Basic Quantum Computing Algorithms and Their Implementation in Cirq

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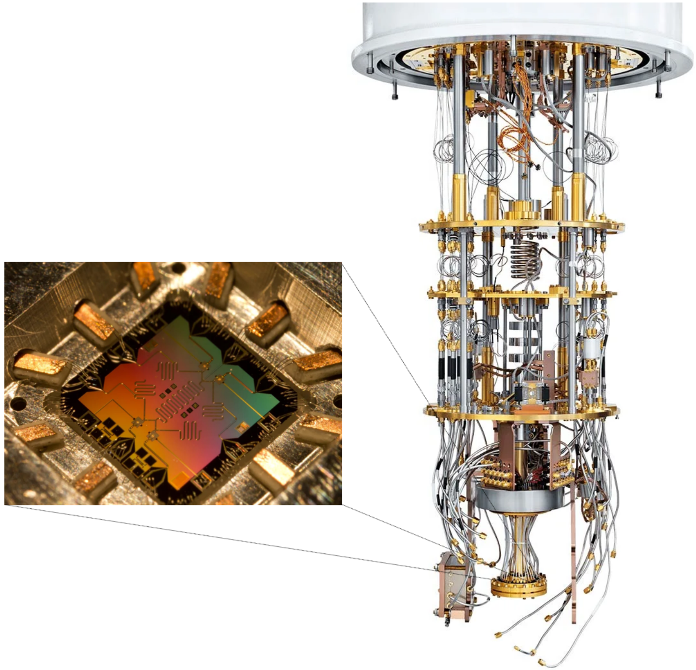
Part I

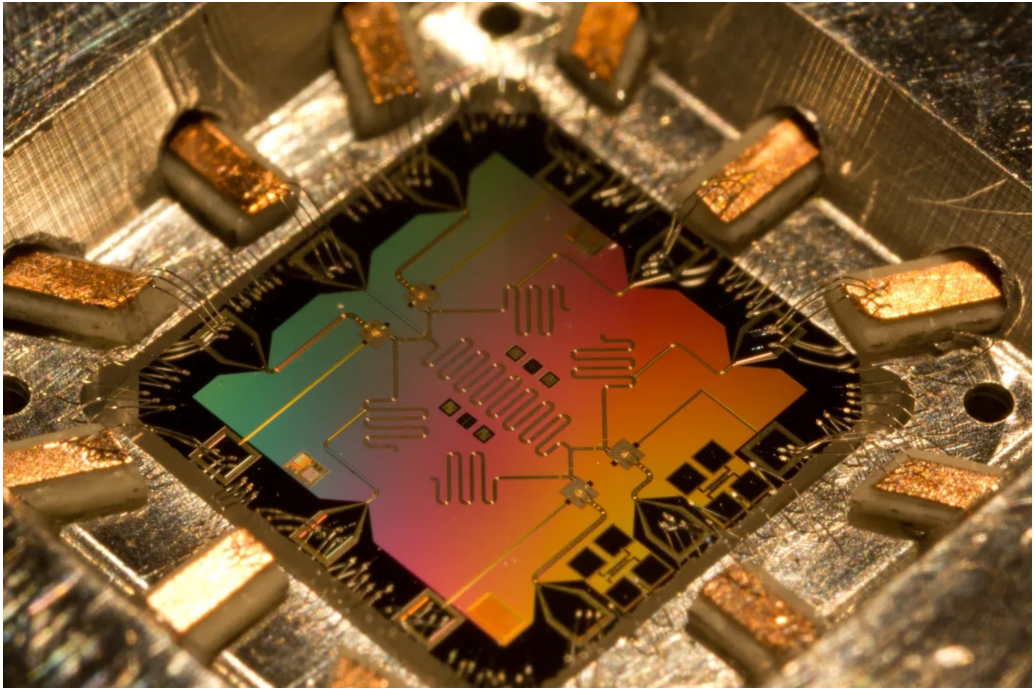
Introduction to Quantum Computing

Hardware

Superconducting technology:

