Analysis on the Binary Game

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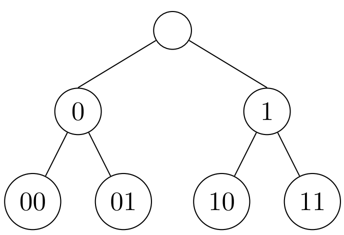
Abstract

We analyze the evolution of the value of the "binary game" from both numerical and analytical perspective. We find numerically that the game value oscillate around zero and seems to have two converged subsequences, which indicates a stable advantage to playing last. Moreover, the empirical variance of the game value also converge to zero. In theoretical analysis, we prove all these observations and find the limits as $n \to \infty$ as an even or odd number sequence. The limits are related to the golden ratio, i.e. $(\sqrt{5}-1)/2$, which is quite beautiful.

1 Introduction

In this project, we are to analyze the value of a sequential zero sum game, which we called Binary Game. The game has n moves, alternating between Paul (first) and Carole. At each move the player selects a bit, zero or one. The starting node, denoted e, is the empty string. After u moves the intermediate node will be a binary string of length u. At the end of the game, the leaves are the 2^n strings of length n. The values $VALUE(x)$ for the leaves x are set in advance. In particular it obeys the uniform distribution on $[-1, 1]$. Paul wants to maximize the value while Carole wants to minimize it. When both of them take their optimal strategy, the value of the game is difined as the value of the root node, i.e. $VALUE(e)$. Our goal is to analyze the value of the game as n increase. Detailed information about the game can be found in [1].

A typical structure of the game looks like the follows.



2 Algorithm and implementation

We follow the DFS algorithm described in [1] and Algorithm 1 shows the pseudocode for the core algorithm.

*This is a group project of Foundamental Algorithm class with Prof. Joel Spencer at New York University.

Algorithm 1 Binary Game's algorithm (an Algorithm of Depth First Search)

D is the distribution from which leaf values are sampled.

procedure GAME VALUE(P, H) ▷ P is position, H is player
  if $P = ROOT$ **then**
    $H \leftarrow Player1$
  else
    $Pass$
  if $P = LEAF$ **then**
    $Value \leftarrow x \sim D$
    return $Value$
  else
    if $F = Player1$ **then**
      $Value \leftarrow \max_{i \in Adj[P]} \{GameValue(i, Player2)\}$ ▷ Max Strategy
      return $Value$
    else
      $Value \leftarrow \min_{i \in Adj[P]} \{GameValue(i, Player1)\}$ ▷ Min Strategy
      return $Value$

The implementation is a little bit tricky. The code with comments can be found in attached python files and we simply describe the implementation here for reference.

we define several classes to facilitate us implementing this algorithm.

1. The tree class. It represents the tree of all possible "board situations" of this binary game, with depth n as game's total unrolled steps, within which contains a nested Position class and a Node Class.

2. The position class. This is an abstraction of a specific, i.e. current position in the whole course of this binary game.

3. The node class. This class is not exposed to the user and stores actual information of a specific position, i.e. position's value, its parent position as well as its child position.

We initialize a game by recursively calling *gen_board* function, which sample the leaves' initialized values from a given distribution. After initializing the game, we scores it by recursively calling two functions, *scoring_max_min_first* and *scoring_max_min_second*. The visualization codes naturally follows.

3 Numerical results

3.1 Trend of the game value: oscillation and convergence of variance

We conducted 500 times experiments for each $n \in [1, 20]$, computed and plotted the mean and variance of the game value with 95% confidence interval for each n, where n is the total steps of our game.

