HYPERBOLIC MODELS FOR CAT(0) SPACES

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Abstract. We introduce analogues of curve graphs and cubical hyperplanes for the class of CAT(0) spaces. This toolkit sheds new light on CAT(0) spaces, allowing us to prove a dichotomy of a rank-rigidity flavour, establish rigidity theorems for isometries of the curve graphs, characterise rank-one isometries both in terms of their actions on the curve graphs and in terms of these hyperplanes, and find Isom-invariant copies of the Gromov boundaries of the curve graphs in the visual boundary of the underlying CAT(0) space.

1. Introduction

Two of the most well-studied topics in geometric group theory are CAT(0) cube complexes and mapping class groups. This is in part because they both admit powerful combinatorial-like structures that encode interesting aspects of their geometry: hyperplanes for the former and curve graphs for the latter. In recent years, analogies between the two theories have become more and more apparent. For instance, there are counterparts of curve graphs for CAT(0) cube complexes [Hag14, Gen20b] and rigidity theorems for these counterparts that mirror the surface setting [Iva97, Fio22]; it has been shown that mapping class groups are quasiisometric to CAT(0) cube complexes [Pet21]; and both can be studied using the machinery of hierarchical hyperbolicity [BHS19]. However, the considerably larger class of CAT(0) *spaces* is left out of this analogy, as the lack of a combinatorial-like structure presents a difficulty in importing techniques from those areas. In this paper, we bring CAT(0) spaces into the picture by developing analogues of hyperplanes and curve graphs for them.

1.1. Hyperplanes and curve graphs

CAT(0) cube complexes have been studied both via group actions and as interesting spaces in their own right [NR98, SW05, CS11, CFI16, Hua17], and led to groundbreaking advances in 3–manifold theory [Wis21, Ago13]. However, their name hides the fact that it is really their combinatorial structure that makes them so tractable: since revolutionary work of Sageev [Sag95, Sag97], it has become increasingly clear that the geometry of a CAT(0) cube complex is entirely encoded by its hyperplanes and the way they interact with one another. Notably, we are not aware of any cases where methods from the world of cube complexes have been successfully exported to the CAT(0) setting. One explanation could be that CAT(0) cube complexes seem to be rather unrepresentative of the more general class of CAT(0) spaces. For instance, many CAT(0) groups have property (T), but no group admitting an unbounded action on a CAT(0) cube complex can have property (T) [NR97].

The main new notion we introduce is that of *curtains*, which are CAT(0) analogues of hyperplanes.

Definition A. Let X be a CAT(0) space. A *curtain* is $\pi_\alpha^{-1}(P)$, where α is a geodesic, π_α the closest-point projection, and P a subinterval of α of length one not containing the endpoints.

Each curtain delimits two natural "halfspaces", which it separates from each other, and just as in CAT(0) cube complexes (with the ℓ^1 or ℓ^∞ metrics), one can use curtains to measure the distance between two points. Curtains also present key differences from hyperplanes. The two most noteworthy are that the set of curtains is uncountable, and that curtains are not convex (Remark 2.4). However, such differences are necessary, as Sageev's construction [Sag97] produces cube complexes under surprisingly weak conditions. If we are to consider the more general class of CAT(0) spaces, curtains must not satisfy such conditions. Moreover, nonconvexity is even to be expected by comparison with complex hyperbolic space, see Remark 2.4.

The other analogy considered is with *curve graphs* of surfaces. The discovery that curve graphs [Har81] are hyperbolic [MM99] has been one of the most influential results in the theory of mapping class groups, and it has had many important repercussions [FM02, Bow08, Mah11, BCM12, MS13, BBF15]. Since then, analogous spaces have been introduced in different settings with great effect, notably $\operatorname{Out} F_n$ [HV98, BF14, Hat95, KL09, HM13, Man14], free products [Hor16a], right-angled Artin groups [KK13], cocompactly cubulated groups [Hag14, Gen20b], and Artin groups of spherical and FC type [CW17, CGGMW19, MW21].

Given an arbitrary CAT(0) space X, we use curtains to define a new family of metric spaces X_L. More precisely, we define a family of metrics d_L on X and we write $X_L = (X, \mathsf{d}_L)$. These spaces are canonical in the sense that their underlying sets are all X, and the metric is defined intrinsically. This construction is inspired by work of Genevois and Hagen on CAT(0) cube complexes [Hag14, Gen20b]. It will be seen from the results described below that these spaces share many fundamental properties with curve graphs. In the first place, we prove the following.

Theorem B. *For any CAT(0) space X and any natural number L, the space X_L is hyperbolic, and* $\operatorname{Isom} X < \operatorname{Isom} X_L$.

As mentioned, these spaces X_L are meant to capture the hyperbolicity in X, or alternatively to collapse the non-hyperbolicity. As L increases, the X_L see an increasing amount of hyperbolicity: it is always the case that $\mathsf{d}_{L+1}(x,y) \geqslant \mathsf{d}_L(x,y)$. In fact, they will eventually be unbounded when X presents some negatively-curved behaviour, for instance in the presence of rank-one isometries or isolated flats, and if X is hyperbolic then X_L is quasiisometric to X for all sufficiently large L.

1.2. Hyperbolic isometries

Both surfaces and CAT(0) spaces have isometries that can naturally be considered hyperbolic-like, namely pseudo-Anosov and rank-one isometries. Pseudo-Anosov isometries are precisely those mapping classes that act loxodromically on the curve graph [MM99], and, in the cubical setting, rank-one elements are those that skewer a pair of *separated* hyperplanes [CS11, Gen20a]. The notion of (L–)separation carries over to the setting of curtains (Definition 2.11), allowing us to bring the two perspectives together in CAT(0) spaces.

Theorem C. *Let g be a semisimple isometry of a proper CAT(0) space X. The following are equivalent.*

- *g is rank-one.*
- *g skewers a pair of separated curtains.*
- *g acts loxodromically on some X_L.*

As a consequence, if G acts properly coboundedly on X and some X_L is unbounded, then Gromov's classification of actions on hyperbolic spaces yields a loxodromic isometry of X_L, which is rank-one by the above theorem.

The main metric tool we use is a characterisation of *contracting* (equivalently *Morse* [CS15]) geodesics in terms of curtains. Recall that, for proper CAT(0) spaces, an isometry is rank-one precisely when it has a contracting axis [BF09].

Theorem D. *A geodesic ray in a CAT(0) space is contracting if and only if it crosses an infinite chain of L–separated curtains at a uniform rate.*

Although our proof is considerably different, the statement of Theorem D exactly parallels the characterisation of contracting geodesic rays in CAT(0) cube complexes, with curtains replacing hyperplanes [CS15, Gen20a].

When considering subgroups instead of single elements, an important notion of hyperbolic-type behaviour is given by *stability*. This was introduced by Durham-Taylor [DT15], who showed that a subgroup of the mapping class group is stable if and only if it is convex cocompact in the sense of Farb–Mosher [FM02]. Hence, a subgroup of the mapping class group is stable exactly when its orbit maps on the curve graph are quasiisometric embeddings [KL08]. Other instances of this perspective include [BBKL20, ADT17, KMT17, ABD21, Che20]. We show that the same result holds for the spaces X_L.

Theorem E. *A subgroup of a group acting geometrically on a CAT(0) space X is stable if and only if it is finitely generated and it has a quasiisometrically embedded orbit in some X_L.*

Because geometric actions on X descend to actions on X_L, one would hope that these actions would be acylindrical, as is the case for the action of the mapping class group on the curve graph [Bow08], the action of a right-angled Artin group on its *extension graph* [KK14], and the action of a cocompactly cubulated group on the *contact graph* when the cube complex has a *factor system* [BHS17]. Although a counterexample is not yet present, a statement of such generality seems hopeless: ongoing work of Shepherd shows that, even in the significantly better-behaved cubical case, there exists a rank-one cocompactly cubulated group that does not act acylindrically on the contact graph [She22]. This means that the following result, which mirrors a result of Genevois for CAT(0) cube complexes [Gen20b] and the situation for $\mathrm{Out}(F_n)$ with respect to its free factor complex [BF14], is in a sense as good as one could hope for. Note that it still implies the existence of an acylindrical action on a hyperbolic space [DGO17, Osi16], and indeed on a quasitree [Bal17].

Theorem F. *If G acts properly on a CAT(0) space X, then G acts* non-uniformly *acylindrically on every X_L. In particular, every rank-one element is a* WPD *isometry of every X_L on which it acts loxodromically.*

In particular, we recover that the existence of a rank-one element implies acylindrical hyperbolicity of the group [Sis18]. Thus it is natural to ask for a geometric criterion implying the existence of a rank-one element. We show that curtains provide such a criterion.

Theorem G. *If G is a group acting properly cocompactly on a CAT(0) space X, then G contains a rank-one element if and only if X contains a pair of separated curtains.*

An alternative phrasing of the above theorem is the following, which makes it clear that cocompact CAT(0) spaces without rank-one isometries have certain product-like features. This perspective will be pursued further in Section 1.4.

Corollary H. *If G is a group acting properly cocompactly on a CAT(0) space X, then precisely one of the following happens.*

- *G has a rank-one isometry.*
- *Every pair of disjoint curtains in X is crossed by an infinite chain of curtains.*

As a direct consequence of [KL98, Prop. 3.3], we obtain the following corollary, which extends a result of Levcovitz for CAT(0) cube complexes [Lev18]. It can also be interpreted in terms of the X_L, generalizing a result of Hagen about contact graphs of CAT(0) cube complexes [Hag13].

Corollary I. *If X is a CAT(0) space admitting a proper cocompact group action and X has a pair of separated curtains, then the divergence of X is at least quadratic.*

1.3. Ivanov's theorem

Aside from hyperbolicity, one of the most important results about the curve graph is *Ivanov's theorem* [Iva97, Kor99, Luo00], which states that every automorphism of the curve graph is induced by some mapping class. This has been the fundamental tool in the proofs of some very strong theorems about mapping class groups, such as quasiisometric rigidity [BKMM12, BHS21] and commensuration of the Johnson kernel, Torelli group, and other more general normal subgroups [BM04, BM19]. Fioravanti has also proved a version of Ivanov's theorem for the contact graph of many CAT(0) cube complexes [Fio22]. We prove the following analogue for a large class of CAT(0) spaces. This may be surprising, because there is no version known for any of the various hyperbolic models of $\mathrm{Out}(F_n)$. Recall that a CAT(0) space has the *geodesic extension property* if every geodesic segment appears in some biinfinite geodesic.

Theorem J. *Let X be a proper CAT(0) space with the geodesic extension property. If any one of the following holds, then* $\mathrm{Isom}\,X = \mathrm{Isom}\,X_L$ *for all L.*

- *X admits a proper cocompact action by a group that is not virtually free.*
- *X is a tree that does not embed in $\mathbf{R}$.*
- *X is one-ended.*

Note that Theorem J gives an exact *equality* between $\mathrm{Isom}\,X$ and $\mathrm{Isom}\,X_L$, rather than just an isomorphism, because $\mathrm{Isom}\,X$ is always a subset of $\mathrm{Isom}\,X_L$.

The statement of Theorem J cannot hold in full generality, as one can see by considering the real line. Indeed, for any order-preserving bijection $\phi : (0,1) \to (0,1)$, there is an element of $\mathrm{Isom}\,\mathbf{R}_L$ whose restriction to each component of $\mathbf{R} \smallsetminus \mathbf{Z}$ is ϕ. If ϕ is not the identity, then this map is not an isometry of $\mathbf{R}$, it is merely a $(1,1)$–quasiisometry. This also shows that the following result is optimal.

Theorem K. *Let X be a CAT(0) space. For each L, every element of* $\mathrm{Isom}\,X_L$ *is a $(1,1)$–quasiisometry of X.*

There is often an important distinction to be made between quasiisometries with multiplicative constant one (also known as *rough isometries*) and more general quasiisometries, as rough isometries tend to preserve more geometric structure. For instance, (four-point) Gromov hyperbolicity is preserved by rough isometries, but not by general quasiisometries [DK18, Eg. 11.36].

Our route to proving Theorem J is to connect the groups $\mathrm{Isom}\,X_L$ with Andreev's work in CAT(0) spaces on Aleksandrov's problem [And06]. The problem asks for which metric spaces it is the case that any self-map that setwise sends unit spheres to unit spheres is necessarily an isometry [Ale70]. This problem originates from the Beckman–Quarles theorem, which shows that this holds for Euclidean n–space ($n > 1$).

1.4. Large-scale properties

In Section 1.2, we have shown that a cocompact CAT(0) space has a rank-one element if and only if some X_L is unbounded. It is therefore natural to wonder what can be said in the case where all the spaces X_L are bounded.

Theorem L. *Let X be a CAT(0) space admitting a proper cocompact group action. If the diameters of the spaces X_L are uniformly bounded, then X is wide, i.e. no asymptotic cone of X has a cut point.*

In particular, we can obtain conclusions both in the case where some X_L is unbounded, and in the case where the X_L are uniformly bounded. This leaves open the situation where the diameters of the X_L diverge. We show that this cannot happen.

Theorem M. *Let G be a group acting properly cocompactly on a CAT(0) space X. If there is an integer L_1 such that* $\operatorname{diam} X_{L_1} > 2$ *then the Tits boundary of X has diameter at least $\frac{3\pi}{2}$. In particular* ([GS13]) *G has a rank-one isometry, so some X_{L_2} is unbounded (Theorem C).*

One of the main open problems in CAT(0) geometry is the *rank-rigidity* conjecture [BB08], which asks for a CAT(0) version of the celebrated theorem for Hadamard manifolds [BBE85, BBS85, Bal85, BS87, EH90]. Even partial progress on this has been quite hard, leading Behrstock–Druţu to ask the simpler question of whether proper cocompact CAT(0) spaces without rank-one isometries are *wide* [BD14, Q. 6.10]. We show the following.

Corollary N (Rank dichotomy)**.** *Let G be a group acting properly cocompactly on a CAT(0) space X. Exactly one of the following holds.*

- *Every X_L has diameter at most two, in which case G is wide.*
- *Some X_L is unbounded, in which case G has a rank-one element, and if G is not virtually cyclic then it is* acylindrically hyperbolic*.*

In particular, the above corollary provides a positive answer to the question of Behrstock–Druţu. After establishing Corollary N, we were surprised to discover that the question was already answered by Kent–Ricks [KR21, p.1467]. However, our proof uses completely different methods, being based on the notions and tools we develop in this paper including curtains, the hyperbolic models X_L and the characterisations of rank-one isometries via separated curtains (Theorem C and Theorem G).

In the setting of CAT(0) cube complexes, the rank-rigidity conjecture was proved by Caprace–Sageev [CS11], with the proof relying heavily on the discrete combinatorial structure of hyperplanes. More recently, Stadler has established an important part of the conjecture for CAT(0) spaces of rank two [Sta22].

The spaces X_L also allow us to gain new insights on the visual boundary ∂X of X. Indeed, each hyperbolic space X_L comes equipped with its Gromov boundary ∂X_L, and one can ask if the spaces ∂X_L can be seen inside ∂X. The answer turns out to be yes, and, surprisingly, their images inside ∂X are $\operatorname{Isom} X$-invariant.

Theorem O. *Let X be a proper CAT(0) space. For each L, the space ∂X_L embeds homeomorphically as an* $\operatorname{Isom} X$*-invariant subspace of ∂X, and every point in the image of ∂X_L is a visibility point of ∂X. The embedding is induced by the change-of-metric map $X_L \to X$.*

In other settings, results of this type have proved to be rather useful. Indeed, Theorem O is related to the situation for the curve graph [Kla99, Ham06] and the free factor complex [BR15], where similar results have been used in the study of random walks [KM96, Hor16b]. Ongoing work of Le Bars relies on Theorem O to analyse the asymptotic behaviour of random walks in CAT(0) spaces, as discussed in Section 1.5. Moreover, for the case of a finite-dimensional CAT(0) cube complex, Fernós–Lécureux–Mathéus [FLM21] showed that the *regular Roller boundary* is equivariantly homeomorphic to the boundary the contact graph and used this to prove a central limit theorem for certain CAT(0) cube complexes. Since Theorem O works for every L, we can consider the subspace $\mathcal{B}$ of ∂X obtained by taking the union of the images of the ∂X_L. This results in a boundary for X that sees all of its negative curvature. Indeed, it follows from Theorem D that $\mathcal{B}$ contains the *Morse boundary* of X, [CS15, Cor17]; generally $\mathcal{B}$ will be a much larger subspace of ∂X.

1.5. Further directions

The framework of curtains and separation allows for results that are strikingly similar to ones for cube complexes and mapping class groups. This opens up a large range of potential directions for further study. We briefly discuss a few of these below.

Random walks.

Thanks to work of Maher–Tiozzo [MT18], much can be said about random walks on groups that act by isometries on (not necessarily proper) hyperbolic spaces. Since actions of groups on a CAT(0) space X descend to actions on the hyperbolic spaces X_L, this now includes the class of CAT(0) groups. As suggested in [LB22b], it would be natural to try to use the X_L to obtain results about random walks on CAT(0) groups, for example a description of the Poisson boundary similar to those for mapping class groups [KM96] and $\mathrm{Out}(F_n)$ [Hor16b]. One could also make use of the cubical perspective, and try to use curtains and Theorem O to emulate the strategy of [FLM18] to prove a central limit theorem for random walks on CAT(0) groups. This strategy is successfully implemented in ongoing work of Le Bars [LB22a].

Stabilisation of the spaces X_L.

One of the downsides of the construction of the spaces X_L is that we end up with a family of hyperbolic spaces that potentially represent infinitely many isometry- (or even quasiisometry-) types. Whilst much can still be said just from knowing that every contracting geodesic embeds in *some* X_L, there are potential applications where it would be useful to know that there is some L_0 beyond which the X_L stop changing, if only up to quasiisometry.

The exact stabilisation happens in the case of universal covers Salvetti complexes of right-angled Artin groups, and the coarse stabilisation occurs for hyperbolic spaces (Corollary 4.7). However, it seems unlikely that this should hold in full generality. Indeed, ongoing work of Shepherd shows that, even in the better-behaved cubical case, there exists a cocompact CAT(0) cube complex where Genevois's cubical hyperbolic models [Gen20b] do not stabilise, even up to quasiisometries [She22]. On the other hand, if the cube complex has a factor system then Genevois's spaces do stabilise [MQZ20]. It would be interesting to have criteria for these stabilisations to occur.