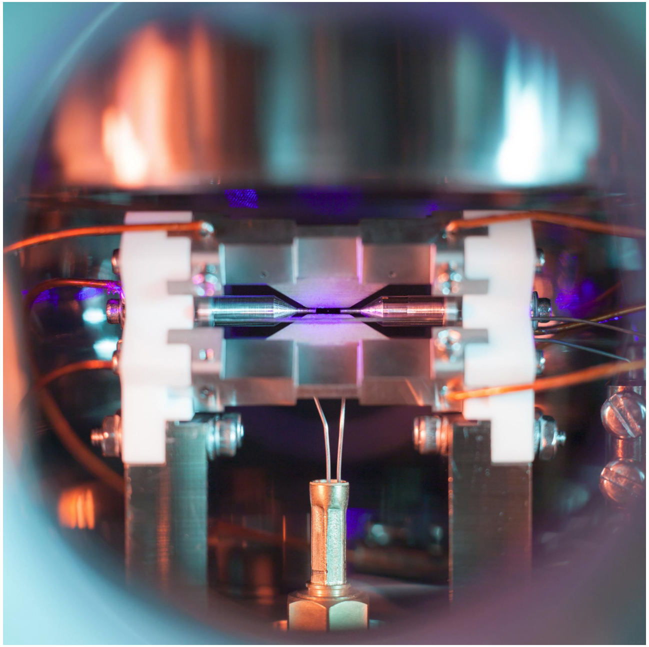


Hardware

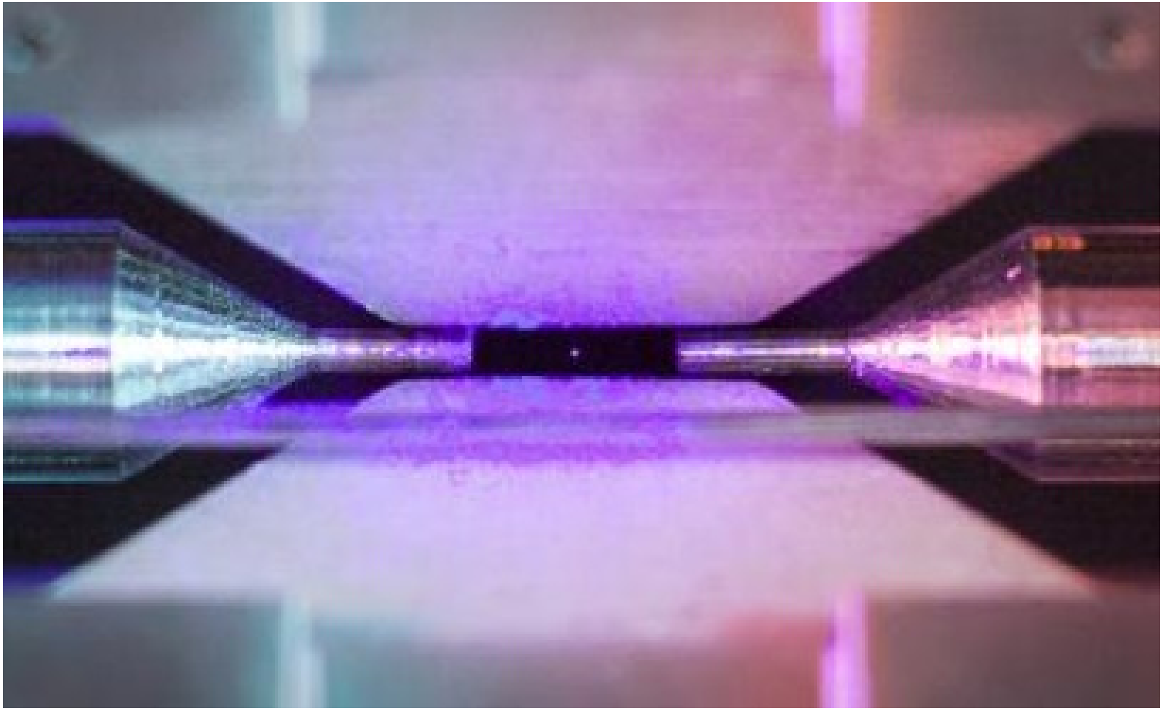


Hardware

Trapped-ion technology:



Hardware



Qubit

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$\alpha = \cos\frac{\theta}{2}$$

$$\beta = e^{i\phi}\sin\frac{\theta}{2} = (\cos\phi + i\sin\phi)\sin\frac{\theta}{2}$$

$$\Pr(|0\rangle) = |\alpha|^2 = \cos^2\frac{\theta}{2}$$

$$\Pr(|1\rangle) = |\beta|^2 = |e^{i\phi}|^2\sin^2\frac{\theta}{2} = \sin^2\frac{\theta}{2}$$

$$\Pr(|0\rangle) + \Pr(|1\rangle) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

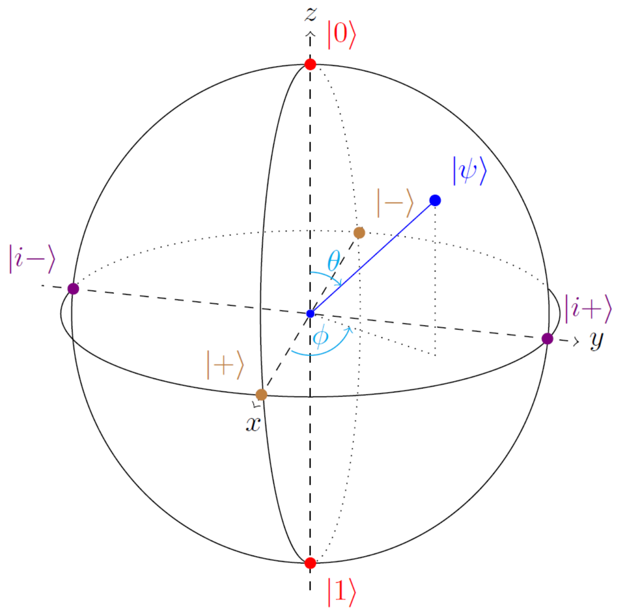


Figure. Bloch sphere.

Qubit

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$\alpha = \cos\frac{\theta}{2}$$

$$\beta = e^{i\phi}\sin\frac{\theta}{2} = (\cos\phi + i\sin\phi)\sin\frac{\theta}{2}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$$

$$|i+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + i\frac{1}{\sqrt{2}}\,|1\rangle$$

$$|i-\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - i\frac{1}{\sqrt{2}}\,|1\rangle$$

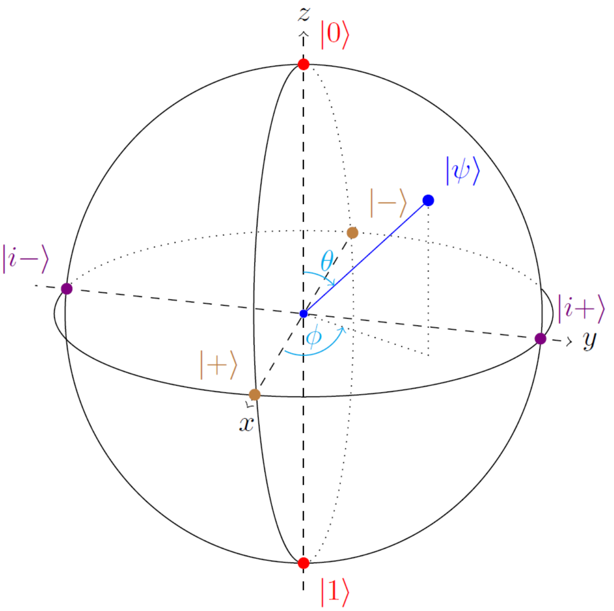


Figure. Bloch sphere.