Value trend: oscillation By taking a look at the original data, we first noticed the relationship between n and the mean value. In Figure 1, we found the mean of game value oscillates in a zigzag version with the

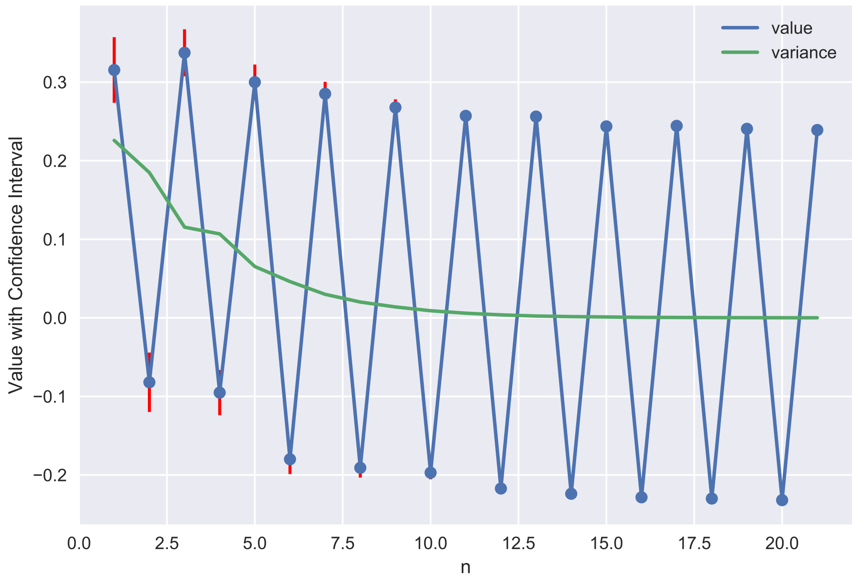


Figure 1: Value trend: oscillation and convergence of variance

increasing of n. Because Paul always play at first, the game value goes down from $n = 1$ to $n = 2$. Assuming we are at $n = i$, if mean value goes up from $i - 1$ to i, then it goes down from i to $i + 1$ and vice versa. If we looked at the range of value and the corresponding n more precisely, we found that with n increasing, the mean of value oscillates in a constant range, changing between positive and negative. When n is odd, the mean value is always positive and goes to negative in the next time; if n is even, the value is negative and will become positive later. Besides, the mean never goes to zero, indicating there is advantage in this game.

Value trend: convergence of variance Our next observation focused on the variance of experiments for each n. With the same sample number for every n, however, we found the variance of the sampled game value decreasing rapidly with the increasing of n. It starts from around 0.25 and converges to 0. The convergence of sample variance has no relationship with the parity of n.

Summary: advantage, variance and two converged sequences Below are our observations on mean and variance of the sampled game value:

1. The relationship between parity of n and the mean of game value suggests us to study the trend in two cases separately, when n is even and odd. This is reasonable as we know when n is odd, Paul will play the last step and has control of the final result by choosing the maximum value of all leaves; while when n is even, Carol will play the last step by choosing the minimal value of all leaves and the only thing for Paul is to receive the game result. That is to say, the advantage comes in the last step.

2. The change of variance is a little bit amazing to us, which implies that in either case, the game value may converge with the increasing of n, regardless to the parity of n.

3. Together with the two points above, we infer that the game value might converge to two different points when n goes to infinity and the points depend on the parity of n.

3.2 Distribution of game value

Convergence to different points The game value converges when n goes to infinity, but the points converged to are different when n is odd and even. We plotted the histograms of the sampled game value in two groups and normed the graph so that the sum of the area of each bin is 1. The x-axis is the value, while y-axis denotes the probability density of the corresponding value. We denote (a_i, b_i) is the i_{th} bin of the histogram, N_i is the number of sample in that bin and N is the number of samples for each n, where $p(value = X)$ is the probability density of value when $value = X$:

$$P(X \in (a_i, b_i)) = \frac{N_i}{N}$$

$$p(value = X, X \in (a_i, b_i)) = \frac{P(X \in (a_i, b_i))}{(b_i, a_i)}$$

Figure 2 shows the odd case of n with $n = 15, 17, 19, 21$. We observed that the distribution of sampled game value in each subplot forms a normal distribution asymptotically, centering around the mean of that sample. When n increases, the distribution shrinks to the corresponding center, which verifies the decreasing variance when n increases. The center for each distribution, however, also converges to some point between $(0.20, 0.30)$.

Figure 3 represents the case when n is even, where $n = 14, 16, 18, 20$. The change of the distribution is similar to the case when n is odd except that the center for each distribution converges to some point in $(-0.25, -0.15)$.