Quasiisometry-invariance.

Given two quasiisometric CAT(0) spaces X and Y, it is natural to ask whether the families $\{X_L\}$ and $\{Y_L\}$ are related to each other. If one considers only a single value of L, then in general Y_L is not quasiisometric to X_L. Indeed $X = \mathbf{R}$ and $Y = \mathbf{R} \times [0,1]$ are quasiisometric, but X_0 is unbounded and Y_0 is bounded. However, X_L and Y_L are quasiisometric for $L \geqslant 1$. This suggests asking: if X and Y are quasiisometric, is it true that for each L there exists L' such that X_L quasiisometrically embeds into $Y_{L'}$ and Y_L quasiisometrically embeds into $X_{L'}$? What if X and Y both admit proper cocompact actions by a common group? Examples of Croke–Kleiner show that such a statement is not a given [CK00].

Acylindricity of the action.

Let G be a group acting properly on X, so that G acts non-uniformly acylindrically on every X_L by Theorem F. In general, these actions may fail to be acylindrical [She22], but are there conditions that guarantee acylindricity? If, in addition, the spaces X_L stabilise, this would provide a candidate for a largest acylindrical action of G [ABO19]. To obtain acylindricity, an obstruction that needs to be addressed is a lower bound on the stable translation length of the G-action on X_L. Indeed, by [Bow08, Lem. 2.2], all acylindrical actions have such a lower bound, otherwise one could find arbitrarily many elements that coarsely stabilise any two points on an axis.

Roller boundaries.

In CAT(0) cube complexes there is a notion of boundary defined using hyperplanes, namely the Roller boundary. In it, a point is defined as a choice of orientation for each halfspace, or, more technically, as a non-principal ultrafilter over the set of orientations of hyperplanes. By including curtains in one of their halfspaces, the construction can be extended to the general CAT(0) setting. One can hope to exploit this boundary to import cubical results. For instance, in [FLM21] it is shown that the regular Roller boundary is homeomorphic to the Gromov boundary of the contact graph. If it turns out that the spaces X_L are preserved under quasiisometry, a statement of the previous type would suggest that (part of) the Roller boundary is also preserved under quasi-isometry, potentially providing a new quasiisometry-invariant.

Sublineraly Morse geodesics.

Recent work of Qing–Rafi–Tiozzo introduced the notion of *sublinearly Morse* geodesics, which generalise contracting geodesics in the CAT(0) setting, and used them to define the sublinearly Morse boundary [QRT20]. In Theorem D we characterise contracting geodesics in terms of separated curtains and the spaces X_L. Is it possible to obtain a similar characterisation for the sublinear case? And what can be said about the topology on the boundary?

Combination theorems.

Given some kind of geometric decomposition of a CAT(0) space X into CAT(0) pieces, what can be said about the hyperbolic models of X from the smaller pieces? A natural candidate would be a space X admitting a geometric action by a group G that admits a graph of group decomposition.

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2. Curtains and the L–separation space

We refer the reader to [BH99, Part II] for a thorough treatment of CAT(0) spaces. The main property that we shall use directly is the fact that CAT(0) spaces are geodesic metric spaces with a *convex* metric. That is, given any two geodesic segments $\alpha, \beta : [0,1] \to X$, parametrised proportional to arc-length, the function $t \mapsto \mathsf{d}(\alpha(t), \beta(t))$ is convex. An immediate consequence of this is that each pair of points (x, y) is joined by a unique geodesic.

We denote this geodesic $[x, y]$. Moreover, for any geodesic $\alpha : I \to X$, where $I \subset \mathbf{R}$ is an interval, the closest-point projection map, written $\pi_\alpha : X \to \alpha$, is 1–Lipschitz.

2.1. Curtains

The main ingredient of this paper is the concept of *curtains*, which morally mimic hyperplanes in a CAT(0) cube complex. The following appears in simplified form in the introduction.

Definition 2.1 (Curtain, pole)**.** Let X be a CAT(0) space and let $\alpha : I \to X$ be a geodesic. For a number r with $[r - 1/2, r + 1/2]$ in the interior of I, the *curtain dual to* α *at* r is

$$h = h_\alpha = h_{\alpha,r} = \pi_\alpha^{-1}(\alpha[r - 1/2, r + 1/2]).$$

The segment $\alpha[r - 1/2, r + 1/2]$ is called the *pole* of the curtain.

Although the analogy between curtains and hyperplanes is not perfect, they do share a number of important properties. For instance, curtains separate the space into two halfspaces.

Definition 2.2 (Halfspaces, separation)**.** Let X be a CAT(0) space and let $h = h_{\alpha,r}$ be a curtain. The *halfspaces* determined by h are $h^- = \pi_\alpha^{-1}\alpha(I \smallsetminus [r - 1/2, \infty))$ and $h^+ = \pi_\alpha^{-1}\alpha(I \smallsetminus (-\infty, r + 1/2])$. Note that $\{h^-, h, h^+\}$ is a partition of X. If A, B are subsets of X such that $A \subseteq h^-$ and $B \subseteq h^+$, then we say that h *separates* A from B.

Remark 2.3 (Curtains are thick and closed)**.** Because π_α is 1–Lipschitz, we have $\mathsf{d}(h_\alpha^-, h_\alpha^+) = 1$. Moreover, curtains are closed subsets because projection to a geodesic is continuous and $[r - 1/2, r + 1/2]$ is closed.

Remark 2.4 (Failure of convexity)**.** Unlike hyperplanes in a CAT(0) cube complex, curtains can fail to be convex: it is possible to have two geodesic segments $\alpha\colon [a, b] \to X$ and β such that $\pi_\beta(\alpha(a)) = \pi_\beta(\alpha(b))$, but $\pi_\beta(\alpha)$ is not constant; in particular, it can happen that $\alpha(a), \alpha(b) \in h_\beta^-$ but $\alpha(t) \in h_\beta^+$ for some $t \in (a, b)$. We thank Michah Sageev for informing us of this fact and providing an example. Another example appears as [Pia13, Ex. 5.1]—it is reproduced in Figure 1. Note that such a configuration appears, for instance, in the product of two (non–2-valent) trees.

Whilst the lack of convexity may seem like a failure, it is actually to be expected, for a similar phenomenon occurs in complex hyperbolic geometry. Indeed, every totally geodesic submanifold of $\mathbf{H}^n_{\mathbf{C}}$ is either totally real or complex-linear [Gol99, §3.1.11], and in particular has real-codimension at least 2 when $n \geqslant 2$. Nonetheless, Mostow considered bisectors $\{z \in \mathbf{H}^n_{\mathbf{C}} : \mathsf{d}(x, z) = \mathsf{d}(z, y)\}$, calling them *spinal surfaces* [Mos80] (for $n = 2$ these were earlier considered by Giraud [Gir21]), and using them in the construction of nonarithmetic lattices in $\mathrm{PU}(2, 1)$. These spinal surfaces are spiritually similar to the curtains we consider here.

Although curtains need not be convex, Lemmas 2.5, 2.6, and 2.7 below show that they do enjoy some convexity-like features.

Lemma 2.5 (Curtains separate)**.** *Let $h = h_{\alpha,r}$ be a curtain, and let $x \in h^-$, $y \in h^+$. For any continuous path $\gamma : [a, b] \to X$ from x to y and any $t \in [r - 1/2, r + 1/2]$, there is some $c \in [a, b]$ such that $\pi_\alpha\gamma(c) = \alpha(t)$.*

Proof. The map $f = \alpha^{-1}\pi_\alpha\gamma : [a, b] \to I$ is continuous, with $f(a) < t < f(b)$, so c is provided by the intermediate value theorem. □

Lemma 2.6 (Star convexity)**.** *Let h be a curtain with pole P. For every $x \in h$, the geodesic $[x, \pi_P x]$ is contained in h. In particular, h is path connected.*

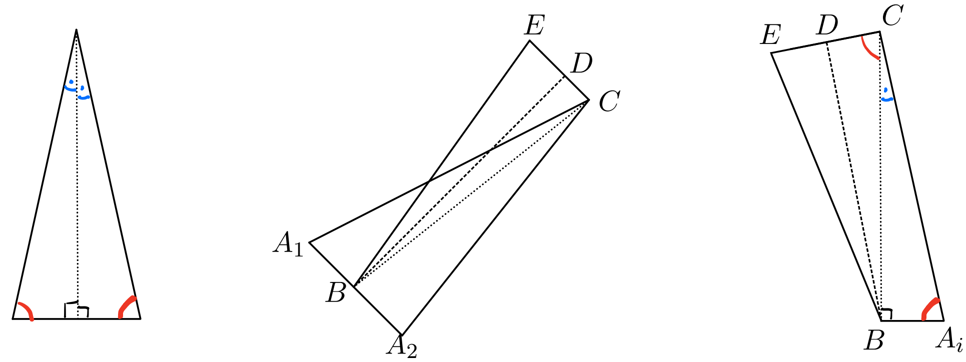

FIGURE 1. Glue together two copies of the isosceles triangle on the left to obtain the central CAT(0) space [Pia13, Ex. 5.1]. The right-hand quadrilateral has angle $\frac{\pi}{2}$ at C, so $\pi_{[E,C]}(A_1) = \pi_{[E,C]}(A_2) = C$, but $\pi_{[E,C]}(B) = D$.

Proof. Let $h = h_\alpha$. Since $x \in h$, we have $\pi_\alpha(x) \in P$. By the triangle inequality, $\pi_\alpha[x, \pi_\alpha(x)] = \pi_\alpha(x)$, so $[x, \pi_\alpha(x)] \subset h$. □

Lemma 2.7 (No bigons for related curtains)**.** *Let $\alpha = [x_1, x_3]$ be a geodesic and let $x \notin \alpha$. For any $x_2 \in \alpha$, if h is a curtain dual to $[x_2, x]$ that meets $[x_1, x_2]$, then h does not meet $[x_2, x_3]$.*

Proof. If h does this then there exist $p_1 \in h \cap [x_1, x_2]$, $p_3 \in h \cap [x_2, x_3]$ with $\pi_{[x,x_2]}(p_1) = \pi_{[x,x_2]}(p_3)$. Because $x_2 \in [x, x_2] \smallsetminus h$, we have $\mathsf{d}(p_i, x_2) > \mathsf{d}(p_i, \pi_{[x,x_2]}(p_i))$. But this contradicts the fact that α is a geodesic. □

We also record the following basic property of closest-point projections that will be used throughout the paper.

Lemma 2.8. *Let α be a geodesic, and let $x \in X$. For any $y \in \alpha$, we have $\pi_{[x,\pi_\alpha x]}(y) = \pi_\alpha(x)$.*

Proof. By the triangle inequality, we have $\pi_\alpha(z) = \pi_\alpha(x)$ for all $z \in [\pi_\alpha x, x]$. Because $\pi_{[x,\pi_\alpha x]}$ is 1–Lipschitz, we therefore have $\mathsf{d}(y, \pi_\alpha x) \leqslant \mathsf{d}(y, z)$ for all $z \in [\pi_\alpha x, x]$, and the inequality must be strict for $z \neq \pi_\alpha(x)$ as balls are strictly convex. □

The main feature of curtains is the fact that they can be used to define a new family of distances on X. The first that we will consider is the *chain distance.*

Definition 2.9 (Chain, chain distance)**.** A set $\{h_i\}$ of curtains is a chain if h_i separates h_{i-1} from h_{i+1} for all i. We say that $\{h_i\}$ separates $A, B \subset X$ if every h_i does. The chain distance from $x \in X$ to $y \in X \smallsetminus \{x\}$ is

$$\mathsf{d}_\infty(x, y) = 1 + \max\{\, |c| : c \text{ is a chain separating } x \text{ from } y\}.$$

The following lemma states that the chain distance d_∞ and the original distance d on X differ by at most 1. Apart from the statement that curtains really do encode "meaningful" distances on X, the fact that d and d_∞ are comparable turns out to be surprisingly useful, and will be used in multiple places.

Lemma 2.10. *For any $x, y \in X$, there is a chain c of curtains dual to $[x, y]$ that realises $\mathsf{d}_\infty(x, y) = 1 + |c| = \lceil \mathsf{d}(x, y) \rceil$.*

Proof. Let $D = \lceil \mathsf{d}(x,y) \rceil$, and let $\delta = \mathsf{d}(x,y) - \lfloor \mathsf{d}(x,y) \rfloor$ be the fractional part. For $i \in \{1,\dots,D-1\}$, let $r_i = i - 1/2 + \frac{i\delta}{D}$. Observe that the intervals $[r_i - 1/2, r_i + 1/2]$ are pairwise disjoint. This implies that the curtains h_{α,r_i} form a chain c of cardinality $\lceil \mathsf{d}(x,y) \rceil - 1$. Hence $\mathsf{d}_\infty(x,y) \geqslant \lceil \mathsf{d}(x,y) \rceil$. On the other hand, for any curtain h we have $\mathsf{d}(h^-, h^+) = 1$. As curtains are closed, it follows that $\mathsf{d}_\infty(x,y) \leqslant \lceil \mathsf{d}(x,y) \rceil$. □

2.2. L–separation

We are now ready to introduce the notion of L–separation, which will be fundamental to this article. If curtains are reminiscent of hyperplanes and wall-spaces, L–separation mimics the behaviour of such objects in hyperbolic spaces, and thus should be thought of as a source of negative curvature. For instance, it is possible to induce a wall-space structure on the hyperbolic plane $\mathbf{H}^2$ by considering a cover $\mathbf{H}^2 \to \Sigma$ and using the lifts of appropriate curves to define walls. Such lifts will enjoy strong separation properties, for instance the closest-point projections between them will have uniformly bounded diameter. In our case, curtains are not convex, so closest-point projections are not well defined. However, one can achieve similar results by considering how curtains interact with one other.

Definition 2.11 (L–separated, L–chain)**.** Let $L \in \mathbf{N}$. Disjoint curtains h and h' are said to be *L–separated* if every chain meeting both h and h' has cardinality at most L. Two disjoint curtains are said to be *separated* if they are L-separated for some L. If c is a chain of curtains such that each pair is L–separated, then we refer to c as an *L–chain*.

L–separation was introduced by Genevois [Gen20a] under the name "L–well-separated", to distinguish it from the earlier notion of Charney–Sultan [CS15]. As the more recent definition is better suited to applications, we feel it deserves the simpler terminology.

Unless otherwise stated, we shall assume that $L < \infty$. The next two lemmas will be a staple asset during the paper—their proofs are purely combinatorial and do not use any CAT(0) geometry.

Lemma 2.12 (Gluing disjoint L–chains)**.** *Suppose that c and c' are L–chains such that every element of c is disjoint from every element of c'. Let h be the maximal element of c, and let h' be the minimal element of c'. If there exists $z \in h^+ \cap h'^-$, then the chain $c \cup c' \smallsetminus \{h\}$ is an L–chain.*

Proof. Let h'' be the maximal element of $c \smallsetminus \{h\}$. It suffices to check that $\{h'', h'\}$ is an L–chain. But any chain meeting both h'' and h' must meet both h'' and h, because h separates h'' from h'. □

Lemma 2.13 (Gluing L–chains)**.** *Suppose that $c = \{h_1, \dots, h_n\}$ and $c' = \{h'_1, \dots . h'_m\}$ are L–chains with $n > 1$ and $m > L+1$. If there exists $z \in h_n^+ \cap {h'_1}^-$, then $c'' = \{h_1, \dots, h_{n-1}, h'_{L+2}, \dots, h'_m\}$ is an L–chain of cardinality $n + m - L - 2$.*

Proof. The existence of z implies that if h_i meets h'_k, then h_j meets h'_l for all $j \geqslant i$ and all $k \leqslant j$. Since c' is an L–chain, this implies that h_{n-1} cannot meet h'_{L+1}. That is, $\{h_1, \dots, h_{n-1}, h'_{L+1}, \dots, h'_m\}$ is a chain. It meets the conditions of Lemma 2.12. □

The following lemma is a good first example of using curtains to obtain strong geometric features via simple combinatorial arguments.

Lemma 2.14 (Bottleneck)**.** *Suppose that A, B are two sets which are separated by an L-chain $\{h_1, h_2, h_3\}$ all of whose elements are dual to a geodesic $b = [x_1, y_1]$ with $x_1 \in A$ and $y_1 \in B$. For any $x_2 \in A$ and $y_2 \in B$, if $p \in h_2 \cap [x_2, y_2]$, we have $\mathsf{d}(p, \pi_b(p)) < 2L + 1$.*

Proof. Let c be a chain dual to $[p, \pi_b(p)]$ that realises $1 + |c| = \lceil \mathsf{d}(p, \pi_b(p)) \rceil$, as provided by Lemma 2.10. According to Lemma 2.8, every $z \in b$ has $\pi_{[p,\pi_b(p)]}(z) = \pi_b(p)$, so no element of c meets b. Furthermore, Lemma 2.7 shows that no element of c can meet both $[x_2, p]$ and $[p, y_2]$. Therefore, each element of c must either meet either h_1 or h_2. Since $\{h_1, h_2, h_3\}$ is an L-chain, this means that c has at most $2L$ elements. By the choice of c we have $\mathsf{d}(p, \pi_b(p)) \leqslant 2L+1$. $\square$

Given a CAT(0) space X, we can now use curtains and L–separation to define a family of metrics on X, similarly to [Gen20a]. The metric spaces produced will be the eponymous hyperbolic models.

Definition 2.15 (L–metric)**.** Given distinct points $x, y \in X$, define

$$\mathsf{d}_L(x, y) = 1 + \max\{\ |c| : c \text{ is an } L\text{–chain separating } x \text{ from } y\}.$$

Remark 2.16. Since L–chains are chains, we have $\mathsf{d}_L(x, y) \leqslant \mathsf{d}_\infty(x, y) < 1 + \mathsf{d}(x, y)$.

Let us show that d_L is a metric. For $L = \infty$, this also follows from Lemma 2.10.

