Embedding quadratization gadgets on Chimera and Pegasus graphs

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Nicholas Chancellor[†]
Durham University

We group all known quadratizations of cubic and quartic terms in binary optimization problems into five and eight unique graphs respectively. We then perform a minor embedding of these graphs onto the well-known Chimera graph, and the brand new *Pegasus* graph. In cases where two or more graphs have a minor embedding with the same overhead in terms of auxiliary variables, we make recommendations for which gadgets are best to use for certain problems.

Discrete optimization problems are often naturally formulated in terms of minimizing some polynomial of degree > 2 [1], which is then 'quadratized' into a quadratic function which can be solved using standard algorithms for universal classical computers [2], using special-purpose classical annealers [3], or using quantum annealers [4]. With dozens of quadratization methods available , one should choose the best quadratization for a given problem, and for a given method for solving the problem.

There are ways to quadratize functions of discrete variables without adding any auxiliary variables [5–8], but when those methods cannot be applied we introduce auxiliary variables. The resulting quadratic functions (called 'gadgets') that accurately or exactly simulate the original high-degree functions, will have some connectivity between the binary variables (or bits, or qubits, herein referred to for convenience only, as qubits) which can be represented by a graph in which vertices represent qubits and edges indicate when two different qubits appear together in a quadratic term. Since this graph incorporates no information about the linear terms, constant term, or the coefficients of the quadratic terms, many different gadgets have the same graph, therefore in this paper we will classify all known quadratization gadgets into categories according to their corresponding graph (herein called their 'gadget graph').
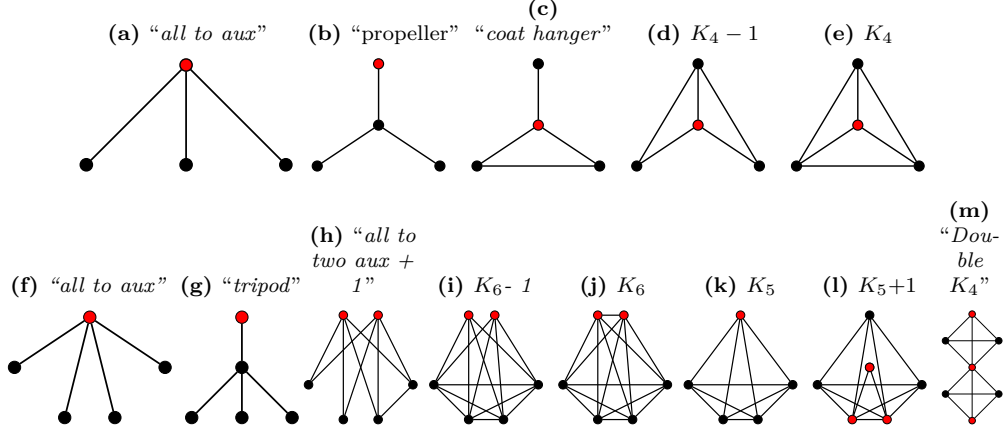
Figure 1: Graph connectivities for D-Wave's Chimera and Pegasus graphs.

(a) Chimera

(b) Pegasus

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* n.dattani@cfa.harvard.edu

† nicholas.chancellor@durham.ac.uk

Figure 2: Gadget graphs. Graphs showing the connectivity between qubits in quadratization gadgets for cubic to quadratic gadgets (top row), and quartic to quadratic gadgets (bottom row). Red vertices represent auxiliary qubits and black vertices represent logical qubits. Black edges denote the existence of a quadratic term in the gadget, involving the two corresponding qubits represented by vertices connected by the edge. *Linear terms in the gadget are completely ignored in these gadget graphs.*

(a) "all to aux" **(b)** "propeller" **(c)** "*coat hanger*" **(d)** $K_4 - 1$ **(e)** K_4

(f) "*all to aux*" **(g)** "*tripod*" **(h)** "*all to two aux + 1*" **(i)** K_6-1 **(j)** K_6 **(k)** K_5 **(l)** K_5+1 **(m)** "*Double K_4*"

Gadget graphs for all known single cubic terms and for all known single quartic terms are given in Figure 2. Gadget graphs tell us a lot about how costly the quadratic optimization problem will be, and those with larger connectivity tend to yield more difficult functions to optimize. Furthermore, some optimization methods only work if their corresponding graph has a certain connectivity, two examples of such connectivities being the ones in D-Wave's well-known Chimera graph, and in their very recently presented *Pegasus* graph, both shown in Figure. 1.