1-qubit quantum gates

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}|0\rangle & |1\rangle\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \Rightarrow \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$|0\rangle = \begin{bmatrix}1 \\ 0\end{bmatrix} \quad |1\rangle = \begin{bmatrix}0 \\ 1\end{bmatrix}$$

$$X|\psi\rangle = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}\beta \\ \alpha\end{bmatrix}$$

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta \\ \alpha-\beta\end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -1\end{bmatrix} = |-\rangle$$

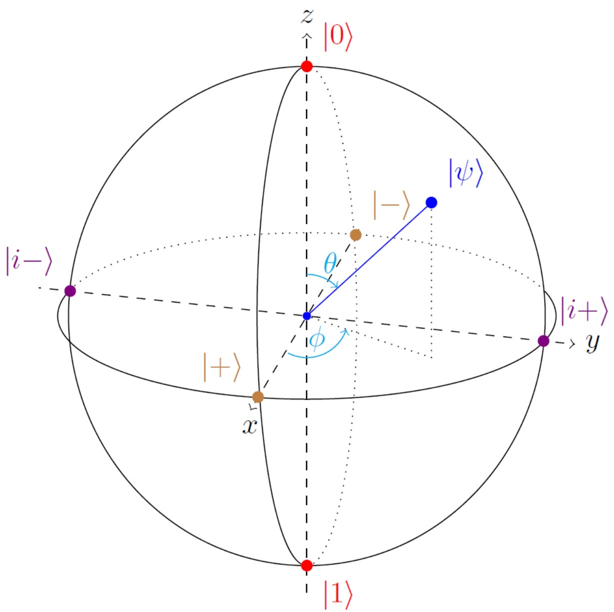


Figure. Bloch sphere.

1-qubit quantum gates

$$P(\lambda)\left|\psi\right\rangle = \begin{bmatrix}1 & 0\\ 0 & e^{i\lambda}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ e^{i\lambda}\beta\end{bmatrix}$$

$$Z\left|\psi\right\rangle = \begin{bmatrix}1 & 0\\ 0 & e^{i\pi}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ -\beta\end{bmatrix}$$

$$S\left|\psi\right\rangle = \begin{bmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{2}}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ i\beta\end{bmatrix}$$

$$T\left|\psi\right\rangle = \begin{bmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{4}}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ e^{i\frac{\pi}{4}}\beta\end{bmatrix} = \begin{bmatrix}\alpha\\ \frac{1}{\sqrt{2}}(1+i)\beta\end{bmatrix}$$

$$Z\left|+\right\rangle = \left|-\right\rangle \quad Z\left|-\right\rangle = \left|+\right\rangle \quad S\left|+\right\rangle = \left|i+\right\rangle$$

$$Z\left|i-\right\rangle = S|S\left|i-\right\rangle = T|T|T|T\left|i-\right\rangle = \left|i+\right\rangle$$

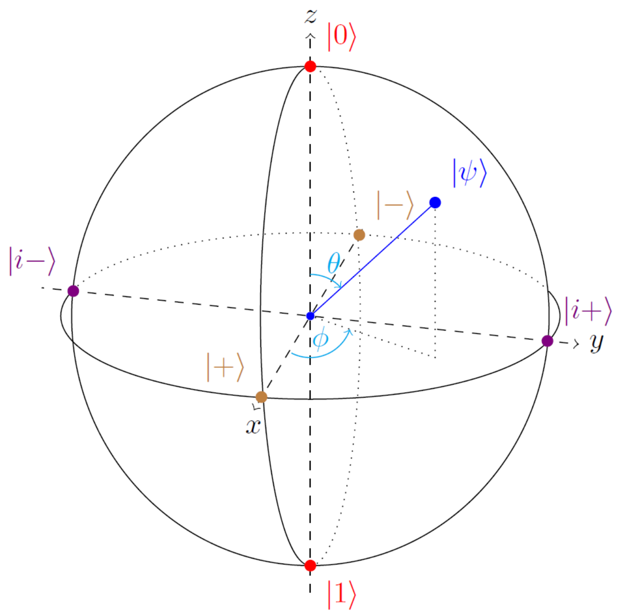
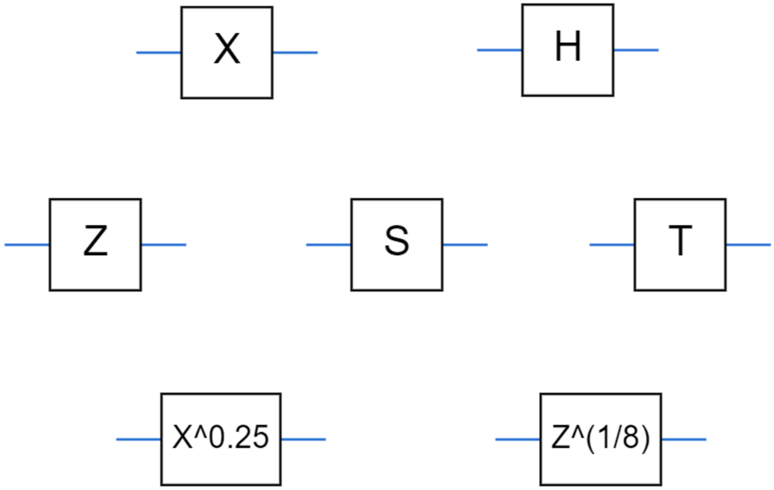


Figure. Bloch sphere.