Lemma 2.17. *d_L is a metric for every $L \in \mathbf{N} \cup \{\infty\}$.*

Proof. The map d_L is clearly symmetric and separates points. Given $x, y, z \in X$, let c be a chain realising $\mathsf{d}_L(x, y) = 1 + |c|$. We have $z \in h$ for at most one $h \in c$, and every other element of c separates z from at least one of x and y. Let $c' \subset c$ be the subchain of curtains separating z from x. We have shown that $\mathsf{d}_L(x, z) + \mathsf{d}_L(z, y) \geqslant (1 + |c'|) + (1 + |c| - |c'|) - 1 = 1 + |c| = \mathsf{d}_L(x, y)$. $\square$

Notation 2.18. We write X_L for the metric space (X, d_L).

The following is a simple consequence of the lemmas on gluing L–chains. Recall that a metric space is *weakly roughly geodesic* if there is a constant C such that for any $x, y \in X$ and any nonnegative $r \leqslant \mathsf{d}(x, y)$, there is a point $z \in X$ such that $|\,\mathsf{d}(x, z) - r| \leqslant C$ and $\mathsf{d}(x, z) + \mathsf{d}(z, y) \leqslant \mathsf{d}(x, y) + C$.

Lemma 2.19. *X_L is weakly roughly geodesic, with constant $L + 5$.*

Proof. Let $\{h_1, \dots, h_n\}$ be an L–chain realising $\mathsf{d}_L(x, y)$. Given $0 \leqslant r \leqslant d(x, y)$, let $z \in h_{\lceil r \rceil}$. Let c, c' be L–chains realising $\mathsf{d}_L(x, z)$ and $\mathsf{d}_L(z, y)$. We know that $|c| \geqslant r - 1$ and $|c'| \geqslant n - r - 1$. According to Lemma 2.13, we also have that $|c| + |c'| - (L + 3) \leqslant n$, and this also shows that $\mathsf{d}_L(x, z) = |c| + 1 \leqslant r + L + 5$. $\square$

Since every weakly roughly geodesic space is quasigeodesic, this implies that X_L is quasigeodesic. In Section 3, we shall give more precise information by showing that CAT(0) geodesics of X are uniform quasigeodesics of X_L.

Example 2.20. An instructive example to consider is the *tree of flats*, i.e. the Cayley complex C of the right-angled Artin group $\mathbf{Z}^2 * \mathbf{Z} = \langle a, b \rangle * \langle c \rangle$. Consider the geodesic $\alpha = [a^{-1}, a]$, with curtain $h = h_{\alpha,0}$. The vertices $C^0 \cap h$ in h are exactly those corresponding to reduced words whose first letter is not a or a^{-1}. See Figure 2.

There are two noteworthy things here. Firstly, h is not Hausdorff–close to any hyperplane of C in the cubical sense. Secondly, h contains points that are arbitrarily far from $h^- \cup h^+$, even in the metric d_L.

Note that there are only three ways that a curtain h can intersect a flat F in this example: the intersection $h \cap F$ is either empty, equal to F, or a strip of width at most 1. From this it can be seen that X_L is quasiisometric to the Bass–Serre tree for every $L \geqslant 2$.

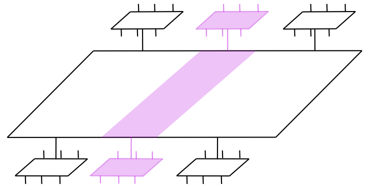

FIGURE 2. A curtain in a CAT(0) cube complex that is not close to any hyperplane.

The fact that curtains and hyperplanes need not be obviously related raises the following question.

Question 2.21. How do the spaces X_L defined here using curtains compare to the spaces defined by Genevois using hyperplanes [Gen20b] in the case where X is a CAT(0) cube complex?

It is possible to construct quasiline subcomplexes of $\mathbf{Z}^2$ where any correspondence depends on some coboundedness constant. Indeed, let $\gamma \subset \mathbf{Z}^2$ be the "zigzag" geodesic that passes through (n, n) and $(n+1, n)$ for all n. For each natural number k, let $X(k)$ be the CAT(0) cube complex bounded by γ and the translation of γ by $(0, k)$, which has a $\frac{k}{2}$–cobounded $\mathbf{Z}$–action. It can easily be seen that the contact graph of $X(k)$ is a quasiline, so Genevois's spaces are all unbounded [Gen20b, Fact 6.50], but the space $X(k)_L$ is bounded for all $L \leqslant \frac{k-3}{2}$.

A disadvantage of the distance d_L is that it provides no information on the family of curtains realizing the distance between two points, a part of its size. We conclude the section by proving that in many situations, up to a linear loss in length, we can replace a given L–chain by one dual to a fixed geodesic. This will simplify arguments in a number of places.

Lemma 2.22 (Dualising chains)**.** *Let $L, n \in \mathbf{N}$, let $\{h_1, \dots, h_{(4L+10)n}\}$ be an L–chain, and suppose that $A, B \subset X$ are separated by every h_i. For any $x \in A$ and $y \in B$, the sets A and B are separated by an L–chain of length at least $n+1$ all of whose elements are dual to $[x, y]$.*

Proof. Let us first prove the statement for $n = 1$, illustrated in Figure 3. Writing $\alpha = [x, y]$, let a_1 and a_2 be the first points of $\alpha \cap h_3$ and $\alpha \cap h_{2L+8}$ respectively, and let b_1 and b_2 be the last points of $\alpha \cap h_{2L+3}$ and h_{4L+8}, respectively. Since curtains are thick, we have $\mathsf{d}(a_i, b_i) > 2L+1$, so Lemma 2.10 provides chains $\{k_0^i, \dots, k_{2L}^i\}$ dual to α that separate a_i from b_i. Since h_1 and h_2 are L–separated, the curtain k_L^1 is disjoint from h_1, and similarly k_L^1 is disjoint from h_{2L+5}. The same argument shows that k_L^2 is disjoint from $h_{2L+6} \cup h_{4L+10}$. This shows that the k_L^i separate A from B, but it also shows that k_L^1 and k_L^2 are L–separated, because any curtain meeting both must also meet h_{2L+5} and h_{2L+6}.

Now suppose that $n > 1$, and again write $\alpha = [x, y]$. For $j \in \{1, \dots, n-1\}$, let $x_j \in \alpha \cap h^+_{(4L+10)j} \cap h^-_{1+(4L+10)j}$. Let $x_0 = x$, $x_n = y$. The $n = 1$ case provides, for each $j < n$, a pair of L–separated curtains k_j^1 and k_j^2 that are dual to α and separate x_j from x_{j+1}. Since the k_j^i are all dual to α, they are pairwise disjoint, so we can repeatedly apply Lemma 2.12 to complete the proof. □

Corollary 2.23. *Let b, c be geodesic rays with $b(0) = c(0)$. If some infinite L–chain is crossed by both b and c, then $b = c$.*

Proof. By Lemma 2.22, b and c cross an infinite L–chain $\{h_i\}$ dual to b. By Lemma 2.14, if $c(t_i) \in h_i$, then $\mathsf{d}(c(t_i), b) < 2L+1$. By convexity of the metric, this implies that $c = b$. □

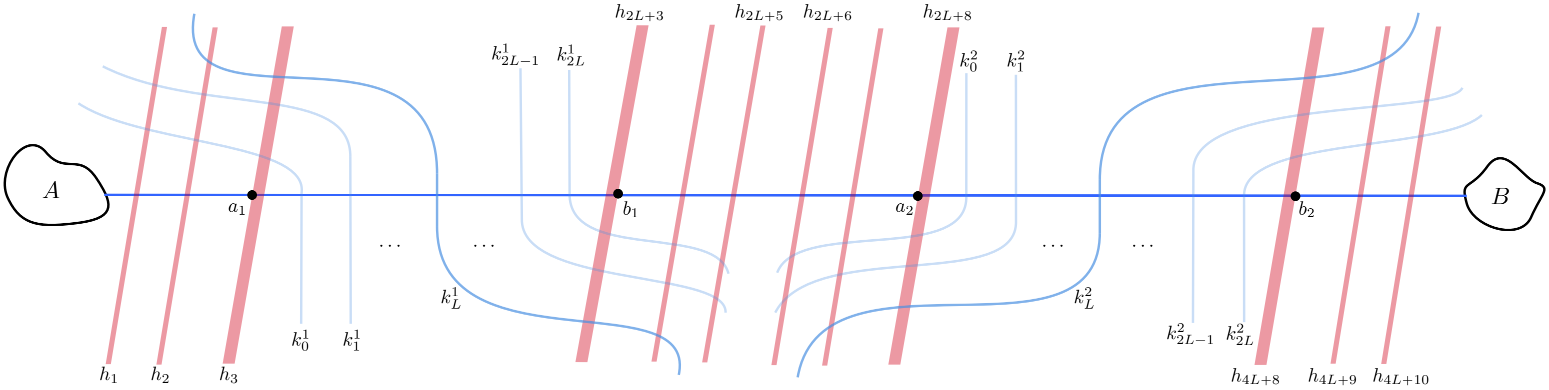

FIGURE 3. The base case of Lemma 2.22

3. Hyperbolicity and isometries

In this section we begin establishing, for a CAT(0) space X, some of the properties of the spaces $X_L = (X, \mathsf{d}_L)$ that mirror those of the curve graph. In Section 3.1, we prove Theorem B, namely that every X_L is a hyperbolic quasigeodesic space, in the sense that every quasigeodesic triangle is thin. The strategy for this is to apply "guessing geodesics", Proposition A.1, to the CAT(0) geodesics of X. As there are different, non-equivalent, definitions of hyperbolicity for quasigeodesic spaces, we note in Proposition 3.6 that X_L is coarsely dense in its *injective hull* $E(X_L)$ (Definition A.2), which is a hyperbolic geodesic space. In Section 3.2, we prove results on CAT(0) analogues of Ivanov's theorem, as discussed in Section 1.3.

3.1. Hyperbolicity of the models

In order to apply the "guessing geodesic" criterion to CAT(0) geodesic triangles, we need to understand their interaction with L–separated curtains. We start by bounding the amount that a CAT(0) geodesic can backtrack through separated curtains. Whilst curtains are not themselves convex, this can be thought of as showing that a pair of separated curtains is (almost) a convex object in its own right.

Lemma 3.1. *Let h and k be L–separated, and let α be a CAT(0) geodesic. If there exist $t_1 < t_2 < t_3 < t_4$ satisfying either*

$$\begin{aligned} &\alpha(t_1) \in h,\ \alpha(t_2) \in k,\ \alpha(t_3) \in k,\ \alpha(t_4) \in h \\ \textit{or}\quad &\alpha(t_1) \in h,\ \alpha(t_2) \in k,\ \alpha(t_3) \in h,\ \alpha(t_4) \in k, \end{aligned}$$

then $t_3 - t_2 \leqslant L + 1$.

Proof. The two cases are treated similarly, so we just consider the former. Let c be a chain dual to α that realises $\mathsf{d}_\infty(\alpha(t_2), \alpha(t_3)) = 1 + |c|$, as given by Lemma 2.10. Every curtain in c separates $\alpha(t_1)$ from $\alpha(t_4)$, and hence meets h because h is path connected. Similarly, every curtain in c meets k. Thus $|c| \leqslant L$, so

$$\mathsf{d}(\alpha(t_2), \alpha(t_3)) \leqslant \mathsf{d}_\infty(\alpha(t_2), \alpha(t_3)) = 1 + |c| \leqslant 1 + L. \qquad \square$$

Corollary 3.2. *If α is a CAT(0) geodesic and $t_1 < t_2 < t_3$, then any L–chain c separating $\alpha(t_2)$ from $\{\alpha(t_1), \alpha(t_3)\}$ has cardinality at most $L' = 1 + \lfloor \frac{L}{2} \rfloor$.*

Proof. Assume that $|c| \geqslant 2$ and that $\alpha(t_2) \in h^+$ for every $h \in c$. Since h is path connected, both $\alpha|_{(t_1,t_2)}$ and $\alpha|_{(t_2,t_3)}$ cross h. Let $h_1, h_2 \in c$ be minimal. According to Lemma 3.1, the length of $\alpha \cap (h_2 \cup h_2^+)$ is at most $L+1$. Recall that curtains are closed, and that $\mathsf{d}(h^-, h^+) = 1$ for every $h \in c$. For $L \in \{0,1\}$, this gives a contradiction with the fact that $\alpha(t_2) \in h_2^+$, so $|c| \leqslant 1$. Otherwise, it implies that $\alpha \cap h_2^+$ has length at most $L-1$, and hence we obtain $|c| \leqslant 2 + \lfloor \frac{L-2}{2} \rfloor$. □

This lack of backtracking allows us to show that, up to parametrisation, CAT(0) geodesics of X are uniform quasigeodesics of X_L.

Proposition 3.3. *There is a constant q such that every CAT(0) geodesic $\alpha : I \to X$ is an unparametrised q–quasigeodesic of X_L.*

Proof. After a translation of $\mathbf{R}$, we may assume that $0 \in I$. Let $t_0 = 0$. For $i > 0$, given t_{i-1}, let t_i be minimal such that $\mathsf{d}_L(\alpha(t_{i-1}), \alpha(t_i)) \geqslant 2L+6$. For $i < 0$, given t_{i+1}, let t_i be maximal such that $\mathsf{d}_L(\alpha(t_{i-1}), \alpha(t_i)) \geqslant 2L+8$.

We claim that $i \mapsto \alpha(t_i)$ is a uniform quasigeodesic in X_L. Clearly it is coarsely Lipschitz. Let $c_i = \{h_i^1, \dots, h_i^{n_i}\}$ be an L–chain realising $\mathsf{d}_L(\alpha(t_i)), \alpha(t_{i+1})) = 1 + |c_i|$. If h_i^j separates $\alpha(t_i)$ from $\alpha(t_l)$, where $l < i$, then so does every h_i^k with $k < j$, so by Corollary 3.2 applied to $\{t_l, t_i, t_{i+1}\}$, we must have $j \leqslant L'$. In other words, every h_i^j with $j > L'$ separates $\alpha(t_i)$ from every $\alpha(t_l)$ with $l < i$. Similarly, every h_i^j with $j < n_i + 1 - L'$ separates $\alpha(t_{i+1})$ from every $\alpha(t_l)$ with $l > i+1$. Let $c_i' = \{h_i^{L'+1}, \dots, h_i^{n_i - L'}\} \subset c_i$. We have $|c_i'| = n_i - 2L' - 2 \geqslant (2L+8-1) - 2 - (2+L) = L+3$.

Moreover, the c_i' are pairwise disjoint as sets (though their elements can intersect). Applying Lemma 2.13 to each pair (c_i', c_{i+1}'), we obtain an L'–chain $\bigcup c_i''$, where $c_i'' \subset c_i'$ has $|c_i' \smallsetminus c_i''| \leqslant L+2$. Since $|c_i'| \geqslant L+3$, this gives the colipschitz property for α with constant $\frac{1}{2L+8}$. □

Now that we know a well-behaved family of quasigeodesics of X_L, we aim to apply the "guessing geodesics" criterion to them to show that X_L is hyperbolic. The main difficulty is to show that triangles in X_L whose edges are CAT(0) geodesics are necessarily thin.

Proposition 3.4. *If $[x,y,z]$ is a CAT(0)-geodesic triangle, then, as subsets of X_L, the set $[x,y]$ is contained in a uniform neighbourhood of $[x,z] \cup [z,y]$.*

Proof. Let $c = \{h_1, \dots, h_n\}$ be an L–chain realising $\mathsf{d}_L(y,z) = 1 + |c|$, numbered from y to z. Note that every h_i must meet at least one of $[x,y]$ and $[x,z]$, for x cannot be on the same side of h_i as both y and z. Moreover, Corollary 3.2 implies that if h_i meets $[x,y]$, then h_j does not meet $[x,z]$ for any $j < i - 2L'$. Similarly, if h_i meets $[x,z]$, then h_j does not meet $[x,y]$ for any $j > i + 2L'$.

<u>Claim:</u> Let $p, p' \in [y,z]$. If at most one h_i separates p from p', then $\mathsf{d}_L(p,p') \leqslant 3L+7$.

<u>Proof:</u> Let c' be a maximal L–chain separating p from p'. According to Corollary 3.2, removing $2L'$ elements of c' results in a subchain c'' of c' that (perhaps after relabelling) separates y and p from p' and z. If $|c''| \geqslant 2L+6$, then applying Lemma 2.13 would contradict maximality of c. Hence $|c'| \leqslant 2L+5+2L'$, so $\mathsf{d}_L(p,p') \leqslant 3L+7$. ◇

Let $p \in [y,z]$. By repeatedly using Claim 3.1, we may assume that p is separated from both y and z by at least $2L'+5$ elements of c. In particular, there exists $i \in [L'+5, n-L'-4]$ such that $p \in [y,z] \smallsetminus (h_i^- \cup h_{i+1}^+)$. If h_i does not meet $[x,y]$, then it meets $[x,z]$, and hence h_{i+1} meets $[x,z]$. Similarly, if h_{i+1} does not meet $[x,z]$, then h_i meets $[x,y]$. As the cases are similar, we shall assume that h_i meets $[x,y]$.

Let $j = i - 2L' - 2$. From the above, we know that none of h_{j-2}, h_{j-1}, h_j, h_{j+1} meet $[x,z]$. Let $p' \in [y,z] \cap h_{j-1}^+ \cap h_j^-$. By Claim 3.1 and the triangle inequality, we know that $\mathsf{d}_L(p,p') \leqslant (2L'+3)(3L+7)$. By construction, there exists $q \in [x,y]$ such that no element of c separates p' from q. We shall bound $\mathsf{d}_L(q,p')$.

Let κ be an L–chain realising $\mathsf{d}_L(p',q) = 1+|\kappa|$. By Corollary 3.2, at most $2L'$ elements of κ either do not separate x from y or do not separate y from z. Because c is an L–chain, at most $2L$ elements of κ meet either h_{j+1} or h_{j-2}. Any other element of κ meets $[y,z]$ and is disjoint from both h_{j-2} and h_{j+1}. According to Lemma 2.12, there are at most three such curtains. Thus $\mathsf{d}_L(p',q) \leqslant 2L'+2L+4$. We have shown that $\mathsf{d}_L(p,[x,y]) \leqslant (2L'+4)(3L+7)$. □

We are now ready to prove hyperbolicity of the spaces X_L. It is clear than any isometry of X is also an isometry of X_L, and we take this opportunity to point out how the actions of $\operatorname{Isom} X$ on the various X_L relate to one another. Recall that if a group G acts on two metric spaces X and Y, then the action on X is said to *dominate* the action on Y if there is a G-equivariant, coarsely Lipschitz map $X \to Y$.

