COARSE OBSTRUCTIONS TO COCOMPACT CUBULATION

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Abstract. We provide geometric methods to both lower- and upper-bound the large-scale dimension of CAT(0) cube complexes quasiisometric to a given group G. When these bounds overlap, this provides obstructions to G being cocompactly cubulated. More strongly, it prevents G from being a coarse median space.

As applications, we show that many free-by-cyclic groups cannot be cocompactly cubulated, and that any tubular group with a coarse median is virtually compact special.

1. Introduction

Cubulation has proved to be an important tool in the study of finitely generated groups. That is, when studying a group G, if one can find an action of G on a CAT(0) cube complex, then one gains access to powerful combinatorial machinery controlling the geometry of G.

The unqualified term *cubulation* generally refers to a proper action of G on a (possibly infinite-dimensional) CAT(0) cube complex; i.e. an action where each ball contains only finitely many orbit points (with multiplicity). Stronger conclusions can be drawn from a *cocompact cubulation*, namely a proper cocompact action of G on a (necessarily finite-dimensional) CAT(0) cube complex. And strongest of all is for G to be *virtually compact special*, meaning that a finite-index subgroup of G is the fundamental group of a finite cube complex that is *special* in the sense of [HW08].

Many groups have been successfully (cocompactly) cubulated, and many of these are even virtually special; indeed, Agol's theorem [Ago13] states that every hyperbolic group that can be cocompactly cubulated is virtually special. However, there are many groups of interest for which cocompact cubulability is unknown. For many, it is expected to be impossible; we simply lack refined criteria to verify this suspicion. In fact, all general methods for obstructing cocompact cubulation known to the authors are simply negations of properties of cocompactly cubulated groups. For instance, if G has super-quadratic Dehn function, property (T), distorted elements, or elements whose centralisers do not virtually split, then G is not cocompactly cubulated [NR97, Ebe82].

Our goal in this article is to provide the first general obstructions to cocompact cubulation that are not a negation of some group-theoretic property. These obstructions are of a coarse-geometric nature, and in fact they provide a strong negation to the possibility of cocompact cubulation, by ruling out the existence of a *coarse median*.

If G acts properly cocompactly on a CAT(0) cube complex X, then G is quasiisometric to X. Merely being quasiisometric to a CAT(0) cube complex (*quasicubical*) is much weaker than being cocompactly cubulated; for instance, all hyperbolic groups are quasicubical [HW12], even though some have property (T) and hence cannot act on CAT(0) cube complexes without global fixed-points [NR97]. If G is quasicubical then one can pass the median of the cube complex along the quasiisometry to equip G with a ternary operator that behaves coarsely the same. This makes G a *coarse median space* in the sense of [Bow13].

Roughly speaking, for each $n \geqslant 2$ we describe geometric configurations whose "geometric rank" is n but that cannot appear in any space of "median rank" less than $n+1$. Showing that G cannot have a coarse median then amounts to finding such a configuration in G "at top rank". This can be summarised as follows (see Theorem 4.3). By the *quasiflat rank* of a metric space X, we mean the supremal integer $\mathrm{qf.rk}\,X$ for which there is a quasiisometric embedding $\mathbf{R}^{\mathrm{qf.rk}\,X} \to X$.

Theorem A. *Let G be a finitely generated group with* $\mathrm{qf.rk}\,G \leqslant n$. *If G contains a quasiisometrically embedded* richly branching flat *of dimension n, then G cannot be a coarse median space. In particular, G cannot be cocompactly cubulated.*

The motivating observation for defining richly branching flats (Definition 4.2) is that in a, say, 2–dimensional CAT(0) cube complex, half-flats can only branch off a flat in two directions, i.e. those parallel to the coordinate axes. This picture becomes muddled when considering spaces only up to quasiisometry. For example, a cyclic gluing of six quarter-planes is quasiisometric to a flat, and half-flats can branch off such an object in more than two directions. However, in a sense, this is the as complicated as it gets: quasiflats in a coarse median space are well-approximated by unions of orthants [Bow19]. This quasiflat rigidity generalises (and strengthens) several previous theorems [BKS16, Hua17, BHS21].

As the name suggests, then, richly branching flats should be thought of as being flats with "too much" branching "all over". Similar configurations were considered in work of Haettel [Hae16], who characterised which symmetric spaces and affine buildings admit coarse medians. However, his arguments rely on facts specific to the setting of symmetric spaces and affine buildings, such as [KL97].

Part of the proof of Theorem A involves relating the quasiflat rank of a coarse median space X to its *coarse-median rank*, $\mathrm{rk}\,X$. Proposition 3.2 in particular shows that these two quantities agree when X is a finitely generated group. In general, though, it is not easy to ascertain the quasiflat rank of a group. In order to facilitate the applications of Theorem A discussed below, we therefore establish the following bound, which is a consequence of Proposition 3.2 and the general Lemma 3.5. Recall that the *virtual geometric dimension* $\mathrm{vgd}\,G$ of a group G is the infimal dimension among classifying spaces of finite-index subgroups of G.

Theorem B. *Let G be a finitely generated group. If G admits a coarse median, then* $\mathrm{rk}\,G \leqslant \mathrm{vgd}\,G$.

Although this bound is far from optimal in general, as can be seen from hyperbolic manifolds, it can be useful in concrete situations where geometric (or cohomological) dimensions are easily computable.

Let us now discuss the applications of Theorem A considered in this paper. We first note that it can be used to recover the main result of [Hae16] mentioned above. The two main classes of groups that we consider are *free-by-cyclic* groups and *tubular* groups.

Free-by-cyclic groups form a heavily-studied class, with rich behaviours in accordance with the theory of free-group automorphisms. Among the numerous known properties of free-by-cyclic groups, we find that they pass the "obvious" tests for cocompact cubulability: they have quadratic isoperimetric functions [BG10] and their abelian subgroups are undistorted [But19], for instance.

It is therefore natural to ask which free-by-cyclic groups can be cocompactly cubulated. As it turns out, all hyperbolic free-by-cyclic groups are cocompactly cubulated, by work of Hagen–Wise [HW16, HW15]. Also, Hagen–Przytycki characterised which graph manifold groups are cocompactly cubulated [HP15], and some of those are free-by-cyclic. Gersten's group [Ger94a] (and small tweaks thereof) does not act properly on a CAT(0) space, for reasons involving translation-length, and so cannot be cocompactly cubulated. Though cocompact cubulability is expected to be rare amongst free-by-cyclic groups, little seems to be known in general. For example, the following question is open.

Question 1. *Are toral relatively hyperbolic free-by-cyclic groups cocompactly cubulated?*

Using Theorem A, we show that many free-by-cyclic groups cannot be cocompactly cubulated; indeed they cannot even admit a coarse median. The class of free-by-cyclic groups we can handle is considerably larger than those for which Gersten's ideas rule out being CAT(0), and only partly because it is not restricted to linearly growing automorphisms. The result is slightly technical, so we give only a partial version of it here, and refer the reader to Theorem 5.8 for the precise statement.

Theorem C. *If a free-by-cyclic group (virtually) has an improved relative train track structure with a Nielsen cycle supporting at least three internal linear strata, then it cannot admit a coarse median.*

The way to think about this theorem is as follows. The *improved relative train track* structure (from [BFH00], refining [BH92]) provides a systematic way to build up $G = F \rtimes_\phi \mathbf{Z}$ in "layers", known as *strata*. The simplest strata are those that are fixed by ϕ; these can give rise to $\mathbf{Z}^2$ subgroups of G. More generally, a *Nielsen cycle* is a subset of the train track that is stabilised by ϕ. Next are the *linear* strata: roughly, the translates of a linear stratum by powers of ϕ grow in complexity at a linear rate. These can give rise to quasiflats (this is the role of the term *internal*, Definition 5.7), which branch off the Nielsen cycle supporting the stratum. The conditions of Theorem C can therefore be understood as saying that there is some quasiflat in G with three directions of branching, allowing the obstructions of Theorem A to kick in.

In the positive direction, one could more leniently ask which free-by-cyclic groups are quasicubical. For instance, it turns out (by a combination of [HRSS22] and [Pet21]) that all graph manifolds are quasicubical, even those that cannot be cocompactly cubulated. In this vein, we combine results from the literature to observe the following. See Proposition 5.3 for a more precise statement.

Theorem D. *If a free-by-cyclic group G is represented by an improved relative train track with no quadratic strata and such that each Nielsen cycle supports at most one linear stratum, then G is quasicubical.*

In the special case of linearly-growing free-by-cyclic groups (or, more generally, those with no quadratic strata), the combination of Theorems C and D leaves quite a restricted set of possible train tracks. It would be interesting to know whether such groups are quasicubical. Even in the presence of higher-degree polynomial strata, one could ask the following.

Question 2. *If a free-by-cyclic group has no quasiisometrically embedded richly branching flats, must it be quasiisometric to a finite-dimensional CAT(0) cube complex?*

Our other main application of Theorem A is to tubular groups. A group is *tubular* if it can be written as a graph of groups with $\mathbf{Z}^2$ vertices and $\mathbf{Z}$ edges. This simple description belies the remarkably varied behaviour that tubular groups display. Indeed, the class includes: Gersten's group that is not a subgroup of any CAT(0) group [Ger94a]; CAT(0) groups that have quadratically diverging rays but no super-quadratically diverging rays [Ger94b]; non-Hopfian CAT(0) groups [Wis96]; Croke–Kleiner groups, which have CAT(0) structures with differing visual boundaries [CK00]; and Brady–Bridson groups, which fill the isoperimetric spectrum [BB00].

We use richly branching flats to prove a rigidity result for cubulations of tubular groups (Theorem 6.10).

Theorem E. *If a tubular group admits a coarse median, then it is virtually compact special.*

In [Wis14], Wise characterised which tubular groups admit free actions on CAT(0) cube complexes. These cubulations were further investigated by Woodhouse [Woo16, Woo18]. Wise also showed that all cocompactly cubulated tubular groups are virtually special, and we use this in the proof of Theorem E.

In their work on the isoperimetric spectrum [BB00], Brady–Bridson established a relationship between distortion and isoperimetry in a single, parametrised family of tubular groups with one vertex group and two edge groups. Along to way to Theorem E, we show the following (Theorem 6.8).