Implementation in Cirq



2-qubit quantum gates

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{bmatrix}|00\rangle & |01\rangle & |10\rangle & |11\rangle\end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} \Rightarrow \begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix}$$

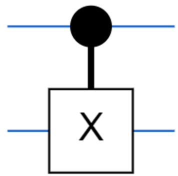
$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

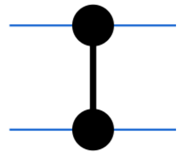
$$CX|\psi\rangle = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{11}\\ \alpha_{10}\end{bmatrix}$$

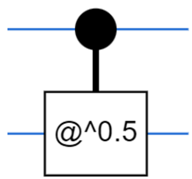
$$CP(\lambda)|\psi\rangle = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&e^{i\lambda}\end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ e^{i\lambda}\alpha_{11}\end{bmatrix}$$

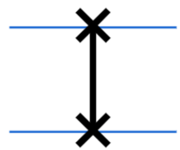
$$SWAP|\psi\rangle = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{10}\\ \alpha_{01}\\ \alpha_{11}\end{bmatrix}$$

Implementation in Cirq





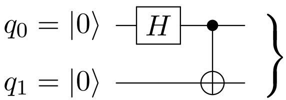




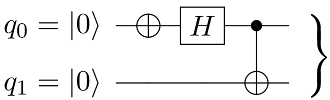
Part II

Quantum entanglement

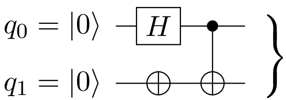
Bell states



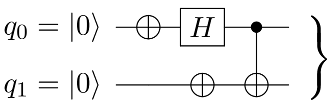
$$|\psi_e\rangle = CX|H\,|00\rangle = CX\left(\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|01\rangle\right) = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle = |\Phi^{+}\rangle$$



$$|\psi_e\rangle = CX|H\,|00\rangle = CX\left(\frac{1}{\sqrt{2}}\,|00\rangle - \frac{1}{\sqrt{2}}\,|01\rangle\right) = \frac{1}{\sqrt{2}}\,|00\rangle - \frac{1}{\sqrt{2}}\,|11\rangle = |\Phi^{-}\rangle$$

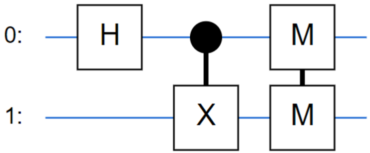


$$|\psi_e\rangle = CX|H\,|00\rangle = CX\left(\frac{1}{\sqrt{2}}\,|10\rangle + \frac{1}{\sqrt{2}}\,|11\rangle\right) = \frac{1}{\sqrt{2}}\,|10\rangle + \frac{1}{\sqrt{2}}\,|01\rangle = |\Psi^{+}\rangle$$

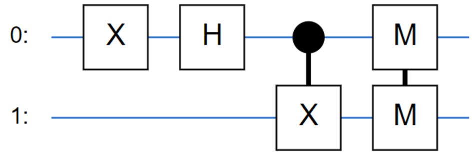


$$|\psi_e\rangle = CX|H\,|00\rangle = CX\left(\frac{1}{\sqrt{2}}\,|10\rangle - \frac{1}{\sqrt{2}}\,|11\rangle\right) = \frac{1}{\sqrt{2}}\,|10\rangle - \frac{1}{\sqrt{2}}\,|01\rangle = -\,|\Psi^{-}\rangle$$

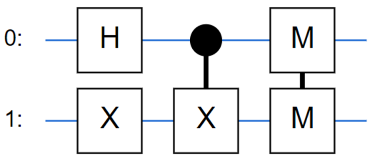
Implementation in Cirq



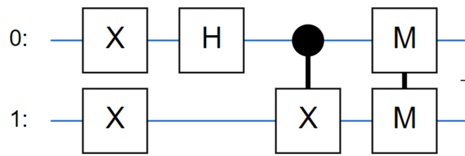
$|\Phi^{+}\rangle$



$|\Phi^{-}\rangle$



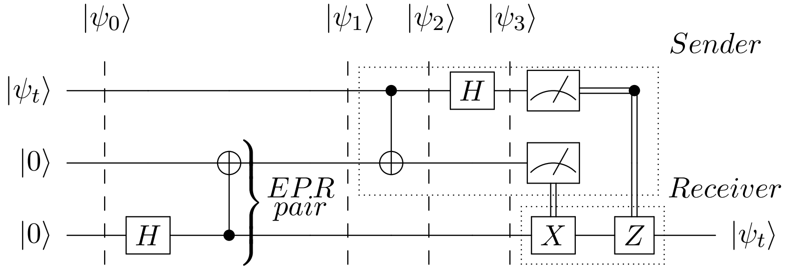
$|\Psi^{+}\rangle$



$-|\Psi^{-}\rangle$

Part III

Quantum teleportation



$$|\psi_t\rangle = \alpha_t |0\rangle + \beta_t |1\rangle \qquad |\psi_0\rangle = |\psi_t\rangle \otimes |00\rangle = \alpha_t |000\rangle + \beta_t |100\rangle$$

$$|\psi_1\rangle = \frac{\alpha_t}{\sqrt{2}} |000\rangle + \frac{\alpha_t}{\sqrt{2}} |011\rangle + \frac{\beta_t}{\sqrt{2}} |100\rangle + \frac{\beta_t}{\sqrt{2}} |111\rangle$$

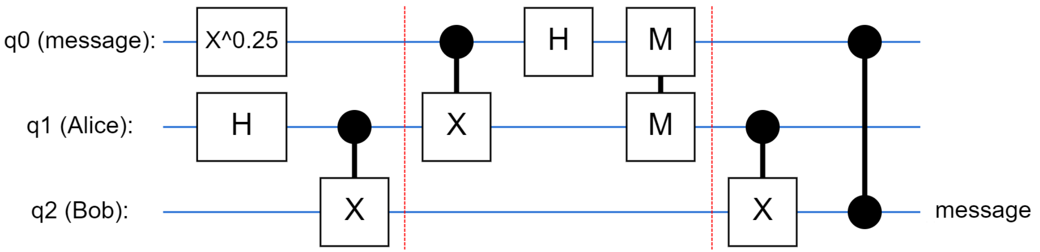
$$|\psi_2\rangle = \frac{\alpha_t}{\sqrt{2}} |000\rangle + \frac{\alpha_t}{\sqrt{2}} |011\rangle + \frac{\beta_t}{\sqrt{2}} |110\rangle + \frac{\beta_t}{\sqrt{2}} |101\rangle$$