Theorem 3.5. *For each $L < \infty$, the space X_L is a quasigeodesic hyperbolic space. Moreover, $\operatorname{Isom} X < \operatorname{Isom} X_L$, and the action of $\operatorname{Isom} X$ on X_L dominates the one on X_{L-1}.*

Proof. X_L is a quasigeodesic space, either by Lemma 2.19 or Proposition 3.3. Given $x,y \in X_L$, let η_{xy} be the unique CAT(0) geodesic in X from x to y. We shall apply Proposition A.1. Remark 2.16 shows that the η_{xy} are coarsely connected. Conditions (G1) and (G2) are immediate from Proposition 3.3 because CAT(0) geodesics are unique. (G3) is provided by Proposition 3.4. Thus the conditions are met, so X_L is hyperbolic. The remainder follows immediately from the definitions. □

Proposition 3.6. *The injective hull $E(X_L)$ is a geodesic hyperbolic space, and X_L is a coarsely dense subspace.*

Proof. X_L is a quasigeodesic hyperbolic space by Theorem 3.5, and it is weakly roughly geodesic by Lemma 2.19. The result is given by Proposition A.3. □

3.2. An Ivanov-type Theorem

We have seen that $\operatorname{Isom} X < \operatorname{Isom} X_L$ for every CAT(0) space X and every integer L. Moreover, since X and X_L have the same underlying set, every isometry of X_L induces a bijection of X. Our purpose here is to address what can be said about these bijections.

Given $x \in X$ and $r \in \mathbf{R}$, write $B(x,r)$ for the closed ball of radius r centred on x, and write $S(x,r)$ for the sphere of radius r centred on x. We say that a collection $\mathcal{C}$ of bijections $X \to X$ *preserves r–balls* if $gB(x,r) = B(gx,r)$ for all $x \in X$, all $r \geqslant 0$, and all $g \in \mathcal{C}$. We use similar terminology for spheres.

Proposition 3.7. *Let X be a CAT(0) space and let $n, L \in \mathbf{N}$. The group $\operatorname{Isom} X_L$ preserves n–balls. If X has the geodesic extension property, then $\operatorname{Isom} X_L$ also preserves n–spheres.*

Proof. First note that for any $x \in X$ we have $B_X(x,1) = B_{X_L}(x,1)$. Hence $\operatorname{Isom} X_L$ preserves 1–balls. Now suppose that $\operatorname{Isom} X_L$ preserves $(n-1)$–balls, and let $z \in B(x,n)$. There is some point $y \in [x,z]$ such that $\mathsf{d}(x,y) \leqslant n-1$ and $\mathsf{d}(y,z) \leqslant 1$, so by assumption we have $\mathsf{d}(gx,gy) \leqslant n-1$ and $\mathsf{d}(gy,gz) \leqslant 1$ for all $g \in \operatorname{Isom} X_L$. This shows that $gB(x,n) \subset B(gx,n)$ for all $g \in \operatorname{Isom} X_L$. But now we have

$$gB(x,n) \subset B(gx,n) = gg^{-1}B(gx,n) \subset gB(x,n),$$

so $\operatorname{Isom} X_L$ preserves n–balls.

Now suppose that X has the geodesic extension property. Given $x, y \in X$ with $\mathsf{d}(x, y) = n \geqslant 1$, let $z \in X$ be such that $y \in [x, z]$ and $\mathsf{d}(y, z) = 1$. This makes y the unique element of $B(x, n) \cap B(z, 1)$, so the fact that $g \in \operatorname{Isom} X_L$ preserves k–balls implies that gy is the unique element of $B(gx, n) \cap B(gz, 1)$. The fact that these balls meet in a single point must mean that $\mathsf{d}(gx, gz) = n + 1$, and so $\mathsf{d}(gx, gy) = n$. That is, $gS(x, n) \subset S(gx, n)$. As in the case of balls, it follows that $\operatorname{Isom} X_L$ preserves n–spheres. □

Some additional assumption is certainly needed for the elements of $\operatorname{Isom} X_L$ to preserve spheres, for if X is a CAT(0) space of diameter at most one then X_L is a clique.

Corollary 3.8. *Let X be a CAT(0) space. Every isometry $g \in \operatorname{Isom} X_L$ is a (1,1)–quasiisometry of X.*

Proof. For any pair $x, y \in X$ there is a unique integer n such that $\mathsf{d}(x, y) \in (n, n+1]$. Since g preserves n–balls and $(n+1)$–balls, we have that $\mathsf{d}(gx, gy) \in (n, n+1]$. This shows that $|\mathsf{d}(x, y) - \mathsf{d}(gx, gy)| < 1$. □

Note that in quasiisometrically rigid CAT(0) spaces, such as symmetric spaces and buildings [KL97], this means that every isometry of X_L is at bounded distance from an isometry of X. It turns out that for symmetric spaces and buildings, every isometry of X_L coincides with an isometry of X (Corollary 3.13). It is natural to wonder whether this is the general picture. However, Corollary 3.8 is optimal in general: consider the real line, as discussed in Section 1.3.

On the other hand, Proposition 3.7 provides a route to obtaining stronger results under additional assumptions that rule out the real line. Our approach relies on Andreev's contributions to Aleksandrov's problem [And06] in CAT(0) spaces, which asks whether every self-bijection that preserves 1–spheres is an isometry.

Definition 3.9 (Diagonal tube)**.** Let (X, d) be a metric space. The *diagonal tube* of d is the set $V_{\mathsf{d}} = \{(x, y) \in X \times X : \mathsf{d}(x, y) \leqslant 1\}$. We say that a metric d' *realises* V if $V = V_{\mathsf{d}'}$.

In the case that (X, d) is a CAT(0) space with the geodesic extension property, let $V = V_{\mathsf{d}}$. For any element ϕ of any $\operatorname{Isom} X_L$, we can consider the pullback metric $\mathsf{d}'(x, y) = \mathsf{d}(\phi(x), \phi(y))$. We know from Proposition 3.7 that d' realises V. Therefore, in order to show that $\operatorname{Isom} X = \operatorname{Isom} X_L$, it is sufficient to show that d is the unique metric realising V.

For the remainder of this section, geodesics will be understood to be biinfinite. Say that geodesics a and b are *asymptotic* if they share an endpoint in the visual boundary ∂X (the collection of all equivalence classes of geodesic rays, where two rays are equivalent if they are at finite Hausdorff-distance). Say that geodesics a and c are *virtually asymptotic* if there is a sequence $a_0 = a, a_1, \dots, a_n = c$ such that a_i is asymptotic to a_{i-1} for all i.

Lemma 3.10 ([And06, Lem. 2.4])**.** *Let X be a proper CAT(0) space with the geodesic extension property and suppose that d' is a metric on X realising V. Let a and c be virtually asymptotic geodesics. If $\mathsf{d}'|_a = \mathsf{d}|_a$, then $\mathsf{d}'|_c = \mathsf{d}|_c$.*

This lemma provides a route to proving that d is the unique metric realising V. Namely, one can try to show that every geodesic of X is virtually asymptotic to some geodesic whose metric is uniquely determined by V.

Following the terminology of [And06], say that a geodesic is *higher-rank* if it bounds a Euclidean strip, and *strictly rank-one* if it is not virtually asymptotic to any higher-rank geodesic.

Proposition 3.11 ([And06, Cor. 3.2])**.** *Let X be a CAT(0) space, and suppose that d' is a metric on X realising V. If a is a higher-rank geodesic, then $\mathsf{d}'|_a = \mathsf{d}|_a$.*

Combined with Lemma 3.10, this shows that if X is a proper CAT(0) space with the geodesic extension property and every geodesic of X is virtually higher-rank, then d is the unique metric realising V, so $\operatorname{Isom} X = \operatorname{Isom} X_L$. This applies in particular to universal covers of Salvetti complexes of non-free right-angled Artin groups. Note that it is also straightforward to show from Proposition 3.7 that $\operatorname{Isom} X = \operatorname{Isom} X_L$ when X is a tree that does not embed in $\mathbf{R}$.

Proposition 3.12 ([And06, Thm 4.7])**.** *Let X be a proper, one-ended CAT(0) space, and suppose that d' is a metric on X realising V. If a is a geodesic that is strictly rank-one, then there is a geodesic b virtually asymptotic to a such that $\mathsf{d}'|_b = \mathsf{d}|_b$.*

Importantly for us, Proposition 3.12 does not assume the geodesic extension property. Combining Propositions 3.11 and 3.12 with Lemma 3.10 and Proposition 3.7 gives the following.

Corollary 3.13. *If X is a proper, one-ended CAT(0) space with the geodesic extension property, then* $\operatorname{Isom} X = \operatorname{Isom} X_L$ *for all L.*

This covers both symmetric spaces and buildings. In fact, if X is cobounded, then it has a rank-one isometry as soon as it is not one-ended, so this covers all higher-rank examples. However, one can generalise this result in the presence of a geometric action. Recall that for a subspace Y of a CAT(0) space X, the *convex hull* of Y is defined to be the intersection of all convex sets containing Y, which is easily verified to be a CAT(0) subspace. Equivalently, let $Y^0 = Y$, and given Y^i, let Y^{i+1} be the union of all geodesics joining points of Y^i. The convex hull of Y is $\bigcup Y^i$.

Theorem 3.14. *Let X be a CAT(0) space with the geodesic extension property, and suppose that a group G acts properly cocompactly on X. If G is not virtually free, then* $\operatorname{Isom} X = \operatorname{Isom} X_L$.

Proof. If G is one-ended, then this is Corollary 3.13. Otherwise, observe that G is finitely presented and hence is accessible by Dunwoody's theorem [Dun85]. Hence, by Stallings' theorem [Sta68, Sta71], there is a nontrivial (finite) graph of groups decomposition of G such that edge groups are finite and, as G is not virtually free, some vertex group G_v is one-ended. Let $f : G \to X$ be an orbit map $g \mapsto gx_0$ with quasiinverse $\bar{f} : X \to G$, and let X_v be the CAT(0) convex hull in X of $f(G_v) = G_v \cdot x_0$. Note that X_v is a proper CAT(0) space, and the action of G_v on $f(G_v)$ extends to an action on X_v.

<u>Claim:</u> The action of G_v on X_v is cocompact.

<u>Proof:</u> Since the edge groups of the decomposition are finite, for any other vertex group G_w there is a ball $B_1 \subset G$ of uniformly finite radius and at uniformly bounded distance from G_v such that any path in G from a point of $\bar{f}f(G_w)$ to a point of $\bar{f}f(G_v)$ must pass through B_1. In particular, there is a ball $B_2 \subset X$ uniformly close to $f(G_v)$ such that if $z \in f(G_w) \cap (f(G_v))^1$, then z lies on a geodesic between two points of B_2. As balls in X are convex, this shows that the intersection $(f(G_v))^1 \cap f(G_w)$ is contained in the convex hull of B_2. By the construction of the convex hull, iterating shows that $X_v \cap f(G_w)$ is uniformly bounded. As $f(G)$ coarsely coincides with X, we obtain that the Hausdorff distance between X_v and $f(G_v)$ is uniformly bounded. Since X_v is proper, the action of G_v on X_v is cocompact. $\diamondsuit$

In particular, $f|_{G_v}$ is a quasiisometry from G_v to X_v, so X_v is one-ended. Moreover, the action of G_v on X_v is semisimple [BH99, Prop. II.6.10] and G_v is not torsion [Swe99, Thm 11]. Hence X_v contains a geodesic, namely an axis a of a hyperbolic element of G_v.

Now, suppose that ξ and ζ are points of the visual boundary of X with the property that there is a compact set B such that any path α with $(\alpha(-n)) \to \xi$ and $(\alpha(n)) \to \zeta$ must pass through B. It is a simple consequence of the Arzelà–Ascoli theorem and convexity of the metric that there is a geodesic β with $\beta(-\infty) = \xi$ and $\beta(\infty) = \zeta$. By repeatedly applying this fact, it can be seen that any geodesic in X is virtually asymptotic to a.

According to Propositions 3.11 and 3.12, for any metric d' realising V there is a geodesic $b \subset X_v$ that is virtually asymptotic to a and has $\mathsf{d}|_b = \mathsf{d}'_b$. Because X has the geodesic extension property, Lemma 3.10 shows that $\mathsf{d}' = \mathsf{d}$. Hence d is the unique metric realising V, so every element of $\operatorname{Isom} X_L$ is an isometry of X. □

4. Contracting geodesics and stability

Let X be a CAT(0) space. In this section, we consider geodesics in, and groups of isometries of, X that can be considered "negatively curved".

Definition 4.1 (Contracting)**.** We say that a geodesic γ is D–contracting if for any ball B disjoint from γ we have $\operatorname{diam} \pi_\gamma(B) \leqslant D$. A hyperbolic isometry of X is contracting if it has a contracting axis.

In fact, the definition of contracting geodesic makes sense in any metric space: given a geodesic γ and a point x, the set of points in γ realising $\mathsf{d}(x,\gamma)$ is nonempty, because one only needs to consider a compact subinterval of γ by the triangle inequality. One then finds that the closest-point projection to a contracting geodesic in a metric space is coarsely unique. Charney–Sultan showed that, in CAT(0) spaces, a geodesic being contracting is equivalent to its being *Morse* [CS15, Thm 2.14]. In the setting of proper CAT(0) spaces, Bestvina–Fujiwara showed an isometry is *rank one* if and only if its axes are contracting [BF09, Thm 5.4].

The first result of this section is the following, which sums up Lemma 4.4 and Proposition 4.6; it is worth noting that the proof does not directly use X_L or the hyperbolicity thereof. We say that a geodesic α meets a chain $\{h_i\}$ of hyperplanes *r–frequently* if there are $\alpha(t_i) \in h_i$ such that $t_{i+1} - t_i \leqslant r$ for all i.

Theorem 4.2. *Let X be a CAT(0) space. If $\alpha \subset X$ is a D–contracting geodesic, then there is a $(10D+3)$–chain of curtains met $8D$–frequently by α. Conversely, if a geodesic β meets an L–chain of curtains T–frequently, where $T \geqslant 1$, then β is $16T(L+4)$–contracting.*

Recall the following version of bounded geodesic image for CAT(0) spaces.

Lemma 4.3 ([CS15, Lem. 4.5])**.** *Let α be a D–contracting geodesic in a geodesic space X. If $x, y \in X$ have $\mathsf{d}(\pi_\alpha(x), \pi_\alpha(y)) \geqslant 4D$, then any geodesic γ from x to y satisfies $\pi_\alpha(\gamma) \subseteq \mathcal{N}_{5D}(\gamma)$.*

We now prove the forward direction of Theorem 4.2. Recall that $h_{\alpha,r}$ denotes the curtain dual to the geodesic α centred around the point $\alpha(r)$.

Lemma 4.4. *Let $\alpha : I \to X$ be a D–contracting geodesic. The chain $\{h_i = h_{\alpha,8Di} : 8Di \in I\}$ is a $(10D+3)$–chain such that $\alpha(8Di) \in h_i$.*

Proof. Let k be a curtain meeting both h_i and h_{i+1}, and let β be its pole. See Figure 4. Because $\operatorname{diam} \pi_\alpha\beta \leqslant \operatorname{diam}\beta = 1$, there exists $t \in [8Di + 4D - 1, 8Di + 4D + 1]$ such that $\alpha(t) \notin \pi_\alpha\beta$. Let $x \in k \cap h_i$ and $y \in k \cap h_{i+1}$. There are points $x', y' \in \beta$ such that $[x, x'] \subset k$ and $[y, y'] \subset k$. Since π_α is continuous, there is some $z \in [x, x'] \cup [y, y']$ such that $\pi_\alpha(z) = \alpha(t)$. The cases are similar, so let us assume that $z \in [x, x']$. Lemma 4.3 tells us that $\alpha(t) = \pi_\alpha(z) \in \pi_\alpha[x, x']$ is $5D$–close to $[x, x'] \subset k$.

Thus $\alpha(8Di + 4D)$ is $(5D+1)$–close to every curtain k meeting both h_i and h_{i+1}. This shows that every chain of curtains meeting h_i and h_{i+1} has cardinality at most $10D+3$. □

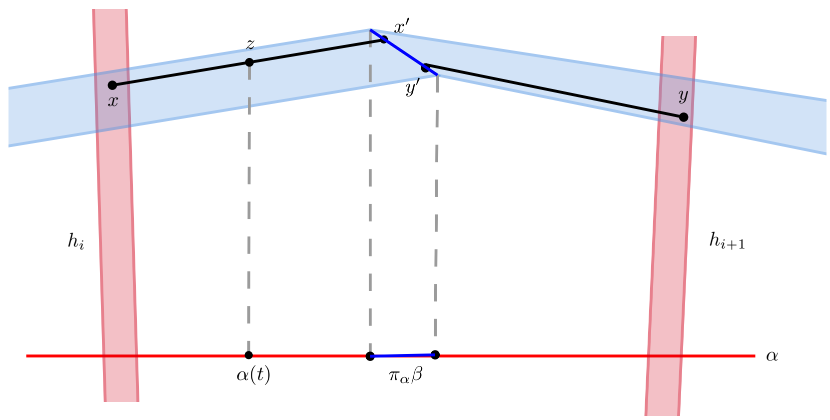

FIGURE 4. The proof of Lemma 4.4.

For the reverse direction of Theorem 4.2, we are given an L–chain of curtains that meet α. We begin by showing that we can assume that they are actually dual to α. This is similar to Lemma 2.22, but includes the extra information of how frequent the crossing is.