Theorem F. *If a tubular group contains a distorted element, then it has super-quadratic Dehn function.*

Acknowledgements. We are pleased to thank each of Naomi Andrew, Mark Hagen, and Monika Kudlinska for several very helpful conversations about free-by-cyclic groups, and Jean Pierre Mutanguha for sharing his preprint [Mut] with us. We also thank the organisers of the 2023 thematic program on geometric group theory in Montreal, where this work began.

2. Preliminaries

2.1. Ultralimits

2.1. **Definition** (Ultrafilter)**.** An *ultrafilter* ω on $\mathbf{N}$ is a set of subsets of $\mathbf{N}$ satisfying:

- If $A \in \omega$ and $A \subset B$, then $B \in \omega$.
- If $A, B \in \omega$, then $A \cap B \in \omega$.
- For all $A \subset S$, either $A \in \omega$ or $S \smallsetminus A \in \omega$.

The ultrafilter ω is *non-principal* if it contains no finite sets.

One can think of sets in ω as having measure 1, and those not in ω as having measure 0. We shall work only with non-principal ultrafilters, so we refer to them simply as "ultrafilters".

2.2. **Definition** (Ultralimit)**.** Given a sequence $(x_n) \in \mathbf{R}$, if there exists $x \in \mathbf{R}$ such that $\{n : |x_n - x| < \epsilon\} \in \omega$ for every $\epsilon > 0$, then we call x the *ultralimit* of (x_n) and write $x = \lim_\omega x_n$. Every sequence has at most one ultralimit. Let (X_n, d_n) be a sequence of metric spaces with basepoints $b_n \in X_n$. The *ultralimit* $\lim_\omega(X_n, b_n)$ can be defined as the metric quotient of the pseudometric space whose elements are sequences (x_n), with $x_n \in X_n$, such that $\lim_\omega \mathsf{d}_n(x_n, b_n)$ exists, and whose pseudometric is $\hat{\mathsf{d}}_\omega((x_n), (y_n)) = \lim_\omega \mathsf{d}_n(x_n, y_n)$.

Every ultralimit of metric spaces is complete [BH99, Lem. I.5.53]. One often considers ultralimits where the terms in the sequence are all derived from the same metric space. For instance, if X is a proper metric space, then X can be written as the ultralimit of a sequence of nested balls; $X = \lim_\omega(B_X(b, n), b)$. Two other important cases are *asymptotic cones* and *tangent cones.*

2.3. **Definition** (Cones)**.** Let X be a metric space, and let $b \in X$. Let (λ_n) be a sequence of real numbers. If $\lambda_n \to 0$, then we call $\hat{X} = \lim_\omega(X, \lambda_n \mathsf{d}, b)$ an *asymptotic cone* of X; it is independent of the choice of b. If $\lambda_n \to \infty$, then we call $T_b X = \lim_\omega(X, \lambda_n \mathsf{d}, b)$ a *tangent cone* of X at b.

2.4. **Lemma.** *Let X and Y be metric spaces and let (λ_n) be a sequence converging to 0. Any quasiisometric embedding $f : X \to Y$ induces a bilipschitz embedding of asymptotic cones $\hat{f} : \hat{X} \to \hat{Y}$.*

Proof. Given $x = (x_n) \in \hat{X}$, let $\hat{f}(x) = (f(x_n))$. The map $\hat{f}$ is well defined, because if $(x_n) = (z_n)$, then $\lim_\omega \lambda_n \mathsf{d}_X(x_n, z_n) = 0$, and hence $\lim_\omega \lambda_n \mathsf{d}_Y(f(x_n), f(z_n)) = 0$ because f is a quasiisometry. Essentially the same argument shows that $\hat{f}$ is bilipschitz. □

2.2. Medians

2.5. **Definition** (Median algebra)**.** A *median algebra* is a set M with a ternary operation μ satisfying:

$$\mu(a,a,x)=a, \quad \mu(a,b,x)=\mu(a,x,b)=\mu(x,a,b), \quad \mu(a,b,\mu(x,y,z))=\mu(\mu(a,b,x),\mu(a,b,y),z)$$

for all $a,b,x,y,z\in M$. The latter equality is called the *five-point condition.*

It can be useful to interchangeably think of μ both as a median operator and as giving "projection" maps $\mu(a,b,\cdot)$ from M to the "hull" of $\{a,b\}$. For instance, the five-point condition can (almost) be described by the slogan "the projection of the median is the median of the projections".

For a subset A of a median algebra M, let $J(A)=\{\mu(a,a',x)\ :\ a,a'\in A,\ x\in M\}$. The subset A is *median-convex* if $J(A)=A$. The median-convex hull of a subset A is the intersection of all median-convex subsets containing A. If M has rank n, then the median-convex hull of A can be obtained as $J^n(A)$ (see [Bow22, Prop. 8.2.3], for instance).

2.6. **Definition** (Median morphism, rank)**.** If (M,μ) and (N,ν) are median algebras, then a map $f:M\to N$ is a *median morphism* if $f\mu(x,y,z)=\nu(fx,fy,fz)$ for all $x,y,z\in M$. The *rank* of a median algebra M, denoted $\operatorname{rk}M$, is the supremal n such that there is a median monomorphism $\{0,1\}^n\to M$.

A *wall* in a median algebra M is a partition of M into two nonempty, median-convex subsets, called *halfspaces.* Two walls are said to *cross* if all four *quarterspaces* (intersections of halfspaces) are nonempty. According to [Bow13, Prop. 6.2], the rank of a median algebra is equal to the supremal cardinality of a set of pairwise crossing walls.

2.7. **Definition** (Median metric space)**.** A metric space (X,d) is a *median metric space* if for every $x_1,x_2,x_3\in X$ there is a unique point μ such that $\mathsf{d}(x_i,x_j)=\mathsf{d}(x_i,\mu)+\mathsf{d}(\mu,x_j)$ for all $i\neq j$.

One basic example of a median metric space is a *panel*, i.e. a direct product of a finite number of nontrivial closed intervals in $[0,\infty)$, equipped with the ℓ^1-metric and thus the component-wise median.

It can be shown that the map $(x_1,x_2,x_3)\mapsto\mu$ makes X into a median algebra [Sho54]. Moreover, this map is 1–Lipschitz in each factor: we have $\mathsf{d}(\mu(x,y,z),\mu(x,y,z'))\leqslant\mathsf{d}(z,z')$, and similarly for the other factors by symmetry of μ. One simple consequence is the following, rather crude, estimate.

2.8. **Lemma.** *Let X be a median metric space of rank n. For every $x\in X$ and every r, the median-convex hull of the ball $B_X(x,r)$ is contained in the ball $B_X(x,2^nr)$.*

Proof. It suffices to show that $J(B_X(x,r))\subset B_X(x,2r)$. But this holds because if $x_1,x_2\in B_X(x,r)$ and $z\in X$, then $\mathsf{d}(x,\mu(x_1,x_2,z))\leqslant\mathsf{d}(x,\mu(x,x,z))+\mathsf{d}(x,x_1)+\mathsf{d}(x,x_2)\leqslant 2r$. □

Every complete, connected median metric space is geodesic (see [Bow22, Lem. 13.3.2], for instance). Since the completion of any median metric space is also a median metric space, we shall always implicitly assume that our median metric spaces are complete.

We say that a subset Y of a metric space X is *r–separated* if $\mathsf{d}(y_1,y_2)\geqslant r$ for every $y_1,y_2\in Y$.

2.9. **Lemma.** *If (X,μ) is a connected median metric space of rank n and $p\in X$, then every tangent cone of X at p is a connected median metric space of rank at most n.*

Proof. Let (λ_n) be a sequence with $\lambda_n\to\infty$, and let T_pX be the corresponding tangent cone of X at p. Given points $x^1=(x^1_n)$, $x^2=(x^2_n)$, and $x^3=(x^3_n)$ in T_pX, set $\mu'(x^1,x^2,x^3)=\lim_\omega\mu(x^1_n,x^2_n,x^3_n)$. The fact that μ is 1–Lipschitz in each factor implies that μ' is independent of the choice of representatives of the x^i, similarly to the proof of Lemma 2.4.

For each $i \neq j$ we have $\mathsf{d}(x^i_n, x^j_n) = \mathsf{d}(x^i_n, \mu(x^1_n, x^2_n, x^3_n)) + \mathsf{d}(\mu(x^1_n, x^2_n, x^3_n), x^j_n)$, so in the ultralimit we get $\mathsf{d}(x^i, x^j) = \mathsf{d}(x^i, \mu'(x^1, x^2, x^3)) + \mathsf{d}(\mu'(x^1, x^2, x^3), x^j)$. In particular, μ' produces a well-defined point of T_pX satisfying the desired equalities. A simple computation shows that any point satisfying those equalities must actually be $\mu'(x^1, x^2, x^3)$, which shows that (T_pX, μ') is a median metric space.

Since X is complete and connected, it is geodesic. As an ultralimit of a geodesic spaces, T_pX is geodesic [KL95, Prop. 3.4]. In particular, it is connected. It remains to bound $\operatorname{rk} T_pX$.

Suppose that there exists a median monomorphism $f : \{0,1\}^k \to T_pX$. Let us write $Q = \{0,1\}^k$, and μ_Q for its median operator. For each $q \in Q$, let (x^q_n) be a sequence in X representing $f(q)$. Since $f(Q)$ is a finite median subalgebra of T_pX, this must be captured by the approximating sequences. More concretely, for every $\varepsilon > 0$ we must have

$$M_\varepsilon = \left\{m \in \mathbf{N} : \lambda_m \, \mathsf{d}\left(\mu(x^{q_1}_m, x^{q_2}_m, x^{q_3}_m),\, x^{\mu_Q(q_1,q_2,q_3)}_m\right) < \varepsilon \text{ for all } q_1, q_2, q_3 \in Q\right\} \in \omega.$$

Let $r > 0$ be such that $f(Q)$ is r–separated. For every $\varepsilon > 0$, we also have

$$N_\varepsilon = \{m \in \mathbf{N} : \{x^q_m : q \in Q\} \text{ is } \frac{r-\varepsilon}{\lambda_m}\text{– separated}\} \in \omega.$$

In particular, there exists $m \in N_{\frac{r}{2}} \cap M_{\frac{r}{2^{n+3}}}$, because ω is an ultrafilter.