Any graph, can be mapped onto the Chimera or Pegasus graphs by minor-embedding [9, 10], where the Chimera graph or the Pegasus graph is a graph minor of the graph representing the problem that needs to be optimized. This

Table I: Grouping of all known quadratization gadgets for singe cubic terms, into categories corresponding to their gadget graphs. For each unique gadget graph, the nuber of auxiliary qubits required for construction of the gadget is listed, followed by the number of auxiliary qubits required to minor-embed the gadget graph onto Chimera and Pegasus

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Gadget Graph	Example Gadgets	N_{aux} Quadratization	N_{aux} Embedding	N_{aux} Total		N_{aux} Embedding	N_{aux} Total
			Chimera			Pegasus	
	Cubic → Quadratic						
All to Aux	NTR-KZFD NTR-ABCG	1	0	**1**		0	**1**
Propeller	NTR-ABCB	1	0	**1**		0	**1**
Coat Hanger	PTR-A	1	1	**2**		0	**1**
$K_4 - 1$	NTR-AC	1	1	**2**		0	**1**
K_4	PTR-Ishikawa NTR-RBL-(3→ 2) PTR-BCR-1,2,3,4 PTR-KZ	1	2	**3**		0	**1**
K_5+1	PTR-RBL-(3→ 2)	3	5	**8**		1	**4**

Figure 3

often means that one binary variable in the quadratic optimization problem needs to be represented by two qubits instead of one, making the number of qubits needed to solve the original problem larger than before, and sometimes completely impossible. For example a quartic function with 1000 binary variables has $\binom{1000}{3} > 166$ million possible cubic terms and $\binom{1000}{4} > 40$ billion possible quartic terms which have to be quadratized, and then minor-embedded. If our minimization method can only be applied for up to 50 billion qubits, we cannot afford for each quartic-to-quadratic gadget to require its own auxiliary qubit for minor-embedding.

We have provided minor-embeddings for all gadget graphs in Figure 2, for both Chimera and Pegasus. We note that **all** cubic to quadratic gadgets involvnig one auxiliary qubit can be embedded onto Pegasus without any further auxiliary qubits for the embedding, because Pegasus contains the K_4 graph, which means any possible connections between the three logical qubits and the one auxiliary qubit are already contained in Pegasus. Since Chimera does not contain K_4, only negative cubic terms are so far known to be quadratizable with gadgets that embed directly onto Chimera without any extra qubits for the embedding.

Table II: Grouping of all known quadratization gadgets for single **quartic** terms, into categories corresponding to their gadget graphs. For each unique gadget graph, the nuber of auxiliary qubits required for construction of the gadget is listed, followed by the number of auxiliary qubits required to minor-embed the gadget graph onto Chimera and Pegasus.

Gadget Graph	Example Gadgets	N_{aux} Quadratization	N_{aux} Embedding	N_{aux} Total	N_{aux} Embedding	N_{aux} Total
			Chimera		Pegasus	
All to Aux	NTR-KZFD	1	0	**1**	0	**1**
Tripod	NTR-ABCB	1	0	**1**	0	**1**
All22Aux+1	PTR	2	1	**3**	0	**2**
All2All - 1	PTR-Ishikawa	2	5	**7**	2	**4**
K_5	PTR-BCR-2 PTR-BCR-4 NTR-RBL-$(4\rightarrow 2)$	1	3	**4**	2	**3**
K_6	PTR-BCR-3	2	8	**10**	2	**4**
Double K_4	PTR-KZ(?)	3	5	**8**	1	**4**

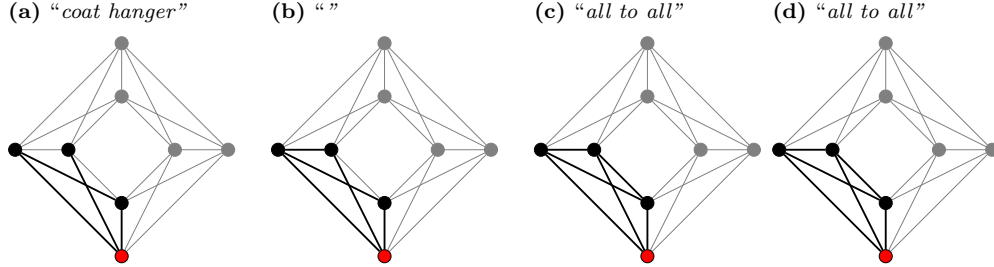# I.   MINOR EMBEDDINGS FOR CUBIC TO QUADRATIC GADGETS

A.   Chimera graph

Figure 4: Minor embeddings of all ***cubic*** to quadratic gadgets onto a 'unit cell' of a Chimera graph. Grey vertices and edges are not used. Thick black edges denote graph minors, in which two physical qubits (two vertices) represent one logical qubit (this is done when logical qubits need to be connected to more qubits than the Chimera unit cell otherwise allows).