Lemma 4.5. *Let α be a geodesic meeting an L–chain of curtains $c = \{h_i\}$ at points $\alpha(t_i)$ with $t_{i+1} - t_i \in [1, T]$. There is an L–chain $\{k_i = h_{\alpha,s_i}\}$ of curtains dual to α such that $s_{i+1} - s_i \leqslant 4T(L+3)$ for all i.*

Proof. Let $m = 2L + 6$. If α has length at most $2T(L+4)$, then let $\{k_i\}$ consist of any one curtain dual to α. Otherwise, there is some i such that both h_i and h_{i+m-1} exist. For each such i, let c'_i be a chain of curtains dual to α realising $|c'_i| + 1 = \mathsf{d}_\infty(\alpha(t_{i+2}), \alpha(t_{i+m-3})) \geqslant m - 5 = 2L + 1$. Since h_i and h_{i+1} are L–separated, at most L elements of c'_i meet h_i, and similarly at most L elements of c'_i meet h_{i+m-1}. Let g_i be any other element of c'_i, so that $\{h_i, g_i, h_{i+m-1}\}$ is a chain.

Define $k_i = g_{mi}$ for every i such that h_{mi} and h_{mi+m-1} exist, and consider the chain $c'' = \{k_i\}$. For each i, let s_i be the real number such that $k_i = h_{\alpha,s_i}$. We have $s_{i+1} - s_i \leqslant \mathsf{d}(h_{im}, h_{im+2m}) \leqslant 2Tm$, so it remains to show that c'' is an L–chain. Since h_{i+m-1} and h_{i+m} separate k_i from k_{i+1}, any curtain meeting k_i and k_{i+1} must meet h_{i+m-1} and h_{i+m}. Because c is an L–chain, this implies that c'' is an L–chain. □

Lemma 4.5 allows us to apply Lemma 2.14 to prove the reverse direction of Theorem 4.2.

Proposition 4.6. *If α is a geodesic meeting an infinite L–chain of curtains $\{h_i\}$ at points $\alpha(t_i)$ such that $t_{i+1} - t_i \leqslant T$, then α is $(16T(L+3)+3)$–contracting.*

Proof. According to Lemma 4.5, after replacing L by $4T(L+3)$, we may assume that $h_i = h_{\alpha,t_i}$. Suppose that $x, y \in X$ have $\mathsf{d}(\pi_\alpha x, \pi_\alpha y) > 2L + 2$. Then (perhaps after relabelling) there exists i such that $x \in h^-_{i-1}$ and $y \in h^+_{i+1}$. Let $z \in [x, y] \cap h_i$. Lemma 2.14 tells us that $\mathsf{d}(z, \pi_\alpha z) < 2L + 1$. As $\pi_\alpha z \in h_i$, we have

$$\mathsf{d}(x, y) \geqslant \mathsf{d}(x, z) \geqslant \mathsf{d}(x, \pi_\alpha z) - 2L - 1 \geqslant \mathsf{d}(x, \pi_\alpha x) - 2L - 1.$$

In particular, any point y with $\mathsf{d}(x, y) \leqslant \mathsf{d}(x, \pi_\alpha x) - 2L - 1$ has $\mathsf{d}(\pi_\alpha x, \pi_\alpha y) \leqslant 2 + 2L$.

Let B be a ball centred on x that is disjoint from α, and let $w \in B$. Let y be the point of $[x, w]$ with $\mathsf{d}(x, y) = \min\{\mathsf{d}(x, w), \mathsf{d}(x, \pi_\alpha x) - 2L - 1\}$. As $\mathsf{d}(y, w) \leqslant 2L - 1$, we have

$$\mathsf{d}(\pi_\alpha x, \pi_\alpha w) \leqslant \mathsf{d}(\pi_\alpha x, \pi_\alpha y) + \mathsf{d}(\pi_\alpha y, \pi_\alpha w) \leqslant (2L+2) + (2L+1),$$

because π_α is 1–Lipschitz. □

Corollary 4.7. *A CAT(0) space X is hyperbolic if and only if X and X_L are quasiisometric for some L.*

Proof. All geodesics in a hyperbolic space are uniformly contracting, so the forward direction follows from Lemma 4.4. The reverse is Theorem 3.5. □

When considering axes of isometries, the conclusion of Theorem 4.2 can be strengthened.

Definition 4.8 (Skewer)**.** An isometry g is said to skewer two curtains h_1, h_2 if, perhaps after flipping, $g^m h_1^+ \subsetneq h_2^+ \subsetneq h_1^+$ for some $m \in \mathbf{N}$.

As discussed in the introduction, the equivalence between contraction and skewering of separated curtains mirrors a characterisation in the cubical setting [CS11, Gen20a].

Theorem 4.9. *Let g be a semisimple isometry of X. The following are equivalent.*
1. *(1) g is contracting.*
2. *(2) g acts loxodromically on some X_L.*
3. *(3) g is hyperbolic and there exist L, n such that $\mathsf{d}_L(x, g^n x) > L + 4$ for some x lying in an axis of g.*
4. *(4) g is skewers a pair of separated curtains.*

Proof. (2) and (4) follow immediately from (1) by Theorem 4.2. If g acts loxodromically on X_L, then g is necessarily a hyperbolic isometry of X, because $\mathsf{d}_L \leqslant 1 + \mathsf{d}$ (Remark 2.16), and (3) is immediate.

Assuming (3), let c be an L–chain realising $|c| + 1 = \mathsf{d}_L(x, g^n x) > L + 4$. By Lemma 2.13, we can find a nonempty subchain c' such that $\bigsqcup g^j c'$ is an L–chain. Let h_1 be the minimal element of c', and let h_2 be the maximal element. Let s_1 satisfy $\alpha(s_1) \in h_1$ and let s_2 satisfy $\alpha(s_2) \in g h_2$. Applying Theorem 4.2, we see that α is contracting, with constant depending only on L and $|s_2 - s_1|$. Hence (3) implies (1).

Finally, we show that (4) implies (1). Let h_1, h_2 be curtains such that $g^m h_1^+ \subsetneq h_2^+ \subsetneq h_1^+$. Because any curtain crossing h_1 and $g^m h_1$ crosses h_2, the curtains $g^{km} h_1$ and $g^{(k+1)m} h_1$ are L–separated for every $k \in \mathbf{Z}$. That is, $\{g^{km} h_1 : k \in \mathbf{Z}\}$ is an L–chain. In particular, we have a nested sequence of halfspaces

$$\cdots g^{2m} h_1^+ \subsetneq g^m h_1^+ \subsetneq h_1^+ \subsetneq g^{-m} h_1^+ \subseteq g^{-2m} h_1^+ \cdots$$

If $p \in h_1$, then $\mathsf{d}(p, g^{nm} p) \geqslant n$ by Remark 2.3, so $\lim_{k \to \infty} \frac{\mathsf{d}(p, g^{km} p)}{k} \geqslant 1$. Hence g is hyperbolic, because it is semisimple.

Any axis α of g necessarily meets h_1. Indeed, let $x \in \alpha$. If $x \in h_1$ then we are done. Otherwise, there is some n such that h_1 separates x from $g^{nm} x$, and we apply Lemma 2.5. Hence $\alpha = g^{km} \alpha$ meets every $g^{km} h_1$. Theorem 4.2 tells us that α is contracting. □

We finish this section by characterising stable subgroups of CAT(0) groups in terms of their orbits on the X_L. As discussed in the introduction, this is the same as the situation in mapping class groups [DT15, FM02, KL08], and, more generally, hierarchically hyperbolic groups [ABD21].

Definition 4.10 (Stable)**.** Let Y be a subset of a CAT(0) space X. We say that Y is *stable* if there exist $\mu, D \geqslant 0$ such that every geodesic between points of Y is D–contracting and stays μ–close to Y. A subgroup of a group acting properly coboundedly on X is *stable* if it has a stable orbit.

We remark that the above definition is specialised to the case of CAT(0) spaces. In general, it is necessary to consider *Morse* geodesics instead of contracting ones, but in CAT(0) spaces the two notions are equivalent [CS15, Thm 2.9].

Proposition 4.11. *Let G be a group acting properly coboundedly on a CAT(0) space X. A subgroup $H \leqslant G$ is stable if and only if it is finitely generated and there is some L such that orbit maps $H \to X_L$ are quasiisometric embeddings.*

Proof. Assume that an orbit $H \cdot x$ is stable in X, and let $g, h \in H$. By assumption, there is a contracting geodesic α connecting gx and hx. By Theorem 4.2, α meets an L–chain of curtains L–frequently, where L is determined by the contracting constant. Thus α uniformly quasiisometrically embeds in X_L by definition, which gives a coarse-linear equivalence between $\mathsf{d}_L(gx, hx)$ and $\mathsf{d}(gx, hx)$. Since H is stable, by [Tra19, Prop. 4.11] it is both finitely generated and undistorted in G. Thus $\mathsf{d}(hx, kx)$ and $\mathsf{d}_H(h, k)$ are comparable, yielding the forward direction.

For the converse, suppose that $H \cdot x$ is quasiisometrically embedded in X_L. Since H is finitely generated, its orbit maps on X are coarsely Lipschitz, and because the remetrisation $X \to X_L$ is coarsely Lipschitz, this means that $H \cdot x$ is also quasiisometrically embedded in X. In particular, for any $g, h \in H$, the CAT(0) geodesic $[gx, hx]$ uniformly quasiisometrically embeds in X_L, and so is contracting by Theorem 4.2. The contracting property implies that the quasiisometric image of any H–geodesic from g to h is uniformly Hausdorff-close to $[gx, hx]$. Hence $[gx, hx]$ stays uniformly close to $H \cdot x$. □

5. Acylindricity and the diameter dichotomy

In this section, we show that each space X_L is either unbounded or uniformly bounded, and investigate the former case. We start with a simple consequence of Theorem 4.9.

Lemma 5.1. *A group G acting coboundedly by isometries on a CAT(0) space X has a contracting element if and only if some X_L is unbounded.*

Proof. If G has a contracting element, then Theorem 4.9 implies that some X_L is unbounded. If X_L is unbounded, then so is its injective hull $E(X_L)$, which contains X_L as a coarsely dense subspace. As G acts coboundedly on the geodesic hyperbolic space $E(X_L)$, Gromov's classification [Gro87, CCMT15] implies that G contains an isometry acting loxodromically on $E(X_L)$ and hence on X_L. That isometry is contracting by Theorem 4.9. □

It turns out that one can say more about the action of G on X_L.

Definition 5.2 (WPD, non-uniform acylindricity)**.** Let G be a group acting on a metric space X. An element $g \in G$ is called *WPD* (weakly properly discontinuous) if for each $\varepsilon > 0$ and each $x \in X$, there exists $m > 0$ such that

$$|\{h \in G \,:\, \mathsf{d}(x, hx),\, \mathsf{d}(g^m x, hg^m x) < \varepsilon\}| < \infty.$$

The action is said to be *non-uniformly acylindrical* if for each $\varepsilon > 0$ there exists R such that for any $x, y \in X$ with $\mathsf{d}(x, y) \geqslant R$, only finitely many $g \in G$ have $\max\{\mathsf{d}(x, gx), \mathsf{d}(y, gy)\} < \varepsilon$.

Observe that every hyperbolic isometry in a non-uniformly acylindrical action is WPD. Moreover, note that an action on a bounded metric space is always non-uniformly acylindrical.

We need the following lemma, which is similar to Lemma 2.14, but does not require the curtains to be dual to a single geodesic.

Lemma 5.3. *Suppose that α and α' are geodesics that cross three pairwise L–separated curtains h_1, h_2, h_3, and let $x_i \in \alpha \cap h_i$, $y_i \in \alpha' \cap h_i$. If $\mathsf{d}(x_1, x_2), \mathsf{d}(x_2, x_3) \leqslant T$, then $\mathsf{d}(x_2, y_2) \leqslant 2L + \lceil T \rceil$.*

Proof. Let c be chain dual to $\beta = [x_2, y_2]$ that realises $\mathsf{d}_\infty(x_2, y_2) = 1 + |c|$. In view of Lemma 2.10, it suffices to show that $|c| \leqslant 2L + \lceil T \rceil - 1$.

Because $x_2, y_2 \in h_2$, every element of c must intersect h_2 by Lemma 2.5. Thus L–separation tells us that at most $2L$ elements of c can intersect $h_1 \cup h_3$. Moreover, by Lemma 2.7, no curtain in c can intersect both $[x_1, x_2]$ and $[x_2, x_3]$. Hence, perhaps after relabelling, all but at most $2L$ elements of c must cross both $[x_1, x_2]$ and $[y_2, y_3]$. But $\mathsf{d}(x_1, x_2) = T$, so $|c| \leqslant 2L + \lceil T \rceil - 1$. □

We now have all the ingredients to prove that the action on the X_L is non-uniformly acylindrical.

Proposition 5.4. *Any group G acting properly on a CAT(0) space X acts non-uniformly acylindrically on every X_L. In particular, if $g \in G$ is contracting, then g is WPD on X_L for every L for which it is loxodromic.*

Proof. If $\operatorname{diam} X_L < \infty$ then there is nothing to prove. Otherwise, given $\varepsilon > 0$, let $\varepsilon' = \lceil \varepsilon \rceil$ and let $R = 4 + 2\varepsilon'$. Suppose that $x, y \in X$ have $\mathsf{d}_L(x, y) \geqslant R$, and let $b = [x, y]$. There is an L–chain $\{k_1, \dots, k_{\varepsilon'}, h_1, h_2, h_3, k'_1, \dots, k'_{\varepsilon'}\}$ separating x from y. Let $x_i \in b \cap h_i$, and let B be the ball in X with centre x_2 and radius $2L + \mathsf{d}(x, y) + 1$. Note that $b \subset B$.

For any $g \in G$ with $\mathsf{d}_L(x, gx), \mathsf{d}_L(y, gy) < \varepsilon$, the curtains h_1, h_2, and h_3 all separate gx from gy. From Lemma 5.3, we deduce that $\mathsf{d}(x_2, gb) \leqslant 2L + \max\{\mathsf{d}(x_1, x_2), \mathsf{d}(x_2, x_3)\} + 1$. In particular, $gb \subset gB$ meets B. By properness of the action of G on X, there are only finitely many such g. □

Remark 5.5. In fact, when considering WPD elements, we could weaken the assumptions of Proposition 5.4 with more work. Namely, if one drops the assumption that the action of G on X is proper, then one can still show that any WPD contracting isometry of X is WPD on X_L.

Since the existence of a loxodromic WPD element for an action on a hyperbolic space is equivalent to acylindrical hyperbolicity [DGO17, Osi16], we obtain the following. This simplifies the proof from [Sis18].

Corollary 5.6. *If G acts properly on a proper CAT(0) space and has a rank-one element, then G is acylindrically hyperbolic or virtually cyclic.*

Now that we understand the case where some X_L is unbounded, it is desirable to have some control on the case where every X_L is bounded. The next result provides this under a natural assumption.

Definition 5.7 (Geodesic extension property)**.** A CAT(0) space has the geodesic extension property if every geodesic segment is a restriction of some biinfinite geodesic.

Proposition 5.8. *Let X be a cobounded CAT(0) space with the geodesic extension property, and let $L \in \mathbf{N}$. Either* $\operatorname{diam} X_L \leqslant 4$, *or X_L is unbounded.*

Proof. We show that the existence of an L–chain of length $n \geqslant 4$ implies the existence of an L–chain of length $n+1$. See Figure 5. Let $c = \{h_1, \dots, h_n\}$ be an L–chain with $n \geqslant 4$. Let B be a ball in X that meets both h_1^- and h_n^+. Let α be a biinfinite geodesic such that h_n is dual to α. Fix a point $p \in B$, and let r be such that the translates of p are r–dense in X. Let $q \in \alpha \cap h_n^+$ satisfy $\mathsf{d}(q, h_n) = r + L + \operatorname{diam} B + 2$, and fix $g \in \operatorname{Isom} X$ such that $\mathsf{d}(gp, q) \leqslant r$.

Let q' be the point in $\alpha \cap h_n^+$ with $\mathsf{d}(q', h_n) = L + 2$, and let $\{h_1', \dots, h_{L+1}'\}$ be a chain dual to α separating q' from $\alpha \cap h_n$. By the choice of g, we have $gB \subset h_{L+1}'^{+}$ Moreover, we have $h_n \subset h_1'^{-}$. As every gh_i meets gB, any gh_i meeting h_n must meet every h_j', so no two elements of gc can meet h_n, as gc is an L–chain. In particular, there is an L–subchain $c' = \{gh_{i_1}, gh_{i_2}\} \subset gc$ contained in h_n^+. Applying Lemma 2.12 to c and c' produces an L–chain of length $n + 1$. □

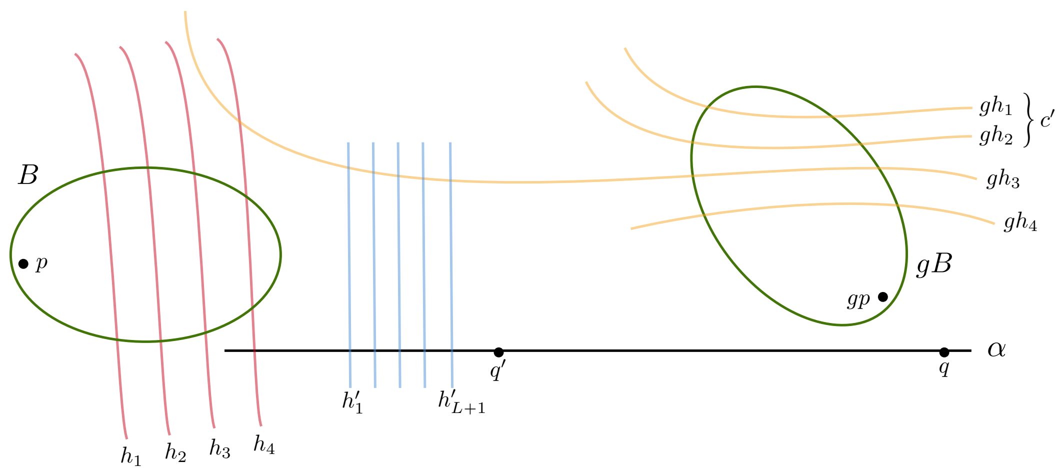

FIGURE 5. The proof of Proposition 5.8, illustrated with $n = 4$.

In particular, if no X_L is unbounded, then every X_L has diameter at most 4. When X is proper (but does not necessarily have the geodesic extension property) we can strengthen the previous statement with Corollary 5.20 below. This does not obsolete Proposition 5.8, though. For one thing, its proof is elementary and entirely self-contained. For another, Corollary 5.20 does not say anything about the diameters of the bounded X_L in the case where some $X_{L'}$ is unbounded.