For $i \in \{1, \dots, k\}$, let e_i denote the point in Q with i^{th} coordinate 1 and all other coordinates 0, and let e_0 denote the point $(0, \dots, 0)$. For $i \in \{0, \dots, k\}$, let B_i denote the median-convex hull of the ball $B_X(x^{e_i}_m, \frac{r}{10n\lambda_m})$. By Lemma 2.8, we have $B_i \subset B_X(x^{e_i}_m, \frac{r}{8\lambda_m})$. Since the set $\{x^q_m : q \in Q\}$ is $\frac{r}{2\lambda_m}$–separated, the B_i are pairwise disjoint. By [Rol98, Thm 2.7], for each $i \in \{1, \dots, k\}$ there is a wall h_i of X separating B_0 from B_i. The h_i must cross pairwise. Indeed, for $q \in Q$ the median $\mu(x^{e_0}_m, x^{e_i}_m, x^q_m)$ lies in B_i if and only if the i^{th} coordinate of q is 1, so the crossing of the h_i is witnessed by the set $\{x^q_m : q \in Q\}$. This shows that $k \leqslant n$. □

In work on the asymptotic cones of mapping class groups [BM08], Behrstock–Minsky introduced a notion of dimension, later called *separation dimension* by Bowditch [Bow13], that is a simple tweak on the more standard notion of *inductive dimension* [HW41, Eng95].

2.10. **Definition** (Separation dimension)**.** Let Y be a Hausdorff topological space. The *separation dimension* of Y is defined inductively as follows.

- If $Y = \varnothing$, then $\operatorname{sepdim} Y = -1$.
- Otherwise, $\operatorname{sepdim} Y \leqslant n$ if for each distinct $x, y \in Y$ there exist closed subsets $A, B \subset Y$ with $x \notin B$, $y \notin A$, and $Y = A \cup B$, such that $\operatorname{sepdim}(A \cap B) \leqslant n - 1$.

Let X be a metric space. By definition, $\operatorname{sepdim} X$ is always bounded above by the inductive dimension of X, which in turn is equal to the *topological* (or *covering*) dimension of X by the Katětov–Morita theorem [Kat52, Mor54] (see also [Eng95, Thm 4.1.3]). If X is proper, then $\operatorname{sepdim} X$ is equal to the topological dimension [HW41, §III.6]. This is not true in general: the rational points of Hilbert space have separation dimension zero but topological dimension one [Erd40]. In our setting we have the following.

2.11. **Lemma** ([Hae16, Cor. 3.7])**.** *If X is a connected median metric space, then* $\operatorname{sepdim} X = \operatorname{rk} X$.

2.3. Coarse medians

2.12. **Definition** (Quasimedian map)**.** Let X and Y be metric spaces equipped with ternary operators μ_X and μ_Y, respectively. A map $f : X \to Y$ is said to be *q–quasimedian* if $\mathsf{d}_Y(f\mu_X(x, y, z), \mu_Y(fx, fy, fz)) \leqslant q$ for all $x, y, z \in X$.

The following definition can be thought of as a higher-rank version of Gromov's tree approximation lemma for hyperbolic spaces [Gro87].

2.13. **Definition** (Coarse median space)**.** Let X be a metric space. A *coarse median* on X is a ternary operator $\mu : X \to X$ such that there is some sequence (h_n) with the following properties.

- μ is h_0–coarsely Lipschitz in each factor.
- For each finite subset $A \subset X$ there is a finite median algebra M with an $h_{|A|}$–quasimedian map $\iota : M \to X$ and a map $o : A \to M$ such that $\mathsf{d}(\iota o(a), a) \leqslant h_{|A|}$ for all $a \in A$.

We call (X, μ) a *coarse median space.* If every M can be chosen to have rank at most n, then we say μ has rank at most n, writing $\operatorname{rk} \mu \leqslant n$. We write $\operatorname{rk} X$ for the infimal rank of coarse medians on X.

If a metric space X admits a coarse median, then we shall often simply refer to X as a coarse median space. The definition of a coarse median space can also be formulated in terms more similar to Definition 2.5 [NWZ19]. The following lemma provides another link with median metric spaces.

2.14. **Lemma** ([Bow18b, Thm 6.9])**.** *Let (X, μ) be a coarse median space with $\operatorname{rk} \mu \leqslant n$, and let $\hat{X}$ be an asymptotic cone of X. After a bilipschitz change of metric, $(\hat{X}, \hat{\mu})$ is a median metric space of rank at most n.*

Whilst the fact that $\hat{X}$ is a median metric space of finite rank is already useful, the exact control on ranks will be refined in Proposition 3.2 below, which shows that if $\hat{X}$ is an asymptotic cone of a coarse median space X, then $\operatorname{rk} X = \operatorname{rk} \hat{X}$. This justifies the similarity in notation between the rank of a coarse median space X and that of a median algebra.

A subset of a coarse median space (X, μ) is a *quasisubalgebra* if it is the image of some median algebra M under a quasimedian map $M \to X$. A subset $A \subset X$ is *k–coarsely convex* if $\mu(a, a', x)$ is k–close to A for all $a, a' \in A$, $x \in X$. As in the setting of median algebras, for $A \subset X$ let $J(A) = \{\mu(a, a', x) : a, a' \in A, x \in X\}$. Following [Bow18a], and in analogy with the setting of median algebras, if $\operatorname{rk} \mu = n$, then the *coarse-median hull* of a subset $A \subset X$ is defined to be $J^n(A)$. It can be checked that coarse-median hulls are uniformly coarsely convex.

3. Dimension bounds

First we establish a technical lemma that allows one to pass from having bilipschitz cubes in the asymptotic cone of a coarse median space X to having large uniformly quasi-isometrically embedded cubes in X. This is similar to part of the argument of Lemma 2.9.

3.1. **Lemma.** *Let X be a coarse median space. For each d there is a constant q such that the following holds. If there is a median monomorphism of $\{0,1\}^d$ in an asymptotic cone $\hat{X}$ of X, then there are q–quasimedian q–quasiisometries $[0, n]^d \to X$ for all n.*

Proof. Suppose that $f : \{0,1\}^d \to \hat{X} = \lim_\omega (X, \lambda_n \mathsf{d})$ is a median monomorphism. After rescaling $\hat{X}$, we can assume that points in the image of f lie at distance at least 1 from each other. For each $p \in \{0,1\}^d$, let (x_n^p) be a sequence in X such that $(x_n^p) = f(p)$. Let $h = h_{2^d}$ be given by the definition of a coarse median space, and let $t = t(d)$, $r = r(d)$ be given by [Bow19, Lem. 7.2]. We are free to assume (for convenience) that $h, t, r \geqslant 1$. Let $\varepsilon > 0$ be such that $\frac{1-\varepsilon}{\varepsilon} > 10htr$. We have

$$N_\varepsilon = \left\{ m \in \mathbf{N} : \{x_m^p : p \in \{0,1\}^d\} \text{ is a } \frac{1-\varepsilon}{\lambda_m}\text{–separated } \frac{\varepsilon}{\lambda_m}\text{–quasisubalgebra} \right\} \in \omega.$$

In particular, N_ε contains arbitrarily large elements.

Given $m \in N_\varepsilon$, if we let $\kappa_m = \frac{\varepsilon}{\lambda_m}$, then we have that $\{x_m^p\}$ is a $10\kappa_m htr$–separated κ_m–quasisubalgebra consisting of 2^d points. According to [Bow19, Lem. 7.2, 9.1], there is a median algebra Π_m of cardinality at most 2^{2^d} such that the following hold.

- Π_m is a d–dimensional h–quasisubalgebra.
- There is a median epimorphism $\pi_m : \Pi_m \to \{0,1\}^d$ such that $\mathsf{d}(q, x_m^{\pi_m(q)}) \leqslant \kappa_m r$.
- Π_m is "s–straight", where s depends only on the coarse median parameters.

By the second of these properties, each Π_m consists of a union of clusters $\{\pi_m^{-1}(x_m^p)\}$, each of which is at a distance of at least $5\kappa_m htr$ from all others. We deduce that Π_m contains a $5\kappa_m htr$–separated h–quasisubalgebra Q_m isomorphic to a d–cube.

By the first and third items above, each Q_m meets the assumptions of [Bow19, Lem. 9.3]. For each m, an application of that lemma now provides a uniform quasimedian quasiisometry from a product of d intervals of length at least $5\kappa_m htr$ to the coarse-median hull of Q_m. The lemma follows by restricting these maps to subcubes. $\square$

Using this, we can obtain the following characterisation of the infimal rank of a coarse median space. By the *quasiflat rank* of a metric space X, we mean the supremal integer $\operatorname{qf.rk} X$ for which there is a quasiisometric embedding $\mathbf{R}^{\operatorname{qf.rk} X} \to X$.

3.2. **Proposition.** *Let X be a coarse median space, and let $\hat{X}$ be an asymptotic cone of X, which is a median algebra of rank* $\operatorname{rk}\hat{X}$. *The following quantities agree.*

- $\operatorname{rk} X$.
- $\operatorname{rk}\hat{X}$.
- *The supremal d such that there is a bilipschitz, median embedding of* $[0,1]^d$ *in* $\hat{X}$.
- $\operatorname{sepdim}\hat{X}$.

If X is proper and has cocompact isometry group, then we also have $\operatorname{rk} X = \operatorname{qf.rk} X$. *In this case, there exists a quasimedian quasiisometric embedding* $\mathbf{R}^{\operatorname{rk} X} \to X$*, for some coarse median on X realising* $\operatorname{rk} X$.

Proof. The agreement of $\operatorname{sepdim}\hat{X}$ with $\operatorname{rk}\hat{X}$ is given by [Hae16, Cor. 3.7]. If $\operatorname{rk}\hat{X} \geqslant d$, then by [Hae16, Thm G] there is a bilipschitz, median embedding of $[0,1]^d$ in $\hat{X}$. By Lemma 3.1, this implies the existence of arbitrarily large uniform quasicubes in X, which shows that $\operatorname{rk} X \geqslant d$. Thus $\operatorname{rk} X \geqslant \operatorname{rk}\hat{X}$. The converse is [Bow13, Thm 2.3].

