

WLOG $x \leq y \leq z \leq w$, and $k \geq 0$ unless otherwise stated (by flipping one bit if necessary). We would like to quadratise

$$f = xb_1b_2b_3 + yb_2b_3b_4 + zb_3b_4b_1 + wb_4b_1b_2 + kb_1b_2b_3b_4.$$

We shall write $\mathbf{b} = b_1b_2b_3b_4$.

List of Cases:

1. An even number of x, y, z, w lie on each side of $-\frac{k}{2}$, i.e. $-\frac{k}{2} \leq x \leq y \leq z \leq w$ or $x \leq y \leq -\frac{k}{2} \leq z \leq w$ or $x \leq y \leq z \leq w \leq -\frac{k}{2}$. Done by lemma 0.1 and flipping pairs of bits.
2. At least three of x, y, z, w lie in $(-\infty, -k] \cup [0, \infty)$. By flipping bits WLOG $y, z, w \geq 0$. Summarised in proposition 0.9.
 - a) $x \geq -\frac{k}{2}$. Then we are in a previous case.
 - b) $-k \leq x \leq -\frac{k}{2}$. Done by lemma 0.6.
 - c) $x \leq -k$. Done by lemma 0.2 and flipping one bit.
3. An odd number of x, y, z, w lie on each side of $-\frac{k}{2}$ and at least two of x, y, z, w lie in $[-k, 0]$. WLOG $x \leq -\frac{k}{2}, y, z, w \geq -\frac{k}{2}$.
 - a) $-\frac{k}{2} \leq y \leq 0 \leq z \leq w$. Note that $x \geq -k$ in this cases because at least two of x, y, z, w lie in $[-k, 0]$.
 - i. $k + x + y \leq 0$. Done by 0.12.
 - ii. $k + x + y \geq 0$. Done by 0.11.
 - b) $-\frac{k}{2} \leq y \leq z \leq 0 \leq w$.
 - i. $x \leq -k$. Done by 0.10.
 - ii. $x \geq -k$ and $k + x + y \leq 0$. Done by 0.12.
 - iii. $x \geq -k$ and $k + x + y \geq 0$. Done by 0.13.
 - c) $-\frac{k}{2} \leq y \leq z \leq w \leq 0$.
 - i. $k + x + y \leq 0$. Done by 0.15.
 - ii. $k + x + y \geq 0$. Done by 0.14.

Lemma 0.1

If $k \geq 0$ and $x, y, z, w \geq -\frac{k}{2}$, then $g = b_a((3k + x + y + z + w) - (2k + x + z + w)b_1 - (2k + x + y + w)b_2 - (2k + x + y + z)b_3 - (2k + y + z + w)b_4) + k(b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4) + x(b_1b_2 + b_1b_3 + b_2b_3) + y(b_2b_3 + b_2b_4 + b_3b_4) + z(b_1b_3 + b_1b_4 + b_3b_4) + w(b_1b_2 + b_1b_4 + b_2b_4)$ is a quadratisation of f .

Proof. Using the symmetry in the condition that x, y, z, w satisfy and the symmetry in the quadratisation, WLOG consider $b_1 \leq b_2 \leq b_3 \leq b_4$, so we only need to check 5 cases: $\mathbf{b} = 0000, 0001, 0011, 0111$, or 1111 .

If $\mathbf{b} = 0000$, then $g = b_a(3k + w + x + y + z)$. Since $k + w + x \geq 0$ and $k + y + z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = b_a(k + x)$. Since $k + x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = k + y + z - b_a(k + y + z)$. Since $-(k + y + z) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = 3k + w + x + 3y + z - b_a(3k + w + x + 2y + z)$. Since $-(k + w + x) \leq 0$ and $-(k + 2y) \leq 0$ and $-(k + z) \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\mathbf{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-k - 2w \leq 0$ and $-k - 2x \leq 0$ and $-k - 2y \leq 0$ and $-k - 2z \leq 0$ and $-k \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.2

If $k, x, y, z, w \leq 0$, then $g = b_a(k(b_1 + b_2 + b_3 + b_4 - 3) + x(b_1 + b_2 + b_3 - 2) + y(b_2 + b_3 + b_4 - 2) + z(b_3 + b_4 + b_1 - 2) + w(b_4 + b_1 + b_2 - 2))$ is a quadratisation of f .

