Embedding quadratization gadgets on Chimera and Pegasus graphs

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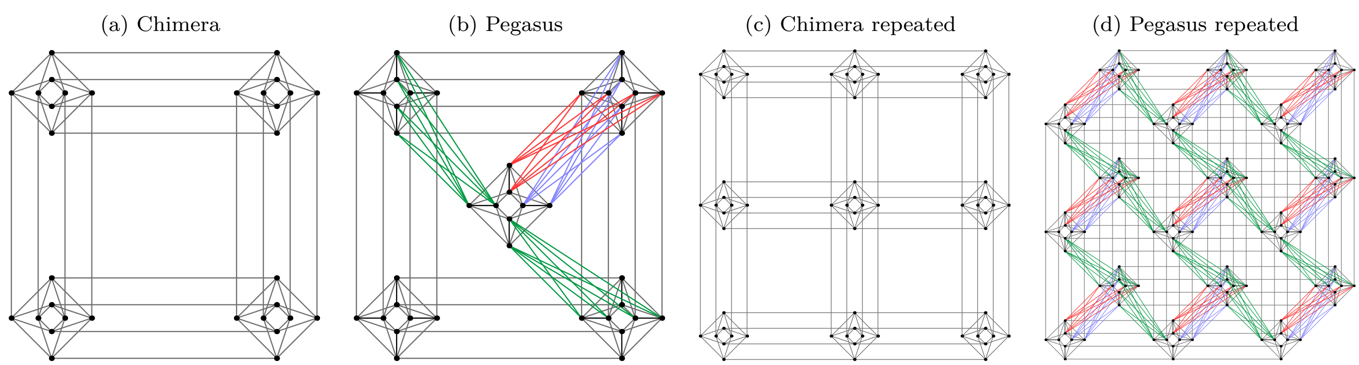
We group all known quadratizations of cubic and quartic terms in binary optimization problems into five and eight unique graphs respectively. We then perform a minor embedding of these graphs onto the well-known Chimera graph, and the brand new *Pegasus* graph. In cases where two or more graphs have a minor embedding with the same overhead in terms of auxiliary variables, we make recommendations for which gadgets are best to use for certain problems.

Discrete optimization problems are often naturally formulated in terms of minimizing some polynomial of degree > 2 Dattani and Bryans [1], which is then 'quadratized' into a quadratic function which can be solved using standard algorithms for universal classical computers Kolmogorov [2], using special-purpose classical annealers [?], or using quantum annealers [?]. With dozens of quadratization methods available , one should choose the best quadratization for a given problem, and for a given method for solving the problem.

There are ways to quadratize functions of discrete variables without adding any auxiliary variables Ishikawa [3], Tanburn *et al.* [4], Okada *et al.* [5], Dridi and Alghassi [6], but when those methods cannot be applied we introduce auxiliary variables. The resulting quadratic functions (called 'gadgets') that accurately or exactly simulate the original high-degree functions, will have some connectivity between the binary variables (or bits, or qubits, herein referred to for convenience only, as qubits) which can be represented by a graph in which vertices represent qubits and edges indicate when two different qubits appear together in a quadratic term. Since this graph incorporates no information about the linear terms, constant term, or the coefficients of the quadratic terms, many different gadgets have the same graph, therefore in this paper we will classify all known quadratization gadgets into categories according to their corresponding graph (herein called their 'gadget graph').

Figure 1: Graph connectivities for D-Wave's Chimera and Pegasus graphs.



Gadget graphs for all known single cubic terms and for all known single quartic terms are given in Figure 2. Gadget graphs tell us a lot about how costly the quadratic optimization problem will be, and those with larger connectivity tend to yield more difficult functions to optimize. Furthermore, some optimization methods only work if their corresponding graph has a certain connectivity, two examples of such connectivities being the ones in D-Wave's well-known Chimera graph [?], and in their very recently presented *Pegasus* graph, both shown in Figure. 1.