If $\operatorname{qf.rk} X \geqslant d$, then $\hat{X}$ contains a bilipschitz copy of $\mathbf{R}^d$ and hence $\operatorname{sepdim}\hat{X} \geqslant d$. By the above, we therefore have $\operatorname{rk} X \geqslant \operatorname{qf.rk} X$. It remains to show that if X is proper and has cocompact isometry group, then X admits a quasimedian quasiisometric embedding of $\mathbf{R}^{\operatorname{rk} X}$, for this in particular implies that $\operatorname{qf.rk} X \geqslant \operatorname{rk} X$.

Suppose that X is proper and has cocompact isometry group. Fix a basepoint $x_0 \in X$. Let μ be a coarse median on X with rank equal to $\operatorname{rk} X$. For $n \in \mathbf{N}$, let Q_n be a uniform quasicube in X of dimension $d = \operatorname{rk} X$ and with diameter at least n. Write z_n for the central point of Q_n. By cocompactness, we can translate z_n into a fixed compact set C containing x_0 by an isometry g_n. This gives uniform quasicubes in the sequence of coarse median spaces $(X, g_n\mu)$ that are all centred in the compact set C.

Now take an (unrescaled) ultralimit. Since X is proper, we have $X = \lim_\omega B(x_0, m)$, where $B(x_0, m)$ is the ball of radius m centred on x_0. Choosing m_n so that $g_nQ_n \subset B(x_0, m_n)$, we get that $Q = \lim_\omega g_nQ_n \subset X$, because g_nQ_n is centred in the fixed compact set C. Because the Q_n are uniform quasiisometric embeddings of increasingly large d–cubes, Q is a quasiisometric embedding of $\mathbf{R}^d$.

It remains to show that there is a coarse median realising $\operatorname{rk} X$ for which the embedding of Q is quasimedian. It is easy to see that the conditions defining a coarse median of rank $\operatorname{rk} X$ hold for the ultralimit $\lim_\omega g_n\mu$, and $Q \to X$ is quasimedian with respect to this because $g_nQ_n \to X$ is uniformly quasimedian with respect to $g_n\mu$. $\square$

Whilst Proposition 3.2 is very precise, it is not always easy to ascertain the quasiflat rank of a group. It is therefore desirable to have a result giving less optimal but more pragmatic control. That is the goal of the remainder of this section, where we give a bound in terms of geometric dimension.

3.3. **Definition.** Let X be a combinatorial cell complex, and let $c \in C_n(X)$ be an n–chain. Letting $c = \sum_{\sigma \in \operatorname{Supp} c} a_\sigma \sigma$, we write $|c| = \sum_{\sigma \in \operatorname{Supp} c} |a_\sigma|$. A function f is called a k^{th}*–order homological isoperimetric function* for X if for each k–boundary b there is a $(k+1)$–chain c with $\partial c = b$ and $|c| \leqslant f(|b|)$. The k^{th}*–order homological Dehn function* of X is the minimal k^{th}–order homological isoperimetric function.

Note that we are not considering these functions up to the usual equivalence. We are interested in slightly more precise control for specific complexes. The following can be extracted from the proof of [Fle98, Thm 2.1].

3.4. **Lemma.** *For each k, q there exists C such that the following holds. Let Y' and Z' be connected combinatorial cell complexes with finite $(k+1)$–skeletons, and let Y and Z be their universal covers. Suppose that Y and Z are k–connected and let $D_{k,Y}$ be the k^{th}–order homological Dehn function of Y. If $f : Y \to Z$ is a q–quasiisometric embedding, then for every k–boundary b in $f(Y)$, if c is a $(k+1)$–chain in Z with $\partial c = b$, then $|c| \geqslant \frac{1}{C} D_{k,Y}(\frac{|b|}{C}) - C|b|$.*

For a finitely generated group G, write $\operatorname{gd} G$ for the *geometric dimension* of G; that is, the infimal dimension of a $K(G,1)$. Recall that G has *type F* it has a finite $K(G,1)$, and *type F_∞* if it has a $K(G,1)$ whose n–skeleton is finite for all n.

3.5. **Lemma.** *If G is a group of type F_∞, then $\operatorname{qf.rk} G \leqslant \operatorname{gd} G$.*

Proof. Suppose that $\operatorname{gd} G < \infty$. Whilst it remains open whether G must be of type F, a combination of Propositions 7.2.13 and 7.2.15 of [Geo08] shows that $G \times \mathbf{Z}$ is of type F. It follows from [Bro94, VIII.7.1] that we can find a finite $K(G \times \mathbf{Z}, 1)$ of dimension $d = 1 + \operatorname{gd} G$. Let X be its universal cover.

If $\operatorname{qf.rk} G > \operatorname{gd} G$, then clearly $\operatorname{qf.rk}(G \times \mathbf{Z}) \geqslant d+1$. Since X is quasiisometric to $G \times \mathbf{Z}$, there is a q–quasiisometric embedding $f : \mathbf{R}^{d+1} \to X$ for some q. Let S_n be the sphere of radius n in $\mathbf{R}^{d+1}$ centred at the origin. Let E_n be its equator: the intersection of S_n with the hyperplane $\{(z_0, \dots, z_d) \in \mathbf{R}^{d+1} : z_0 = 0\}$. Let H_n^+ and H_n^- be the two hemispheres of S_n that meet in E_n.

Up to a uniformly bounded perturbation, simplicial approximation implies that $b_n = f(E_n)$ is a $(d-1)$–cycle in $f(\mathbf{R}^{d+1})$. As X is contractible, b_n is a $(d-1)$–boundary. By Lemma 3.4, there is a constant $C = C(d,q)$ such that for every n, every d–chain c_n in X with $\partial c_n = b_n$ has $|c_n| \geqslant \frac{1}{C} D_{d-1}\big(\frac{|b_n|}{C}\big) - C|b_n|$, where D_{d-1} is the $(d-1)^{\text{th}}$–order homological Dehn function of $\mathbf{R}^{d+1}$. In particular, $|c_n|$ is bounded below by a fixed superlinear function of $|b_n|$.

By the construction of b_n, up to a small perturbation it is filled by $f(H_n^+)$. Let us write c_n^+ for this filling. Since f is a quasiisometric embedding, there is a divergent function $\delta : \mathbf{N} \to \mathbf{R}_{>0}$ such that $\operatorname{Supp} c_n^+$ contains a d–cell $z_n \subset X$ at a distance of at least $\delta(n)$ from b_n. We can also fill b_n with (a perturbation of) $f(H_n^-)$. Let us write c_n^- for this filling.

The d–chain $c_n^+ \cup c_n^-$ has zero boundary, so since $H_d(X, \mathbf{Z}) = 0$ and X has no $(d+1)$–cells, the coefficient of every d–cell must be zero. In particular, $z_n \in \operatorname{Supp} c_n^-$. Since the distance from z_n to c_n^- diverges, this eventually contradicts the assumption that f is a quasiisometric embedding. □

For a finitely generated group G, write $\operatorname{vgd} G$ for the *virtual geometric dimension* of G; that is, the infimal geometric dimension of a finite-index subgroup of G.

3.6. **Theorem.** *Let G be a finitely generated group. If G admits a coarse median, then $\operatorname{rk} G \leqslant \operatorname{vgd} G$.*

Proof. By Proposition 3.2, we have $\operatorname{rk} G = \operatorname{qf.rk} G$. According to [Bow22, Prop. 12.4.7], every asymptotic cone of G is n–connected for all n. By [Ril03, Thm D], we find that G is of type F_∞. Suppose that $\operatorname{vgd} G < \infty$, and let H be a finite-index subgroup of G realising $\operatorname{vgd} G$. It is of type F_∞, for instance by [Bro94, IX.6.1]. The result follows from Lemma 3.5, because H is quasiisometric to G. □

3.7. **Remark.** Since coarse medians can be passed along quasiisometries, Theorem 3.6 could equally well be stated in terms of the infimal geometric dimension of a group quasiisometric to G.

4. Richly branching flats

Here we describe certain geometric configurations whose appearance in a metric space X prevents X from admitting a coarse median of low rank. The functional root of these obstructions is the following.

4.1. **Lemma.** *Let $n \geqslant 2$ and equip $(\mathbf{R}^n, \ell^1)$ with its standard median. Let $v \in \mathbf{R}^n$. Let H^+ and H^- be the two halfspaces of $\mathbf{R}^n$ bounded by $v^\perp$. Let X be obtained from $\mathbf{R}^n$ by gluing a copy I of $[0,\infty) \times \mathbf{R}^{n-1}$ to $\mathbf{R}^n$ along $v^\perp$. If X is a median metric space such that $I \cup H^+$ is median isometric to $(\mathbf{R}^n, \ell^1)$, then v is parallel to some coordinate axis.*

Proof. Suppose that $I \cup H^+$ is median isometric to $\mathbf{R}^n$, but that v is not parallel to any coordinate axis. Let $\mathbf{0}$ denote the origin of $\mathbf{R}^n$. There is some point $\mathbf{1} \in v^\perp$ whose coordinates are all nonzero. Inside $\mathbf{R}^n$, the median interval from $\mathbf{0}$ to $\mathbf{1}$ is an n–box, i.e. a product of n nontrivial intervals. Let a^+ be one of its vertices in H^+, and let a^- be the opposite vertex, which lies in H^-.

