z \ge 0$, then $g = b_a((3k + x + y + z + w) - (2k + x + z + w)b_1 - (2k + x + y + w)b_2 - (2k + x + y + z)b_3 - (2k + y + z + w)b_4) + k(b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4) + x(b_1b_2 + b_1b_3 + b_2b_3) + y(b_2b_3 + b_2b_4 + b_3b_4) + z(b_1b_3 + b_1b_4 + b_3b_4) + w(b_1b_2 + b_1b_4 + b_2b_4)$ is a quadratisation of f.

Proof. By symmetry, it suffices to check the quadratisation for $b_1 \le b_2 \le b_3$ (we cannot make assumption on b_4 because x is special among x, y, z, w). Notice that in both cases, we have $k + a + b \ge 0$ for any two distinct variables a, b among x, y, z, w.

If $\boldsymbol{b} = 0000$, then $g = b_a(3k + w + x + y + z)$. Since $k + x + y \ge 0$ and $k + w + z \ge 0$ and $k \ge 0$, so the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a(k + x)$. Since $k + x \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w)$. Since $k + w \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = -(b_a - 1)(k + y + z)$. Since $-(k + y + z) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = -(b_a - 1)(k + x + y)$. Since $-(k + x + y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = 3k + w + x + 3y + z - b_a(3k + w + x + 2y + z)$. Since $-(k + x) \leq 0$ and $-(k + w + y) \leq 0$ and $-(k + y + z) \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1110$, then $g = 3k+w+3x+y+z-b_a(3k+w+2x+y+z)$. Since $-(k+x+y) \leq 0$ and $-(k + x + z) \leq 0$ and $-(k + w) \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-2(k+x+y) \leq 0$ and $-2(k+w+z) \leq 0$ and $-k \leq 0$, $b_a^* = 1$, so $\min g = k+w+x+y+z = f$. □

Lemma 0.2

If $k, x, y, z, w \leq 0$, then $g = b_a(k(b_1 + b_2 + b_3 + b_4 - 3) + x(b_1 + b_2 + b_3 - 2) + y(b_2 + b_3 + b_4 - 2) + z(b_3 + b_4 + b_1 - 2) + w(b_4 + b_1 + b_2 - 2))$ is a quadratisation of f.

Remark 0.3. Using the standard quadratisation for the negative monomial, we can quadratise $-b_1b_2b_3b_4$ as $(3-b_1-b_2-b_3-b_4)b_a$, and quadratise $-b_1b_2b_3$ as $(2-b_1-b_2-b_3)b'_a$. Here we are saying that we can add them together and use the *same* auxiliary variable.

Proof. By symmetry, it suffices to check the cases when $\boldsymbol{b} = 0000, 0001, 0011, 0111$, or 1111.

If $\boldsymbol{b} = 0000$, then $g = -b_a(3k + 2w + 2x + 2y + 2z)$. Since $-k \geq 0$ and $-x \geq 0$ and $-y \geq 0$ and $-z \geq 0$ and $-w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_a(2k + w + 2x + y + z)$. Similar to the case when $\boldsymbol{b} = 0000$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = -b_a(k + w + x)$. Since $-k \geq 0$ and $-w \geq 0$ and $x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = b_a y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1111$, then $g = b_a(k + w + x + y + z)$. Since $k \leq 0$ and $w \leq 0$ and $x \leq 0$ and $y \leq 0$ and $z \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$.

□

This has a natural generalisation to n variables.

