We would like to quadratise

$$f = xb_1b_2b_3 + yb_2b_3b_4 + zb_3b_4b_1 + wb_4b_1b_2 + kb_1b_2b_3b_4.$$

Also assume $k \geq 0$ for now and write $\boldsymbol{b} = b_1b_2b_3b_4$.

Lemma 0.1

If $x \leq -k$ and $-\frac{k}{2} \leq y \leq z \leq 0 \leq w$, then $g = b_a(x(b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + zb_1b_4 + wb_1b_4 + kb_1b_4$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $w \geq 0$ and $k + z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_ay - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_ax - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z + b_a(-k - w + x + y - z)$. Since $-k \leq 0$ and $-w \leq 0$ and $y \leq z$ and $-x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z + b_a(-k - w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k+w+z-b_a(k+z)$. Since $-(k+z) \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w + z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x) \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.2

If $x \leq -k$ and $-k/2 \leq y \leq z \leq w \leq 0$, then $b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1b_2 + b_1b_3) + y(-b_2 - b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 + b_3 - b_4 + b_1b_4) + w(+1 - b_1 + b_2 - b_4 + b_1b_4) + k(+1 - b_1 - b_4 + b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = k+w+z-b_a(k+w+z)$. Since $-\frac{k}{2} - w \leq 0$ and $-\frac{k}{2} - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_ay$. Since $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - w \leq 0$ and $y \leq z$ and $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \leq 0$ and $x \leq 0$ and $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k - 2w \leq 0$ and $x \leq 0$ and $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \leq z$ and $y \leq w$ and $x \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $\frac{k}{2} + w \geq 0$ and $\frac{k}{2} + z \geq 0$ and $x \geq 0$ and $y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \leq w$ and $y \leq z$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x \leq 0$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

□

Lemma 0.3

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq z \leq w$ and $k + x + y \geq 0$, then $b_a(x(-1 + b_1 + b_2 + b_3) + y(-1 + b_2 + b_3 + b_4) + z(-1 + b_1 + b_3 + b_4) + w(-1 + b_1 + b_2 + b_4) + k(-3 + 2b_1 + 2b_2 + 2b_3 + 2b_4)) + x(+1 - b_1 - b_2 - b_3 + b_1b_2 + b_1b_3 + b_2b_3) + y(+1 - b_2 - b_3 - b_4 + b_2b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 - b_3 - b_4 + b_1b_3 + b_1b_4 + b_3b_4) + w(+1 - b_1 - b_2 - b_4 + b_1b_2 + b_1b_4 + b_2b_4) + k(+3 - 2b_1 - 2b_2 - 2b_3 - 2b_4 + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$. Since $-x - k \leq 0$ and $-y - k \leq 0$ and $-k \leq 0$ and $-z \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(k + y + z)$. Since $k + y \geq 0$ and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k \leq 0$ and $-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w + z)$. Since k and w and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(k + x + z)$. Since $k + x \geq 0$ and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(3k + w + x + y + 2z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(k + w + x)$. Since $k + x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(3k + 2w + x + y + z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(3k + w + 2x + y + z)$. Since $2k + 2x \geq 0$ and $k + y \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z + b_a(5k + 2w + 2x + 2y + 2z)$. Since $2k + 2x \geq 0$ and $2k + 2y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.4

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq z \leq w$ and $k + x + y \leq 0$, then $b_a(x(+b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + z(+b_1b_4) + w(+b_1b_4) + k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since k and w and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $-x, -y, z, w,$ and k are all non-negative, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k, -w, x, y$ and $-z$ are all non-positive, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k$ and $-z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. For the same reason as with $\boldsymbol{b} = 0010$ we have $\min g = y = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.5

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq 0 \leq w$ and $k + x + y \leq 0$, then $b_a(x(+b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + z(+b_1b_4) + w(+b_1b_4) + k(+b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(k + w + z)$. Since $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $w \geq 0$ and $k + 2z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\boldsymbol{b} = 0010$, $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k - z \leq 0$ and $-w \leq 0$ and $-x \leq 0$ and $-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.6