Any graph, can be mapped onto the Chimera or Pegasus graphs by minor-embedding Choi [7, 8], where the Chimera graph or the Pegasus graph is a graph minor of the graph representing the problem that needs to be optimized. This often means that one binary variable in the quadratic optimization problem needs to be represented by two qubits

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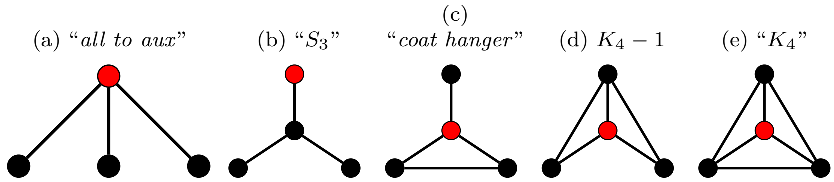
Figure 2



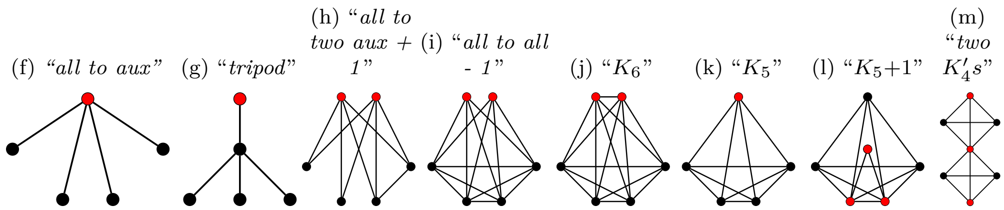


Figure 3

instead of one, making the number of qubits needed to solve the original problem larger than before, and sometimes completely impossible. For example a quartic function with 1000 binary variables has $\binom{1000}{3} > 166$ million possible cubic terms and $\binom{1000}{4} > 40$ billion possible quartic terms which have to be quadratized, and then minor-embedded. If our minimization method can only be applied for up to 50 billion qubits, we cannot afford for each quartic-to-quadratic gadget to require its own auxiliary qubit for minor-embedding.

We have provided minor-embeddings for all gadget graphs in Figure 2, for both Chimera and Pegasus. We note that ***all*** cubic to quadratic gadgets involvnig one auxiliary qubit can be embedded onto Pegasus without any further auxiliary qubits for the embedding, because Pegasus contains the K_4 graph, which means any possible connections between the three logical qubits and the one auxiliary qubit are already contained in Pegasus. Since Chimera does not contain K_4, only negative cubic terms are so far known to be quadratizable with gadgets that embed directly onto Chimera without any extra qubits for the embedding.

Table I

Gadget Graph	Example Gadgets	N_{aux} **Quadratization**	N_{aux} **Embedding** (**Chimera**)	N_{aux} **Total** (**Chimera**)	N_{aux} **Embedding** (**Pegasus**)	N_{aux} **Total** (**Pegasus**)
	Cubic → Quadratic					
All to Aux	NTR-KZFD NTR-ABCG	1	0	**1**	0	**1**
$\boldsymbol{S_3}$	NTR-ABCB	1	0	**1**	0	**1**
Coat Hanger	PTR-A	1	1	**2**	0	**1**
$\boldsymbol{K_4} - 1$	NTR-AC	1	1	**2**	0	**1**
$\boldsymbol{K_4}$	PTR-Ishikawa NTR-RBL-(3→ 2) PTR-BCR-1,2,3,4 PTR-KZ	1	2	**3**	0	**1**
K_5**+1**	PTR-RBL-(3→ 2)	3	5	**8**	1	**4**

I. MINOR EMBEDDINGS FOR CUBIC TO QUADRATIC GADGETS

A. Chimera graph

Figure 4: Minor embeddings of all cubic to quadratic gadgets onto a 'unit cell' of a Chimera graph. Transparent vertices and edges are not used. Thick black edges denote graph minors, in which two physical qubits (two vertices) represent one logical qubit (this is done when logical qubits need to be connected to more qubits than the Chimera unit cell otherwise allows).