Theorem 0.4

Let $a_0, a_1, \cdots a_n \leq 0$ and $f = a_0 \prod_{i=1}^n b_i + \sum_{i=1}^n \left(a_i \prod_{j \neq i} b_j \right)$ has quadratisation $g = b_a \left(a_0 \left(\sum_{i=1}^n b_i - (n-1) \right) + \sum_{i=1}^n a_i \left(\sum_{j \neq i} b_j - (n-2) \right) \right)$

Proof. Again WLOG $b_1 = b_2 = \cdots = b_k = 0$ and $b_{k+1} = b_{k+2} = \cdots = b_n = 1$ for some $k \in \{0, 1, \cdots, n\}$. Then

$$f = \begin{cases} 0, & \text{if } k \geq 2, \\ a_1, & \text{if } k = 1, \\ \displaystyle\sum_{i=0}^n a_i, & \text{if } k = 0. \end{cases}$$

For $k \geq 2$, we have

$$\sum_{j \neq i} b_j \leq n-2 \quad \forall j, \text{ and} \sum_{j=1}^{n} b_j \leq n-2,$$

so the coefficient of b_a in g is non-negative, so the minimiser is $b_a^* = 0$ and $\min g = 0 = f$.

If $k = 1$, then $g = b_a a_1$, so $b_a^* = 1$ and $\min g = a_1 = f$.

If $k = 0$, then $g = b_a \sum_{i=0}^{n} a_i$, so $b_a^* = 1$ and $\min g = \sum_{i=0}^{n} a_i = f$.

□

Next we consider some substitutions that reduce other cases to the two cases above that we know how to quadratise.

If we consider the substitution $b_1' = 1 - b_1$ and $b_2' = 1 - b_2$, then

$$\begin{aligned} f &= (kb_3b_4 - kb_2'b_3b_4 - kb_1'b_3b_4 + kb_1'b_2'b_3b_4) + (xb_3 - xb_1'b_3 - xb_2'b_3 + xb_1'b_2'b_3) \\ &\quad + (yb_3b_4 - yb_2'b_3b_4) + (zb_3b_4 - zb_1'b_3b_4) + (wb_4 - wb_1'b_4 - wb_2'b_4 + wb_1'b_2'b_4), \end{aligned}$$

so ignoring all linear and quadratic terms it is

$$\begin{aligned} f' &= kb_1'b_2'b_3b_4 + xb_1'b_2'b_3 + (-y-k)b_2'b_3b_4 + (-z-k)b_3b_4b_1' + wb_4b_1'b_2' \\ &= k'b_1'b_2'b_3b_4 + x'b_1'b_2'b_3 + y'b_2'b_3b_4 + z'b_3b_4b_1' + w'b_4b_1'b_2' \end{aligned}$$

This is of the original form with $y' = -y - k$ and $z' = -z - k$ and other coefficients unchanged. If we have a 1-auxiliary quadratisation for f in terms of b_1, b_2, b_3, b_4, b_a, then after the substitution and taking care of the linear and quadratic terms in f, we obtain a 1-aux quadratisation for f' in terms of $b_1', b_2', b_3, b_4, b_a$.

If f has $k \geq 0$ and $w, x, y, z \geq -\frac{k}{2}$ as in 0.1, then f' has $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$. And this correspondence is invertible, so given any f' with $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$, we know that it has a 1-aux quadratisation. We can also do the same substitution on the other pair of variables b_3, b_4 to prove that any f'' with $k'' \geq 0$ and $w'', x'', y'', z'' \leq -\frac{k}{2}$ has a 1-aux quadratisation.

To sum up, if $k \geq 0$, an even number of x, y, z, w are at least $-\frac{k}{2}$, and an even number of them are most $-\frac{k}{2}$, then f has a quadratisation in 1 auxiliary.

We can also consider substituting $b_1' = 1 - b_1$ but not flipping b_2, then

$$f = k(b_2b_3b_4 - b_1'b_2b_3b_4) + x(b_2b_3 - b_1'b_2b_3) + yb_2b_3b_4 + z(b_3b_4 - b_3b_4b_1') + w(b_2b_4 - b_4b_1'b_2),$$

so ignoring quadratic and linear terms we are left with

$$f' = -kb_1'b_2b_3b_4 - xb_1'b_2b_3 + (k+y)b_2b_3b_4 - zb_3b_4b_1' - wb_4b_1'b_2.$$

If $k \geq 0$ and $y \leq -k$ and $x, z, w \geq 0$, then f' is of the form in 0.2, so we can quadratise f'. Putting back in the quadratic and linear terms and substitute back, we can obtain a quadratisation of f.