Remark 0.3. Using the standard quadratisation for the negative monomial, we can quadratise $-b_1b_2b_3b_4$ as $(3 - b_1 - b_2 - b_3 - b_4)b_a$, and quadratise $-b_1b_2b_3$ as $(2 - b_1 - b_2 - b_3)b'_a$. Here we are saying that we can add them together and use the *same* auxiliary variable.

Proof. By symmetry, it suffices to check the cases when $\mathbf{b} = 0000, 0001, 0011, 0111$, or 1111 .

If $\mathbf{b} = 0000$, then $g = -b_a(3k + 2w + 2x + 2y + 2z)$. Since $-k \geq 0$ and $-x \geq 0$ and $-y \geq 0$ and $-z \geq 0$ and $-w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = -b_a(2k + w + 2x + y + z)$. Similar to the case when $\mathbf{b} = 0000$ we have $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = -b_a(k + w + x)$. Since $-k \geq 0$ and $-w \geq 0$ and $x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = b_ay$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\mathbf{b} = 1111$, then $g = b_a(k + w + x + y + z)$. Since $k \leq 0$ and $w \leq 0$ and $x \leq 0$ and $y \leq 0$ and $z \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$.

□

This has a natural generalisation to n variables.

Theorem 0.4

Let $a_0, a_1, \dots, a_n \leq 0$ and $f = a_0 \prod_{i=1}^n b_i + \sum_{i=1}^n \left(a_i \prod_{j \neq i} b_j \right)$ has quadratisation $g = b_a \left(a_0 \left(\sum_{i=1}^n b_i - (n-1) \right) + \sum_{i=1}^n a_i \left(\sum_{j \neq i} b_j - (n-2) \right) \right)$

Proof. Again WLOG $b_1 = b_2 = \dots = b_k = 0$ and $b_{k+1} = b_{k+2} = \dots = b_n = 1$ for some $k \in \{0, 1, \dots, n\}$. Then

$$f = \begin{cases} 0, & \text{if } k \geq 2, \\ a_1, & \text{if } k = 1, \\ \sum_{i=0}^n a_i, & \text{if } k = 0. \end{cases}$$

For $k \geq 2$, we have

$$\sum_{j \neq i} b_j \leq n-2 \quad \forall j, \text{ and } \sum_{j=1}^n b_j \leq n-2,$$

so the coefficient of b_a in g is non-negative, so the minimiser is $b_a^* = 0$ and $\min g = 0 = f$.

If $k = 1$, then $g = b_a a_1$, so $b_a^* = 1$ and $\min g = a_1 = f$.

If $k = 0$, then $g = b_a \sum_{i=0}^n a_i$, so $b_a^* = 1$ and $\min g = \sum_{i=0}^n a_i = f$.

□

Next we consider some substitutions that reduce other cases to the two cases above that we know how to quadratise.

If we consider the substitution $b'_1 = 1 - b_1$ and $b'_2 = 1 - b_2$, then

$$\begin{aligned} f = & (kb_3b_4 - kb'_2b_3b_4 - kb'_1b_3b_4 + kb'_1b'_2b_3b_4) + (xb_3 - xb'_1b_3 - xb'_2b_3 + xb'_1b'_2b_3) \\ & + (yb_3b_4 - yb'_2b_3b_4) + (zb_3b_4 - zb'_1b_3b_4) + (wb_4 - wb'_1b_4 - wb'_2b_4 + wb'_1b'_2b_4), \end{aligned}$$

so ignoring all linear and quadratic terms it is

$$\begin{aligned} f' = & kb'_1b'_2b_3b_4 + xb'_1b'_2b_3 + (-y - k)b'_2b_3b_4 + (-z - k)b_3b_4b'_1 + wb_4b'_1b'_2 \\ = & k'b'_1b'_2b_3b_4 + x'b'_1b'_2b_3 + y'b'_2b_3b_4 + z'b_3b_4b'_1 + w'b_4b'_1b'_2 \end{aligned}$$

This is of the original form with $y' = -y - k$ and $z' = -z - k$ and other coefficients unchanged. If we have a 1-auxiliary quadratisation for f in terms of b_1, b_2, b_3, b_4, b_a , then after the substitution and taking care of the linear and quadratic terms in f , we obtain a 1-aux quadratisation for f' in terms of $b'_1, b'_2, b_3, b_4, b_a$.