Since $I \cup H^+$ is median isometric to $\mathbf{R}^n$, the median interval in $I \cup H^+$ from $\mathbf{0}$ to $\mathbf{1}$ is also an n–box, with a^+ as one of its vertices. Let b be the vertex opposite a^+, which lies in I. We have the following identities.

$$\mu(\mathbf{0},\mathbf{1},a^\pm) = a^\pm, \quad \mu(a^+,a^-,\mathbf{0}) = \mathbf{0}, \quad \mu(a^+,a^-,\mathbf{1}) = \mathbf{1},$$
$$\mu(\mathbf{0},\mathbf{1},b) = b, \quad \mu(a^+,b,\mathbf{0}) = \mathbf{0}, \quad \mu(a^+,b,\mathbf{1}) = \mathbf{1}.$$

By repeatedly applying the five-point condition and these identities, we can now make the following computation.

$$\begin{aligned} b \;&=\; \mu(\mathbf{0},\mathbf{1},b), \;=\; \mu\big(\mu(a^+,a^-,\mathbf{0}),\mu(a^+,a^-,\mathbf{1}),b\big) \;=\; \mu\big(a^+,a^-,\mu(\mathbf{0},\mathbf{1},b)\big) \\ &=\; \mu\big(\mu(a^+,a^-,b),\mu(a^+,a^-,\mathbf{0}),\mathbf{1}\big) \;=\; \mu\big(\mu(a^+,a^-,b),\mathbf{0},\mathbf{1}\big) \\ &=\; \mu\big(\mu(a^+,b,a^-),\mu(a^+,b,\mathbf{0}),\mathbf{1}\big) \;=\; \mu\big(a^+,b,\mu(a^-,\mathbf{0},\mathbf{1})\big) \\ &=\; \mu\big(\mu(a^+,b,\mathbf{0}),\mu(a^+,b\mathbf{1}),a^-\big) \;=\; \mu(\mathbf{0},\mathbf{1},a^-) \;=\; a^-. \end{aligned}$$

This contradiction shows that v must be parallel to some coordinate axis. □

The idea for the configurations we shall consider is that they contain enough branching to force the condition of Lemma 4.1 to fail when one passes to the asymptotic cone. Indeed, there are only n coordinate axes in $\mathbf{R}^n$, so one would expect $n+1$ directions of branching to be enough. One needs to be a little more careful in order to get coarse obstructions, because quasiflats only yield bilipschitz flats in the asymptotic cone. For instance, the cyclic union of six quarterplanes is a median metric space bilipschitz to $\mathbf{R}^2$, and there can be branching along three lines through the origin.

4.2. **Definition** (RBF)**.** For a natural number $n \geqslant 2$, an *n–dimensional richly branching flat*, or *n–RBF*, is a piecewise linear space R constructed as follows. Let B, the *base flat*, be an isometric copy of $\mathbf{R}^n$. Let $v_0, \dots, v_n$ be pairwise linearly independent vectors in B.

For each i, choose a coarsely dense subset $P_i \subset \mathbf{R}$. To obtain R from B, glue, along its boundary, a copy of the half-flat $\mathbf{R}^{n-1} \times [0, \infty)$ along each codimension-1 affine subspace of the form $pv_i + v_i^\perp$ with $p \in P_i$.

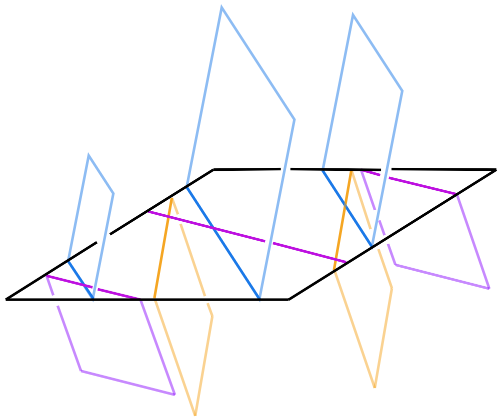


FIGURE 1. A 2–RBF. Note that the only intersection between half-planes glued to B occurs in B itself.

4.3. **Theorem.** *Let X be a coarse median space with $\operatorname{rk} X \leqslant n$. There is no quasiisometric embedding of an n–RBF into X.*

Proof. Suppose that $f : R \to X$ is a quasiisometric embedding in X of an n–RBF. Let $\hat{R}$ be the asymptotic cone of R with respect to some ultrafilter ω, some sequence (λ_m), and some basepoint b. The base flat $B \subset R$ yields a subspace $\hat{B} \subset \hat{R}$ that is isometric to $\mathbf{R}^n$. Let $\hat{X}$ be the asymptotic cone of X with respect to ω, (λ_m), and $f(b)$. Lemma 2.4 shows that f induces a bilipschitz embedding $\hat{f} : \hat{R} \to \hat{X}$. In particular, $\hat{X}$ contains bilipschitz copies of $\mathbf{R}^n$, and so has separation dimension at least n. By Proposition 3.2, we must have $\operatorname{rk} X = \operatorname{rk} \hat{X} = n$.

According to [Bow19, Lem. 5.2], $\hat{f}\hat{B}$ is a finite union of isometric, median embedded panels. There must be a panel P of $\hat{f}\hat{B}$ that is the image of an isometric, median embedding $\phi : [0, \infty)^n \to \hat{f}\hat{B} \subset \hat{X}$. Composing gives a bilipschitz embedding $g = \hat{f}^{-1}\phi : [0, \infty)^n \to \hat{B}$. Let $U \subset \hat{B}$ be an open ball in the image of g. By Rademacher's theorem, both g and g^{-1} are almost everywhere differentiable on their domains. Hence there is a point $p \in U$ in the image of g such that g^{-1} is differentiable at p and g is differentiable at $g^{-1}(p)$.

For small $\varepsilon > 0$, every line segment $\gamma_i : t \mapsto p + tv_i$ defined on $(-\varepsilon, \varepsilon)$ is contained in U. The v_i are pairwise independent, so the fact that g^{-1} is bilipschitz means that the tangent vectors $\frac{d}{dt}(g^{-1}\gamma_i)(0)$ must be pairwise independent as well. Consequently, there must be some i such that $\frac{d}{dt}(g^{-1}\gamma_i)(0)$ is not parallel to any coordinate axis of $[0, \infty)^n$.

Let $F \subset \hat{R}$ be an n–flat consisting of a half-flat in $\hat{B}$ glued to a half-flat meeting $\hat{B}$ along $p + v_i^\perp$. By a similar argument to the above, we can make a small perturbation of p within U so that

- $\hat{f}(p)$ lies in the interior of P and in the interior of a panel of $\hat{f}F$;
- g^{-1} is differentiable at p, and g is differentiable at $g^{-1}(p)$;
- $\frac{d}{dt}(g^{-1}\gamma_i)(0)$ is not parallel to any coordinate axis;
- the analogous maps for F are differentiable at p and the respective preimage.

Consider the tangent cone $T_{\hat{f}(p)}\hat{X}$. According to Lemma 2.9, $T_{\hat{f}(p)}\hat{X}$ is a median metric space of rank at most n. The panel P in which $\hat{f}(p)$ lives induces a flat subspace B' of $T_{\hat{f}(p)}\hat{X}$ that is median isometric to $(\mathbf{R}^n, \ell^1)$. Because $\hat{f}(p)$ is in the interior of a panel of

$\hat{f}F$ and F is glued to $\hat{B}$ along $p+v_i^{\perp}$, the above property of derivatives produces a half-flat I glued to B' along a codimension-1 subspace whose normal vector is not parallel to any coordinate axis. Moreover, the union of I with a halfspace of B' is median isometric to $(\mathbf{R}^n, \ell^1)$. But this contradicts Lemma 4.1. $\square$

5. Free-by-cyclic groups

All free groups considered in this section will be finitely generated. Let F be a (finitely generated) free group, and let $\phi \in \operatorname{Aut} F$. The *free-by-cyclic* group corresponding to ϕ is the group $G = F \rtimes_\phi \mathbf{Z}$. For each $k > 0$, the group $F \rtimes_{\phi^k} \mathbf{Z}$ is a finite-index subgroup of G.

An important perspective on free-by-cyclic groups is to view them as mapping tori of homotopy equivalences of graphs, where the free-group kernel is identified with the fundamental group of the graph. By important work of Bestvina–Feighn–Handel, this can be done very effectively using "improved relative train track maps" [BFH00, Thm 5.1.5]. Below we summarise the parts of this machinery that are needed for our application.

A *filtration* of a graph Γ is a sequence of nested subgraphs $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$. The *strata* of the filtration are the subgraphs $\Gamma_i \smallsetminus \Gamma_{i-1}$. Note that the Γ_i are not necessarily connected.

5.1. **Improved relative train tracks.** Let ϕ be an automorphism of a free group F. After replacing ϕ by some positive power, there is a connected finite graph Γ with $\pi_1\Gamma = F$, a filtration $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n$, and a particularly nice homotopy equivalence $f : \Gamma \to \Gamma$ inducing ϕ. More specifically, f has the properties described below.

- Some edges of Γ are fixed by f. These are referred to as *invariant* edges. Let k be the number of invariant edges. The graph Γ_k is the union of the invariant edges, and if $i \leqslant k$, then the i^{th} stratum consists of a single edge.
- A path $\alpha \subset \Gamma$ is a *Nielsen path* if $f(\alpha) = \alpha$. Note that α may be a concatenation of smaller Nielsen paths. By a *Nielsen cycle*, we mean a Nielsen path that is an immersed (nontrivial) cycle.
- Some edges e of Γ have the property that $f(e) = ep$, where p is a Nielsen cycle, called the *suffix* of e. Such edges are called *linear* edges if the length of $f^n(e)$ grows linearly in n. Distinct linear edges have distinct suffixes [BFH05, Rem 3.12].
- Let l be such that there are exactly $l - k$ linear edges. The graph Γ_l is the union of the invariant and the linear edges. If $i \leqslant l$, then the i^{th} stratum is a single edge.
- Every vertex of Γ_l is fixed by f.
- In addition, Γ may have *polynomial strata* of higher degrees, and it may have *exponential strata.*

The map f is called an *improved relative train track map*, and we refer to the entire package of data above as the *IRTT structure* of $F \rtimes_\phi \langle t \rangle$, or simply of ϕ.

5.2. **Definition** (Support)**.** We say that a Nielsen cycle p *supports* a linear stratum e if the suffix of e is a nonzero power of p.

The following result, which largely collates facts from the literature, shows that, if there are not very many linear strata, then G has fairly strong hyperbolic- and cubical-like features.

5.3. **Proposition.** *Let $G = F \rtimes_\phi \mathbf{Z}$, and fix an IRTT structure for (a power of) ϕ.*

- *If there are no invariant strata, then G is hyperbolic and cocompactly cubulated.*
- *If there are no linear strata, then G is virtually hyperbolic relative to groups of the form $F' \times \mathbf{Z}$, where F' is free.*
- *If there are no quadratic strata and each Nielsen cycle supports at most one linear stratum, then G is virtually* a colourable hierarchically hyperbolic group.