Density of the converged points (limits) The probability density of the converged points, which are limits, goes to infinity with n going to infinity. Figure 4 shows a more clear trend for the distribution of game value under different n. With y-axis representing the probability density estimated from the histograms, all red lines denote the game value distribution when n is odd, while all green lines represent that when n is even. For all the red lines, we could see while the center of each distribution converges to one point in $(0.20, 0.30)$, the peak of each distribution increases rapidly. This implies that the density of the center increases quickly with the increasing of n. Denote the point which the game value converges to as X_{odd} when when n is odd, the trend also suggests that the probability density of X_{odd} goes to infinity. The trend for distribution of game value when n is even is rather similar. With the center of all distribution converges to a point X_{even} in $(-0.25, -0.15)$, the peak of the distribution increases fast and goes to infinity.

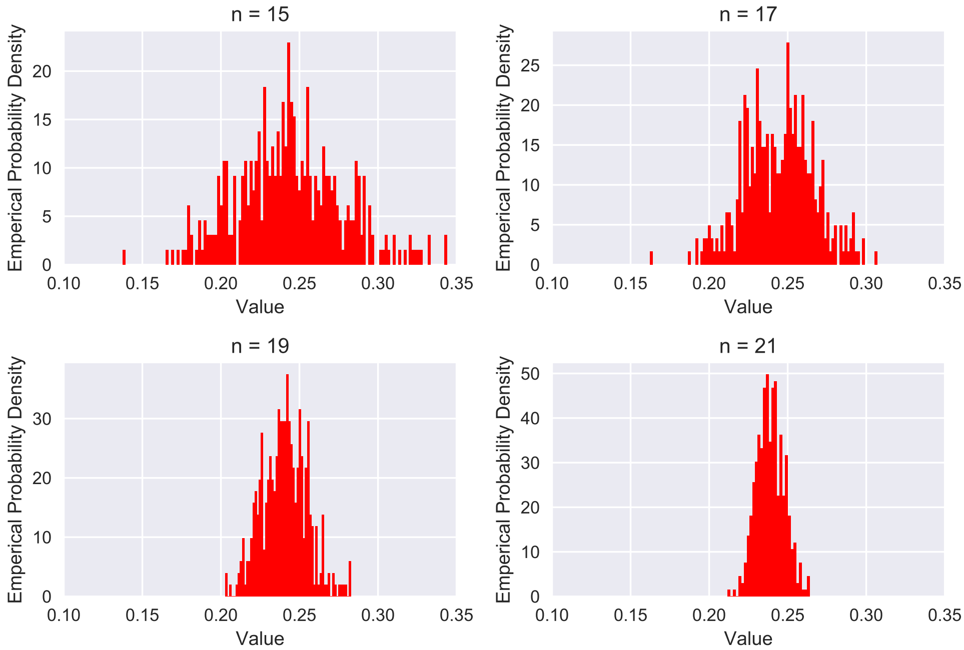


Figure 2: Game Value Distribution: n is odd

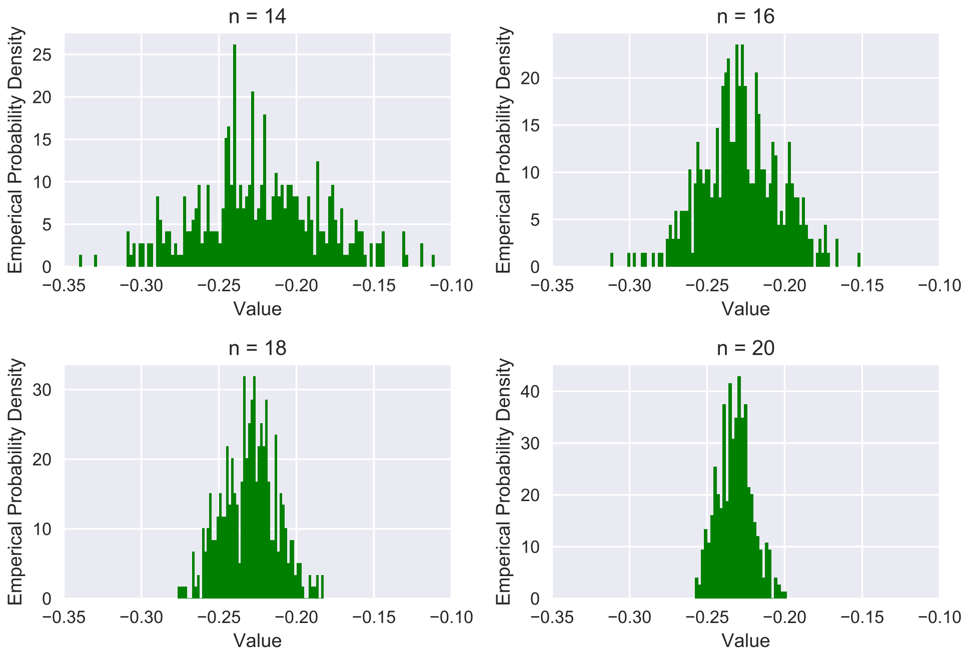


Figure 3: Game Value Distribution: n is even

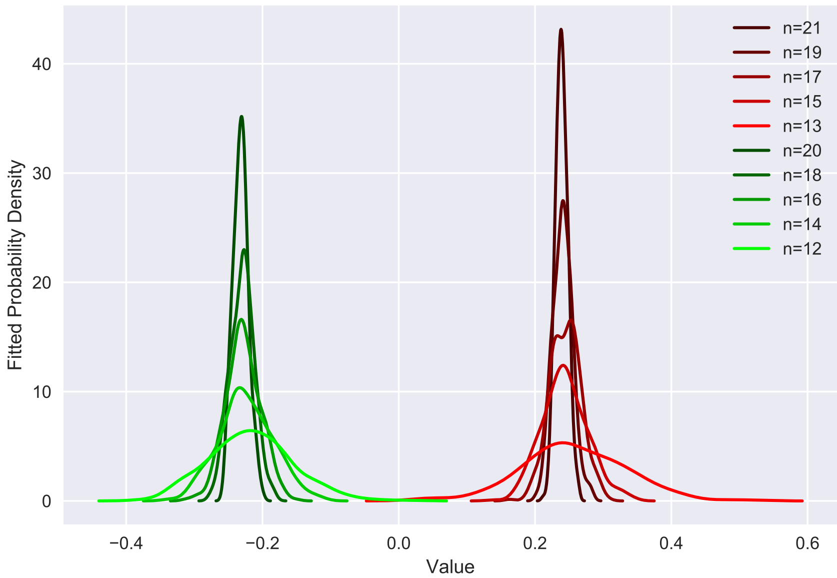