If f has $k \geq 0$ and $w, x, y, z \geq -\frac{k}{2}$ as in 0.1, then f' has $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$. And this correspondence is invertible, so given any f' with $k' \geq 0$ and $w', x' \geq -\frac{k}{2}$ and $y', w' \leq -\frac{k}{2}$, we know that it has a 1-aux quadratisation. We can also do the same substitution on the other pair of variables b_3, b_4 to prove that any f'' with $k'' \geq 0$ and $w'', x'', y'', z'' \leq -\frac{k}{2}$ has a 1-aux quadratisation.

To sum up, if $k \geq 0$, an even number of x, y, z, w are at least $-\frac{k}{2}$, and an even number of them are most $-\frac{k}{2}$, then f has a quadratisation in 1 auxiliary.

We can also consider substituting $b'_1 = 1 - b_1$ but not flipping b_2 , then

$$f = k(b_2b_3b_4 - b'_1b_2b_3b_4) + x(b_2b_3 - b'_1b_2b_3) + yb_2b_3b_4 + z(b_3b_4 - b_3b_4b'_1) + w(b_2b_4 - b_4b'_1b_2),$$

so ignoring quadratic and linear terms we are left with

$$f' = -kb'_1b_2b_3b_4 - xb'_1b_2b_3 + (k + y)b_2b_3b_4 - zb_3b_4b'_1 - wb_4b'_1b_2.$$

If $k \geq 0$ and $y \leq -k$ and $x, z, w \geq 0$, then f' is of the form in 0.2, so we can quadratise f' . Putting back in the quadratic and linear terms and substitute back, we can obtain a quadratisation of f .

By flipping another pair $b'_2 = 1 - b_2$ and $b'_3 = 1 - b_3$ as before, we also see that if $k \geq 0$ and three of x, y, z, w are at most $-k$ and the remaining one is at least 0, then f can be quadratised with 1 auxiliary.

Therefore, we have proved the following.

Proposition 0.5

If $k \geq 0$ and either of the following holds:

1. $x, y, z, w \geq -\frac{k}{2}$;
2. three of them are at least 0 and the other one is at most $-k$;
3. two are at least $-\frac{k}{2}$ and the other two are at most $-\frac{k}{2}$;
4. one is at least 0 and the other three are at most $-k$; or
5. $x, y, z, w \leq -\frac{k}{2}$;

Then f has a quadratisation with 1 auxiliary.

Lemma 0.6

If $k \geq 0$, $-k \leq x \leq -\frac{k}{2}$, and $w, y, z \geq 0$, then $g = b_a((3k + x + y + z + w) - (2k + x + z + w)b_1 - (2k + x + y + w)b_2 - (2k + x + y + z)b_3 - (2k + y + z + w)b_4) + k(b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4) + x(b_1b_2 + b_1b_3 + b_2b_3) + y(b_2b_3 + b_2b_4 + b_3b_4) + z(b_1b_3 + b_1b_4 + b_3b_4) + w(b_1b_2 + b_1b_4 + b_2b_4)$ is a quadratisation of f .

Remark 0.7. This is the same quadratisation as in 0.1.

Proof. By symmetry, WLOG $b_1 \leq b_2 \leq b_3$ (we cannot make assumption on b_4 because x is special among x, y, z, w). If $\mathbf{b} = 0000$, then $g = b_a(3k + w + x + y + z)$. Since $k + x \geq 0$ and $k, y, z, w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = b_a(k + x)$. Since $k + x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = b_a(k + w)$. Since $k, w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = -(b_a - 1)(k + y + z)$. Since $-k \leq 0$ and $-y \leq 0$ and $-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = -(b_a - 1)(k + x + y)$. Since $-(k + x) \leq 0$ and $-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = 3k + w + x + 3y + z - b_a(3k + w + x + 2y + z)$. Since $-(k + x) \leq 0$ and $-k, -w, -y, -z \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\mathbf{b} = 1110$, then $g = 3k + w + 3x + y + z - b_a(3k + w + 2x + y + z)$. Since $-2(k + x) \leq 0$ and $-k, -w, -y, -z \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-2(k + x) \leq 0$ and $-k, -w, -y, -z \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. \square

