MULT90063 Introduction to Quantum Computing

Assignment 1

Due: 5pm, 6th April 2023

Instructions: Work on your own, attempt all questions. Submit your completed written work electronically as a pdf (no other formats accepted) to LMS, with name and student number on the front, on or before the due date. Please show all working. The QUI circuits you create for this project should be saved (and shared) with the indicated filenames and a link to the relevant circuits included in your question answer.

Total marks = 30. Number of questions = 6

Question 1 [5 marks = 1 + 3 + 1]

(a) Consider the following single qubit state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ where the amplitudes are given in polar form as:

$$|a_0| = 0.383, \theta_0 = \pi$$

$$|a_1| = 0.924, \theta_1 = -\pi/2$$

As per the definition given in the lectures, convert to Bloch sphere form

$$|\psi\rangle = \cos\frac{\theta_\mathrm{B}}{2}|0\rangle + \sin\frac{\theta_\mathrm{B}}{2}e^{i\phi_\mathrm{B}}|1\rangle,$$

specifying the Bloch angles θ_B and ϕ_B. Plot on the Bloch sphere.

(b) In the following we have written single-qubit operations in matrix form, corresponding to rotations by angle θ_R about X, Y or Z axes (note these expressions include a global phase of $\pi/2$):

$$R_X(\theta_\mathrm{R}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_\mathrm{R}}{2} & -i\sin\frac{\theta_\mathrm{R}}{2} \\ -i\sin\frac{\theta_\mathrm{R}}{2} & \cos\frac{\theta_\mathrm{R}}{2} \end{bmatrix}, \quad R_Y(\theta_\mathrm{R}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_\mathrm{R}}{2} & -\sin\frac{\theta_\mathrm{R}}{2} \\ \sin\frac{\theta_\mathrm{R}}{2} & \cos\frac{\theta_\mathrm{R}}{2} \end{bmatrix},$$

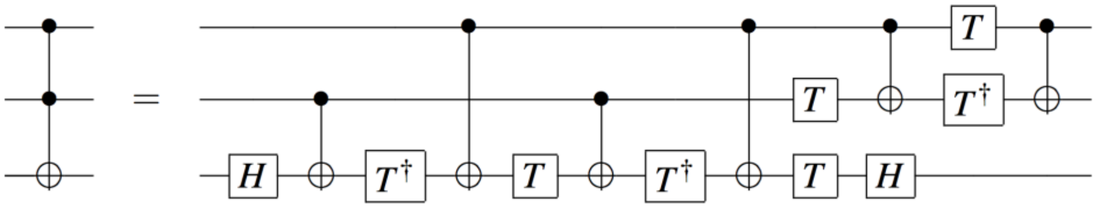
$$R_Z(\theta_\mathrm{R}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_\mathrm{R}}{2} - i\sin\frac{\theta_\mathrm{R}}{2} & 0 \\ 0 & \cos\frac{\theta_\mathrm{R}}{2} + i\sin\frac{\theta_\mathrm{R}}{2} \end{bmatrix}.$$

Explain how these operators are related to the familiar Pauli matrices, X, Y and Z given in lectures.

(c) In the QUI, consider a rotation around the (1,1,0)-axis by $\pi/2$ with global phase set to zero. Use the QUI to perform the rotation to the computational states $|0\rangle$ and $|1\rangle$ individually and hence determine the 2x2 matrix representation for this rotation. Submit appropriate screenshots with your answers. Explain quantitatively (using diagrams) how the operation moves the state on the Bloch Sphere.

Question 2 [4 marks = 1.5 + 2.5]

Below is shown the decomposition of the 3-qubit Toffoli gate into an equivalent circuit comprising only 1- qubit and 2-qubit gates.



(a) Write down the action of H, T and $T^{\dagger}$ on an arbitrary single qubit state in ket notation.

(b) Starting from the initial 3-qubit state $|111\rangle$ (ordering: top-middle-lower), and staying in ket notation, trace through each step of the circuit to verify the output produced.

Question 3 [6 marks = 2 + 1 + 3]

(a) Construct an oracle with five qubits, which identifies (i.e. applies a phase to) numbers in the set (1, 3, 7, 22, and 25). You may use multiply controlled operations.