Figure 4: Game value distribution and fitted probability density

n	15	17	19	21
X_{odd}	0.2509	0.2564	0.2389	0.2353
n	14	16	18	20
X_{even}	-0.2369	-0.2237	-0.2249	-0.2312

Table 1: Game values on X_{odd} and X_{even}

3.3 Properties of the limits (converged points)

Estimation of limits Besides the convergence, we also want to find some properties of the two limits, X_{odd} and X_{even}. We first estimated the value of limits with n in different parity by choosing corresponding point of the maximum estimated density. Table 1 shows our estimation of the limits for each n.

Relative position to value range We also searched the relative position of the game value to the range of the game value. We denote d_{odd} to be the relative position of X_{odd}, d_{even} to be the relative position of X_{even}, $[a, b]$ to be the range of the game value, M to be the middle of the range and r to be the ratio of $|X_{odd} - M|$ and $|X_{even} - M|$. We define d_{odd}, d_{even} and r:

$$d_{odd} = \frac{|X_{odd} - a|}{|b - a|} \quad d_{even} = \frac{|X_{even} - a|}{|b - a|}$$

$$r = \frac{|X_{odd} - M|}{|X_{even} - M|}$$

n	15	17	19	21
d_{odd}	0.6254	0.6282	0.6194	0.6177
n	14	16	18	20
d_{even}	0.3816	0.3882	0.3876	0.3844

Table 2: Relative position of X_{odd} and X_{even}

Table 2 presents the relative positions of limits.