In all three cases, G is quasiisometric to a finite-dimensional CAT(0) cube complex.

Proof. Hyperbolicity when there are no invariant strata is a special case of Brinkmann's theorem [Bri00], and it was proved by Hagen–Wise that hyperbolic free-by-cyclic groups are cocompactly cubulated [HW16, HW15].

More generally, G is (virtually) hyperbolic relative to the mapping tori of its polynomially growing strata [DL22, Thm 4]. If there are no linear strata, then all peripheral subgroups are trivial mapping tori of graphs, which gives the second item.

Suppose that there are no quadratic strata and that each Nielsen cycle supports at most one linear stratum. Since the property of being a colourable hierarchically hyperbolic group is preserved by relative hyperbolicity [BHS19], using [DL22, Thm 4] lets us assume that ϕ is linearly growing. In this case, and assuming that there is at least one linear stratum, after replacing G by a finite-index subgroup, [AM22, Prop. 5.2.2] shows that G can be written as an *admissible* graph of groups in the sense of [CK02]. Let us be more precise. The canonical action on a tree provided by [AM22] gives a nontrivial graph-of-groups decomposition with $\mathbf{Z}^2$ edge groups, and with vertex groups being maximal subgroups of the form $F' \times \mathbf{Z}$, where F' is a (possibly cyclic) free subgroup. The maximality ensures that conjugates of edge groups incident to a common vertex group are non-commensurable. Also by maximality, if E is an edge group, then the preimages in E of the centres of its two incident vertex groups must together generate a finite-index subgroup of E.

Having found that G is virtually an admissible graph of groups, [HRSS22, Thm 4] shows that G is a colourable hierarchically hyperbolic group. Though it is not explicitly stated there, the colourability can be directly seen from the construction of the hierarchy in [HRSS22, §5].

Finally, [Pet21, Thm B] shows that every colourable hierarchically hyperbolic group is quasiisometric to a finite-dimensional CAT(0) cube complex. This applies in all three cases, which concludes the proof. □

The latter case of Proposition 5.3 includes all free-by-cyclic graph manifold groups, as we now discuss. By [BDM09, Thm 11.1], graph manifolds are *thick of order 1* in the sense defined in that paper. According to [BDM09, Cor. 7.9] and [DL22, Thm 4], this means that free-by-cyclic graph manifold groups are polynomially growing, and hence linearly growing by [Hag19, Thm 1.2]. The fact that no Nielsen cycle can support two linear strata is a consequence of the fact that we are considering a manifold.

It is natural to ask whether more of the groups considered in Proposition 5.3 are cocompactly cubulated. Results outside the hyperbolic setting seem to be fairly limited. Hagen–Przytycki characterised which graph manifolds are cocompactly cubulated [HP15]. Button established which *tubular* groups (see Section 6) are free-by-cyclic [But17, Prop. 2.1], and cubulation of tubular groups is well understood [Wis14, Woo18]. However, it is unknown whether all toral relatively hyperbolic free-by-cyclic groups are cocompactly cubulated.

We now turn to the case where ϕ has more linear strata. Our goal will be to show that the the equivalent of Proposition 5.3 fails in a strong way for many such ϕ, by finding RBFs. We shall need the following result, which seems to be known to experts. We are grateful to Monika Kudlinska for informing us of it, and to Jean Pierre Mutanguha for sharing his preprint [Mut] with us.

5.4. **Proposition** ([Mut, Lem. 4.1, 4.2])**.** *Free-by-cyclic subgroups of free-by-cyclic groups are quasiisometrically embedded.*

This includes the degenerate cases of cyclic groups and cyclic-by-cyclic groups.

Let $G = F \rtimes_\phi \mathbf{Z}$. We retain the notation from Item 5.1. In particular, there are k invariant strata, and $l-k$ linear strata in the IRTT decomposition of ϕ. For an (oriented) edge e of a graph, we shall write e^- for its initial vertex and e^+ for its terminal vertex.

5.5. **Definition** (Source)**.** We call a vertex $v \in \Gamma$ a *source* if there is a linear stratum e with $e^- = v$. Consider the directed graph obtained from Γ_l by bi-orienting each invariant edge, and orienting each linear edge e from e^- to e^+. We say that a vertex v *has a nearby source* if there is a directed path from v to some source. We say that a Nielsen cycle has a nearby source if its vertices do.

Note that if a Nielsen cycle p is a union of smaller Nielsen cycles, then it may be the case that one such Nielsen cycle witnesses that p has a nearby source. Indeed, any Nielsen cycle containing a linear stratum has a nearby source. It can also happen, if $e^+ = e^-$, that p is the suffix of a linear stratum witnessing that it has a nearby source.

For $i \leqslant l$, the mapping torus of a connected component of Γ_i can be understood as an amalgamated product, as we now discuss. If $i \leqslant k$, then it is a direct product.

5.6. **Splitting linear strata.** Suppose that $k < i \leqslant l$, so that $f(e_i) = e_i p_i$, where p_i is a Nielsen cycle supported on some component Λ_{i-1} of Γ_{i-1}. Let Λ_i be the component of Γ_i containing e_i. Let M_j be the mapping torus of Λ_j.

The universal cover $\widetilde{M}_{i-1}$ contains a quasiflat Q obtained from a line covering p_i by pushing along the covers of fibres: at "height zero" we see a concatenation of lifts of p_i, and at "height n" we see a concatenation of lifts of $f^n(p_i)$, each of which has length linear in n. By Proposition 5.4, Q is undistorted in $\widetilde{M}_i$. The mapping torus of e_i embeds in M_i, and cutting $\widetilde{M}_i$ along its core curve gives a cyclic splitting of $\pi_1 M_i$. Geometrically, one boundary component of the edge-strip is glued to the ϕ–orbit of e_i^-, whilst the other is glued to a quasigeodesic in Q "of slope one".

Depending on whether e_i^- lies in Λ_{i-1} or not, $\pi_1 M_i$ is either an HNN extension of $\pi_1 M_{i-1}$ or an amalgamated product with the fundamental group of the mapping torus of another component Λ'_{i-1} of $\Gamma_i \smallsetminus e_i$. In some cases, Λ'_{i-1} may be contractible, in which case $\pi_1 M_i \cong \pi_1 M_{i-1}$. More generally, if the component of $\Gamma_l \smallsetminus e_i$ containing e_i^- is contractible, then the edge strip of the corresponding splitting cannot be extended to a half-quasiflat whose coarse intersection with Q is a neighbourhood of the ϕ–orbit of e_i^+.

Suppose that the component of $\Gamma_l \smallsetminus e_i$ containing e_i^- is not contractible. If e_i^- lies in a non-contractible component of Γ_k, then the edge strip can be extended to a half-quasiflat by taking a sub-half-flat in the universal cover of the mapping torus of a loop in that component. The coarse intersection of this half-quasiflat with Q is a neighbourhood of the ϕ–orbit of e_i^+. Because suffixes are immersed cycles, if e_i^- lies in a contractible component of Γ_k, then no linear stratum e_j can have $e_j^+ = e_i^-$. Thus there must be some linear e_j with $e_j^- = e_i^-$, and by the same reasoning, e_j^+ must lie in a non-contractible component of Γ_k. We can again extend the edge strip from e_i to a half-quasiflat meeting Q in a quasiline: follow along a lift of e_j and then take a half-flat coming from a loop in the component of Γ_k containing e_j^+.

5.7. **Definition.** We call a linear stratum e_i *internal* if there is a non-contractible component of Γ_k containing either e_i^- or e_j^+, where e_j is a linear stratum with $e_j^- = e_i^-$.

In view of the above discussion, the stratum e_i being internal is equivalent to the corresponding edge-strip extending to a quasiflat, with one half attached to the ϕ–orbit of e_i^+, and the other half to the orbit of e_i^-.

The following is the main result of this section.

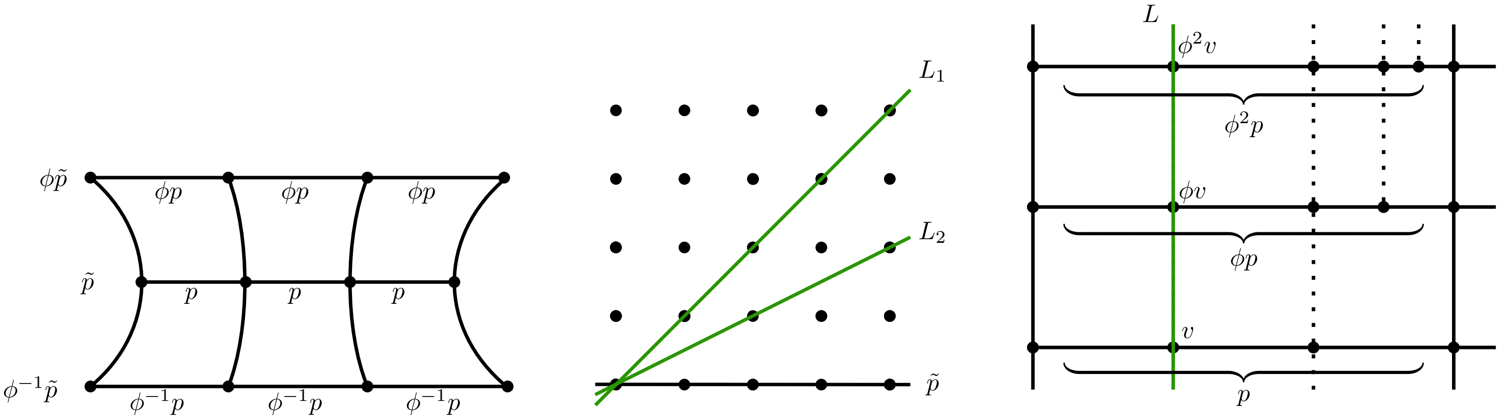
5.8. **Theorem.** *Let $G = F \rtimes_\phi \langle t \rangle$. Suppose that the IRTT structure of (a power of) ϕ has a Nielsen cycle p that either*

- *supports three internal linear strata, or*
- *supports two internal linear strata and has a nearby source.*

Then G does not admit a coarse median.

