Embedding quadratization gadgets for reduction by substitution

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Consider the function:

$$Ab_1b_2b_3 + Bb_2b_3b_4 + Cb_3b_4b_1 + Db_4b_1b_2 \tag{1}$$

We will *re-use* auxiliary variables by pairing variables appropriately:

$$b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_1 + b_4b_1b_2 \tag{2}$$

Here only 2 auxiliary variables are required in total. **If 2 auxiliaries is the best we can do for this function, is the above quadratization the best we can do in terms of graph connectivity?**
Let us continue with a cyclic function of cubic terms, but with 5 variables instead of 4:

$$Ab_1b_2b_3 + Bb_2b_3b_4 + Cb_3b_4b_5 + Db_4b_5b_1 + Eb_5b_1b_2 \tag{3}$$

Now 3 auxuliary variables are required in total:

$$b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_5 + b_4b_5b_1 + b_5b_1b_2 \tag{4}$$

Is it possible to quadratize this with fewer qubits? Maybe we need to do a brute force check.

Let us now try with $n = 6$ and $k = 3$:

$$Ab_1b_2b_3 + Bb_2b_3b_4 + Cb_3b_4b_5 + Db_4b_5b_6 + Eb_5b_6b_1 + Fb_6b_1b_2 \tag{5}$$
$$=b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_5 + b_4b_5b_6 + b_5b_6b_1 + b_6b_1b_2 \tag{6}$$

We therefore only 3 need auxiliaries. **Is it possible to quadratize this with fewer qubits?** Maybe we need to do a brute force check.

This analysis is staight-forward to contunue to larger n and $k = 3$.
Let's now look at $n = 5$ and $k = 4$:

$$Ab_1b_2b_3b_4 + Bb_2b_3b_4b_5 + Cb_3b_4b_5b_1 + Db_4b_5b_1b_2 + Eb_5b_1b_2b_3. \tag{7}$$

Here we will pair some variables again (2 auxiliaries):

$$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_1 + b_4b_5b_1b_2 + b_5b_1b_2b_3 \tag{8}$$
$$=b_{a_{12}}b_{a_{34}} + b_2b_{a_{34}}b_5 + b_{a_{34}}b_5b_1 + b_4b_5b_{a_{12}} + b_5b_{a_{12}}b_3 \tag{9}$$

We cannot pair anymore variables, so we need 4 more auxiliaries. In total we have 6 auxiliaries.
Is it possible to quadratize this with fewer than 6 qubits? Some symmetric functions of 5 variables only need 3 auxiliaries

Interestingly, by adding a degre-5 term ($n = 5, k = 5$), we can do it with 4 auxiliaries:

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$$Ab_1b_2b_3b_4b_5 + Bb_1b_2b_3b_4 + Cb_2b_3b_4b_5 + Db_3b_4b_5b_1 + Eb_4b_5b_1b_2 + Fb_5b_1b_2b_3. \tag{10}$$

Here we will pair some variables again (2 auxiliaries):

$$\begin{aligned} & b_1b_2b_3b_4b_5 + b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_1 + b_4b_5b_1b_2 + b_5b_1b_2b_3 && (11)\\ =& b_{a_{12}}b_{a_{34}}b_5 + b_{a_{12}}b_{a_{34}} + b_2b_{a_{34}}b_5 + b_{a_{34}}b_5b_1 + b_4b_5b_{a_{12}} + b_5b_{a_{12}}b_3 && (12)\\ =& b_{a_{12}}b_{a_{34,5}} + b_{a_{12}}b_{a_{34}} + b_2b_{a_{34,5}} + b_{a_{34,5}}b_1 + b_4b_{a_{12,5}} + b_{a_{12,5}}b_3 && (13) \end{aligned}$$

Is it possible to quadratize this with fewer than 4 qubits? Some symmetric functions of 5 variables only need 3 auxiliaries.

Now let's look at $n = 6$ and $k = 5$:

$$Ab_1b_2b_3b_4b_5 + Bb_2b_3b_4b_5b_6 + Cb_3b_4b_5b_6b_1 + Db_4b_5b_6b_1b_2 + Eb_5b_6b_1b_2b_3 + Fb_6b_1b_2b_3b_4. \tag{14}$$

Let us introduce 3 auxuliuaries:

$$\begin{aligned} & b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_1 + b_4b_5b_6b_1b_2 + b_5b_6b_1b_2b_3 + b_6b_1b_2b_3b_4 && (15)\\ =& b_{a_{12}}b_{a_{34}}b_5 + b_2b_{a_{34}}b_{a_{56}} + b_{a_{34}}b_{a_{56}}b_1 + b_4b_{a_{56}}b_{a_{12}} + b_{a_{56}}b_{a_{12}}b_3 + b_6b_{a_{12}}b_{a_{34}} && (16) \end{aligned}$$

Let us now pair up some of the auxiliary qubits!

$$b_{a_{12}a_{34}}b_5 + b_2b_{a_{34}a_{56}} + b_{a_{34}a_{56}}b_1 + b_4b_{a_{56}a_{12}} + b_{a_{56}a_{12}}b_3 + b_6b_{a_{12}a_{34}}$$

This is now quadratic, with 6 total auxiliaries. **Can it be done with fewer?**