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq 0 \leq w$ and $k + x + y \geq 0$, then $b_a(x(-1 + b_1 + b_2 + b_3) + y(-1 + b_2 + b_3 + b_4) + z(-1 + b_1 + b_3 + b_4) + w(-1 + b_1 + b_2 + b_4) + k(-3 + 2b_1 + 2b_2 + 2b_3 + 2b_4)) + x(+1 - b_1 - b_2 - b_3 + b_1 b_2 + b_1 b_3 + b_2 b_3) + y(+1 - b_2 - b_3 - b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4) + z(+1 - b_1 - b_3 - b_4 + b_1 b_3 + b_1 b_4 + b_3 b_4) + w(+1 - b_1 - b_2 - b_4 + b_1 b_2 + b_1 b_4 + b_2 b_4) + k(+3 - 2b_1 - 2b_2 - 2b_3 - 2b_4 + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$. Since $-k - x \leq 0$ and $-\frac{k}{2} - y \leq 0$ and $-\frac{k}{2} - z \leq 0$ and $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = b_a(k + y + z)$. Since $\frac{k}{2} + y \geq 0$ and $\frac{k}{2} + z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k+w+z)$. Since $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = b_a(k+x+z)$. Since $k+x+z \geq k+x+y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(3k+w+x+y+2z)$. Since $k+x \geq 0$ and $k+y \geq 0$ and $k+2z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = b_a(k+w+x)$. Since $k+x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(3k+2w+x+y+z)$. Since $k+x \geq 0$ and $k+y \geq 0$ and $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = x + b_a(3k+w+2x+y+z)$. Since $2k+2x \geq 0$ and $\frac{k}{2}+y \geq 0$ and $\frac{k}{2}+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = k+w+x+y+z+b_a(5k+2w+2x+2y+2z)$. Since $2k+2x \geq 0$ and $k+2y \geq 0$ and $k+2z \geq 0$ and $k \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = k+w+x+y+z = f$. □

Lemma 0.7

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k+x+y \geq 0$, then $b_a(x(1-b_1-b_2-b_3) + y(1-b_2-b_3-b_4) + z(1-b_1-b_3-b_4) + w(1-b_1-b_2-b_4) + k(3-2b_1-2b_2-2b_3-2b_4)) + x(+b_1b_2+b_1b_3+b_2b_3) + y(+b_2b_3+b_2b_4+b_3b_4) + z(+b_1b_3+b_1b_4+b_3b_4) + w(+b_1b_2+b_1b_4+b_2b_4) + k(+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = b_a(3k+w+x+y+z)$. Since $k+x \geq 0$ and $2k+w+y+z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = b_a(k+x)$. Since $k+x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = b_a(k+w)$. Since $k+w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = k+y+z-b_a(k+y+z)$. Since $-\frac{k}{2}-y \leq 0$ and $-\frac{k}{2}-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = b_a(k+z)$. Since $k+z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = k+w+y-b_a(k+w+y)$. Since $-\frac{k}{2}-w \leq 0$ and $-\frac{k}{2}-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k+x+y-b_a(k+x+y)$. Since $-(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = 3k+w+x+3y+z-b_a(3k+w+x+2y+z)$. Since $-k-x \leq 0$ and $-k-2y \leq 0$ and $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = b_a(k+y)$. Since $k+y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = k+w+z-b_a(k+w+z)$. Since $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = k+x+z-b_a(k+x+z)$. Since $-(k+x+z) \leq -(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = 3k+w+x+y+3z-b_a(3k+w+x+y+2z)$. Since $-k-x \leq 0$ and $-k-w-y \leq 0$ and $-k-2z \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = k+w+x-b_a(k+w+x)$. Since $-(k+w+x) \leq -(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = 3k+3w+x+y+z-b_a(3k+2w+x+y+z)$. Since $-k-2w \leq 0$ and $-k-x \leq 0$ and $-k-y-z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = 3k+w+3x+y+z-b_a(3k+w+2x+y+z)$. Sincde $-(k+x+y) \leq 0$ and $-k-x \leq 0$ and $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-2k - 2x \leq 0$ and $-k - 2y \leq 0$ and $-k - 2z \leq 0$ and $-k - 2w \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$.

$\square$

Lemma 0.8

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k + x + y \leq 0$, then $b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1b_2 + b_1b_3) + y(-b_2 - b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 + b_3 - b_4 + b_1b_4) + w(+1 - b_1 + b_2 - b_4 + b_1b_4) + k(+1 - b_1 - b_4 + b_1b_4)$ is a quadratisation.

Proof. If $\boldsymbol{b} = 0000$, then $g = k + w + z - b_a(k + w + z)$. Since $-k - w - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0001$, then $g = -b_a y$. Since $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - 2z \leq 0$ and $x \leq w$ and $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \leq 0$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k \leq 0$ and $x \leq z$ and $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \leq w$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\boldsymbol{b} = 1000$, then $g = -b_a x$. Since $-x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $k \geq 0$ and $w \geq x$ and $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\boldsymbol{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\boldsymbol{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\boldsymbol{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \leq w$ and $y \leq z$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\boldsymbol{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x + y \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. $\square$