We would like to quadratise

$$f = xb_1b_2b_3 + yb_2b_3b_4 + zb_3b_4b_1 + wb_4b_1b_2 + kb_1b_2b_3b_4.$$

We shall write $\boldsymbol{b} = b_1b_2b_3b_4$.

Lemma 0.1

If $k \geq 0$ and $x, y, z, w \geq -\frac{k}{2}$, then $g = b_a((3k+x+y+z+w) - (2k+x+z+w)b_1 - (2k+x+y+w)b_2 - (2k+x+y+z)b_3 - (2k+y+z+w)b_4) + k(b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4) + x(b_1b_2+b_1b_3+b_2b_3) + y(b_2b_3+b_2b_4+b_3b_4) + z(b_1b_3+b_1b_4+b_3b_4) + w(b_1b_2+b_1b_4+b_2b_4)$ is a quadratisation of f.

Proof. Using the symmetry in the condition that x, y, z, w satisfy and the symmetry in the quadratisation, WLOG consider $b_1 \leq b_2 \leq b_3 \leq b_4$, so we only need to check 5 cases: $\boldsymbol{b} = 0000, 0001, 0011, 0111$, or 1111.

If $\boldsymbol{b} = 0000$, then $g = b_a(3k+w+x+y+z)$. Since $k+w+x \geq 0$ and $k+y+z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a(k+x)$. Since $k+x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = k+y+z-b_a(k+y+z)$. Since $-(k+y+z) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = 3k+w+x+3y+z-b_a(3k+w+x+2y+z)$. Since $-(k+w+x) \leq 0$ and $-(k+2y) \leq 0$ and $-(k+z) \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1111$, then $g = 6k+3w+3x+3y+3z-b_a(5k+2w+2x+2y+2z)$. Since $-k-2w \leq 0$ and $-k-2x \leq 0$ and $-k-2y \leq 0$ and $-k-2z \leq 0$ and $-k \leq 0$, $b_a^* = 1$, so $\min g = k+w+x+y+z = f$.

□

Lemma 0.2

If $k, x, y, z, w \leq 0$, then $g = b_a(k(b_1+b_2+b_3+b_4-3) + x(b_1+b_2+b_3-2) + y(b_2+b_3+b_4-2) + z(b_3+b_4+b_1-2) + w(b_4+b_1+b_2-2))$ is a quadratisation of f.

Remark 0.3. Using the standard quadratisation for the negative monomial, we can quadratise $-b_1b_2b_3b_4$ as $(3-b_1-b_2-b_3-b_4)b_a$, and quadratise $-b_1b_2b_3$ as $(2-b_1-b_2-b_3)b_a'$. Here we are saying that we can add them together and use the *same* auxiliary variable.

Proof. By symmetry, it suffices to check the cases when $\boldsymbol{b} = 0000, 0001, 0011, 0111$, or 1111.

If $\boldsymbol{b} = 0000$, then $g = -b_a(3k+2w+2x+2y+2z)$. Since $-k \geq 0$ and $-x \geq 0$ and $-y \geq 0$ and $-z \geq 0$ and $-w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_a(2k+w+2x+y+z)$. Similar to the case when $\boldsymbol{b} = 0000$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = -b_a(k+w+x)$. Since $-k \geq 0$ and $-w \geq 0$ and ≥ 0, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = b_ay$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1111$, then $g = b_a(k+w+x+y+z)$. Since $k \leq 0$ and ≤ 0 and $x \leq 0$ and $y \leq 0$ and $z \leq 0$, $b_a^* = 1$, so $\min g = k+w+x+y+z = f$.

□

Next we consider some substitutions that reduce other cases to the two cases above that we know how to quadratise.

If we consider the substitution $b_1' = 1 - b_1$ and $b_2' = 1 - b_2$, then

$$\begin{aligned} f &= (kb_3b_4 - kb_2'b_3b_4 - kb_1'b_3b_4 + kb_1'b_2'b_3b_4) + (xb_3 - xb_1'b_3 - xb_2'b_3 + xb_1'b_2'b_3) \\ &\quad + (yb_3b_4 - yb_2'b_3b_4) + (zb_3b_4 - zb_1'b_3b_4) + (wb_4 - wb_1'b_4 - wb_2'b_4 + wb_1'b_2'b_4), \end{aligned}$$

so ignoring all linear and quadratic terms it is

$$\begin{aligned} f' &= kb_1'b_2'b_3b_4 + xb_1'b_2'b_3 + (-y-k)b_2'b_3b_4 + (-z-k)b_3b_4b_1' + wb_4b_1'b_2' \\ &= k'b_1'b_2'b_3b_4 + x'b_1'b_2'b_3 + y'b_2'b_3b_4 + z'b_3b_4b_1' + w'b_4b_1'b_2' \end{aligned}$$

This is of the original form with $y' = -y - k$ and $z' = -z - k$ and other coefficients unchanged. If we have a 1-auxiliary quadratisation for f in terms of b_1, b_2, b_3, b_4, b_a, then after the substitution and taking care of the linear and quadratic terms in f, we obtain a 1-aux quadratisation for f' in terms of $b_1', b_2', b_3, b_4, b_a$.