Proof. As G has geometric dimension two, Theorem 3.6 shows that if G admits a coarse median, then it admits one of rank at most two. Our goal is to find a 2–RBF in G, for then Theorem 4.3 will show that G can admit no such coarse median.

To do this, there is no loss in replacing ϕ by a positive power, because this replaces G by a finite-index subgroup. Moreover, by Proposition 5.4, we can replace G by the fundamental group of the mapping torus M of the maximal linear subgraph Γ_l in the IRTT hierarchy of ϕ. Figure 2 illustrates the construction of the RBF.



Part of the quasiflat Q.

Gluing lines in Q from strata supported on p, with $n_1 = 1$, $n_2 = 2$.

Gluing line in Q from a nearby source v, illustrated with $v \in p$.

FIGURE 2. Parts of the construction of the 2–RBF in Theorem 5.8. The base flat Q has half-quasiflats glued along the green lines.

Consider a line $\tilde{p} \subset \tilde{M}$ covering p. Pushing $\tilde{p}$ along fibres yields a cocompact quasiflat $Q \subset \tilde{M}$. Since distinct linear strata have distinct suffixes, each internal stratum e_i supported by p has suffix p^{n_i}, with the n_i distinct. The edge-strip corresponding to e_i is thus glued to Q along a line L_i of slope $\frac{1}{n_i}$. Since the n_i are distinct, the L_i are non-parallel. Since each e_i is internal, the edge strip extends to a half-flat that is glued to Q along L_i. By Proposition 5.4 and since Q is cocompact, we have shown that if p supports three internal linear strata, then there is a 2–RBF inside G, as desired.

In the remaining case, p supports only two internal linear strata, but it has a nearby source v. Let e be a linear stratum with $e^- = v$, and let p_e be its suffix. Let q be a directed path in Γ_l from a vertex u of p to v. Consider the half-quasiflat with boundary the ϕ–orbit of u where from each point in that orbit there is a single lift of q, followed by a single lift of e, followed by a ray consisting of lifts of p_e. This half-quasiflat is attached to Q along a quasigeodesic L parallel to the fibres. In particular, its attaching line is not parallel to either of the L_i. (Note that this is the case even if p is the suffix of e.) Again, Proposition 5.4 and cocompactness of Q give a 2–RBF inside G. As discussed above, this completes the proof. □

6. TUBULAR GROUPS

In this section we consider *tubular groups*. Our goal will be to understand which tubular groups admit coarse medians.

6.1. **Definition** (Tubular group). A *tubular group* is the fundamental group of a graph of groups with $\mathbf{Z}^2$ vertices and $\mathbf{Z}$ edges.

Each tubular group G has an associated graph of spaces $\bar{X}$, with torus vertices and circle edges, such that $G = \pi_1\bar{X}$. We enumerate the vertex tori $\bar{F}_1, \dots, \bar{F}_n$, and the edge circles $\bar{E}_1, \dots, \bar{E}_m$. The universal cover X of $\bar{X}$ is a tree of spaces with 2–flat vertex spaces and line edge spaces. Given a vertex flat $F \subset X$, we shall also write $\bar{F}$ for the vertex torus in $\{\bar{F}_1, \dots, \bar{F}_n\}$ covered by F.

6.2. **Definition** (Excursion decomposition)**.** Let F be a vertex flat of X. A path $\delta : I \to X$ is an *$\bar{E}_j$-excursion on F* if it has initial and terminal segments both traversing an edge E covering $\bar{E}_j$, and only the endpoints of δ meet F. A path γ in X with endpoints in some vertex flat F has a unique *excursion decomposition* $\gamma = \sigma_1\delta_1\sigma_2 \cdots \delta_n\sigma_{n+1}$, where $\sigma_i \subset F$ and δ_i is an excursion.

6.3. **Lemma.** *Let γ be a geodesic in X joining points in a vertex flat F. For each $\bar{E}_j$ incident to $\bar{F}$, there is at most one $\bar{E}_j$–excursion in the excursion decomposition of γ.*

Proof. Let γ be a path with two $\bar{E}_j$–excursions in its excursion decomposition of γ. Write $\gamma = \alpha\delta_1\beta\delta_2\varepsilon$, where the δ_i are $\bar{E}_j$–excursions and $\alpha, \beta, \varepsilon$ are subpaths of γ. There is a path $\gamma' = \alpha\delta_1\delta_2'\beta'\varepsilon$ with the same endpoints as γ, where δ_2' and β' are $\mathrm{Stab}_G F$–translates of δ_2 and β, respectively. We have $|\gamma'| = |\gamma|$. As $\delta_1\delta_2$ is a concatenation of $\bar{E}_j$–excursions, γ' is not a geodesic, and hence nor is γ. ☐

6.1. Distortion and isoperimetry

Recall that a subspace Y of a metric space X is *distorted* if the inclusion map $Y \to X$ is not a quasiisometric embedding when Y is given the path metric. Let us say that a tubular group is *distorted* if one of its vertex groups is distorted. We shall prove that distorted tubular groups have super-quadratic Dehn functions.

Let $\bar{E} \in \{\bar{E}_1, \ldots, \bar{E}_m\}$, and let F cover a vertex torus to which $\bar{E}$ is incident. An *$\bar{E}$–line* in F is the image in F of an incident edge space E covering $\bar{E}$. An $\bar{E}$–line is a biinfinite F–geodesic, but may be distorted in X. Note that, for fixed $\bar{E}$, any two $\bar{E}$–lines are isometric in X. An *$\bar{E}$–segment* is a segment in an $\bar{E}$–line.

For a path p in X, write $\mathbf{i}p$ and $\mathbf{t}p$ for the initial and terminal vertices of p, respectively.

6.4. **Lemma.** *If G is a distorted tubular group, then there exists a distorted $\bar{E}$–line in X.*

Proof. Let $F \subset X$ be a distorted flat. There is a sequence (γ_i) of X–geodesics with endpoints in F such that $\frac{|\gamma_i|_X}{\mathsf{d}_F(\mathbf{i}\gamma_i,\mathbf{t}\gamma_i)} \to 0$. For each i, let $\gamma_i = \sigma_1^i\delta_1^i \ldots \delta_{n_i}^i\sigma_{n_i+1}^i$ be the excursion decomposition of γ_i. By Lemma 6.3, we have $n_i \leqslant m$ for all i.

Since subpaths γ_i are geodesics, we have $|\gamma_i|_X = \sum_j |\sigma_j^i|_F + \sum_j |\delta_j^i|_X$. In particular, the above convergence implies that $\sum_j |\sigma_j^i|_F < \frac{1}{2}\,\mathsf{d}_F(\mathbf{i}\gamma_i, \mathbf{t}\gamma_i)$ for all sufficiently large i. By the triangle inequality, $\mathsf{d}_F(\mathbf{i}\gamma_i, \mathbf{t}\gamma_i) \leqslant \sum_j |\sigma_j^i|_F + \sum_j \mathsf{d}_F(\mathbf{i}\delta_j^i, \mathbf{t}\delta_j^i)$. Thus, for each sufficiently large i there is some k_i such that $\mathsf{d}_F(\mathbf{i}\delta_{k_i}^i, \mathbf{t}\delta_{k_i}^i) \geqslant \frac{1}{2m}(\sum_j |\sigma_j^i|_F + \sum_j \mathsf{d}_F(\mathbf{i}\delta_j^i, \mathbf{t}\delta_j^i))$. After passing to a subsequence and relabelling the $\bar{E}_j$, we may assume that $k_i = k$ and δ_k^i is an $\bar{E}_k$–excursion for all i. But now we compute

$$\frac{|\delta_k^i|_X}{\mathsf{d}_F(\mathbf{i}\delta_k^i, \mathbf{t}\delta_k^i)} \leqslant \frac{|\gamma_i|_X}{\mathsf{d}_F(\mathbf{i}\delta_k^i, \mathbf{t}\delta_k^i)} \leqslant \frac{2m|\gamma_i|_X}{\sum_j |\sigma_j^i|_F + \sum_j \mathsf{d}_F(\mathbf{i}\delta_J^i, \mathbf{t}\delta_j^i)} \leqslant \frac{2m|\gamma_i|_X}{\mathsf{d}_F(\mathbf{i}\gamma_i, \mathbf{t}\gamma_i)} \to 0,$$

and we conclude that $\bar{E}_k$–lines are distorted. ☐

We now define a labelled, directed graph that encodes the distortion caused by edge spaces. For each i, fix a basis for the vertex group $\mathrm{Stab}_G \bar{F}_i$, and let $|\cdot|_i$ be the corresponding word norm.

6.5. **Definition** (Distortion graph)**.** The vertex set of Δ is the set of flats $\bar{F}_i$. There is a directed edge e from $\bar{F}_{i_1}$ to $\bar{F}_{i_2}$ if and only if there is some distorted edge $\bar{E}_j$ whose endpoints are $\bar{F}_{i_1}$ and $\bar{F}_{i_2}$. This directed edge e is given a label ℓ_e as follows. There exist $w_k \in \mathrm{Stab}_G \bar{F}_{i_k}$ such that the edge $\bar{E}_j$ identifies w_1 with w_2. Set $\ell_e = \frac{|w_2|_{i_2}}{|w_1|_{i_1}}$.

Note that all labels are rational. Also, if $e = uv$ is an edge of Δ, then there is another edge $e' = vu$, with $\ell_{e'} = \frac{1}{\ell_e}$. We say that a directed cycle in Δ is *balanced* if the product of the labels of its edges is 1.

6.6. **Lemma.** *If G is distorted, then Δ has an unbalanced cycle.*

Proof. Suppose that all cycles in Δ are balanced. By proceeding along a spanning tree of Δ, one can label the vertices of Δ with positive integers N_i in such a way that for each directed edge $e \subset \Delta$ from $\bar{F}_{i_1}$ to $\bar{F}_{i_2}$ we have $\ell_e = \frac{N_{i_2}}{N_{i_1}}$.