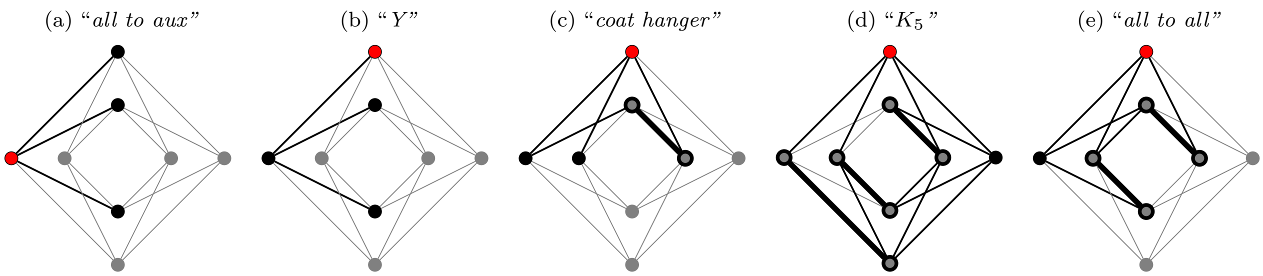


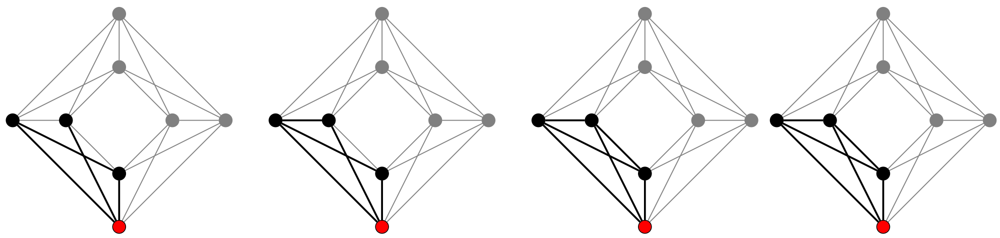
Table II

Gadget Graph	Example Gadgets	N_{aux} **Quadratization**	N_{aux} **Embedding** (Chimera)	N_{aux} **Total** (Chimera)	N_{aux} **Embedding** (Pegasus)	N_{aux} **Total** (Pegasus)
All to Aux	NTR-KZFD	1	0	**1**	0	**1**
Tripod	NTR-ABCB	1	0	**1**	0	**1**
All22Aux+1	PTR	2	1	**3**	0	**2**
All2All - 1	PTR-Ishikawa	2	5	**7**	2	**4**
K_5	PTR-BCR-2 PTR-BCR-4 NTR-RBL-(4→ 2)	1	3	**4**	2	**3**
K_6	PTR-BCR-3	2	8	**10**	2	**4**
Two $K_4's$	PTR-KZ(?)	3	5	**8**	1	**4**

B. Pegasus graph

Figure 5: Minor embeddings of all cubic to quadratic gadgets onto a 'unit cell' of a Pegasus graph. Transparent vertices and edges are not used. Figs 4a and 4b do not need to be altered since the Chimera graph is a sub-graph of Pegasus, so only the gadgets that required auxiliary qubits for minor embedding onto Chimera are embedded for Pegasus here to show that with Pegasus no auxiliary qubits are needed for cubic to quadratic gadgets.

(a) "*coat hanger*" (b) "" (c) "*all to all*" (d) "*all to all*"