5.1. TITS BOUNDARIES

Whilst the proof of Proposition 5.8 is elementary, it requires that we begin with an L–chain of length at least four to conclude that X_L is unbounded, leaving some mystery as to the significance of L–chains of length two and three. By using more advanced machinery, we shall show that the existence of even two separated curtains implies the existence of an unbounded X_L.

Recall that the *visual boundary* ∂X as a set is defined to be the collection of all equivalence classes of geodesic rays, where two rays are equivalent if they are at finite Hausdorff-distance. Equivalently, it is the set of all geodesic rays emanating from a fixed basepoint $\mathfrak{o}$. Because of this, if the basepoint is understood then we shall often fail to distinguish between an element of ∂X and the representative with that basepoint. We write α^∞ for the element of the visual boundary represented by a geodesic ray α.

Definition 5.9 (Angle)**.** If σ_1 and σ_2 are geodesics with $\sigma_1(0) = \sigma_2(0) = x$, then $\angle_x(\sigma_1, \sigma_2) = \lim_{t \to 0} \gamma(t)$, where $\gamma(t)$ is the angle at x in the comparison triangle for $[x, \sigma_1(t), \sigma_2(t)]$. If $\xi_1, \xi_2 \in \partial X$, then $\angle(\xi_1, \xi_2) = \sup\{\angle_x(\xi_1, \xi_2) : x \in X\}$.

The angle defines a distance and hence a topology on the visual boundary. As one of the prominent features of curtains is that they separate the space, it is natural to wonder if there is a well-defined notion of limit of a curtain, and if it would disconnect the boundary. This turns out to be the case.

Definition 5.10 (Limit of a curtain)**.** For a curtain $h = h_\alpha$ with pole P, write

$$\Lambda(h) = \{\xi \in \partial X \,:\, \text{there exists } p \in P \text{ such that } \pi_\alpha[p, \xi] = p\}.$$

Lemma 5.11. *If ∂X is connected, then $\Lambda(h)$ is non-empty and separates ∂X into two components.*

Proof. Let $x \in P$, and let $\gamma\colon [a, b] \to \partial X$ be a path from $\alpha^{-\infty}$ to α^∞. By [BH99, Prop. II.9.2], the map $\phi\colon [a, b] \to [0, \pi]$ defined by $\phi(t) = \angle_x(\alpha^\infty, \gamma(t))$ is continuous. Since $\phi(a) = \pi$ and $\phi(b) = 0$, there is some $\eta = \gamma(t_0)$ such that $\angle_x(\alpha^\infty, \eta) = \frac{\pi}{2}$. We claim that $\pi_\alpha[x, \eta] = x$, which implies that $\gamma(t_0) \in \Lambda(h)$.

Suppose that there is $y \in [x, \alpha(t_0)]$ such that $\pi_\alpha(y) \neq x$. By [BH99, Prop. II.2.4] we have $\angle_{\pi_\alpha(y)}(y, x) \geqslant \frac{\pi}{2}$. Moreover, as $\alpha^{-\infty}$ and α^∞ are opposite ends of a geodesic, we have $\angle_x(\eta, \alpha^{-\infty}) = \frac{\pi}{2}$ [Bal95, I.3.9, I.3.10]. Thus, the triangle $[x, y, \pi_\alpha(y)]$ has two angles of size at least $\frac{\pi}{2}$ and no ideal vertex, which contradicts the CAT(0)–inequality. Thus any path from $\alpha^{-\infty}$ to α^∞ must intersect $\Lambda(h)$, providing the result. □

Our goal now is to obtain lower bounds on the angles between various points in the visual boundary. We first note in Corollary 5.13 that the angle between the endpoints of a geodesic α and points of $\Lambda(h_\alpha)$ is at least $\frac{\pi}{2}$.

Lemma 5.12. *Let α be a geodesic ray. If β is a geodesic ray based at $\alpha(t_0)$ such that $\pi_\alpha\beta \subset \alpha|_{[0,t_0]}$, then $\angle(\alpha^\infty, \beta^\infty) \geqslant \frac{\pi}{2}$.*

Proof. We have $\angle(\alpha^\infty, \beta^\infty) \geqslant \angle_{\alpha(t_0)}(\alpha^\infty, \beta^\infty)$ by definition. As $t \to 0$, the fact that $\pi_\alpha\beta(t) \subset \alpha|_{[0,t_0]}$ implies that the comparison angle for $[\alpha(t_0), \beta(t), \alpha(t_0 + t)]$ at $\alpha(t_0)$ is at least $\frac{\pi}{2}$. □

Corollary 5.13. *If ∂X is connected and $h = h_\alpha$ is a curtain, then $\angle(\alpha^\infty, \xi) \geqslant \frac{\pi}{2}$ for all $\xi \in \Lambda(h)$.*

Next we aim to bound the angle between a pair of separated curtains; this is done in Proposition 5.16. We recall the following lemma.

Lemma 5.14 ([Bal95, Thm II.4.4])**.** *Let $\xi_1, \xi_2 \in \partial X$, $x \in X$. Let σ_i be the geodesic ray from x to ξ_i. The quantity $c = \lim_{t\to\infty} \frac{1}{t}\,\mathsf{d}(\sigma_1(t), \sigma_2(t))$ is independent of x. In the Euclidean triangle with sides of length 1, 1, and c, the angle opposite c is $\angle(\xi_1, \xi_2)$.*

We use the following version of Lemma 5.12 that does not require any information on basepoints.

Lemma 5.15. *Let α and β be geodesic rays. If $\pi_\alpha\beta \subset \alpha|_{[0,t_0]}$, then*

$$\lim_{t\to\infty} \frac{1}{t}\,\mathsf{d}(\alpha(t), \beta(t)) \geqslant \sqrt{2}.$$

Proof. Let $\beta(t_1) = \pi_\beta\alpha(t_0)$ and write $\delta = \mathsf{d}(\alpha(t_0), \beta(t_1))$. By the reverse triangle inequality, $\mathsf{d}(\alpha(t_0), \beta(t)) \geqslant \mathsf{d}(\beta(t_1), \beta(t)) - \delta$ for all t.

Now, for $t \geqslant t_1$ let γ_t be the geodesic from $\alpha(t_0)$ to $\beta(t)$. By convexity of the metric, γ_t lies in the $\mathsf{d}(\alpha(t_0), \pi_\alpha\beta(t))$–neighbourhood of $[\beta(t), \pi_\alpha\beta(t)]$, so the fact that $\pi_\alpha[\pi_\alpha\beta(t), \beta(t)] =$

$\pi_\alpha\beta(t)$ means that $\pi_\alpha\gamma_t \subset \alpha|_{[0,t_0]}$. It follows that if $t > \max\{t_0, t_1\}$, then $\angle_{\alpha(t_0)}(\alpha(t), \beta(t)) \geqslant \frac{\pi}{2}$. Using convexity of the metric, we use this to compute

$$\begin{aligned} \mathsf{d}(\alpha(t), \beta(t))^2 &\geqslant \mathsf{d}(\alpha(t), \alpha(t_0))^2 + \mathsf{d}(\alpha(t_0), \beta(t))^2 \\ &\geqslant (t-t_0)^2 + (t-t_1-\delta)^2, \end{aligned}$$

and the result follows immediately. □

Proposition 5.16. *Suppose that X has connected visual boundary. Let h, h' be curtains with respective poles P and P'. If h and h' are separated, then $\angle(\xi, \xi') \geqslant \frac{\pi}{2}$ for all $\xi \in \Lambda(h)$, $\xi' \in \Lambda(h')$.*

Proof. Let $\xi \in \Lambda(h)$ and let $p \in P$ be such that $\pi_P[p, \xi] = p$. Let $\alpha = [p, \xi]$. Take any point $\xi' \in \Lambda(h')$, and let $\beta \subset h'$ be a geodesic ray based in P' with $\beta^\infty = \xi'$.

For $i \in \{0, 1\}$, consider the chains $c_i = \{h_{\alpha,2n+i} : n \in \mathbf{N}\}$. Since h and h' are L–separated for some L, at most L of each can meet h', and hence β can meet at most L of each. Since $c_0 \cup c_1$ is a cover of α and $\pi_\alpha\beta$ is nonempty, this means that there exists $t_0 \geqslant 0$ such that $\pi_\alpha\beta \subset \alpha|_{[0,t_0]}$.

Let $\delta = \mathsf{d}(\alpha(t_0), \beta(0))$. Now let γ be the geodesic ray based at $\alpha(t_0)$ with $\gamma^\infty = \xi'$. By the flat strip theorem, $\mathsf{d}(\gamma(t), \beta(t)) \leqslant \delta$ for all t. In particular, $c = \lim_{t\to\infty} \frac{1}{t} \mathsf{d}(\alpha(t), \gamma(t)) = \lim_{t\to\infty} \frac{1}{t} \mathsf{d}(\alpha(t), \beta(t))$. According to Lemma 5.15, we have $c \geqslant \sqrt{2}$. Lemma 5.14 now tells us that $\angle(\xi, \xi')$ is equal to the isosceles angle in the Euclidean triangle with side lengths 1, 1, and c, which is at least $\frac{\pi}{2}$. □

Combining these results gives us information about the *Tits boundary* in the case where X has a pair of separated curtains.

Definition 5.17 (Tits boundary)**.** The *Tits metric* on ∂X is the path-metric induced by $\angle(\cdot, \cdot)$. The *Tits boundary* $\partial_T X$ of X is the (extended) metric space obtained in this way.

Corollary 5.18. *Let X be a CAT(0) space with connected visual boundary. If X has a pair of separated curtains, then the Tits boundary $\partial_T X$ of X has diameter at least $\frac{3\pi}{2}$.*

Proof. Let $h = h_\alpha$ and $k = k_\beta$ be L–separated curtains. We may assume that $k \subset h^-$ and $h \subset k^-$. By Lemma 5.11, any path in ∂X from α^∞ to β^∞ must pass through both $\Lambda(h)$ and $\Lambda(k)$. Corollary 5.13 and Proposition 5.16 show that, in the angle metric on ∂X, the length of such a path must be at least $\frac{3\pi}{2}$. □

Bringing in group actions, we are now in a position to prove the main result of this section. The proof relies on work of Guralnick–Swenson [GS13], which itself relies on ideas of [PS09].

Theorem 5.19. *Let X be a CAT(0) space, and let G be a group acting properly cocompactly on X. If X has a pair of separated curtains, then G has a rank-one element.*

Proof. If G has no rank-one element, then ∂X is connected [BB08], so [GS13, Thm 3.12] shows that $\operatorname{diam} \partial_T X < \frac{3\pi}{2}$. According to Corollary 5.18, X cannot have a pair of separated curtains. □

In view of Theorem 4.9, we get the following dichotomy for the diameters of the X_L.

Corollary 5.20. *Let X be a CAT(0) space admitting a proper cocompact group action. If $\operatorname{diam} X_L > 2$ for some L, then only finitely many X_L are bounded.*

6. Higher-rank CAT(0) spaces

In this section, we apply our machinery to the coarse geometry of CAT(0) spaces without rank-one isometries. The main goal of the section is to complete the proof of the weak rank-rigidity statement Corollary N by showing that if X is a CAT(0) space admitting a proper cocompact group action and every X_L is bounded, then X is wide. This is Proposition 6.6.

We start with the main technical lemma, which allows us to shrink polygons to efficiently avoid balls.

Lemma 6.1 (Circumnavigation Lemma). *Let $x_1, \dots, x_n \in X$, with $n \geqslant 3$, and write $x_{n+1} = x_1$. Let $p \in [x_1, x_2]$ and let B be the closed r–ball about p. Suppose that every $[x_i, x_{i+1}]$ with $i > 1$ is disjoint from the interior $\mathring{B}$ of B. There is a path from x_1 to x_2 that avoids $\mathring{B}$ and has length at most $8(nr + D)$, where $D = \mathsf{d}(x_1, x_2)$.*

Proof. We begin by modifying the set of x_i. See Figure 6. Let $x'_3 = x_3$. Given x'_i for $i > 2$, if x_{i+2} exists then proceed as follows. If $[x'_i, x_{i+2}]$ is disjoint from B, then delete x_{i+1}, relabel x_j as x_{j-1} for every $j > i + 1$, and repeat with the new x_{i+2} (if it exists). Otherwise, fix a point $x'_{i+1} \in [x_{i+1}, x_{i+2}]$ such that $\mathsf{d}([x'_i, x'_{i+1}], m) = r$. After this process, we have points $x_1, x_2, x_3 = x'_3, x'_4, \dots, x'_m$ with $m \leqslant n$ such that $\mathsf{d}([x'_i, x'_{i+1}], p) = r$ for all $i \in \{3, \dots, m-1\}$.

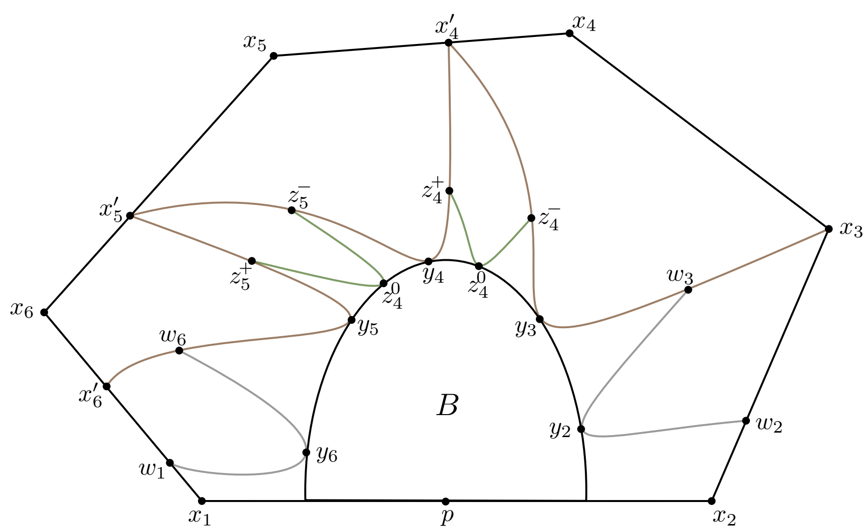

Figure 6. The construction in the proof of Lemma 6.1, illustrated with $n = m = 6$.

Because B is convex, there are unique points $y_i \in [x'_i, x'_{i+1}]$ with $\mathsf{d}(y_i, p) = r$ for $i \in \{3, \dots, m-1\}$. (In the case $m = 3$, set $y_2 = x_2$, $y_3 = x_1$.) Since $[x_1, y_{m-1}]$ meets B, by considering the family of geodesics with endpoints on $[x'_m, x_1]$ and $[x'_m, y_{m-1}]$, we can find $w_1 \in [x'_m, x_1]$ and $w_m \in [x'_m, y_{m-1}]$ such that $[w_1, w_m]$ meets B at a single point y_m. Similarly, we can find $w_2 \in [x'_3, x_2]$ and $w_3 \in [x'_3, y_3]$ such that $[w_2, w_3]$ meets B at a single point y_2 (in the case $m = 3$, these two paths are the same, so label it $[w_1, w_3]$ and write $w_2 = w_3$). Note that by convexity of the metric, $\mathsf{d}(w_1, w_m) \leqslant \mathsf{d}(x_1, y_{m-1})$ and $\mathsf{d}(w_2, w_3) \leqslant \mathsf{d}(x_2, y_3)$, both of which are at most $r + D$. By the triangle inequality, we consequently get that $\mathsf{d}(x_1, w_1)$ and $\mathsf{d}(x_2, w_2)$ are at most $2(r + D)$.

Now, if $m \geqslant 5$, for each $i \in \{4, \dots, m-1\}$ consider a family of geodesics with endpoints in $[x'_i, y_{i-1}]$ and $[x'_i, y_i]$. Because $y_i \in [x'_i, x'_{i+1}]$, from each one of these families we obtain points

$z_i^- \in [x'_{i-1}, x'_i]$ and $z_i^+ \in [x'_i, x'_{i+1}]$ such that the geodesic $[z_i^-, z_i^+]$ meets B at a single point z_i^0. Again, convexity of the metric implies that $\mathsf{d}(z_i^-, z_i^+) \leqslant \mathsf{d}(y_{i-1}, y_i) \leqslant 2r$. Moreover, the triangle inequality gives $\mathsf{d}(z_i^+, z_{i+1}^-) \leqslant \mathsf{d}(z_i^+, z_i^0) + \mathsf{d}(z_i^0, z_{i+1}^0) + \mathsf{d}(z_{i+1}^0, z_{i+1}^-) \leqslant 6r$.

Consider the path P defined as the concatenation

$$[x_1, w_1] \cup [w_1, w_m] \cup [w_m, z_{m-1}^+] \cup [z_{m-1}^+, z_{m-1}^-] \cup \bigcup_{i=2}^{m-4} \Big([z_{m-i+1}^-, z_{m-i}^+] \cup [z_{m-i}^+, z_{m-i}^-]\Big) \cup [z_4^-, w_3] \cup [w_3, w_2] \cup [w_2, x_2],$$

ignoring any terms that are undefined if $m \leqslant 4$. It connects x_1 to x_2 and avoids $\mathring{B}$. It suffices to bound the length $\ell(P)$. We have so far seen that

$$\mathsf{d}(x_1, w_1) \leqslant 2(r + D), \quad \mathsf{d}(w_1, w_m) \leqslant r + D, \quad \mathsf{d}(z_i^+, z_i^-) \leqslant 2r,$$
$$\mathsf{d}(z_i^-, z_{i-1}^+) \leqslant 6r, \quad \mathsf{d}(w_3, w_2) \leqslant r + D, \quad \mathsf{d}(w_2, x_2) \leqslant 2(r + D).$$

This leaves us needing to bound $\mathsf{d}(w_m, z_{m-1}^+)$ and $\mathsf{d}(z_4^-, w_3)$. By the triangle inequality, $\mathsf{d}(z_4^-, w_3) \leqslant \mathsf{d}(z_4^-, z_4^0) + \mathsf{d}(z_4^0, y_2) + \mathsf{d}(y_2, w_3) \leqslant 5r + D$, and similarly $\mathsf{d}(w_m, z_{m-1}^+) \leqslant 5r + D$. Combining all of these, we get that

$$\begin{aligned}\ell(P) &\leqslant 2(r + D) + (r + D) + (5r + D) + (2r) + (m - 5)(8r) + (5r + D) + (r + D) + 2(r + D)\\ &\leqslant 18r + 8D + 8r \max\{0, m - 5\} \leqslant 18r + 8D + 8r(n - 3).\end{aligned}$$
□

We now show that if $\operatorname{diam} X_L$ is uniformly bounded, we can verify the hypotheses of the circumnavigation lemma.