$$|\psi_3\rangle = \frac{1}{2} |00\rangle \otimes (\alpha_t |0\rangle + \beta_t |1\rangle) + \frac{1}{2} |01\rangle \otimes (\alpha_t |1\rangle + \beta_t |0\rangle) +$$

$$+ \frac{1}{2} |10\rangle \otimes (\alpha_t |0\rangle - \beta_t |1\rangle) + \frac{1}{2} |11\rangle \otimes (\alpha_t |1\rangle - \beta_t |0\rangle) =$$

$$= \frac{1}{2} |00\rangle \otimes |\psi_t\rangle + \frac{1}{2} |01\rangle \otimes |\overline{\psi_t}\rangle + \frac{1}{2} |10\rangle \otimes \left|\psi_t^\dagger\right\rangle + \frac{1}{2} |11\rangle \otimes \left|\overline{\psi_t^\dagger}\right\rangle$$

Implementation in Cirq



Part IV

Bernstein-Vazirani + Deutch-Jozsa algorithm

Bernstein-Vazirani algorithm

The problem statement: Find the secret string s if implemented function f is of the form $f(x) = x \cdot s$.

$$|0\rangle^n \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \xrightarrow{f} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle$$

$$\xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)+x\cdot y} |y\rangle = |s\rangle$$

$$f(x) + x\cdot y = x\cdot s + x\cdot y = x\cdot(s\oplus y) = \begin{cases} 0 & (s=y) \\ 0,1,0,1\ldots & (s\neq y) \end{cases}$$

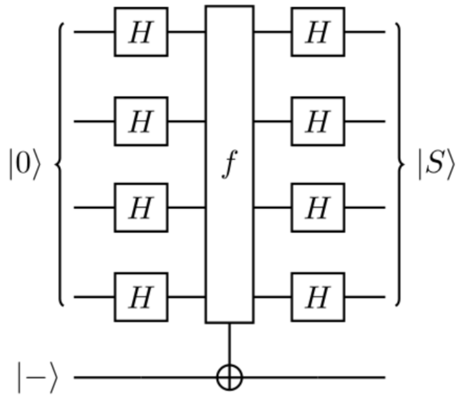
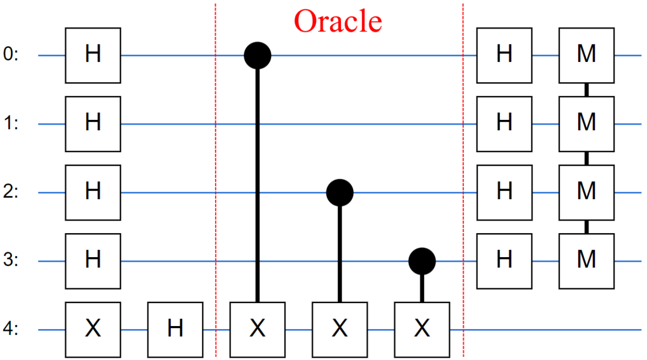


Figure. Bernstein-Vazirani circuit.

Implementation in Cirq



Deutch-Jozsa algorithm

The problem statement: Decide whether the implemented function f is constant or balanced.

$$|0\rangle^n \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \xrightarrow{f} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle$$

$$\xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)+x\cdot y} |y\rangle \; = \; |s\rangle$$

$$|s\rangle \begin{cases} = 0 & \rightarrow f \text{ is constant} \\ \neq 0 & \rightarrow f \text{ is balanced} \end{cases}$$

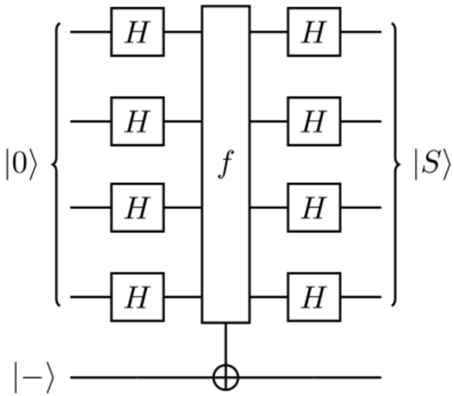
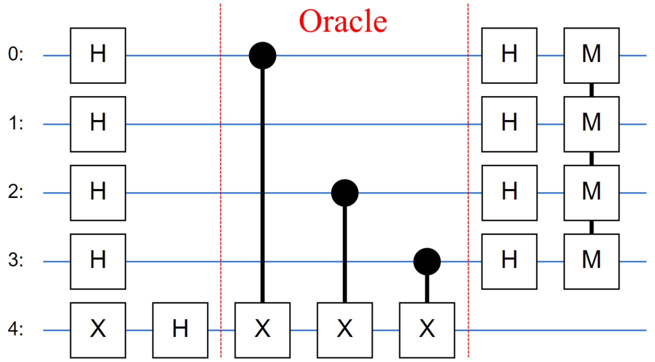


Figure. Deutch-Jozsa circuit.

Implementation in Cirq



Part V

Simon's algorithm

Simon's algorithm

The problem statement: Decide whether the implemented function f is periodic or not.

$$|0\rangle^{\otimes n}|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle |0\rangle^{\otimes n}$$

$$\xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle |f(x)\rangle$$

$$\xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \sum_{x\in\{0,1\}^n} (-1)^{x\cdot y} |y\rangle |f(x)\rangle$$

