P4.3 We have a classification problem with four classes of input vector. The four classes are

$$\text{class 1:}\left\{\mathbf{p}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix}1\\2\end{bmatrix}\right\}, \text{class 2: }\left\{\mathbf{p}_3 = \begin{bmatrix}2\\-1\end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix}2\\0\end{bmatrix}\right\},$$

$$\text{class 3:}\left\{\mathbf{p}_5 = \begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{p}_6 = \begin{bmatrix}-2\\1\end{bmatrix}\right\}, \text{class 4: }\left\{\mathbf{p}_7 = \begin{bmatrix}-1\\-1\end{bmatrix}, \mathbf{p}_8 = \begin{bmatrix}-2\\-2\end{bmatrix}\right\}.$$

Design a perceptron network to solve this problem.

To solve a problem with four classes of input vector we will need a perceptron with at least two neurons, since an S-neuron perceptron can categorize 2^S classes. The two-neuron perceptron is shown in Figure P4.2.

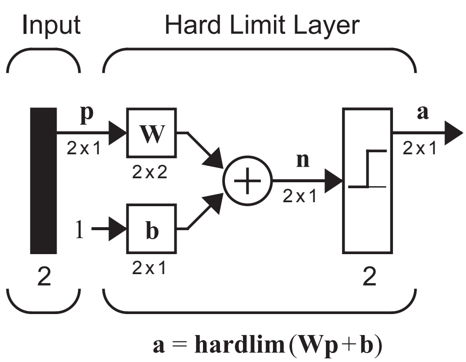

Figure P4.2 Two-Neuron Perceptron

Let's begin by displaying the input vectors, as in Figure P4.3. The light circles ○ indicate class 1 vectors, the light squares □ indicate class 2 vectors, the dark circles ● indicate class 3 vectors, and the dark squares ■ indicate class 4 vectors.

A two-neuron perceptron creates two decision boundaries. Therefore, to divide the input space into the four categories, we need to have one decision boundary divide the four classes into two sets of two. The remaining boundary must then isolate each class. Two such boundaries are illustrated in Figure P4.4. We now know that our patterns are linearly separable.

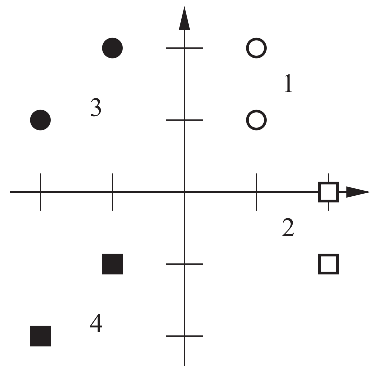

Figure P4.3 Input Vectors for Problem P4.3

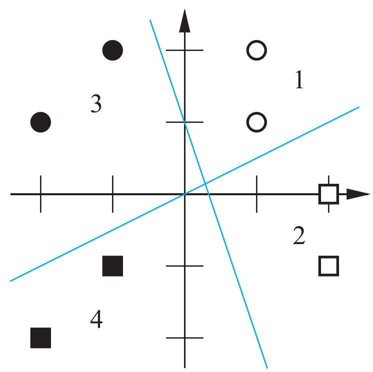

Figure P4.4 Tentative Decision Boundaries for Problem P4.3

The weight vectors should be orthogonal to the decision boundaries and should point toward the regions where the neuron outputs are 1. The next step is to decide which side of each boundary should produce a 1. One choice is illustrated in Figure P4.5, where the shaded areas represent outputs of 1. The darkest shading indicates that both neuron outputs are 1. Note that this solution corresponds to target values of

$$\text{class 1:}\left\{\mathbf{t}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix}0\\0\end{bmatrix}\right\}, \text{class 2: }\left\{\mathbf{t}_3 = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix}0\\1\end{bmatrix}\right\},$$

$$\text{class 3:}\left\{\mathbf{t}_5 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{t}_6 = \begin{bmatrix}1\\0\end{bmatrix}\right\}, \text{class 4: }\left\{\mathbf{t}_7 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{t}_8 = \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

We can now select the weight vectors:

$$_1\mathbf{w} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \text{ and } _2\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Note that the lengths of the weight vectors is not important, only their directions. They must be orthogonal to the decision boundaries. Now we can calculate the bias by picking a point on a boundary and satisfying Eq. (4.15):

$$b_1 = -{}_1\mathbf{w}^T\mathbf{p} = -\begin{bmatrix} -3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1,$$

$$b_2 = -{}_2\mathbf{w}^T\mathbf{p} = -\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

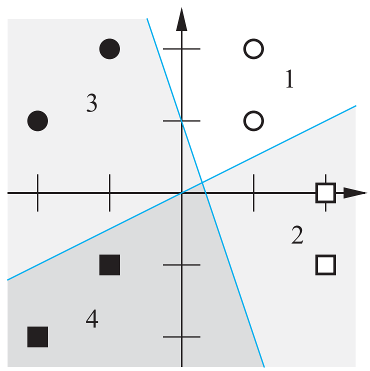

Figure P4.5 Decision Regions for Problem P4.3

In matrix form we have

$$\mathbf{W} = \begin{bmatrix} _1\mathbf{w}^T \\ _2\mathbf{w}^T \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which completes our design.

P4.4 Solve the following classification problem with the perceptron rule. Apply each input vector in order, for as many repetitions as it takes to ensure that the problem is solved. Draw a graph of the problem only after you have found a solution.