For $k \geq 0$, we have shown that f has a 1-auxiliary quadratisation if $x, y, z, w \geq -\frac{k}{2}$, or if $x \leq -k$ and $w, y, z \geq 0$ (both cases are summarised in 0.5), so we can combine them with 0.6 (for $-k \leq x \leq -\frac{k}{2}$, and $w, y, z \geq 0$) to conclude:

Proposition 0.8

f can be quadratised with 1 auxiliary if $k, y, z, w \geq 0$, whatever the value of x is.

Recall that we can substitute $b'_1 = 1 - b_1$ and $b'_2 = 1 - b_2$ to transform y to $-k - y$ and z to $-k - z$. This means we can flip a pair of the cubic coefficients in the range $[0, +\infty)$ to $(-\infty, -k]$ or vice versa. (e.g. since we have a quadratisation for $w \geq 0, y \geq 0, z \geq 0$,

we can find a quadratisation for $w \leq -k, y \leq -k, z \geq 0$) Or we may as well flip x and flip one other coefficient in the proposition above. Since the above proposition does not place any condition on x , we can e.g. go from $w \geq 0, y \geq 0, z \geq 0$ to $w \leq -k, y \geq 0, z \geq 0$. This gives the following

Proposition 0.9

If $k \geq 0$ and three of the cubic coefficients are in $(-\infty, -k] \cup [0, +\infty)$, then f has a quadratisation with 1 auxiliary.

If some f has $k \geq 0$ but is not covered by 0.9, then at least two of the cubic coefficients are in $[-k, 0]$. If it is not covered by 0.5 either, then an odd number of cubic coefficients lie on each side of $-\frac{k}{2}$ (WLOG $x \leq -\frac{k}{2}$ and $y, z, w \geq -\frac{k}{2}$). This is what we will break down into a further 6 cases.

Below we assume $k \geq 0$.

Cases	$x \leq -k$	$-k \leq x \leq -\frac{k}{2}$	
		$k + x + y \leq 0$	$k + x + y \geq 0$
$-\frac{k}{2} \leq y \leq 0 \leq z \leq w$	0.9: only $y \in [-k, 0]$	0.12	0.11
$-\frac{k}{2} \leq y \leq z \leq 0 \leq w$	0.10		0.13
$-\frac{k}{2} \leq y \leq z \leq w \leq 0$	0.15		0.14

Lemma 0.10

If $x \leq -k$ and $-\frac{k}{2} \leq y \leq z \leq 0 \leq w$, then $g = b_a(x(b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + zb_1b_4 + wb_1b_4 + kb_1b_4$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = b_a(k + w + z)$. Since $w \geq 0$ and $k + z \geq 0$, the minimiser is $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. Since $k + 2z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = k + w - x - y + z + b_a(-k - w + x + y - z)$. Since $-k \leq 0$ and $-w \leq 0$ and $y \leq z$ and $-x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = k + w + z + b_a(-k - w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-(k + z) \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w + z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x) \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.11