If f has $k \geq 0$ and $w, x, y, z \geq -\frac{k}{2}$ as in 0.1, then f' has $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$. And this correspondence is invertible, so given any f' with $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$, we know that it has a 1-aux quadratisation. We can also do the same substitution on the other pair of variables b_3, b_4 to prove that any f'' with $k'' \geq 0$ and $w'', x'', y'', z'' \leq -\frac{k}{2}$ has a 1-aux quadratisation.

To sum up, if $k \geq 0$, an even number of x, y, z, w are at least $-\frac{k}{2}$, and an even number of them are most $-\frac{k}{2}$, then f has a quadratisation in 1 auxiliary.

some cases to add here

Lemma 0.4

Lemma 0.5

Below we assume $k \geq 0$.

Lemma 0.6

If $x \leq -k$ and $-\frac{k}{2} \leq y \leq z \leq 0 \leq w$, then $g = b_a(x(b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + zb_1b_4 + wb_1b_4 + kb_1b_4$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $w \geq 0$ and $k + z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z + b_a(-k - w + x + y - z)$. Since $-k \leq 0$ and $-w \leq 0$ and $y \leq z$ and $-x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z + b_a(-k - w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k+w+z-b_a(k+z)$. Since $-(k+z) \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w + z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x) \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.7

If $x \leq -k$ and $-k/2 \leq y \leq z \leq w \leq 0$, then $b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1 b_2 + b_1 b_3) + y(-b_2 - b_3 + b_2 b_4 + b_3 b_4) + z(+1 - b_1 + b_3 - b_4 + b_1 b_4) + w(+1 - b_1 + b_2 - b_4 + b_1 b_4) + k(+1 - b_1 - b_4 + b_1 b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = k+w+z-b_a(k+w+z)$. Since $-\frac{k}{2} - w \leq 0$ and $-\frac{k}{2} - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_a y$. Since $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - w \leq 0$ and $y \leq z$ and $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \leq 0$ and $x \leq 0$ and $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k - 2w \leq 0$ and $x \leq 0$ and $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \leq z$ and $y \leq w$ and $x \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $\frac{k}{2} + w \geq 0$ and $\frac{k}{2} + z \geq 0$ and $x \geq 0$ and $y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \leq w$ and $y \leq z$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x \leq 0$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

□

Lemma 0.8

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq z \leq w$ and $k+x+y \geq 0$, then $b_a(x(-1+b_1+b_2+b_3)+y(-1+b_2+b_3+b_4)+z(-1+b_1+b_3+b_4)+w(-1+b_1+b_2+b_4)+k(-3+2b_1+2b_2+2b_3+2b_4))+x(+1-b_1-b_2-b_3+b_1b_2+b_1b_3+b_2b_3)+y(+1-b_2-b_3-b_4+b_2b_3+b_2b_4+b_3b_4)+z(+1-b_1-b_3-b_4+b_1b_3+b_1b_4+b_3b_4)+w(+1-b_1-b_2-b_4+b_1b_2+b_1b_4+b_2b_4)+k(+3-2b_1-2b_2-2b_3-2b_4+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=3k+w+x+y+z-b_a(3k+w+x+y+z)$. Since $-x-k\leq 0$ and $-y-k\leq 0$ and $-k\leq 0$ and $-z\leq 0$ and $-w\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=0001$, then $g=k+x-b_a(k+x)$. Since $-k-x\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=0010$, then $g=k+w-b_a(k+w)$. Since $-k\leq 0$ and $-w\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=0011$, then $g=b_a(k+y+z)$. Since $k+y\geq 0$ and $z\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=0100$, then $g=k+z-b_a(k+z)$. Since $-k\leq 0$ and $-z\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=0101$, then $g=b_a(k+w+y)$. Since $k+y\geq 0$ and $w\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=0110$, then $g=b_a(k+x+y)$. Since $k+x+y\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=0111$, then $g=y+b_a(3k+w+x+2y+z)$. Since $k+x\geq 0$ and $k+2y\geq 0$ and $k\geq 0$ and $w\geq z\geq 0$, $b_a^*=0$, so $\min g=y=f$.