Optimize your circuit to use as few operations as possible. Briefly describe your implementation of this circuit including any optimizations which you have made.
Save and submit this circuit as "<student number> Assignment 1 Q3a".

(b) Use the oracle which you constructed in part (a) to implement a single iteration of Grover's algorithm, starting from an equal superposition, and calculate the probability of measuring a number which is in the given set of numbers at the output. Show your working.

(c) How many iterations of Grover's algorithm are required to give the highest probability of success using your oracle? Use the geometric picture and show your working.

Question 4 [3 marks]

A four-qubit GHZ state is given by:

$$|\mathrm{GHZ}\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}$$

Using QUI, create a circuit which constructs the four qubit GHZ-state, using only single qubit and two-qubit operations. Optimize the circuit as much as possible. Write down the state at each time-step through the circuit.
Save and submit this circuit as "<student number> Assignment 1 Q4".

Question 5 [8 marks = 2 + 3 + 3]

In this question we will investigate the famous Clauser-Horne-Shimony-Holt (CHSH) inequality from the perspective of quantum information. (The CHSH is closely related closely related to Bell's Theorem, leading to much debate about our everyday perception of locality and realism).

Consider Alice and Bob, each having their own measurement apparatus. Alice can measure either one of two observables *Q* or *R*, while Bob can measure either of the observables *S* or *T*.

(a) Suppose that the observables *Q, R, S,* and *T* take values $\{\pm 1\}$. Consider the quantity composed of combined operators: *QS* + *RS* + *RT* – *QT*. Note that there is an implied tensor product in these combined operators. In this case, what is the upper bound on the expectation (mean value) of the quantity *QS* + *RS* + *RT* – *QT*? This is called Bell's inequality.

(b) Next we consider measuring the expectation value of the quantity *QS* + *RS* + *RT* – *QT* when Alice and Bob have access to quantum states. Consider the case when Alice and Bob have access to the observables comprising Pauli operators as:

$$Q = Z \qquad S = -\frac{Z + X}{\sqrt{2}}$$

$$R = X \qquad T = \frac{Z - X}{\sqrt{2}}$$

and the individual quantum states:

$$|\psi_{\mathrm{Alice}}\rangle = \cos(\theta_A)|0\rangle + \sin(\theta_A)|1\rangle$$
$$|\psi_{\mathrm{Bob}}\rangle = \cos(\theta_B)|0\rangle - \sin(\theta_B)|1\rangle.$$

What are the eigenvalues for each of the observables *Q, R, S,* and *T*? What is the expectation value $\langle QS + RS + RT - QT\rangle$ for the product state $|\psi_{\mathrm{Alice}}\rangle \otimes |\psi_{\mathrm{Bob}}\rangle$? What is the max possible value?

(c) Now consider the case where Alice and Bob share the state (Alice qubit first, Bob qubit second): $|\psi_{\mathrm{AB}}\rangle = \cos(\phi)|01\rangle - \sin(\phi)|10\rangle$. For what values of ϕ is the quantity $\langle QS + RS + RT - QT\rangle$ maximum (given the observables in part (b))? What is the significance of such ϕ for the state $|\psi_{\mathrm{AB}}\rangle$? How does this value for $\langle QS + RS + RT - QT\rangle$ compare to the value in part (a)? This is the basis of the CHSH inequality.

Question 6 [4 marks = 1.5 + 2.5]

Consider a function *f* (*x*) where *x* is a *n*-bit input. The function returns "1" if there are strictly more bits set to one in the input than there are bits set to zero, otherwise it returns "0".

(a) Design a quantum circuit that implements *f* (*x*) for *x* represented by *n*=3 qubits (in ket notation, define the left-most qubit corresponding to the most significant bit). Draw the quantum circuit using one-qubit gates, two-qubit gates and as few Toffoli gates as possible. Explain its working. You may use ancilla qubits.

(b) Work out a 5-bit quantum implementation of *f* (*x*) – i.e. if there are more 1's than 0's in the input, your function should return 1, otherwise returns 0. Draw a schematic quantum circuit which demonstrates the working of this function. You do not need to draw the full quantum circuit, a clear strategy that demonstrates the working for 5-qubit inputs is sufficient. You must use only one, two and three-qubit gates. Hint: You may use your 3-bit implementation from part (a) as a black box.