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq z \leq w$ and $k + x + y \geq 0$, then $g = b_a(x(-1 + b_1 + b_2 + b_3) + y(-1 + b_2 + b_3 + b_4) + z(-1 + b_1 + b_3 + b_4) + w(-1 + b_1 + b_2 + b_4) + k(-3 + 2b_1 + 2b_2 + 2b_3 + 2b_4)) + x(+1 - b_1 - b_2 - b_3 + b_1b_2 + b_1b_3 + b_2b_3) + y(+1 - b_2 - b_3 - b_4 + b_2b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 - b_3 - b_4 + b_1b_3 + b_1b_4 + b_3b_4) + w(+1 - b_1 - b_2 - b_4 + b_1b_2 + b_1b_4 + b_2b_4) + k(+3 - 2b_1 - 2b_2 - 2b_3 - 2b_4 + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$. Since $-x - k \leq 0$ and $-y - k \leq 0$ and $-k \leq 0$ and $-z \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = b_a(k + y + z)$. Since $k + y \geq 0$ and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k \leq 0$ and $-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = b_a(k + w + z)$. Since k and w and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = b_a(k + x + z)$. Since $k + x \geq 0$ and $z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = z + b_a(3k + w + x + y + 2z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = b_a(k + w + x)$. Since $k + x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = w + b_a(3k + 2w + x + y + z)$. Since $k + x \geq 0$ and $k + y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = x + b_a(3k + w + 2x + y + z)$. Since $2k + 2x \geq 0$ and $k + y \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = k + w + x + y + z + b_a(5k + 2w + 2x + 2y + 2z)$. Since $2k + 2x \geq 0$ and $2k + 2y \geq 0$ and $k \geq 0$ and $w \geq z \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.12

If $-k \leq x \leq -\frac{k}{2} \leq y \leq 0 \leq w$ and $y \leq z \leq w$ and $k + x + y \leq 0$, then $g = b_a(x(+b_1 - b_2 - b_3) + y(-b_2 - b_3 + b_4) + z(1 - b_1 + b_3 - b_4) + w(1 - b_1 + b_2 - b_4) + k(1 - b_1 - b_4)) + x(-b_1 + b_1b_2 + b_1b_3) + y(-b_4 + b_2b_4 + b_3b_4) + z(+b_1b_4) + w(+b_1b_4) + k(+b_1b_4)$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = b_a(k + w + z)$. Since $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = b_a y - y$. Since $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = b_a(k + w - x - y + 2z)$. Since $w \geq 0$ and $k + 2z \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = b_a(z - x)$. Since $z \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = b_a(k + 2w - x - y + z)$. Since $k + z \geq 0$ and $w \geq 0$ and $-x \geq 0$ and $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = b_a(w - x)$. Since $w \geq x$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = b_a(k + 2w - 2x - 2y + 2z)$. For the same reason as with $\mathbf{b} = 0010$, $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = y + b_a(w - 2x - y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = b_a x - x$. Since $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = k + w - x - y + z - b_a(k + w - x - y + z)$. Since $-k - z \leq 0$ and $-w \leq 0$ and $-x \leq 0$ and $-y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = b_a(z - y)$. Since $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = k + w + z - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = b_a(w - y)$. Since $w \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = k + w + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = x + b_a(w - x - 2y + z)$. Since $w \geq 0$ and $z - y \geq 0$ and $-y \geq 0$ and $-x \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = k + w + x + y + z - b_a(k + x + y)$. Since $-(k + x + y) \geq 0$, $b_a^* = 0$, so $\min g = k + w + x + y + z = f$.

