A mathematical definition of the Pegasus graph

September 8, 2018

1 Preliminaries

Start with three LxL chimera graphs with the following associated indexes

- $c \in \{0, 2\}$ corresponding to the three chimera graphs
- $i \in \{0, L-1\}$corresponding to the column which the chimera unit cell is on
- $j \in \{0, L-1\}$corresponding to the column of the chimera unit cell
- $u \in \{0, 1\}$ corresponding to which side of the chimera $k_{4,4}$ the vertex is

Within each side of the $k_{4,4}$, we further define two more indexes,

- $p \in \{0, 1\}$
- $q \in \{0, 1\}$

A node within the Pegasus graph is therefore defined by the following 6 indexes (c, i, j, u, p, q)

2 Periodic definition

We now define each of the additional edges which make the three chimera graphs into the Pegasus graph, to define this we denote edges in terms of the node indexes the connect, so that $(c, i, j, u, p, q) \rightarrow (c', i', j', u', p', q')$ indicated an edge between the node defined by (c, i, j, u, p, q) and the one defined by (c', i', j', u', p', q') in this notation, we will often add or subtract numbers to node indexes, for c this is taken to be addition modulo 3, while for u, p, and q,it is taken to be addition modulo 2 the graph is not periodic in i, j, so addition in these directions is *not* modulo L. To start with, let us consider the two edges added to each $k_{4,4}$, these are defined as

$$(c, i, j, u, p, q = 0) \rightarrow (c, i, j, u, p, q = 1) \tag{1}$$

Next, we consider edges which connect to unit cells with the same (i, j), but in a different chimera graph, these are defined as

$$(c, i, j, u, p = 0, q) \to (c + 1, i, j, u + 1, *, *), \tag{2}$$

where $*$ indicate that it connects to all valid indexes, the above equation therefore actually defines four "outgoing" edges from each vertex. Additionally, there for the $p = 1$ vertexes, there will be the following connections

$$(c, i, j, u = 1, p = 1, *) \to (c + 1, i, j + 1, u = 0, *, *) \tag{3}$$

and

$$(c, i, j, u = 0, p = 1, *) \to (c + 1, i + 1, j, u = 1, *, *). \tag{4}$$

Applying Eqs. 1, 2, 3, and 4 to obtain additional connections to every chimera unit cell (and summing over u, p, and q) will produce a Pegasus graph, without producing the same edge twice

3 Incoming edges

Based on Eqs. 2, 3, and 4, we also have "incoming" edges from other unit cells, for instance the incoming edges defined by Eq. 2 are

$$(c, i, j, u, *, *) \leftarrow (c + 2, i, j, u + 1, p = 0, *). \tag{5}$$

Similarly from Eq. 3, we obtain

$$(c, i, j, u = 0, *, *) \leftarrow (c + 2, i, j - 1, u = 1, p = 1, *). \tag{6}$$

Finally from Eq. 4 we obtain

$$(c, i, j, u = 1, *, *) \leftarrow (c + 2, i - 1, j, u = 0, p = 1, *). \tag{7}$$

From these rules, we are now able to count the number of edges incident on a vertex which is not near the border of the graph, to start off, each vertex will have 6 edges coming from its chimera graph. The vertex will also have an additional edge from Eq. 1, leading to a total of 7 edges If $p = 0$, the vertex will additionally have four incident edges from Eq. 2, and an additional two edges from Eqs. 6 or 7, as well as two from Eq. 5, leading to to a total degree of 15. If instead $p = 1$, then the degree is still 15, but with four edges from Eq. 3 or 4, two from Eqs. 6 or 7 and two from Eq. 2.

4 Chimera Edges

For completeness, it is worth defining the chimera edges with the same definitions as before, to start with, the $k_{4,4}$ can be defined as

$$(c, i, j, u = 0, *, *) \to (c, i, j, u = 1, *, *). \tag{8}$$

Next, we define the vertical coupling between unit cells with different j values

$$(c, i, j, u = 0, p, q) \to (c, i, j + 1, u = 0, p, q). \tag{9}$$

Finally, the horizontal coupling with different i values

$$(c, i, j, u = 1, p, q) \to (c, i + 1, j, u = 1, p, q). \tag{10}$$