By flipping another pair $b_2' = 1 - b_2$ and $b_3' = 1 - b_3$ as before, we also see that if $k \geq 0$ and three of x, y, z, w are at most $-k$ and the remaining one is at least 0, then f can be quadratised with 1 auxiliary.

Therefore, we have proved the following.

Proposition 0.5

If $k \geq 0$ and either of the following holds:

1. $x, y, z, w \geq -\frac{k}{2}$;
2. three of them are at least 0 and the other one is at most $-k$;
3. two are at least $-\frac{k}{2}$ and the other two are at most $-\frac{k}{2}$;
4. one is at least 0 and the other three are at most $-k$; or
5. $x, y, z, w \leq -\frac{k}{2}$;

Then f has a quadratisation with 1 auxiliary.

For $k \geq 0$, we have shown that f has a 1-auxiliary quadratisation if $x, y, z, w \geq -\frac{k}{2}$, or if $x \leq -k$ and $w, y, z \geq 0$ (both cases are sumarised in 0.5), so we can combine them with 0.1 (for $-k \leq x \leq -\frac{k}{2}$, and $w, y, z \geq 0$) to conclude:

Proposition 0.6

f can be quadratised with 1 auxiliary if $k, y, z, w \geq 0$, whatever the value of x is.

Recall that we can substitute $b_1' = 1 - b_1$ and $b_2' = 1 - b_2$ to transform y to $-k - y$ and z to $-k - z$. This means we can flip a pair of the cubic coefficients in the range $[0, +\infty)$ to $(-\infty, -k]$ or vice versa. (e.g. since we have a quadratisation for $w \geq 0$, $y \geq 0$, $z \geq 0$, we can find a quadratisation for $w \leq -k, y \leq -k, z \geq 0$) Or we may as well flip x and flip one other coefficient in the proposition above. Since the above proposition does not place any condition on x, we can e.g. go from $w \geq 0, y \geq 0, z \geq 0$ to $w \leq -k, y \geq 0, z \geq 0$. This gives the following

Proposition 0.7

If $k \geq 0$ and three of the cubic coefficients are in $(-\infty, -k] \cup [0, +\infty)$, then f has a quadratisation with 1 auxiliary.

If some f has $k \geq 0$ but is not covered by 0.7, then at least two of the cubic coefficients are in $[-k, 0]$. If it is not covered by 0.5 either, then an odd number of cubic coefficients lie on each side of $-\frac{k}{2}$ (WLOG $x \leq -\frac{k}{2}$ and $y, z, w \geq -\frac{k}{2}$). This is what we will break down into a further 6 cases.

Below we assume $k \geq 0$.

<table>
<tr><th rowspan="2">Cases</th><th rowspan="2">$x \leq -k$</th><th colspan="2">$-k \leq x \leq -\frac{k}{2}$</th></tr>
<tr><th>$k + x + y \leq 0$</th><th>$k + x + y \geq 0$</th></tr>
<tr><td>$-\frac{k}{2} \leq y \leq 0 \leq z \leq w$</td><td>0.7: only $y \in [-k, 0]$</td><td rowspan="2">0.10</td><td rowspan="2">0.9</td></tr>
<tr><td>$-\frac{k}{2} \leq y \leq z \leq 0 \leq w$</td><td>0.8</td></tr>
<tr><td>$-\frac{k}{2} \leq y \leq z \leq w \leq 0$</td><td colspan="2">0.12</td><td>0.11</td></tr>
</table>