□

Lemma 0.13

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq 0 \leq w$ and $k + x + y \geq 0$, then $g = b_a(x(-1 + b_1 + b_2 + b_3) + y(-1 + b_2 + b_3 + b_4) + z(-1 + b_1 + b_3 + b_4) + w(-1 + b_1 + b_2 + b_4) + k(-3 + 2b_1 + 2b_2 + 2b_3 + 2b_4)) + x(+1 - b_1 - b_2 - b_3 + b_1b_2 + b_1b_3 + b_2b_3) + y(+1 - b_2 - b_3 - b_4 + b_2b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 - b_3 - b_4 + b_1b_3 + b_1b_4 + b_3b_4) + w(+1 - b_1 - b_2 - b_4 + b_1b_2 + b_1b_4 + b_2b_4) + k(+3 - 2b_1 - 2b_2 - 2b_3 - 2b_4 + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = 3k + w + x + y + z - b_a(3k + w + x + y + z)$. Since $-k - x \leq 0$ and $-\frac{k}{2} - y \leq 0$ and $-\frac{k}{2} - z \leq 0$ and $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = k + x - b_a(k + x)$. Since $-k - x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = k + w - b_a(k + w)$. Since $-k \leq 0$ and $-w \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = b_a(k + y + z)$. Since $\frac{k}{2} + y \geq 0$ and $\frac{k}{2} + z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = k + z - b_a(k + z)$. Since $-k - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = b_a(k + w + y)$. Since $k + y \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = b_a(k + x + y)$. Since $k + x + y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = y + b_a(3k + w + x + 2y + z)$. Since $k + x \geq 0$ and $k + 2y \geq 0$ and $k + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = k + y - b_a(k + y)$. Since $-k - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = b_a(k+w+z)$. Since $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = b_a(k+x+z)$. Since $k+x+z \geq k+x+y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = z + b_a(3k+w+x+y+2z)$. Since $k+x \geq 0$ and $k+y \geq 0$ and $k+2z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = b_a(k+w+x)$. Since $k+x \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = w + b_a(3k+2w+x+y+z)$. Since $k+x \geq 0$ and $k+y \geq 0$ and $k+z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = x + b_a(3k+w+2x+y+z)$. Since $2k+2x \geq 0$ and $\frac{k}{2} + y \geq 0$ and $\frac{k}{2} + z \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = k+w+x+y+z + b_a(5k+2w+2x+2y+2z)$. Since $2k+2x \geq 0$ and $k+2y \geq 0$ and $k+2z \geq 0$ and $k \geq 0$ and $w \geq 0$, $b_a^* = 0$, so $\min g = k+w+x+y+z = f$. \square

Lemma 0.14

If $-k \leq x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k+x+y \geq 0$, then $g = b_a(x(1-b_1-b_2-b_3) + y(1-b_2-b_3-b_4) + z(1-b_1-b_3-b_4) + w(1-b_1-b_2-b_4) + k(3-2b_1-2b_2-2b_3-2b_4)) + x(+b_1b_2+b_1b_3+b_2b_3) + y(+b_2b_3+b_2b_4+b_3b_4) + z(+b_1b_3+b_1b_4+b_3b_4) + w(+b_1b_2+b_1b_4+b_2b_4) + k(+b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = b_a(3k+w+x+y+z)$. Since $k+x \geq 0$ and $2k+w+y+z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = b_a(k+x)$. Since $k+x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = b_a(k+w)$. Since $k+w \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = k+y+z - b_a(k+y+z)$. Since $-\frac{k}{2} - y \leq 0$ and $-\frac{k}{2} - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = b_a(k+z)$. Since $k+z \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = k+w+y - b_a(k+w+y)$. Since $-\frac{k}{2} - w \leq 0$ and $-\frac{k}{2} - y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = k+x+y - b_a(k+x+y)$. Since $-(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = 3k+w+x+3y+z - b_a(3k+w+x+2y+z)$. Since $-k-x \leq 0$ and $-k-2y \leq 0$ and $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = b_a(k+y)$. Since $k+y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = k+w+z - b_a(k+w+z)$. Since $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = k+x+z - b_a(k+x+z)$. Since $-(k+x+z) \leq -(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = 3k+w+x+y+3z - b_a(3k+w+x+y+2z)$. Since $-k-x \leq 0$ and $-k-w-y \leq 0$ and $-k-2z \leq 0$, $b_a^* = 1$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = k+w+x - b_a(k+w+x)$. Since $-(k+w+x) \leq -(k+x+y) \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = 3k+3w+x+y+z - b_a(3k+2w+x+y+z)$. Since $-k-2w \leq 0$ and $-k-x \leq 0$ and $-k-y-z \leq 0$, $b_a^* = 1$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = 3k+w+3x+y+z - b_a(3k+w+2x+y+z)$. Since $-(k+x+y) \leq 0$ and $-k-x \leq 0$ and $-k-w-z \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = 6k + 3w + 3x + 3y + 3z - b_a(5k + 2w + 2x + 2y + 2z)$. Since $-2k - 2x \leq 0$ and $-k - 2y \leq 0$ and $-k - 2z \leq 0$ and $-k - 2w \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. □

Lemma 0.15

If $x \leq -\frac{k}{2} \leq y \leq z \leq w \leq 0$ and $k + x + y \leq 0$, then $g = b_a(x(-b_1 + b_2 + b_3) + y(+b_2 + b_3 - b_4) + z(-1 + b_1 - b_3 + b_4) + w(-1 + b_1 - b_2 + b_4) + k(-1 + b_1 + b_4)) + x(-b_2 - b_3 + b_1b_2 + b_1b_3) + y(-b_2 - b_3 + b_2b_4 + b_3b_4) + z(+1 - b_1 + b_3 - b_4 + b_1b_4) + w(+1 - b_1 + b_2 - b_4 + b_1b_4) + k(+1 - b_1 - b_4 + b_1b_4)$ is a quadratisation of f .