We construct a metric space X' with underlying set X. For points x and y in a flat covering $\bar{F}_i$, let $D(x,y) = N_i \, \mathsf{d}_F(x,y)$. Set the thickness of the edge-strips to be 1. Let $X' = (X, \mathsf{d}')$ be the path-metric space induced from the partially-defined function D. Note that by the choice of the N_i, every vertex-flat of X' is convex, and in particular undistorted. On the other hand, given an X–geodesic between points x and y of a flat F, by viewing it as a union of segments in edge-strips and segments in vertex-flats we see that it is a coarsely Lipschitz path in X'. Hence $\mathsf{d}'(x,y)$ is coarsely bounded above by $\mathsf{d}(x,y)$. But this shows that $\mathsf{d}(x,y)$ is coarsely lower-bounded by $\mathsf{d}_F(x,y)$. We obtain a contradiction, as we have shown X is quasiisometric to X', whose flats are obviously undistorted. $\square$

To facilitate understanding the Dehn function of G, we first consider the case where each flat of X has at most one parallelism class of distorted $\bar{E}$–lines. In general this is not the same as there being only one j for which there are distorted $\bar{E}_j$–lines, but the graph Δ can still be used to find that some $\bar{E}$–lines are highly distorted.

6.7. **Proposition.** *Let G be a distorted tubular group. If each flat of X has at most one parallelism class of distorted lines, then G has exponential Dehn function.*

Proof. By Lemma 6.6, there is an unbalanced cycle in Δ, and hence an unbalanced embedded cycle $\gamma \subset \Delta$. Let α denote the product of the labels on the edges of γ. Perhaps after reversing γ, we have $\alpha < 1$. Since there is only one parallelism class of distorted lines in each flat, there must be some vertex-flat $\bar{F}$ of γ in which the incoming edge-gluing and the outgoing edge-gluing are (conjugates of) distinct powers of the same group element, with the outgoing one being a proper power. In X, this ensures that there are two conjugate edge-strips glued along the same outgoing $\bar{E}$–line. See Figure 3. After relabelling, the edges of γ are $\bar{E}_1, \dots, \bar{E}_n$, and the initial flat is $\bar{F}$.

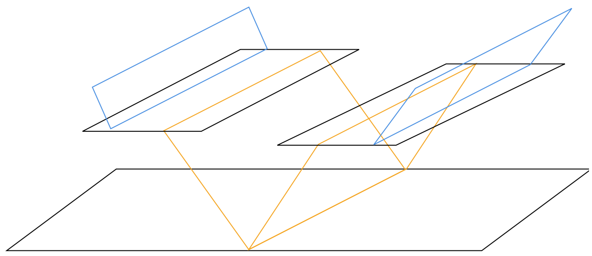


FIGURE 3. Two conjugate edge-strips in orange glued along an $\bar{E}$–line.

Let x and y be sufficiently far-apart points lying in an $\bar{E}_1$–line inside a flat F of X covering $\bar{F}$. For each $n \geqslant 0$, our choice of $\bar{F}$ means that γ^n gives at least two sequences of vertex flats of X, starting with F and ending with $F_{1,n}$ and $F_{2,n}$, respectively, which are translates of F. Moreover, the paths in the Bass–Serre tree of G from F to the $F_{i,n}$ meet only in F. Among all points in $F_{i,n}$ lying in $\bar{E}_1$–lines, let $z^-_{i,n}$ be a closest such point to x. Define $z^+_{i,n}$ similarly with y. In particular, $z^-_{i,0} = x$ and $z^+_{i,0} = y$.

Let $p_{i,n}$ be the path from x to y consisting of: a geodesic $\delta^-_{i,n}$ from x to $z^-_{i,n}$; the affine path in $F_{i,n}$ from $z^-_{i,n}$ to $z^+_{i,n}$; and a geodesic $\delta^+_{i,n}$ from $z^+_{i,n}$ to y. The length of $p_{i,n}$ is $2n + \alpha^n \, \mathsf{d}_F(x,y)$. When $n \sim \log \mathsf{d}_F(x,y)$, this is coarse-linearly equivalent to $\log \mathsf{d}_F(x,y)$.

Together, $p_{1,n}$ and $p_{2,n}$ form a loop of length coarse-linearly equivalent to $\alpha^n \, \mathsf{d}_F(x,y)$, and they bound an embedded disc with at least $\mathsf{d}_F(x,y)$ 2–cells. See Figure 4. Because

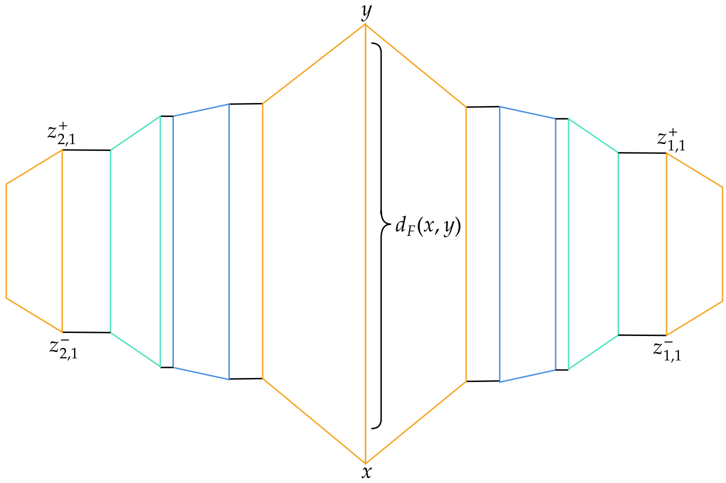


FIGURE 4. A example diagram where γ is length three and $n = 1$.

X is a contractible 2–complex, the fact that the disc is embedded means that any disc with the same boundary must have at least as many 2–cells. In particular, by considering diagrams with $\mathsf{d}_F(x, y) \to \infty$, we find that G has exponential Dehn function. □

6.8. **Theorem.** *Distorted tubular groups have super-quadratic Dehn function.*

Proof. Let G be a distorted tubular group. By Lemma 6.4, X has distorted $\bar{E}$-lines. If each flat of X has at most one parallelism class of distorted $\bar{E}$–lines, then Proposition 6.7 shows that G has exponential Dehn function. Otherwise there is a flat $F \subset X$ with non-parallel, distorted $\bar{E}_1$– and $\bar{E}_2$–lines.

For $j = 1, 2$, let (δ_i^j) be a sequence of increasingly long X–geodesics that are $\bar{E}_j$–excursions from F and witness the distortion of $\bar{E}_j$. Let σ_i^j be the affine path in F from $\mathbf{i}\delta_i^j$ to $\mathbf{t}\delta_i^j$. For each i, one can form a rhombus in $R_i \subset F$ from two translated copies of σ_i^1 and two translated copies of σ_i^2. The area of R_i is quadratic in its perimeter $P_i = 2|\sigma_i^1| + 2|\sigma_i^2|$. By the choice of the δ_i^j, this is super-quadratic in $|\delta_i^1| + |\delta_i^2|$

Taking the same translations of the δ_i^j gives a loop L_i in X meeting R_i in exactly four points. Since X is a contractible 2–complex, any disc filling L_i must contain the embedded disc R_i, and hence have area that is super-quadratic in its perimeter. Thus G has no quadratic isoperimetric function. □

6.2. Coarse median tubular groups

Here we use the results of Section 6.1 together with the construction of RBFs in undistorted tubular groups to prove Theorem E, characterising which tubular groups have coarse medians.

6.9. **Lemma.** *Let G be an undistorted tubular group. If some vertex group has three commensurability classes of incident edge groups, then G has a quasiisometrically embedded 2–RBF.*

Proof. Let X be the tree of spaces for G, and let F be a flat stabilised by a vertex group as in the assumption. Let $\bar{E}_1, \bar{E}_2, \bar{E}_3$ be the images of three pairwise non-commensurable incident edges groups in $\bar{F}$. For each i and each line $E_i \subset F$ covering $\bar{E}_i$, there exists a rough half-flat attached to F along E_i which is the union of an edge-strip together with a half-flat in the vertex-space on the other end of the strip. As G is undistorted, this yields a 2–RBF in X, and hence in G. □

6.10. **Theorem.** *Let G be a rank-2 tubular group. If G admits a coarse median, then G is virtually compact special.*

Proof. According to [Bow13, Cor. 8.3], if G admits a coarse median then it has a quadratic isoperimetric function, and so Theorem 6.8 shows that G must be undistorted. Moreover, G has geometric dimension two since it is a graph of two dimensional groups (see [SW79, Prop. 3.6]), so Theorem 3.6 shows that $\operatorname{rk} G \leqslant 2$. In particular, Theorem 4.3 implies that G cannot have a quasiisometrically embedded 2–RBF. By Lemma 6.9, this means that no vertex group of G can have more than two commensurability classes of incident edge groups. Since G is undistorted, it contains no Baumslag–Solitar subgroups $\mathrm{BS}(m,n)$ with $m \neq \pm n$. Consequently, [Wis14, Cor 5.10, 5.9] implies that G is virtually compact special. □

By a result of Woodhouse [Woo18], it follows that, for tubular groups, admitting a coarse median is equivalent to the existence of a free action on a locally finite CAT(0) cube complex. By Wise's characterisation [Wis14], there are tubular groups that are freely cubulable but do not admit coarse medians.

6.11. **Example** (A freely cubular, tubular, non-quasicubular group)**.** Let P denote the graph isomorphic to the 1-skeleton of a tetrahedron. Associating a $\mathbf{Z}^2$ with each vertex of P, and a $\mathbf{Z}$ with each edge, it is possible to choose attaching maps so that at each vertex, the incident edges are attached along the subgroups generated by $(1,0)$, $(0,1)$, and $(1,1)$. Let G be the tubular group defined by this graph of groups.

It is easy to see from Wise's characterisation that G acts freely on a CAT(0) cube complex. However, since G is undistorted and contains a vertex with three incident, non-commensurable edge groups, Lemma 6.9 implies G is not coarse median (in particular, not quasicubical). Groups such as G can be viewed as "opposite" to hyperbolic groups with property (T), which are quasicubical yet do not act freely on any CAT(0) cube complex.

There is some overlap between Theorem 6.10 and Theorem 5.8. By combining [BD14, Cor. 4.17] and [Mac02, Thm 1.2], it can be seen that any tubular group that is free-by-cyclic is linearly growing. For such groups, one should then compare Proposition 5.3 and Theorem 5.8 with [Wis14, Cor. 5.10].

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