If $\boldsymbol{b}=1000$, then $g=k+y-b_a(k+y)$. Since $-k-y\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=1001$, then $g=b_a(k+w+z)$. Since k and w and $z\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=1010$, then $g=b_a(k+x+z)$. Since $k+x\geq 0$ and $z\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=1011$, then $g=z+b_a(3k+w+x+y+2z)$. Since $k+x\geq 0$ and $k+y\geq 0$ and $k\geq 0$ and $w\geq z\geq 0$, $b_a^*=0$, so $\min g=z=f$.

If $\boldsymbol{b}=1100$, then $g=b_a(k+w+x)$. Since $k+x\geq 0$ and $w\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=1101$, then $g=w+b_a(3k+2w+x+y+z)$. Since $k+x\geq 0$ and $k+y\geq 0$ and $k\geq 0$ and $w\geq z\geq 0$, $b_a^*=0$, so $\min g=w=f$.

If $\boldsymbol{b}=1110$, then $g=x+b_a(3k+w+2x+y+z)$. Since $2k+2x\geq 0$ and $k+y\geq 0$ and $w\geq z\geq 0$, $b_a^*=0$, so $\min g=x=f$.

If $\boldsymbol{b}=1111$, then $g=k+w+x+y+z+b_a(5k+2w+2x+2y+2z)$. Since $2k+2x\geq 0$ and $2k+2y\geq 0$ and $k\geq 0$ and $w\geq z\geq 0$, $b_a^*=0$, so $\min g=k+w+x+y+z=f$. □

Lemma 0.9

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq z \leq w$ and $k+x+y \leq 0$, then $b_a(x(+b_1-b_2-b_3)+y(-b_2-b_3+b_4)+z(1-b_1+b_3-b_4)+w(1-b_1+b_2-b_4)+k(1-b_1-b_4))+x(-b_1+b_1b_2+b_1b_3)+y(-b_4+b_2b_4+b_3b_4)+z(+b_1b_4)+w(+b_1b_4)+k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=b_a(k+w+z)$. Since k and w and $z\geq 0$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=0001$, then $g=b_ay-y$. Since $y\leq 0$, $b_a^*=1$, so $\min g=0=f$.

If $\boldsymbol{b}=0010$, then $g=b_a(k+w-x-y+2z)$. Since $-x,-y,z,w,$ and k are all non-negative, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b}=0011$, then $g=b_a(z-x)$. Since $z\geq x$, $b_a^*=0$, so $\min g=0=f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k, -w, x, y$ and $-z$ are all non-positive, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k$ and $-z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = y = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.10

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq 0 \leq w$ and $k + x + y \leq 0$, then $b_a(x(+b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + z(+b_1b_4) + w(+b_1b_4) + k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $w \geq 0$ and $k + 2z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\boldsymbol{b} = 0010$, $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k - z \leq 0$ and $-w \leq 0$ and $-x \leq 0$ and $-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

$\square$

Lemma 0.11

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq 0 \leq w$ and $k + x + y \geq 0$, then $b_a(x(-1+b_1+b_2+b_3) + y(-1+b_2+b_3+b_4) + z(-1+b_1+b_3+b_4) + w(-1+b_1+b_2+b_4) + k(-3+2b_1+2b_2+2b_3+2b_4)) + x(+1-b_1-b_2-b_3+b_1b_2+b_1b_3+b_2b_3) + y(+1-b_2-b_3-b_4+b_2b_3+b_2b_4+b_3b_4) + z(+1-b_1-b_3-b_4+b_1b_3+b_1b_4+b_3b_4) + w(+1-b_1-b_2-b_4+b_1b_2+b_1b_4+b_2b_4) + k(+3-2b_1-2b_2-2b_3-2b_4+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = 3k+w+x+y+z-b_a(3k+w+x+y+z)$. Since $-k-x \leq 0$ and $-\frac{k}{2} - y \leq 0$ and $-\frac{k}{2} - z \leq 0$ and $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(k + y + z)$. Since $\frac{k}{2} + y \geq 0$ and $\frac{k}{2} + z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k+w+z)$. Since $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(k + x + z)$. Since $k + x + z \geq k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(3k + w + x + y + 2z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k + 2z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(k + w + x)$. Since $k + x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(3k + 2w + x + y + z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(3k + w + 2x + y + z)$. Since $2k + 2x \geq 0$ and $\frac{k}{2} + y \geq 0$ and $\frac{k}{2} + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k+w+x+y+z+b_a(5k+2w+2x+2y+2z)$. Since $2k+2x \geq 0$ and $k+2y \geq 0$ and $k+2z \geq 0$ and $k \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = k+w+x+y+z = f$.