Proof. If $\mathbf{b} = 0000$, then $g = k + w + z - b_a(k + w + z)$. Since $-k - w - z \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0001$, then $g = -b_a y$. Since $-y \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 0010$, then $g = k + w - x - y + 2z + b_a(-k - w + x + y - 2z)$. Since $-k - 2z \leq 0$ and $x \leq w$ and $y \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0011$, then $g = z - x + b_a(x - z)$. Since $x \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0100$, then $g = k + 2w - x - y + z + b_a(-k - 2w + x + y - z)$. Since $-k - 2w \leq 0$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0101$, then $g = w - x + b_a(x - w)$. Since $x \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0110$, then $g = k + 2w - 2x - 2y + 2z + b_a(-k - 2w + 2x + 2y - 2z)$. Since $-k \leq 0$ and $x \leq z$ and $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 0111$, then $g = w - 2x + z + b_a(-w + 2x + y - z)$. Since $x \leq w$ and $y \leq z$ and $x \leq 0$, $b_a^* = 1$, so $\min g = y = f$.

If $\mathbf{b} = 1000$, then $g = -b_a x$. Since $-x \geq 0$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1001$, then $g = b_a(k + w - x - y + z)$. Since $k \geq 0$ and $w \geq x$ and $z \geq y$, $b_a^* = 0$, so $\min g = 0 = f$.

If $\mathbf{b} = 1010$, then $g = z - y + b_a(y - z)$. Since $y \leq z$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1011$, then $g = z + b_a(k + w)$. Since $k + w \geq 0$, $b_a^* = 0$, so $\min g = z = f$.

If $\mathbf{b} = 1100$, then $g = w - y + b_a(y - w)$. Since $y \leq w$, $b_a^* = 1$, so $\min g = 0 = f$.

If $\mathbf{b} = 1101$, then $g = w + b_a(k + z)$. Since $k + z \geq 0$, $b_a^* = 0$, so $\min g = w = f$.

If $\mathbf{b} = 1110$, then $g = w - 2y + z + b_a(-w + x + 2y - z)$. Since $x \leq w$ and $y \leq z$ and $y \leq 0$, $b_a^* = 1$, so $\min g = x = f$.

If $\mathbf{b} = 1111$, then $g = w + z + b_a(k + x + y)$. Since $k + x + y \leq 0$, $b_a^* = 1$, so $\min g = k + w + x + y + z = f$. □

Hence we have shown that every f with $k \geq 0$ has a quadratisation with 1 auxiliary. Using the substitution $b'_1 = 1 - b_1$, we can go between cases with $k \geq 0$ and $k \leq 0$, so all 4-variable pseudo-boolean functions has a quadratisation with 1 auxiliary.

Theorem 0.16 (A sufficient condition for existence of 1-aux quadratisation)

Let $f = f(\mathbf{b})$ be fully symmetric in the n bits b_1, \dots, b_n . Write $|\mathbf{b}| = \sum b_i$ and $h(|\mathbf{b}|) = f(\mathbf{b})$. (Note that h is well-defined because f is symmetric. Suppose $h(0), \dots, h(n)$ is an arithmetic progression but with all positive terms replaced by 0, i.e. $h(i) = (ai + d)^- = \min(ai + d, 0)$ for some constant a, d , then f is quadratisable with one auxiliary.

Proof. In fact we have a quadratisation that is also symmetric in b_1, \dots, b_n . Let $g = b_a(a|\mathbf{b}| + d) = b_a(a(\sum b_i) + d)$. This is quadratic in b, b_a and

$$\min_{b_a} g = f \quad \forall \mathbf{b} \in \{0, 1\}^n$$

using the property of f . □