Summary of properties Combining Table 1 and Table 2, we found the relative position of the limits is near to 0.38 when n is even while near to 0.62 when n is odd. Besides, the distances between points and the middle of the range are nearly same, implying the ratio r defined above is near to 1.

4 Theoretical analysis

Inspired by the numerical results, we further investigate the problem from the theoretical perspective. In this section, we prove that the game value converges for even or odd sequence of n respectively and further find the limits.

4.1 recurrence relation

Let's start with recurrence relation between the node values. As defined in the game, each node has a value and it is a random variable. It is easy to show that if the values of all leaf nodes are i.i.d., the values of nodes having the same height are also i.i.d. Assume the leaf nodes have height 0 and denote the cumulative distribution function (CDF) of the values on nodes having height i as $F_i(x)$, and the corresponding random variable as V_i. In the algorithm, the value of a internal node is calculated by taking the maximum or minimum of its two children. So we have

$$\begin{cases} F_{i+1} = F_i^2, & \text{if take maximum} \\ F_{i+1} = 1 - (1 - F_i)^2 = 2F_i - F_i^2, & \text{if take minimum} \end{cases} \tag{1}$$

Combining the order of playing of the game, we conclude that

$$\begin{cases} F_{i+1} = F_i^2, & \text{if } i + n \text{ odd}, i \leq n - 1 \\ F_{i+1} = 1 - (1 - F_i)^2 = F_i(2 - F_i), & \text{if } i + n \text{ even}, i \leq n - 1 \end{cases} \tag{2}$$

4.2 convergence

Now let's combine two successive updates and derive the relationship as follows.

$$\begin{cases} F_{i+2} = L(F_i) \equiv F_i^2(2 - F_i)^2, & \text{if } i + n \text{ even}, i \leq n - 2 \quad (3) \\ F_1 = F_0^2, & \text{if } n \text{ odd} \quad (4) \end{cases}$$

What we concern about is the value of the root node, i.e. F_n. And the reccurence above is enough for this purpose. If n is even, we apply (3) on F_0 and finally we will reach F_n. If n is odd, we apply (4) on F_0 and then (3) and finally we will also reach F_n.

Now we can state our main result.

Theorem 4.1. *The CDF of the value of the root node, i.e. F_n, converges pointwisely for even n or odd n sequence. Specifically,*

1. *for even n, F_n converge to $\mathbb{1}_{x \geq x_1}$, where x_1 is the solution to $F_0(x_1) = 1 - \phi$;*
2. *for odd n, F_n converge to $\mathbb{1}_{x \geq x_2}$, where x_2 is the solution to $F_1(x_2) = F_0^2(x_2) = 1 - \phi$, or equivalently, $F_0(x_2) = \phi$;*

where $\mathbb{1}$ is the indicator function and $\phi = \frac{\sqrt{5}-1}{2}$ is the Golden ratio.

Proof. 1. Since n is an even number, after applying $n/2$ times of reccurence (3) on F_0 we get F_n. Now let's look at the difference after one reccurence

$$D(F_i) = F_{i+2} - F_i = L(F_i) - F_i = F_i^2(2 - F_i)^2 - F_i \tag{5}$$

For simplicity, let's ignore the subscript and simply write

$$D(F) = F^2(2 - F)^2 - F. \tag{6}$$

The function $D(F)$ has only three zeros in $[0, 1]$, which are

$$0, \quad 1 - \phi, \quad 1, \tag{7}$$

and its graph is shown in Figure 5.

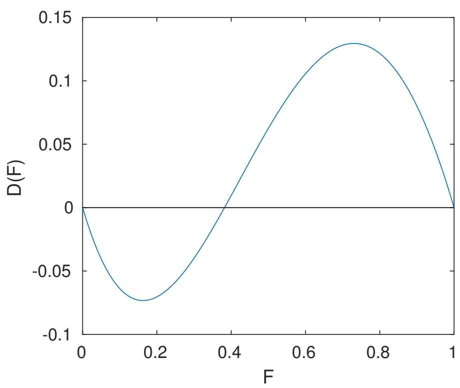


Figure 5: The graph of $D(F)$.