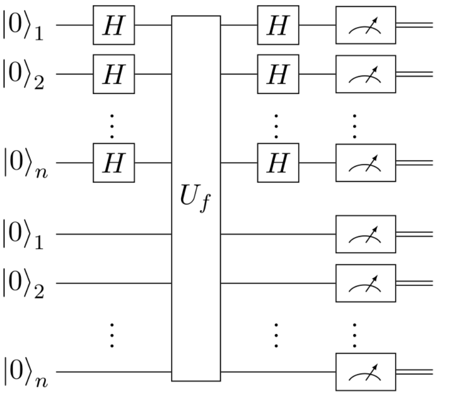


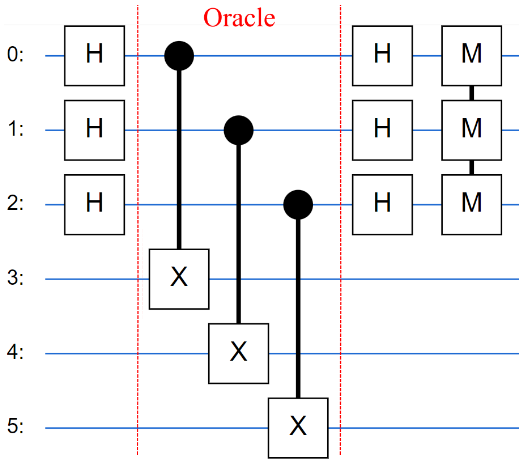
Figure. Simon's circuit.

Quantum state after measuring the lower register:

$$f \text{ is not periodic } \rightarrow \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x_1\cdot y} |y\rangle |f(x_1)\rangle$$

$$f \text{ is periodic } \rightarrow \frac{1}{\sqrt{2^{n+1+\dots}}} \sum_{y\in\{0,1\}^n} \left[(-1)^{x_1\cdot y}+(-1)^{x_2\cdot y}+\dots\right] |y\rangle |f(x_1)\rangle$$

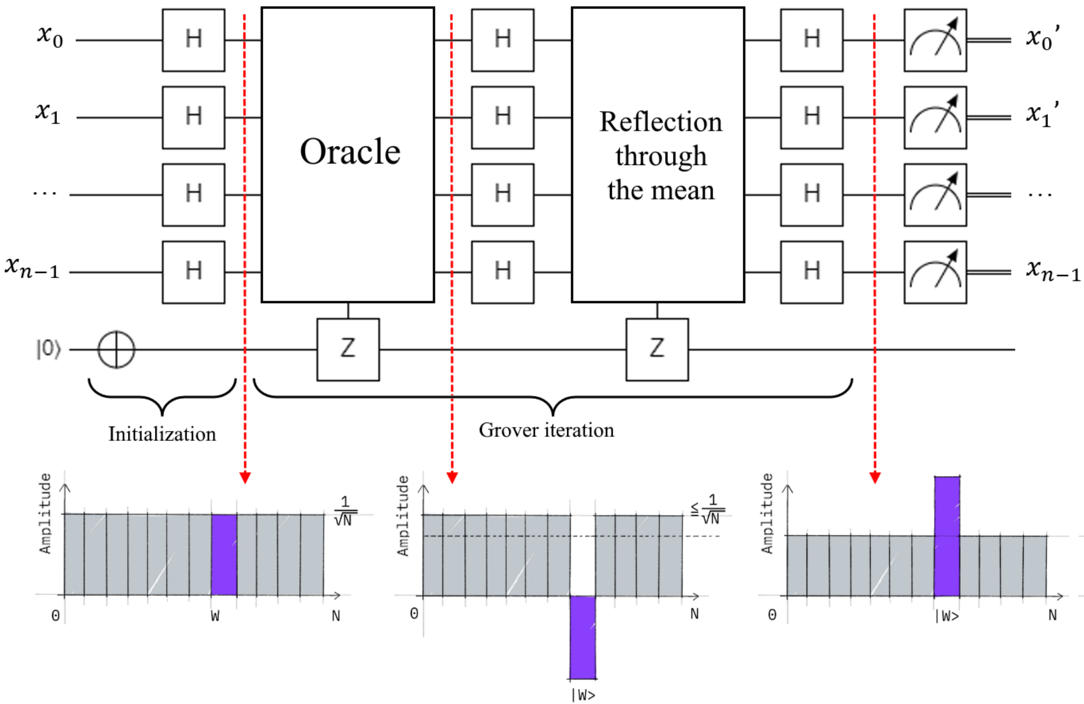
Implementation in Cirq



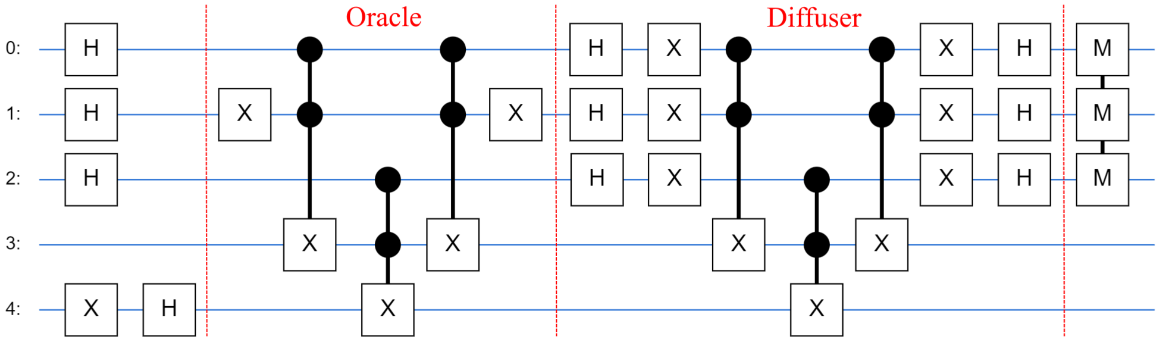
Part VI

Grover's algorithm

Grover's algorithm



Implementation in Cirq



Part VII

Quantum Fourier transform

Quantum Fourier transform

$$\text{IDFT: } x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot e^{2\pi i \frac{kn}{N}}$$

$$\text{QFT}\,|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}}\,|y\rangle$$

$$\frac{y}{N} = \frac{y_1 y_2 \dots y_n}{2^n} = \sum_{k=1}^{n}\frac{y_k}{2^k} \quad \longrightarrow \quad \text{QFT}\,|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i x \sum_{k=1}^{n}\frac{y_k}{2^k}}\,|y_1 y_2 \dots y_n\rangle$$