(a) "*all to aux*"   **(b)** "*Y*"   **(c)** "*coat hanger*"   **(d)** "*K$_5$*"   **(e)** "*all to all*"
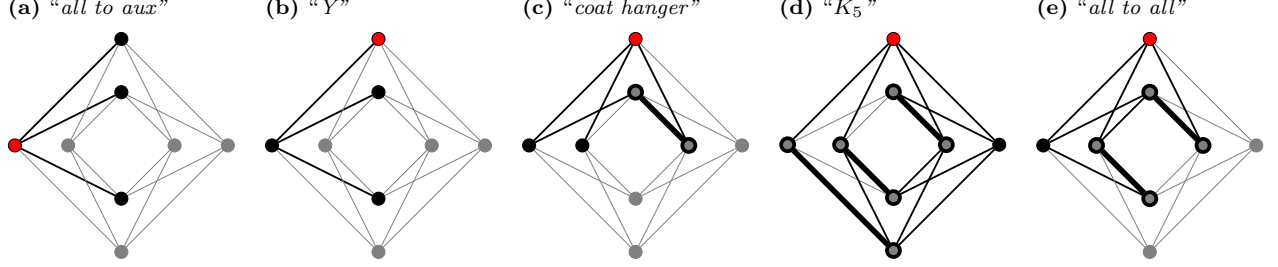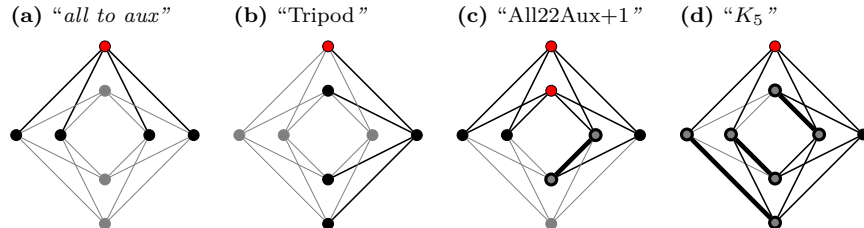

B.   Pegasus graph

Figure 5: Minor embeddings of all ***cubic*** to quadratic gadgets onto a 'unit cell' of a Pegasus graph. Grey vertices and edges are not used. Figures 4a and 4b do not need to be altered since the Chimera graph is a sub-graph of Pegasus, so only the gadgets that required auxiliary qubits for minor embedding onto Chimera are embedded for Pegasus here to show that with Pegasus no auxiliary qubits are needed for cubic to quadratic gadgets.

(a) "*coat hanger*"   **(b)** " "   **(c)** "*all to all*"   **(d)** "*all to all*"

# II.   MINOR EMBEDDINGS FOR QUARTIC TO QUADRATIC GADGETS

A.   Chimera graph

Figure 6: Minor embeddings of all ***quartic*** to quadratic gadgets onto a 'unit cell' of a Chimera graph. Grey vertices and edges are not used. Thick black edges denote graph minors, in which two physical qubits (two vertices) represent one logical qubit (this is done when logical qubits need to be connected to more qubits than the Chimera unit cell otherwise allows).

(a) "*all to aux*"   **(b)** "Tripod"   **(c)** "All22Aux+1"   **(d)** "*K$_5$*"

B. Pegasus graph

Figure 7: Minor embeddings of all **quartic** to quadratic gadgets onto a 'unit cell' of a Pegasus graph. Grey vertices and edges are not used.

(a) *"all to aux"*   **(b)** *" "*   **(c)** *"all to all"*   **(d)** *"K_5+1"*   **(e)** *"K_6"*   **(f)** *"K_5"*   **(g)** *"Two $K_4's$"*
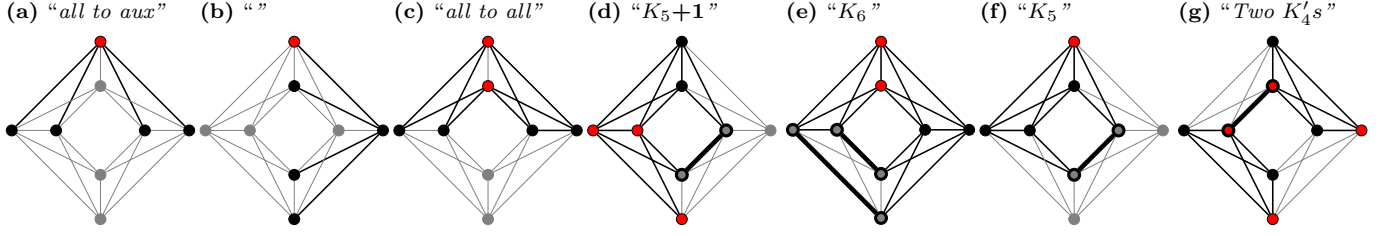
III. RECOMMENDED GADGETS

All gadgets described in this work for quadratizing negative terms (whether cubic or quartic) can be quadratized with only one auxiliary qubit, and can be chimerized and pegasized wihout any futher auxiliary qubits. On Pegasus, there is one 4-local to 2-local gadget for positive terms which stands out over the rest, and it is PTR (which has the "all22aux+1" gadget graph). This is the *only* gadget which embeds positive quartic terms onto Pegasus with only two total auxiliaries (two for the quadratization, and none for the embedding onto the Pegasus graph). All other gadgets for positive quartic terms require three or four total auxiliaries to embed onto Pegasus.

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