Lemma 6.2. *Let $L \geqslant 2$. Suppose that curtains h_1, h_2 are not L–separated, and let B be a ball in X with radius r. If $r \leqslant \frac{L-1}{2}$, then there is a curtain that meets h_1 and h_2 but not B.*

Proof. By Remark 2.3, any chain of curtains all of whose elements meet B must have cardinality at most $\lceil 2r \rceil + 1$. By assumption, there is a chain c of curtains of cardinality $L + 1$ such that every element of c meets both h_1 and h_2. If $r \leqslant \frac{L-1}{2}$, then $\lceil 2r \rceil + 1 \leqslant L$, so some element of c is disjoint from B. □

Lemma 6.3. *Let $L \geqslant 2$ and suppose that γ is a geodesic with dual curtains h_1 and h_2 that are not L–separated. Let $x_1 \in h_1$ and $x_2 \in h_2$ be the points of γ with $\mathsf{d}(x_1, x_2) = \mathsf{d}(h_1, h_2) = D$. If $p \in [x_1, x_2]$ is such that the interior of $B = B(p, \frac{L-1}{2})$ is disjoint from h_1 and h_2, then there is a path from x_1 to x_2 of length at most $8(3L + D)$ that avoids the interior of B.*

Proof. By Lemma 6.2, there is a curtain h meeting both h_1 and h_2 but not B. By star convexity there exist $x_3 \in h \cap h_2$ and $x_6 \in h \cap h_1$ with $[x_6, x_1] \in h_1$ and $[x_2, x_3] \in h$. Moreover, there are points x_4 and x_5 in the pole of h such that $[x_3, x_4] \cup [x_4, x_5], [x_5, x_6] \subset h$. In particular, the conditions of Lemma 6.1 are met with $r = \frac{L-1}{2}$. □

Let us now set up the necessary notation for asymptotic cones. Asymptotic cones were introduced by Gromov in [Gro81] and later clarified by van den Dries and Wilkie [vdDW84]. We refer the reader to [DS05] for a more thorough treatment.

Definition 6.4 (Asymptotic cone)**.** Let ω be a *non-principal ultrafilter* and let (λ_n) be a divergent sequence of positive numbers. Let (X, d) be a metric space, and consider the sequence of metric spaces $X_n = (X, \frac{1}{\lambda_n}\mathsf{d})$. Define an extended pseudometric δ_ω on $\prod_{i=1}^\infty X_n$ by setting $\delta_\omega((x_n), (y_n)) = r$ if for all $\varepsilon > 0$ we have $\{n : r - \varepsilon < \frac{1}{\lambda_n}\mathsf{d}(x_n, y_n) < r + \varepsilon\} \in \omega$, and $\delta_\omega((x_n), (y_n)) = \infty$ if there is no such r. Fix a basepoint $\mathfrak{o} \in X$. The metric quotient of the pseudometric space consisting of all (x_n) with $\delta_\omega((x_n), (\mathfrak{o})) < \infty$ is an *asymptotic cone* of X. We denote this metric space by $(X_\omega, \mathsf{d}_\omega)$, suppressing both the scaling sequence and the basepoint. If (x_n) is a sequence of points in X and $x_\omega \in X_\omega$, then we write $(x_n) \to_\omega x_\omega$ if (x_n) is a representative of x_ω.

Definition 6.5 (Wide)**.** Following [DS05], we say that a metric space is *wide* if none of its asymptotic cones have cut-points.

We are now ready to prove Proposition 6.6. Note that this can also be obtained as a consequence of [DMS10, Prop. 1.1] and the observation that Lemma 6.3 essentially provides *linear divergence*. We provide a proof in the interests of self-containment.

Proposition 6.6. *Let X be a CAT(0) space admitting a proper cocompact group action. If no X_L is unbounded, then X is wide.*

Proof. By Corollary 5.20, every X_L has diameter at most 2. In other words, no pair of curtains are L–separated for any L.

Suppose that for some ultrafilter ω and some scaling sequence (λ_n), the asymptotic cone X_ω has a cut-point p_ω. Note that X_ω is a CAT(0) space [BH99, Cor. II.3.10]. Let $x_\omega, y_\omega \in X_\omega$ be separated by p_ω, and let $\varepsilon = \min\{\mathsf{d}_\omega(x_\omega, p_\omega), \mathsf{d}_\omega(p_\omega, y_\omega)\}$. Fix sequences $(x_n) \to_\omega x_\omega$ and $(y_n) \to_\omega y_\omega$. Note that the set $U = \{n : \mathsf{d}(x_\omega, y_\omega) - \frac{\varepsilon}{4} < \frac{1}{\lambda_n}\mathsf{d}(x_n, y_n) < \mathsf{d}(x_\omega, y_\omega) + \frac{\varepsilon}{4}\}$ is an element of ω. By removing finitely many elements of U, we may also assume that $\frac{\varepsilon}{4}\lambda_n \geqslant \frac{1}{2}$ for all $n \in U$.

For each $n \in U$, we can fix a point $p_n \in [x_n, y_n]$ with $\mathsf{d}(x_n, p_n) = \lambda_n \mathsf{d}_\omega(x_\omega, p_\omega)$. Note that we have $\mathsf{d}(p_n, y_n) > \frac{3\varepsilon}{4}\lambda_n$. Moreover, by construction we have $(p_n) \to_\omega p_\omega$. Let z_n^1 be the point on $[x_n, p_n]$ with $\mathsf{d}(z_n^1, p_n) = \frac{\varepsilon}{2}\lambda_n$, and let z_n^2 be the point on $[p_n, y_n]$ with $\mathsf{d}(p_n, z_n^2) = \frac{\varepsilon}{2}\lambda_n$, which exists because $n \in U$. We use these points to define curtains: for $i \in \{1, 2\}$, let h_n^i be the curtain dual to $[x_n, y_n]$ at z_n^i. Because $\frac{\varepsilon}{4}\lambda_n \geqslant \frac{1}{2}$, the curtain h_n^1 separates x_n from p_n, and because $\mathsf{d}(z_n^2, y_n) > \frac{\varepsilon}{4}\lambda_n$, the curtain h_n^2 separates p_n from y_n. Because no pair of curtains is L–separated for any L, the curtains h_n^1 and h_n^2 are not $\varepsilon\lambda_n$–separated. Moreover, the h_n^i were constructed so that they are disjoint from the interior of the ball $B_n = B(p_n, \frac{\varepsilon\lambda_n - 1}{2})$. By Lemma 6.3, there is a path from z_n^1 to z_n^2 that avoids B_n and has length at most $1 + 8(3\varepsilon\lambda_n + (\varepsilon\lambda_n - 1))$.

In the limit we obtain a path in X_ω from $(z_n^1)_\omega$ to $(z_n^2)_\omega$ that avoids the interior of $B(p_\omega, \frac{\varepsilon}{2})$ and has length at most 32ε. Concatenating this with subintervals of $[x_\omega, y_\omega]$, we see that p_ω cannot separate x_ω from y_ω, a contradiction. □

Remark 6.7. By using Proposition 5.8 instead of Corollary 5.20, a similar argument to the above proof of Proposition 6.6 can be used to show that X is wide under the assumptions that X is cobounded (not necessarily proper), has the geodesic extension property, and no X_L is unbounded.

Theorem 6.8. *Let G be a group acting properly cocompactly on a CAT(0) space X. One of the following holds.*

- *Some X_L is unbounded, in which case G has a rank one element and is either virtually cyclic or acylindrically hyperbolic.*
- *Every X_L has diameter at most 2, in which case G is wide.*

Proof. If every X_L has diameter at most 2, then G is wide by Proposition 6.6. If some X_L has diameter more than 2, then Corollary 5.20 shows that some X_L is unbounded. In this case, the consequences come from Lemma 5.1 and Proposition 5.4. □

7. Curtain boundaries

The main goal of this section is to investigate the relationship the between Gromov boundaries of the spaces X_L and the visual boundary of the corresponding CAT(0) space X. We shall consider the visual boundary as being equipped with the *cone topology*, which is determined by the neighbourhood basis consisting of the following sets. Given a geodesic ray b emanating from a fixed basepoint $\mathfrak{o}$, for constants $r \geqslant 0$ and $\epsilon > 0$, let

$$U(b^\infty, r, \epsilon) := \{c^\infty \in \partial X \ : \ c(0) = \mathfrak{o} \text{ and } \mathsf{d}(c(r), b) < \epsilon\}.$$

Let $\mathcal{B}_L$ be the subspace of the visual boundary ∂X consisting of all geodesic rays b emanating from $\mathfrak{o}$ such that there is an infinite L-chain crossed by b, and let $\mathcal{B} = \bigcup_{L \geqslant 0} \mathcal{B}_L$.

Theorem 7.1. *Let X be a proper CAT(0) space. For each $L \geqslant 0$, we have the following.*
1. *Each point in $\mathcal{B}_L$ is a visibility point of the visual boundary ∂X.*
2. *The subspace $\mathcal{B}_L \subseteq \partial X$ is* $\operatorname{Isom} X$*–invariant.*
3. *The identity map $\iota : X \to X_L$ induces an* $\operatorname{Isom} X$*–equivariant homeomorphism $\partial\iota : \mathcal{B}_L \to \partial X_L$.*

In other words, the visual boundary ∂X contains $\operatorname{Isom} X$–invariant copies of the Gromov boundaries ∂X_L.

Definition 7.2 (Separation from boundary points, crossing)**.** Let $b : [0, \infty) \to X$ be a geodesic ray and let h be a curtain. We say that h *separates* b^∞ from $A \subset X$ if there exists t_0 such that h separates A from $b|_{(t_0,\infty)}$. We say that a geodesic ray b based at $\mathfrak{o}$ *crosses* an infinite chain $\{h_i\}$ of curtains if every h_i separates $\mathfrak{o}$ from b^∞.

Remark 7.3. Because curtains may not be convex, it is *a priori* possible for a geodesic ray b to meet every element an infinite chain of curtains $\{h_i\}$, none of which separates $b(0)$ from b^∞. However, Lemma 3.1 ensures that if b meets every element of an infinite L-chain, then it crosses it.

Recall that $b^\infty \in \partial X$ is said to be a *visibility point* if for any other c^∞ in ∂X there exists a geodesic line l at finite Hausdorff-distance from $b \cup c$. The following lemma establishes part (1) of Theorem 7.1.

Lemma 7.4. *If X is proper, then every point $b^\infty \in \mathcal{B}$ is a visibility point of ∂X.*

Proof. Let $\{h_i\}$ be an infinite L–chain dual to b, which exists by Lemma 2.22, and orient the h_i so that $\mathfrak{o} \in h_i^-$. Let c be any other geodesic ray with $c(0) = b(0) = \mathfrak{o}$. According to Corollary 2.23, there is some k such that $c \subset h_{k-1}^-$. Let $x_n = [c(n), b(n)]$. Lemma 2.14 tells us that there exist $p \in h_k \cap b$, an integer $m \geqslant 1$, and points $p_n \in [c(n), b(n)]$ such that $\mathsf{d}(p_n, p) \leqslant 2L + 2$ for all $n \geqslant m$. Since balls in X are compact, the statement follows from [BH99, Lem. II.9.22]. □

The action of $\operatorname{Isom} X$ on X induces an action on ∂X. Indeed, if α is a geodesic ray based at $\mathfrak{o}$ and $g \in \operatorname{Isom} X$, then $g\alpha$ is a geodesic ray, and there is a unique ray β based at $\mathfrak{o}$ with $\beta^\infty = (g\alpha)^\infty$ [BH99, II.8.2] We declare $g(\alpha^\infty) = \beta^\infty$.

Lemma 7.5. *For any CAT(0) space X, the set $\mathcal{B}_L$ is* $\operatorname{Isom} X$*–invariant.*

Proof. By Lemma 2.22, any geodesic ray b with $b^\infty \in \mathcal{B}_L$ crosses an infinite L–chain dual to b. For any $g \in \operatorname{Isom} X$, the geodesic ray gb crosses an infinite L–chain $\{h_i\}$ dual to gb. The unique geodesic ray c emanating from $\mathfrak{o}$ with $c^\infty = gb^\infty$ lies at finite Hausdorff-distance from gb, so $\pi_{gb}(c)$ is unbounded, and hence c crosses all but finitely many h_i. □

As a step towards Theorem 7.1, and for its own interest, we use curtains to introduce a new topology on ∂X. We relate it to the standard cone topology in Theorem 7.8.

Definition 7.6 (Curtain topology)**.** Let b be a geodesic ray emanating from $\mathfrak{o}$. For each curtain h dual to b, let

$$U_h(b^\infty) = \{a^\infty \in \partial X \,:\, h \text{ separates } \mathfrak{o} = a(0) \text{ from } a^\infty\}.$$

We define the *curtain topology* as follows: a set $U \subset \partial X$ is open if for each $b^\infty \in U$, there is some $U_h(b^\infty)$ with $U_h(b^\infty) \subset U$. It is immediate that such a description yields a topology on ∂X.

Example 7.7. As an example of the difference between the curtain and cone topologies, consider the Euclidean plane $\mathbf{E}^2$. A simple computation shows that the curtain topology on $\partial\mathbf{E}^2$ is the trivial topology, in contrast to the cone topology, which is that of the circle S^1. Note that $\mathcal{B} = \varnothing$ in this example.

Example 7.7 fits the ideology that the curtain topology should detect only negative curvature. Our next result essentially shows that it sees all of it. It also shows that the cone topology on ∂X is either equal to or finer than the curtain topology.

Theorem 7.8. *The identity map $(\partial X, \mathcal{T}_{\text{Cone}}) \to (\partial X, \mathcal{T}_{\text{Curtain}})$ is continuous. Moreover, the curtain and cone (subspace) topologies agree on $\mathcal{B}$.*

Proof. We start by showing that every open set O in $\mathcal{T}_{\text{Curtain}}$ is also open in $\mathcal{T}_{\text{Cone}}$. Let $b^\infty \in O$. By definition, there is some h dual to b such that $U_h(b^\infty) \subseteq O$. Let $r = \mathsf{d}(\mathfrak{o}, h) + 5$. We shall show that $U(b^\infty, r, 1) \subset U_h(b^\infty)$. For this, let $c^\infty \in U(b^\infty, r, 1)$, so that $\mathsf{d}(c(r), b) < 1$, and note that $\mathsf{d}(b(r), c(r)) < 2$. By the choice of r, we have $\mathsf{d}(h, c(r)) > \mathsf{d}(h, b(r)) - 2 = 2$, and h separates $\mathfrak{o}$ from $\pi_b(c)$. Suppose that $c|_{(r,\infty)}$ meets h at $c(t)$. In this case, there is some $s \in (0, r)$ such that $\pi_b c(s) = \pi_b c(t) \in h$. By convexity of the metric, $\mathsf{d}(c(s), \pi_b c(s)) < 1$, so

$$\begin{aligned} \mathsf{d}(c(s), c(t)) &\leqslant \mathsf{d}(c(s), \pi_b c(s)) + \mathsf{d}(\pi_b c(s), c(t)) \\ &= \mathsf{d}(c(s), \pi_b c(s)) + \mathsf{d}(b, c(t)) \\ &\leqslant 1 + \mathsf{d}(b, c(r)) + \mathsf{d}(c(r), c(t)) \\ &< 2 + \mathsf{d}(c(r), c(t)). \end{aligned}$$

But $c(s) \in h$, so $\mathsf{d}(c(s), c(r)) > 2$, contradicting the fact that c is a geodesic. Thus h separates $\mathfrak{o}$ from c^∞, so $c^\infty \in U_h(b^\infty)$. This shows that $\mathcal{T}_{\text{Curtain}} \subset \mathcal{T}_{\text{Cone}}$.

We now show that, when restricted to $\mathcal{B}$, the curtain topology is at least as fine as the cone topology. Fix some $U(b^\infty, r, \epsilon)$ for some geodesic ray b with $b^\infty \in \mathcal{B}$. Let $\{h_i\}$ be an infinite L-chain dual to b, which exists by Lemma 2.22, and let t_i be such that $b(t_i) \in h_i$. Fix k large enough that $\frac{r}{t_k - 1}(2L+1) < \epsilon$. We shall show that $U_{h_{k+1}}(b^\infty) \subset U(b^\infty, r, \epsilon)$. For this, let $b'^\infty \in U_{h_{k+1}}(b^\infty)$. According to Lemma 2.14, if $b'(t'_k) \in h_k$ then $\mathsf{d}(b'(t'_k), b) \leqslant 2L+1$. By convexity of the metric, we have $\mathsf{d}(b'(r), b) \leqslant \frac{r}{t'_k}(2L+1)$. Because h_k is dual to b, we also have $t'_k \geqslant t_k - 1$. Hence $\mathsf{d}(b'(r), b) < \epsilon$, so $b'^\infty \in U(b^\infty, r, \epsilon)$. □

This gives a purely combinatorial description of the subspace topology on $\mathcal{B}$ (and hence on each $\mathcal{B}_L$) via curtains. We show in Corollary 7.13 below that the topology on the *Morse boundary* can also be described combinatorially via curtains.

It remains to establish item (3) of Theorem 7.1, which we do in the next proposition.

Proposition 7.9. *For a proper CAT(0) space X, the identity map $\iota : X \to X_L$ induces an* $\operatorname{Isom} X$*–equivariant homeomorphism $\partial\iota : \mathcal{B}_L \to \partial X_L$.*

Proof. The proof consists of the following four steps.

(1) (Existence and continuity.) Since the map $\iota : X \to X_L$ is $(1,1)$–coarsely Lipschitz and X_L is coarsely dense in its injective hull $E(X_L)$, which is a geodesic hyperbolic space, the existence and continuity of $\partial\iota$ are exactly given by [IMZ21, Lem. 6.18].