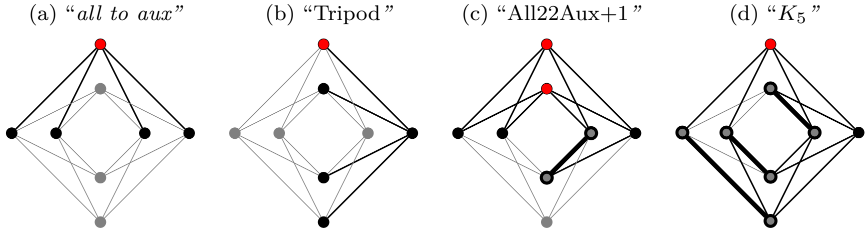


II. MINOR EMBEDDINGS FOR QUARTIC TO QUADRATIC GADGETS

A. Chimera graph

Figure 6: Minor embeddings of all cubic to quadratic gadgets onto a 'unit cell' of a Chimera graph. Transparent vertices and edges are not used. Thick black edges denote graph minors, in which two physical qubits (two vertices) represent one logical qubit (this is done when logical qubits need to be connected to more qubits than the Chimera unit cell otherwise allows).

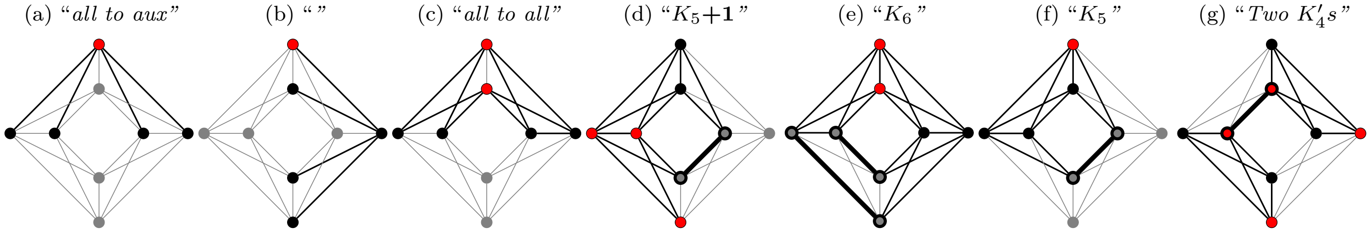
(a) "*all to aux*" (b) "Tripod" (c) "All22Aux+1" (d) "K_5"



B. Pegasus graph

Figure 7: Minor embeddings of all cubic to quadratic gadgets onto a 'unit cell' of a Pegasus graph. Transparent vertices and edges are not used. Figures 6a and 6b do not need to be altered since the Chimera graph is a sub-graph of Pegasus, so only the gadgets that required auxiliary qubits for minor embedding onto Chimera are embedded for Pegasus here to show that with Pegasus no auxiliary qubits are needed for cubic to quadratic gadgets.

(a) "*all to aux*" (b) "" (c) "*all to all*" (d) "K_5**+1**" (e) "K_6" (f) "K_5" (g) "*Two* K_4'*s*"



III. RECOMMENDED GADGETS

All gadgets described in this work for quadratizing negative terms (whether cubic or quartic) can be quadratized with only one auxiliary qubit, and can be chimerized and pegasized wihout any futher auxiliary qubits. On Pegasus, there is one 4-local to 2-local gadget for positive terms which stands out over the rest, and it is PTR (which has the "all22aux+1" gadget graph). This is the *only* gadget which embeds positive quartic terms onto Pegasus with only two total auxiliaries (two for the quadratization, and none for the embedding onto the Pegasus graph). All other gadgets for positive quartic terms require three or four total auxiliaries to embed onto Pegasus.

[1] N. S. Dattani and N. Bryans, http://arxiv.org/abs/1411.6758 (2014), arXiv:1411.6758.
[2] V. Kolmogorov, "QPBO,".
[3] H. Ishikawa, in *2014 IEEE Conference on Computer Vision and Pattern Recognition* (IEEE, 2014) pp. 1362–1369.
[4] R. Tanburn, E. Okada, and N. Dattani, Physical Review A (submitted) (2015), arXiv:1508.04816.
[5] E. Okada, R. Tanburn, and N. S. Dattani, , 5 (2015), arXiv:1508.07190.
[6] R. Dridi and H. Alghassi, Scientific Reports **7**, 43048 (2017).
[7] V. Choi, Quantum Information Processing **7**, 193 (2008).
[8] V. Choi, Quantum Information Processing **10**, 343 (2011).