$\square$

Lemma 0.12

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k+x+y \geq 0$, then $b_a(x(1-b_1-b_2-b_3)+y(1-b_2-b_3-b_4)+z(1-b_1-b_3-b_4)+w(1-b_1-b_2-b_4)+k(3-2b_1-2b_2-2b_3-2b_4))+x(+b_1b_2+b_1b_3+b_2b_3)+y(+b_2b_3+b_2b_4+b_3b_4)+z(+b_1b_3+b_1b_4+b_3b_4)+w(+b_1b_2+b_1b_4+b_2b_4)+k(+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=b_a(3k+w+x+y+z)$. Since $k+x \geq 0$ and $2k+w+y+z \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0001$, then $g=b_a(k+x)$. Since $k+x \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0010$, then $g=b_a(k+w)$. Since $k+w \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0011$, then $g=k+y+z-b_a(k+y+z)$. Since $-\frac{k}{2}-y \leq 0$ and $-\frac{k}{2}-z \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0100$, then $g=b_a(k+z)$. Since $k+z \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0101$, then $g=k+w+y-b_a(k+w+y)$. Since $-\frac{k}{2}-w \leq 0$ and $-\frac{k}{2}-y \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0110$, then $g=k+x+y-b_a(k+x+y)$. Since $-(k+x+y) \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0111$, then $g=3k+w+x+3y+z-b_a(3k+w+x+2y+z)$. Since $-k-x \leq 0$ and $-k-2y \leq 0$ and $-k-w-z \leq 0$, $b_a^*=1$, so $\min g = y = f$.

If $\boldsymbol{b}=1000$, then $g=b_a(k+y)$. Since $k+y \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b}=1001$, then $g=k+w+z-b_a(k+w+z)$. Since $-k-w-z \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=1010$, then $g=k+x+z-b_a(k+x+z)$. Since $-(k+x+z) \leq -(k+x+y) \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=1011$, then $g=3k+w+x+y+3z-b_a(3k+w+x+y+2z)$. Since $-k-x \leq 0$ and $-k-w-y \leq 0$ and $-k-2z \leq 0$, $b_a^*=1$, so $\min g = z = f$.

If $\boldsymbol{b}=1100$, then $g=k+w+x-b_a(k+w+x)$. Since $-(k+w+x) \leq -(k+x+y) \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=1101$, then $g=3k+3w+x+y+z-b_a(3k+2w+x+y+z)$. Since $-k-2w \leq 0$ and $-k-x \leq 0$ and $-k-y-z \leq 0$, $b_a^*=1$, so $\min g = w = f$.

If $\boldsymbol{b}=1110$, then $g=3k+w+3x+y+z-b_a(3k+w+2x+y+z)$. Sincde $-(k+x+y) \leq 0$ and $-k-x \leq 0$ and $-k-w-z \leq 0$, $b_a^*=1$, so $\min g = x = f$.

If $\boldsymbol{b}=1111$, then $g=6k+3w+3x+3y+3z-b_a(5k+2w+2x+2y+2z)$. Since $-2k-2x \leq 0$ and $-k-2y \leq 0$ and $-k-2z \leq 0$ and $-k-2w \leq 0$, $b_a^*=1$, so $\min g = k+w+x+y+z = f$.

□

Lemma 0.13

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k+x+y \leq 0$, then $b_a(x(-b_1+b_2+b_3)+y(+b_2+b_3-b_4)+z(-1+b_1-b_3+b_4)+w(-1+b_1-b_2+b_4)+k(-1+b_1+b_4))+x(-b_2-b_3+b_1b_2+b_1b_3)+y(-b_2-b_3+b_2b_4+b_3b_4)+z(+1-b_1+b_3-b_4+b_1b_4)+w(+1-b_1+b_2-b_4+b_1b_4)+k(+1-b_1-b_4+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b}=0000$, then $g=k+w+z-b_a(k+w+z)$. Since $-k-w-z \leq 0$, $b_a^*=1$, so $\min g = 0 = f$.

If $\boldsymbol{b}=0001$, then $g=-b_ay$. Since $-y \geq 0$, $b_a^*=0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - 2z \leq 0$ and $x \leq w$ and $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \leq 0$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k \leq 0$ and $x \leq z$ and $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \leq w$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $k \geq 0$ and $w \geq x$ and $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \leq w$ and $y \leq z$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x + y \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. □