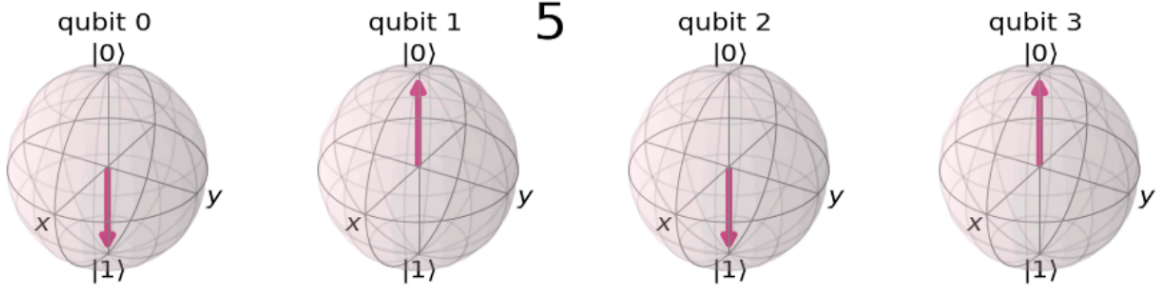
$$\text{QFT}\,|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}\prod_{k=1}^{2^n} e^{2\pi i x \frac{y_k}{2^k}}\,|y_1 y_2 \dots y_n\rangle$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\text{QFT}\,|x\rangle = \frac{1}{\sqrt{2^n}}\Big(|0\rangle + e^{i\pi x}\,|1\rangle\Big) \otimes \Big(|0\rangle + e^{i\frac{\pi}{2}x}\,|1\rangle\Big) \otimes \Big(|0\rangle + e^{i\frac{\pi}{4}x}\,|1\rangle\Big) \otimes \cdots \cdots \otimes \Big(|0\rangle + e^{i\frac{\pi}{2^{n-1}}x}\,|1\rangle\Big)$$

Quantum Fourier transform

5



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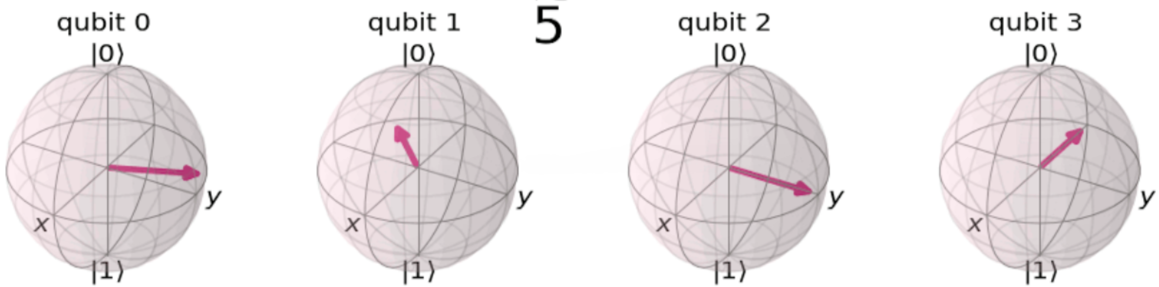
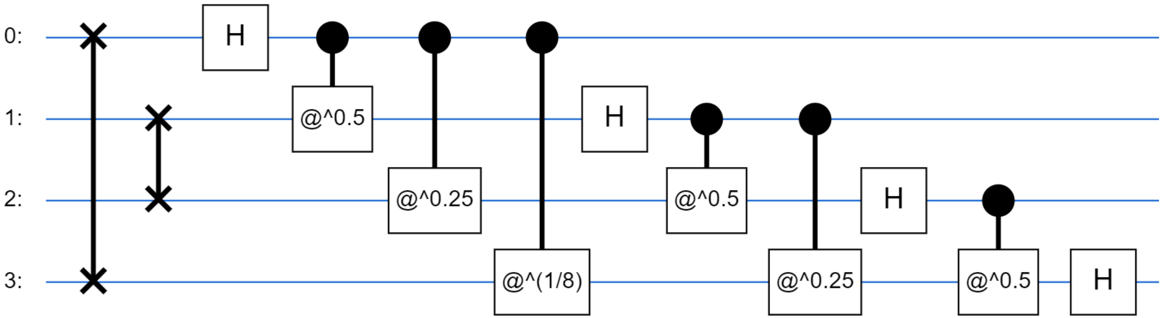


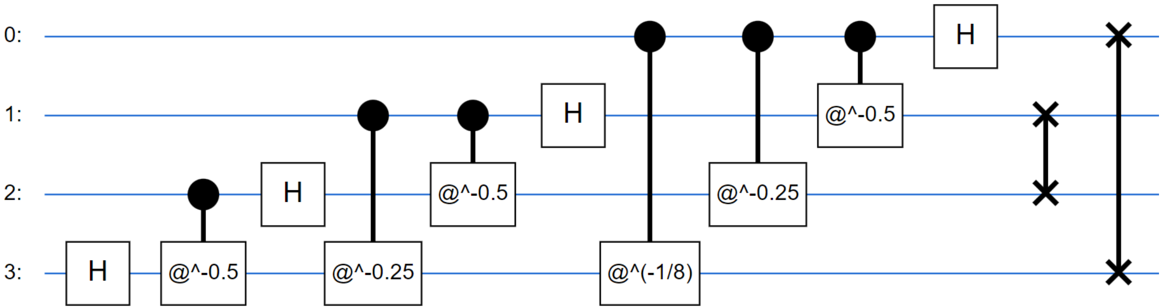
Implementation in Cirq

Direct QFT:



Implementation in Cirq

Inverse QFT:



Part VIII

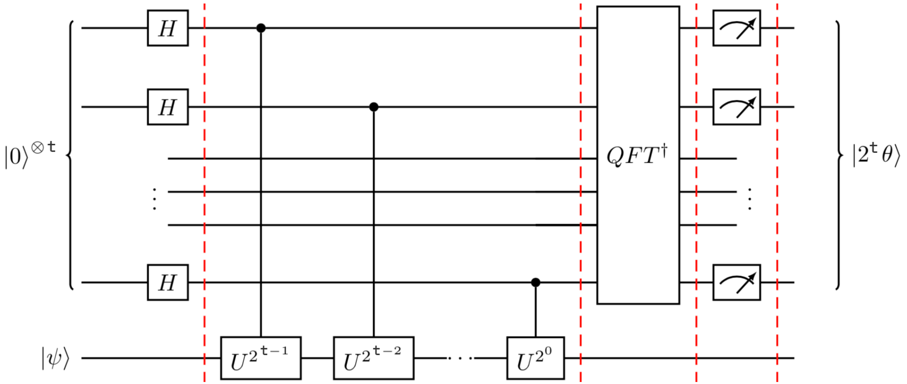
Quantum phase estimation

Quantum phase estimation

The problem statement:

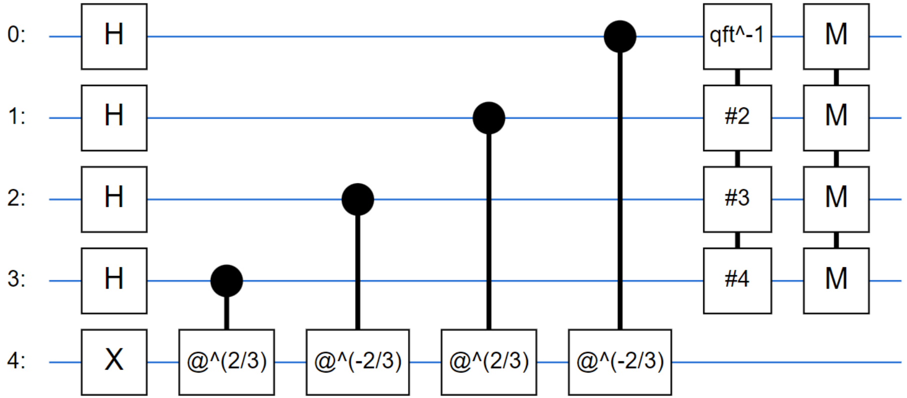
Estimate the phase of an eigenvalue $e^{2\pi i\theta}$ of a unitary operator U, provided with the corresponding eigenstate ψ:

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$



$$|0\rangle^{\otimes t} \to \frac{1}{\sqrt{2^t}}\Big(|0\rangle+|1\rangle\Big)^{\otimes t} \to \frac{1}{\sqrt{2^t}}\Big(|0\rangle+e^{2\pi i\theta 2^{t-1}}|1\rangle\Big)\otimes\Big(|0\rangle+e^{2\pi i\theta 2^{t-2}}|1\rangle\Big)\otimes\cdots\otimes\Big(|0\rangle+e^{2\pi i\theta 2^{0}}|1\rangle\Big) = \mathrm{QFT}\left|2^t\theta\right\rangle$$

Implementation in Cirq



Part IX

Shor's algorithm

Shor's algorithm

The problem statement:

Find factors P, R of number N.

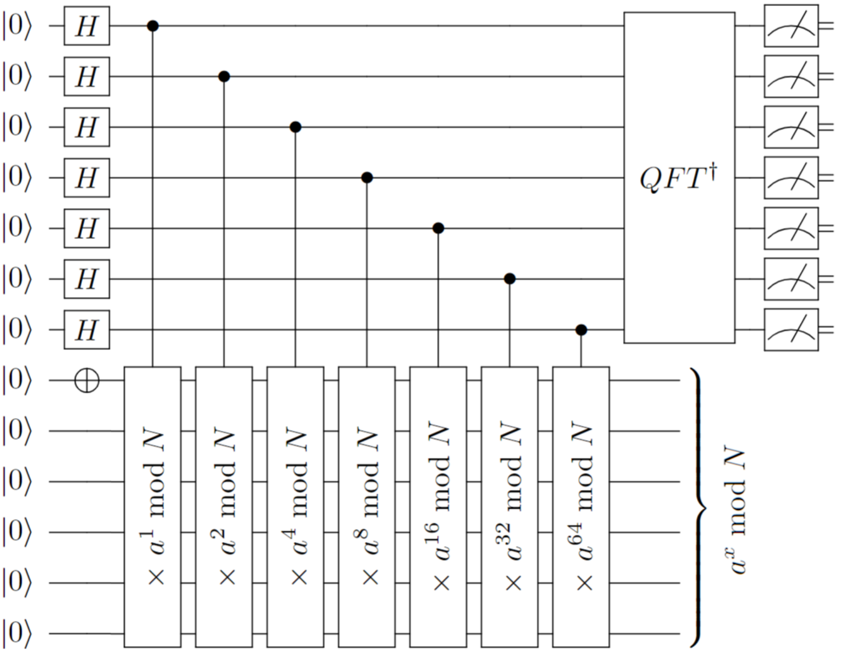
Shor's algorithm procedure:

1. Pick a random integer number a such that: $1 < a < N$.
2. If $\gcd(a, N) \neq 1$ then $P = a$ and $R = N/a$.
3. Otherwise, find the period r of function $f(x) = a^x \bmod N$.
4. If r is odd then go back to step 1 and choose different a.
5. Otherwise, factors $P, R = \gcd(a^{r/2} \pm 1, N)$.

A quantum computer can be used for step 3, in which it is necessary to create a quantum circuit implementing the modular exponentiation function $f(x) = a^x \bmod N$ and use this circuit instead of the U operator in the quantum phase estimation circuit.
The resulting circuit is called a period-finder circuit and the measured result at the output can then be used to determine the searched period.

Shor's algorithm

Period-finder cirquit:



Implementation in Cirq

Implementation of the function $g(y) = (y \times 6) \bmod 35$ (on the left) and period-finder circuit (on the right) designed to find the period of the function $f(x) = 6^x \bmod 35$: