A mathematical definition of the Bulk Pegasus graph

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1 Preliminaries

Firstly note that these rules are only intended to reproduce the bulk (i.e. far from the edge of the graph) behavior of the Pegasus graph, and not the boundaries, which are different in the definition used in the Pegasus whitepaper. In particular, our L is not the same as the M defined in the whitepaper. This work is motivated by trying to gain a better understanding of the structure of the Pegasus graph, rather than designing alternative algorithms to produce a $P(M)$.

Start with three $L \times L$ chimera graphs with the following associated indices

- $c \in \{0, 2\}$ corresponding to the three chimera graphs
- $i \in \{0, L-1\}$ corresponding to the column which the chimera unit cell is on
- $j \in \{0, L-1\}$ corresponding to the row of the chimera unit cell
- $u \in \{0, 1\}$ corresponding to which side of the chimera $k_{4,4}$ the vertex is

Within each side of the $k_{4,4}$, we further define two more indices,

- $p \in \{0, 1\}$
- $q \in \{0, 1\}$

A node within the Pegasus graph is therefore defined by the following 6 indices (c, i, j, u, p, q)

2 Periodic definition

We now define each of the additional edges which make the three chimera graphs into the Pegasus graph, to define this we denote edges in terms of the node indices they connect, so that $(c, i, j, u, p, q) \to (c', i', j', u', p', q')$ indicates an edge between the node defined by (c, i, j, u, p, q) and the one defined by

(c', i', j', u', p', q') in this notation, we will often add or subtract numbers to node indices, for c this is taken to be addition modulo 3, while for u, p, and q, it is taken to be addition modulo 2 the graph is not periodic in i, j, so addition in these directions is *not* modulo L. To start with, let us consider the two edges added to each $k_{4,4}$, these are defined as

$$(c, i, j, u, p, q = 0) \rightarrow (c, i, j, u, p, q = 1) \tag{1}$$

Next, we consider edges which connect two unit cells with the same (i, j), but in a different layer of chimera graph, these are defined as, note that these connections only happen when $c = 0$ or $c = 1$, there is a shift in i and j when we wrap around the periodic boundaries which will be discussed later.

$$(c \in \{0, 1\}, i, j, u, p = 0, q) \rightarrow (c + 1, i, j, u + 1, *, *), \tag{2}$$

where $*$ indicate that it connects to all valid indices, the above equation therefore actually defines four "outgoing" edges from each vertex. Additionally, for the $c \in \{0, 1\}$ vertexes, there will be the following connections

$$(c \in \{0, 1\}, i, j, u = 1, p = 1, *) \rightarrow (c + 1, i, j - 1, u = 0, *, *) \tag{3}$$

and

$$(c \in \{0, 1\}, i, j, u = 0, p = 1, *) \rightarrow (c + 1, i - 1, j, u = 1, *, *). \tag{4}$$

We now need to worry about what happens when we wrap around the periodic boundaries, in this case both i and j get shifted by one, resulting in

$$(2, i, j, u, p = 0, q) \rightarrow (0, i + 1, j + 1, u + 1, *, *), \tag{5}$$

Finally, there need to be special rules for 3 and 4 in the case of $c = 2$, in this case there needs to be a diferent shift in i or j, due to the peroidic wrap notably, the shift needs to be by $+1$ rather than -1 to the appropriate index:

$$(2, i, j, u = 1, p = 1, *) \rightarrow (0, i, j + 1, u = 0, *, *) \tag{6}$$

and

$$(2, i, j, u = 0, p = 1, *) \rightarrow (0, i + 1, j, u = 1, *, *). \tag{7}$$

Applying Eqs. 1, 2, 3, 4, 5, 6, and 7 to obtain additional connections to every chimera unit cell (and summing over u, p, and q) will produce a Pegasus graph, without producing the same edge twice.

Note that Eqs. 3, 4, 6, and 7 have all changed

3 Incoming edges

Note that the incoming edge equations have not been corrected yet.

Based on Eqs. 2, 3, and 4, we also have "incoming" edges from other unit cells, for instance the incoming edges defined by Eq. 2 are ($c \in 1, 2$)

$$(c \in \{1, 2\}, i, j, u, *, *) \leftarrow (c + 2, i, j, u + 1, p = 0, *). \tag{8}$$

Similarly from Eq. 3, we obtain

$$(c, i, j, u = 0, *, *) \leftarrow (c + 2, i, j - 1, u = 1, p = 1, *). \tag{9}$$

From Eq. 4 we obtain

$$(c, i, j, u = 1, *, *) \leftarrow (c + 2, i - 1, j, u = 0, p = 1, *). \tag{10}$$

Finally, we need to consider the incoming edges which go across the periodic boundaries from Eq. 5,

$$(0, i, j, u, *, *) \leftarrow (2, i - 1, j - 1, u + 1, p = 0, *). \tag{11}$$

From these rules, we are now able to count the number of edges incident on a vertex which is not near the border of the graph, to start off, each vertex will have 6 edges coming from its chimera graph. The vertex will also have an additional edge from Eq. 1, leading to a total of 7 edges If $p = 0$, the vertex will additionally have four incident edges from Eq. 2, and an additional two edges from Eqs. 9 or 10, as well as two from Eqs. 8 or 11, leading to to a total degree of 15. If instead $p = 1$, then the degree is still 15, but with four edges from Eq. 3 or 4, two from Eqs. 9 or 10 and two from Eqs. 2 or 5.

4 Chimera Edges

For completeness, it is worth defining the chimera edges with the same definitions as before, to start with, the $k_{4,4}$ can be defined as

$$(c, i, j, u = 0, *, *) \rightarrow (c, i, j, u = 1, *, *). \tag{12}$$

Next, we define the vertical coupling between unit cells with different j values

$$(c, i, j, u = 0, p, q) \rightarrow (c, i, j + 1, u = 0, p, q). \tag{13}$$

Finally, the horizontal coupling with different i values

$$(c, i, j, u = 1, p, q) \rightarrow (c, i + 1, j, u = 1, p, q). \tag{14}$$

Note that we are only defining 'outgoing' edges, eqch qubit will have an additional edge coming from $i - 1$ or $j - 1$.