(2) (Injectivity.) If $b^\infty \in \mathcal{B}$, then by Lemma 2.22 there is an infinite L–chain $\{h_i\}$ dual to b with $\mathfrak{o} \in h_i^-$ for all i. For any other $c^\infty \in \mathcal{B}$, Corollary 2.23 shows that $c \subset h_k^-$ for some k. In particular, if $x_n \in b \cap h_{k+n}$, then $\mathsf{d}_L(x_n, c) \geqslant n$, so b does not lie in a finite X_L–neighbourhood of c.

(3) (Surjectivity.) Let $q : [0,\infty) \to X_L$ be any quasigeodesic ray with $q(0) = \iota(\mathfrak{o})$, and let $(x_n) \subset q$ be an unbounded sequence with $x_0 = \iota(\mathfrak{o})$. Consider the path $q' = \bigcup_{n\in\mathbf{N}}[x_{n-1}, x_n] \subset X$, which crosses an infinite L–chain $\{h_i\}$ by Corollary 7.11. Let $p_i \in q' \cap h_i$. As X is proper, there is some geodesic ray b in X emanating from $\mathfrak{o}$ and some subsequence (p_{i_j}) such that the geodesics $[\mathfrak{o}, p_{i_j}]$ converge uniformly to b on compact sets.

Let us show that b meets every h_i. Since $\{h_i\}$ is a chain, it suffices to show that b meets infinitely many h_i. Suppose that, on the contrary, there is some k such that $b \in h_i^-$ for all $i \geqslant k$. Let $k' = k + (4L+10)(2L+3) + 2$. The curtains h_k, $h_{k'}$ are separated by an L–chain of length $(4L+10)(2L+3)$, and hence, using Lemma 2.22 with $A = h_k$ and $B = h_{k'}$, they are separated by an L–chain $\{m_1, \dots, m_{2L+4}\}$ whose elements are all dual to $[\mathfrak{o}, p_{k'}]$. Lemma 2.14 now yields a point $p \in [\mathfrak{o}, p_{k'}] \cap m_{2L+3}$ and points $y_i \in [\mathfrak{o}, p_i] \cap m_{2L+3}$ such that $\mathsf{d}(y_i, p) \leqslant 2L+2$ for all $i \geqslant k'$. Since $[\mathfrak{o}, p_i] \to b$, the geodesic b must contain a point y with $\mathsf{d}(y,p) \leqslant 2L+2$. On other hand, since $b \in h_k^-$ and $\{m_1, \cdots m_{2L+2}\}$ is an L–chain separating h_k from y_i, we must have $\mathsf{d}(p,y) \geqslant 2L+3$. This is a contradiction, so b meets every h_i.

Since each $p_i \in q'$, the unparametrised quasigeodesics $\iota[\mathfrak{o}, p_i]$ lie in a uniform neighborhood of q. Therefore, the unparametrised quasigeodesic $\iota(b)$ is also in a uniform neighborhood of q, by Lemma 2.14, which concludes the proof of surjectivity. Note that Corollary 2.23 means that b is the unique geodesic ray crossing all the $\{h_i\}$.

(4) (Continuity of the inverse map.) We prove sequential continuity, which is enough because $\mathcal{B}_L$ and ∂X_L are first-countable. Let δ_L be a hyperbolicity constant for X_L, and let $q_n, q : [0,\infty) \to X_L$ be $(1, 10\delta_L)$–quasigeodesics such that $q_n \to q$ uniformly on compact subsets of $[0,\infty)$ (see [KB02, Rem. 2.16]). From items (2) and (3), there are unique geodesic rays b_n, b in X based at $\mathfrak{o}$ with $\partial\iota(b_n^\infty) = q_n^\infty$ and $\partial\iota(b^\infty) = q^\infty$. Moreover, there is an integer K depending only on L, δ_L such that the quasigeodesic rays ιb and q are at Hausdorff-distance at most K, and similarly for the pairs $(\iota b_n, q_n)$. In light of Theorem 7.8, to prove continuity we must show that for any curtain h dual to b, we have $b_n^\infty \in U_h(b^\infty)$ for all but finitely many n.

Since b crosses an infinite L-chain, by Lemma 2.22 it must cross an infinite L-chain $\{h_i\}$ dual to b, with every $h_i \in h^+$. Let $m = 2K + L + 4$, let $p \in h_m \cap b$, and note that $\mathsf{d}(p, h_{L+2}^-) \geqslant 2K+2$. There exists $p' \in q$ with $\mathsf{d}_L(p,p') \leqslant K$. As $q_n \to q$ in X_L, there exists k such that for each $n \geqslant k$ there is some $p_n' \in q_n$ with $\mathsf{d}_L(p', p_n') \leqslant 1$. For each $n \geqslant k$, there is some $p_n'' = b_n(t_n)$ with $\mathsf{d}_L(p_n', p_n'') \leqslant K$. By the triangle inequality, $\mathsf{d}_L(p, p_n'') \leqslant 2K+1$. This shows that no p_n'' lies in h_{L+2}^-. Lemma 3.1 now tells us that $b_n|_{(t_n,\infty)}$ is disjoint from h_1, hence from h. Thus $b_n^\infty \in U_h(b^\infty)$ for all $n \geqslant k$. $\square$

The proof of Item (3) of Proposition 7.9 uses Corollary 7.11, which is a consequence of the following lemma.

Lemma 7.10. *For all λ there exists M as follows. If $P\colon [a,b] \to X$ is an unparametrised λ–quasigeodesic of X_L, then any L–chain $\{c_i\}$ separating $\{x_1, x_3\}$ from x_2, where $x_1, x_2, x_3 \in P$ are consecutive points, has length at most M.*

Proof. Let $\gamma = [x_1, x_3]$ in X. By Corollary 3.2, γ meets at most $1 + \lfloor \frac{L}{2} \rfloor$ elements of $\{c_i\}$, so $\mathsf{d}_L(\gamma, x_2) \geqslant |\{c_i\}| - (1 + \lfloor \frac{L}{2} \rfloor)$. Proposition 3.3 states that γ is an unparametrised q–quasigeodesic of X_L. Since X_L is hyperbolic, the Hausdorff-distance between γ and P is uniformly bounded in terms of λ and q. Thus the distance between γ and x_2 is uniformly bounded, bounding $|\{c_i\}|$. $\square$

Corollary 7.11. *If $P\colon [0,\infty) \to X$ is an unparametrised λ–quasigeodesic ray of X_L, then there is a sequence $(x_i)_{i=0}^{\infty} \subset P$ and an L–chain $\{c_i : i \in \mathbf{N}\}$ such that $c_1, \dots, c_n$ separate x_0 from x_n.*

7.1. Relation to the Morse boundary

We observe here that the curtain topology provides a combinatorial description of the strata of the *Morse boundary.*

Definition 7.12 (Morse boundary)**.** For a fixed base point $\mathfrak{o} \in X$ and $D \geqslant 0$, define $\partial^D X = \{b^\infty : b(0) = \mathfrak{o}$ and b is D–contracting$\}$ as a set, and equip it with the subspace topology inherited from the cone topology on ∂X. The *Morse boundary* of X is defined to be

$$\partial^\star X := \bigcup_{D \geqslant 0} \partial^D X,$$

where $U \subset \partial^\star X$ is open if and only if $U \cap \partial^D X$ is open for every $D \geqslant 0$.

Corollary 7.13. *A set $U \subset \partial^\star X$ is open if and only if for every $D \geqslant 0$ and every $b \in U \cap \partial^D X$, there exists a curtain h dual to b with $U_h(b^\infty) \subset U \cap \partial^D X$.*

Proof. By Theorem 4.2, we have $\partial^\star X \subset \mathcal{B}$ as sets, and Theorem 7.8 shows that the curtain topology agrees with the cone topology on $\partial^D X \subset \partial^\star X$. In particular, U is open in $\partial^\star X$ if and only if $U \cap \partial^D X$ is open in the curtain topology for every D. $\square$

More explicitly, the Morse boundary consists of the points in $\mathcal{B}$ that are represented by rays that cross an infinite L–chain *at a uniform rate.*

Remark 7.14. In [Cas16], Cashen exhibited two quasiisometric CAT(0) spaces whose Morse boundaries, with the subspace topologies inherited from the respective visual boundaries, are not homeomorphic. As the curtain boundary $\mathcal{B}$ contains the Morse boundary as a subspace, it cannot be preserved by quasiisometries.

Remark 7.15. In [Mur19], Murray showed that the Morse boundary is dense ∂X with respect to the cone topology whenever X admits a proper cocompact group action. In particular $\mathcal{B}$ is dense in ∂X in this case.

Appendix A. Auxiliary statements

The following "guessing geodesics" criterion for hyperbolicity is used in Section 3. Since the spaces we deal with are not geodesic, we have included a proof here in the appendix; it is a simple modification of variants for geodesic spaces. Bowditch states a version in terms of a ternary map that ends up being the median of the hyperbolic space [Bow06, Prop. 3.1]. We follow Hamenstädt's formulation [Ham07, Prop. 3.5] in terms of paths, though we cannot assume the paths to be continuous.

For a path $\alpha : [0, n] \to X$, write $\ell(\alpha) = n$ for the parametrisation-length of α.

Proposition A.1. *Let (X, d) be a q–quasigeodesic space. Assume that for some constant $D > 0$ there are D–coarsely-connected paths $\eta_{xy} = \eta(x, y) : [0, 1] \to X$ from x to y, for each pair $x, y \in X$, such that the following are satisfied.*

(G1) If $\mathsf{d}(x, y) \leqslant 2q$ then the diameter of $\eta_{xy} = \eta_{xy}[0, 1]$ is at most D.

(G2) For any $s \leqslant t$, the Hausdorff-distance between $\eta_{xy}[s, t]$ and $\eta(\eta_{xy}(s), \eta_{xy}(t))$ is at most D.

(G3) For any $x, y, z \in X$ the set η_{xy} is contained in the D-neighborhood of $\eta_{xz} \cup \eta_{zy}$.

Then (X, d) is a δ-hyperbolic quasigeodesic space, where δ depends only on D and q. Moreover, the η_{xy} are uniformly Hausdorff-close to q–quasigeodesics.

Proof. It suffices to bound the Hausdorff-distance between η_{xy} and an arbitrary q–quasigeodesic from x to y, for then (G3) implies that q–quasigeodesic triangles are thin.

Let $\alpha : [0, 2^n] \to X$ be any q–coarsely-Lipschitz path. Write $\eta^0 = \eta(\alpha(0), \alpha(2^{n-1}))$ and $\eta^1 = \eta(\alpha(2^{n-1}), \alpha(2^n))$. By (G2), the D–neighbourhood of $\eta^0 \cup \eta^1$ contains $\eta(\alpha(0), \alpha(2^n))$. Repeat this subdivision for η^0 and η^1, and inductively we find that $\eta(\alpha(0), \alpha(2^n))$ lies in the nD–neighbourhood of the concatenation of the paths $\eta(\alpha(i), \alpha(i+1))$. According to (G1), each of these has diameter at most D, so is contained in the D–neighbourhood of $\alpha[i, i+1]$. We therefore have that $\eta(\alpha(0), \alpha(2^n))$ is contained in the $(D \log_2 \ell(\alpha) + D)$–neighbourhood of α.

Now let $\gamma : [0, n] \to X$ be a q–quasigeodesic from x to y. Let t maximise the distance from $\eta_{xy}(t)$ to γ. Write r for this distance, and let s satisfy $\mathsf{d}(\eta_{xy}(t), \gamma(s)) = r$. If $\mathsf{d}(x, \eta_{xy}(t)) \leqslant qr + q + r$, then let $t_1 = 0$. Otherwise, consider $\eta(x, \eta_{xy}(t))$. Coarse connectivity and (G2) imply that there is some $t_1 < t$ such that $\mathsf{d}(\eta_{xy}(t_1), \eta_{xy}(t)) \in (qr + q + r, qr + q + r + D]$. Define $t_2 > t$ similarly. By the choice of r and t, there are s_i such that $\mathsf{d}(\eta_{xy}(t_i), \gamma(s_i)) \leqslant r$. (If $t_i \in \{0, 1\}$ then take $s_i = t_i$.) Observe that $\mathsf{d}(\gamma(s_1), \gamma(s_2)) \leqslant 2qr + 2q + 4r + 2D$.

Let α be the q–coarsely Lipschitz path obtained by concatenating: a q–quasigeodesic from $\eta_{xy}(t_1)$ to $\gamma(s_1)$, the subpath $\gamma[s_1, s_2]$, and a q–quasigeodesic from $\gamma(s_2)$ to $\eta_{xy}(t_2)$. Since α is comprised of q–quasigeodesics, we have $\ell(\alpha) \leqslant q(2qr + 2q + 6r + 2D + q)$. As we have seen, this implies that the $(D \log_2(2q^2 r + 2q^2 + 6qr + 2Dq + q^2) + D)$–neighbourhood of α contains $\eta(\eta_{xy}(t_1), \eta_{xy}(t_2))$. In turn, this is at Hausdorff-distance at most D from $\eta_{xy}[t_1, t_2]$, by (G2). Thus the $(D \log_2(2q^2 r + 2q^2 + 6qr + 2Dq + q^2) + 2D)$–neighbourhood of α contains $\eta_{xy}[t_1, t_2]$. However, we also know from the choice of t_i that $\mathsf{d}(\eta_{xy}(t), \alpha) = \mathsf{d}(\eta_{xy}(t), \gamma(s)) = r$. Thus r satisfies the inequality $r \leqslant D \log_2(2q^2 r + 2q^2 + 6qr + 2Dq + q^2) + 2D$, and so is bounded above by some universal constant $\kappa = \kappa(D, q)$. We have shown that η_{xy} lies in a uniform neighbourhood of any q–quasigeodesic from x to y.

It remains to show that every q–quasigeodesic $\gamma = \gamma[0, n]$ from x to y lies in a uniform neighbourhood of η_{xy}. We know that the set

$$S = \{s \in [0, n] \ : \ \text{there exists } t \text{ satisfying } \mathsf{d}(\gamma(s), \eta_{xy}(t)) \leqslant \kappa\}$$

is nonempty. Given $s \in S$, let t be maximal such that there exists $s'' \leqslant s$ with $\mathsf{d}(\gamma(s''), \eta_{xy}(t)) \leqslant \kappa$. If $\mathsf{d}(\eta_{xy}(t), y) \leqslant D$, then $\mathsf{d}(\gamma(s), y) \leqslant \kappa + D$, so $\gamma[s, n]$ lies in the $q^2(\kappa + D + q + 1)$–neighbourhood of η_{xy}.

Otherwise, consider $\eta(\eta_{xy}(t), y)$. Coarse connectivity and (G2) imply that there is some $t' > t$ for which $\mathsf{d}(\eta_{xy}(t), \eta_{xy}(t')) \leqslant D$. Fix $s' \in [0, n]$ such that $\mathsf{d}(\gamma(s'), \eta_{xy}(t')) \leqslant \kappa$. We have

$s' > s$ by the choice of t. Moreover, we have

$$\mathsf{d}(\gamma(s), \gamma(s')) \leqslant \mathsf{d}(\gamma(s), \eta_{xy}(t)) + \mathsf{d}(\eta_{xy}(t), \eta_{xy}(t')) + \mathsf{d}(\eta_{xy}(t'), \gamma(s')) \leqslant 2\kappa + D.$$

Since γ is a q–quasigeodesic, this implies that $|s - s'| \leqslant q(2\kappa + D + q)$. This shows that S is $q(2\kappa + D + q)$–dense in $[0, n]$. Hence $\gamma(S)$ is $q^2(2\kappa + D + q + 1)$–dense in γ, so γ is contained in the $(q^2(2\kappa + D + q + 1) + \kappa)$–neighbourhood of η_{xy}. □

Section 3 also used a simple fact about injective hulls of hyperbolic spaces that does not seem to appear in the literature.

Definition A.2 (Injective, injective hull)**.** A geodesic space is *injective* if every family of pairwise intersecting balls has nonempty total intersection [AP56]. Every metric space X has an essentially unique *injective hull*: an injective space $E(X)$ with an isometric embedding $e : X \to E(X)$ such that any other isometric embedding of X in an injective space factors via e [Isb64].

Lang proved hyperbolicity of the injective hull of any space that is hyperbolic in the four-point sense of Gromov, and used this to deduce that geodesic hyperbolic spaces are coarsely dense in their injective hulls [Lan13, Prop. 1.3]. These conclusions do not hold for quasi-geodesic spaces that are hyperbolic in the sense that all quasigeodesic triangles are thin (the sense used in this paper): consider the graph of $|x|$ inside $(\mathbf{R}^2, \ell^1)$, for example.

Recall that a metric space X is *coarsely injective* if there is a constant ε such that for any collection of balls $B(x_i, r_i)$ with $r_i + r_j \geqslant \mathsf{d}(x_i, x_j)$ for all i, j, the intersection $\bigcap B(x_i, r_i + \varepsilon)$ is nonempty.

Proposition A.3. *A quasigeodesic hyperbolic space X is coarsely dense in its injective hull if and only if it is weakly roughly geodesic. In particular, the injective hull $E(X)$ of a weakly roughly geodesic hyperbolic space X is a geodesic hyperbolic space.*

Proof. Clearly, if X is coarsely dense in the geodesic space $E(X)$, then it is roughly geodesic. For the reverse, suppose that X is weakly roughly geodesic. According to [CCG⁺20, Prop. 3.12] (also see [HHP20, Prop. 1.1]), it suffices to show that X is coarsely injective.

By hyperbolicity, balls in X are uniformly quasiconvex. Let $\{B_i = B(x_i, r_i)\}$ be a collection of balls in X such that $\mathsf{d}(x_i, x_j) \leqslant r_i + r_j$ for all i, j. According to weak rough geodesicity, uniform thickenings B_i' of the B_i intersect pairwise. Let $f : X \to X'$ be a quasiisometry to a hyperbolic graph, for example that of [CdlH16, Prop. 3.B.6], and let $\bar{f}$ be a quasiinverse. The fB_i' are uniformly quasiconvex and intersect pairwise. According to [CDV17, Thm 5.1], this implies that there is some point $z \in X'$ that is uniformly close to every fB_i'. Thus $\bar{f}(z)$ is uniformly close to every B_i', and hence to every B_i. This shows that X is coarsely injective. □

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