If $F > 1 - \phi$, after applying the operator L on F, $L(F) > F > 1 - \phi$. So $L^k(F)$ is a strictly increasing function of k, if $F > 1 - \phi$, where $L^k(F)$ means applying the operator L k time. In addition, since

$L^k(F)$ is a restriction of the CDF F_{2k}, we have $L^k(F) \le 1$. Therefore $F_{2k}(x)$ $(F_0(x) > 1-\phi)$ converges as k goes to infinity. Moreover, since there is only one zero of $D(F)$ in $(1-\phi, 1]$, which is 1, the only stationary point of $L(F)$ in $(1-\phi, 1]$ is 1. Thus $F_{2k}(x)$ $(F_0(x) > 1-\phi)$ converges to 1.

Similarly, we have $F_{2k}(x)$ $(F_0(x) < 1-\phi)$ converges to 0.

Combining all above, we conclude

$$F_n(x) \to \mathbb{1}_{F_0(x) \ge 1-\phi}, \text{ as } n \to \infty, \quad n \text{ even} \tag{8}$$

2. When n is odd, applying the same analysis, but notice now the initial value of the reccurence (3) is F_1 and $\sqrt{1-\phi} = \phi$, we derive

$$F_n(x) \to \mathbb{1}_{F_1(x) \ge 1-\phi} \equiv \mathbb{1}_{F_0^2(x) \ge 1-\phi} \equiv \mathbb{1}_{F_0(x) \ge \phi}, \text{ as } n \to \infty, \quad n \text{ odd} \tag{9}$$

□

Then we have the following corollaries immediately.

Corollary 4.1. *Denote the probability distribution function corresponding to $F_n(x)$ as $f_n(x)$. Then*

1. *for even n, f_n converge to $\delta(x - x_1)$, where $F_0(x_1) = 1 - \phi$;*
2. *for odd n, F_n converge to $\delta(x - x_2)$, where $F_0(x_2) = \phi$;*

where $\delta(x)$ is the Dirac delta function.

Corollary 4.2. *Denote $\mathbb{E}(V_n)$ and $var(V_n)$ as the expectation and variance of V_n respectively.*

1. *For even n, $\mathbb{E}(V_n) \to x_1$ and $var(V_n) \to 0$, where $F_0(x_1) = 1 - \phi$.*
2. *For odd n, $\mathbb{E}(V_n) \to x_2$ and $var(V_n) \to 0$, where $F_0(x_2) = \phi$.*

Corollary 4.3. *If the PDF corresponding to F_0 is even, then $x_1 = -x_2$, where x_1, x_2 defined as above.*

4.3 Uniform distribution

Assume $V_0 \sim U([-1, 1])$, which is the setting in this game problem. Now $F_0(x) = (x + 1)/2$ and $F_1(x) = (x + 1)^2/4$ (if n odd). So

$$x_1 = 1 - 2\phi, \quad x_2 = 2\phi - 1 \tag{10}$$

and

1. For even n, $\mathbb{E}(V_n) \to 1 - 2\phi \approx -0.236$ and $var(V_n) \to 0$,
2. For odd n, $\mathbb{E}(V_n) \to 2\phi - 1 \approx 0.236$ and $var(V_n) \to 0$,

which is consistent with the numerical results.

As we can see in Figure 6, the theoretical and numerical results are consistent. The black lines mark the limit distribution and red lines(odd case) and green lines(even case) are the empirical distributions.