Lemma 0.8

If $x \le -k$ and $-\frac{k}{2} \le y \le z \le 0 \le w$, then $g = b_a(x(b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + zb_1b_4 + wb_1b_4 + kb_1b_4$ is a quadratisation of f.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $w \ge 0$ and $k + z \ge 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $k + 2z \ge 0$ and $w \ge 0$ and $-x \ge 0$ and $-y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \ge x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + 2z \ge 0$ and $w \ge 0$ and $-x \ge 0$ and $-y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \ge x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. Since $k + 2z \ge 0$ and $w \ge 0$ and $-x \ge 0$ and $-y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \ge 0$ and $z - y \ge 0$ and $-x \ge 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z + b_a(-k - w + x + y - z)$. Since $-k \le 0$ and $-w \le 0$ and $y \le z$ and $-x \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \ge y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z + b_a(-k - w)$. Since $-k \le 0$ and $-w \le 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \ge y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k+w+z-b_a(k+z)$. Since $-(k+z) \le 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w + z \ge 0$ and $-x \ge 0$ and $-y \ge 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x) \ge 0$ and $-y \ge 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.9

If $-k \le x \le -\frac{k}{2} \le y \le 0 \le w$, $y \le z \le w$, and $k + x + y \ge 0$, then $g = b_a(x(-1 + b_1 + b_2 + b_3) + y(-1 + b_2 + b_3 + b_4) + z(-1 + b_1 + b_3 + b_4) + w(-1 + b_1 + b_2 + b_4) + k(-3 + 2b_1 + 2b_2 + 2b_3 + 2b_4)) + x(+1 - b_1 - b_2 - b_3 + b_1b_2 + b_1b_3 + b_2b_3) + y(+1 - b_2 - b_3 - b_4 + b_2b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 - b_3 - b_4 + b_1b_3 + b_1b_4 + b_3b_4) + w(+1 - b_1 - b_2 - b_4 + b_1b_2 + b_1b_4 + b_2b_4) + k(+3 - 2b_1 - 2b_2 - 2b_3 - 2b_4 + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)$ is a quadratisation of f.

Proof. If $\boldsymbol{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$. Since $-x - k \le 0$ and $-\frac{k}{2} - y \le 0$ and $-\frac{k}{2} - z \le 0$ and $-k \le 0$ and $-w \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \le 0$ and $-w \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(k + y + z)$. Since $k + y + z \ge k + 2y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k - z \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k+w+z)$. Since $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(k + x + z)$. Since $k + x + z \geq k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(3k + w + x + y + 2z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k + 2z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(k + w + x)$. Since $k + x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(3k + 2w + x + y + z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(3k + w + 2x + y + z)$. Since $2k + 2x \geq 0$ and $k + y + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z + b_a(5k + 2w + 2x + 2y + 2z)$. Since $2k + 2x \geq 0$ and $2k + 2y + 2z \geq 0$ and $k \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.10

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq w$ and $y \leq z \leq w$ and $k + x + y \leq 0$, then $g = b_a(x(+b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + z(+b_1b_4) + w(+b_1b_4) + k(+b_1b_4)$ is a quadratisation of f.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $w \geq 0$ and $k + 2z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\boldsymbol{b} = 0010$, $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k - z \leq 0$ and $-w \leq 0$ and $-x \leq 0$ and $-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.11