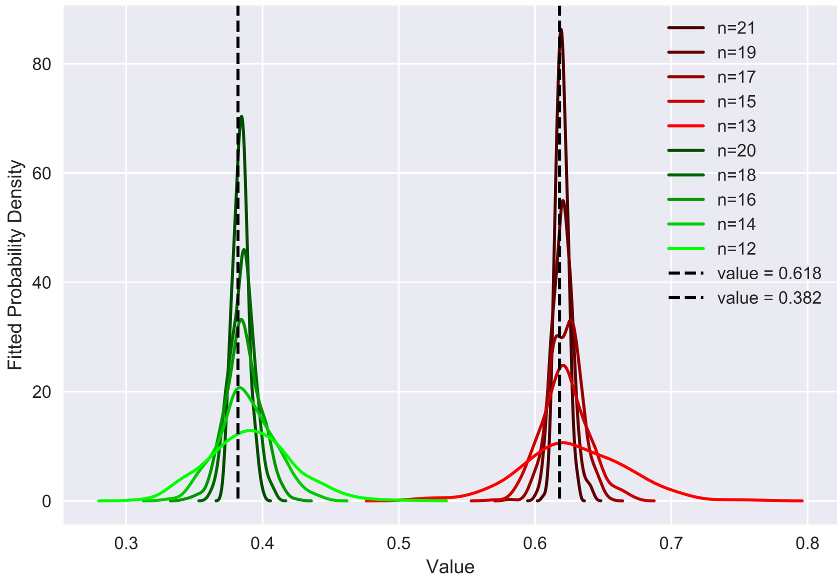


Figure 6: Theoretical and experiment results for uniform distribution.

4.4 Gaussian distribution

For the Gaussian distribution $V_0 \sim \mathcal{N}(0,1)$, we get

$$F_0(x) = \frac{1}{2}\left(1 + erf(\frac{x}{\sqrt{2}})\right), \quad F_1(x) = \frac{1}{4}\left(1 + erf(\frac{x}{\sqrt{2}})\right)^2 \text{ (if } n \text{ odd)}$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the error function. So

$$x_1 \approx -0.3003, \quad \approx 0.3003 \tag{11}$$

and

1. For even n, $\mathbb{E}(V_n) \to x_1 \approx -0.3003$ and $var(V_n) \to 0$,
2. For odd n, $\mathbb{E}(V_n) \to x_2 \approx 0.3003$ and $var(V_n) \to 0$,

As we can see in Figure 7, the theoretical and numerical results are consistent again.

5 Conclusion

In this project, we analyze the evolution of the value of the game, observe and prove it oscillates around zero and converges in subsequences, which indicates a stable advantage to playing last. The limits of the

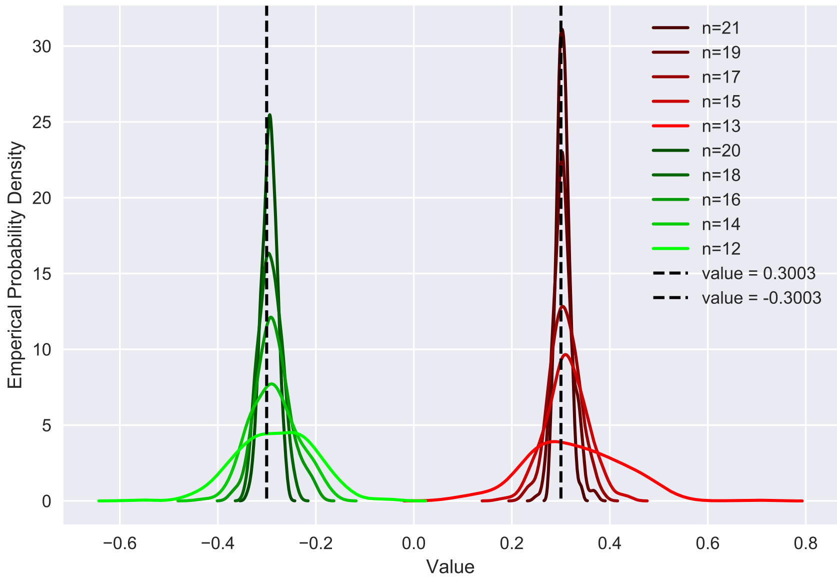


Figure 7: Theoretical and experiment results for Gaussian distribution.

game value are related to the golden ratio, which is very beautiful.

References

[1] Joel Spencer. Game Analysis and Project. 2018. link.