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k + x + y \geq 0$, then $g = b_a(x(1 - b_1 - b_2 - b_3) + y(1 - b_2 - b_3 - b_4) + z(1 - b_1 - b_3 - b_4) + w(1 - b_1 - b_2 - b_4) + k(3 - 2b_1 - 2b_2 - 2b_3 - 2b_4)) + x(+b_1b_2 + b_1b_3 + b_2b_3) + y(+b_2b_3 + b_2b_4 + b_3b_4) + z(+b_1b_3 + b_1b_4 + b_3b_4) + w(+b_1b_2 + b_1b_4 + b_2b_4) + k(+b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)$ is a quadratisation of f.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(3k+w+x+y+z)$. Since $k+x \geq 0$ and $2k+w+y+z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a(k + x)$. Since $k + x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = k + y + z - b_a(k + y + z)$. Since $-\frac{k}{2} - y \leq 0$ and $-\frac{k}{2} - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = k + w + y - b_a(k + w + y)$. Since $-\frac{k}{2} - w \leq 0$ and $-\frac{k}{2} - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + x + y - b_a(k + x + y)$. Since $-(k + x + y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = 3k + w + x + 3y + z - b_a(3k + w + x + 2y + z)$. Since $-k - x \leq 0$ and $-k - 2y \leq 0$ and $-k - w - z \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a(k + y)$. Since $k + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w + z - b_a(k + w + z)$. Since $-k - w - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = k + x + z - b_a(k + x + z)$. Since $-(k + x + z) \leq -(k + x + y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = 3k + w + x + y + 3z - b_a(3k + w + x + y + 2z)$. Since $-k - x \leq 0$ and $-k - w - y \leq 0$ and $-k - 2z \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = k + w + x - b_a(k + w + x)$. Since $-(k + w + x) \leq -(k + x + y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = 3k + 3w + x + y + z - b_a(3k + 2w + x + y + z)$. Since $-k - 2w \leq 0$ and $-k - x \leq 0$ and $-k - y - z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = 3k+w+3x+y+z-b_a(3k+w+2x+y+z)$. Sincde $-(k+x+y) \leq 0$ and $-k - x \leq 0$ and $-k - w - z \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-2k - 2x \leq 0$ and $-k - 2y \leq 0$ and $-k - 2z \leq 0$ and $-k - 2w \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.12

If $x \le -\frac{k}{2} \le y \le z \le w \le 0$ and $k + x + y \le 0$, then $g = b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1b_2 + b_1b_3) + y(-b_2 - b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 + b_3 - b_4 + b_1b_4) + w(+1 - b_1 + b_2 - b_4 + b_1b_4) + k(+1 - b_1 - b_4 + b_1b_4)$ is a quadratisation of f.

Proof. If $\boldsymbol{b} = 0000$, then $g = k + w + z - b_a(k + w + z)$. Since $-k - w - z \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_a y$. Since $-y \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - 2z \le 0$ and $x \le w$ and $y \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \le 0$ and $y \le z$ and $x \le 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \le w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k \le 0$ and $x \le z$ and $y \le w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \le w$ and $y \le z$ and $x \le 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \ge 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $k \ge 0$ and $w \ge x$ and $z \ge y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \le z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \ge 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \le w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \ge 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \le w$ and $y \le z$ and $y \le 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x + y \le 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. □

Hence we have shown that every f with $k \ge 0$ has a quadratisation with 1 auxiliary. Using the substitution $b_1' = 1 - b_1$, we can go between cases with $k \ge 0$ and $k \le 0$, so all 4-variable pseudo-boolean functions has a quadratisation with 1 auxiliary.

Theorem 0.13 (A sufficient condition for existence of 1-aux quadratisation)

Let $f = f(\boldsymbol{b})$ be fully symmetric in the n bits $b_1, \cdots, b_n$. Write $|\boldsymbol{b}| = \sum b_i$ and $h(|b|) = f(\boldsymbol{b})$. (Note that h is well-defined because f is symmetric. Suppose $h(0), \cdots, h(n)$ is an arithmetic progression but with all positive terms replaced by 0, i.e. $h(i) = (ai + d)^- = \min(ai + d, 0)$ for some constant a, d, then f is quadratisable with one auxiliary.

Proof. In fact we have a quadratisation that is also symmetric in $b_1, \cdots, b_n$. Let $g = b_a(a|\boldsymbol{b}| + d) = b_a(a(\sum b_i) + d)$. This is quadratic in $\boldsymbol{b}, b_a$ and

$$\min_{b_a} g = f \quad \forall \boldsymbol{b} \in \{0, 1\}^n$$

using the property of f. □