MULT90063: Assignment 1

Hassan Andrabi (913838)

April 4, 2023

Problem 1a: Bloch sphere form. Consider the following single qubit state $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$ where the amplitudes are given in polar form as: $(|a_0| = 0.383, \theta_0 = \pi)$ and $(|a_1| = 0.924, \theta_0 = -\pi/2)$ As per the definition given in the lectures, convert to Bloch sphere form specifying the Bloch angles θ_B and ϕ_B. Plot on the Bloch sphere.

Answer. The Bloch sphere form of a quantum state $|\psi\rangle$ is parameterised by the angles θ_B and ϕ_B, which respectively represent the colatitude with respect to the z-axis, and the longitude with respect to the x-axis.

$$\begin{aligned}
|\psi\rangle &= 0.383 \cdot e^{i\pi} |0\rangle + 0.924 \cdot e^{-i\frac{\pi}{2}} |1\rangle \\
&= e^{i\pi}(0.383 |0\rangle + 0.924 \cdot e^{-i\frac{3\pi}{2}} |1\rangle) \\
&= \cos\left(\frac{0.7498\pi}{2}\right) |0\rangle + \sin\left(\frac{0.7498\pi}{2}\right) e^{-i\frac{3\pi}{2}} |1\rangle
\end{aligned}$$

Thus, $\theta_B = 0.7498\pi$ and $\phi_B = -\frac{3\pi}{2}$. $|\psi\rangle$ is visualised on the Bloch sphere below

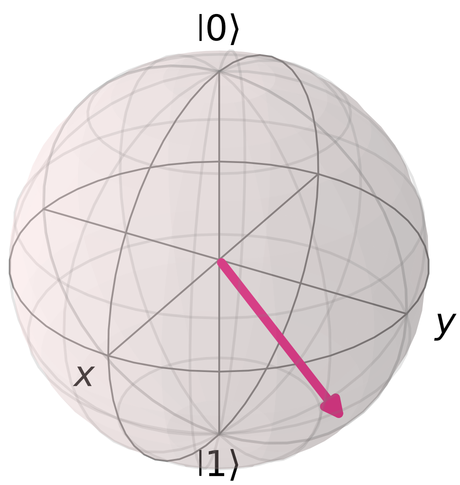

Figure 1: Bloch Sphere representation of $|\psi\rangle = \cos\left(\frac{0.7498\pi}{2}\right) |0\rangle + \sin\left(\frac{0.7498\pi}{2}\right) e^{-i\frac{3\pi}{2}} |1\rangle$

Problem 1b: General rotation. Consider single-qubit operations in matrix form, corresponding to rotations by angle θ_R about X, Y or Z axes. Explain how these operators are related to the familiar Pauli matrices, X, Y and Z given in lectures.

Answer. The relation between the provided general rotation operators and Pauli matrices is demonstrated by the following expression, achieved through exponentiation of the Pauli vector:

$$e^{i\theta(\hat{n}\cdot\sigma)} = I\cos(\theta) + i(\hat{n}\cdot\sigma)\sin(\theta), \tag{1}$$

where $(\sigma = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z})$ is the Pauli vector, and $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$ is a unit vector.

The above equation (1) can be derived using taylor series expansion of $e^{i\theta(\hat{n}\cdot\sigma)}$.

$$\begin{aligned} e^{i\theta(\hat{n}\cdot\sigma)} &= I + i\theta\hat{n}\cdot\sigma - \frac{\theta^2}{2!}(\hat{n}\cdot\sigma)^2 - i\frac{\theta^3}{3!}(\hat{n}\cdot\sigma)^3 + \cdots + i^n\frac{\theta^n}{n!}(\hat{n}\cdot\sigma)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(\theta\hat{n}\cdot\sigma)^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n(\theta\hat{n}\cdot\sigma)^{2n+1}}{(2n+1)!} \end{aligned}$$

Note that all even powers of the inner product of $\hat{n}$ and σ are equal to the identity matrix, such that $(\hat{n}\cdot\sigma)^{2n} = I$, for all $n \in \mathbb{Z}_0^+$, and all odd powers give $(\hat{n}\cdot\sigma)^{2n+1} = \hat{n}\cdot\sigma$, for all $n \in \mathbb{Z}_0^+$. Thus, we have

$$\begin{aligned} e^{i\theta(\hat{n}\cdot\sigma)} &= I\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!} + i(\hat{n}\cdot\sigma)\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!} \\ &= I\cos(\theta) + i(\hat{n}\cdot\sigma)\sin(\theta) \end{aligned}$$

Using this expression, one may recover the general rotation operators by substituting in appropriate values for the unit vector $\hat{n}$, and applying a global phase of $e^{i\frac{\pi}{2}}$ (let $\theta = \frac{\theta_R}{2}$, for consistency of notation).

$$\begin{aligned} \hat{n} = (1,0,0) &\rightarrow e^{i\theta(\hat{n}\cdot\sigma)} = I\cos(\theta) + iX\sin(\theta) = e^{i\frac{\pi}{2}}\begin{bmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{bmatrix} \\ \hat{n} = (0,1,0) &\rightarrow e^{i\theta(\hat{n}\cdot\sigma)} = I\cos(\theta) + iY\sin(\theta) = e^{i\frac{\pi}{2}}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ \hat{n} = (0,0,1) &\rightarrow e^{i\theta(\hat{n}\cdot\sigma)} = I\cos(\theta) + iZ\sin(\theta) = e^{i\frac{\pi}{2}}\begin{bmatrix} \cos(\theta) - i\sin(\theta) & 0 \\ 0 & \cos(\theta) + i\sin(\theta) \end{bmatrix} \end{aligned}$$

Problem 1c: Rotations on the Quantum User Interface (QUI). In the QUI, consider a rotation around the $(1,1,0)$-axis by $\pi/2$ with global phase set to 0. Use the QUI to perform the rotation to the computational states $|0\rangle$ and $|1\rangle$ individually and hence determine the 2x2 matrix representation for this rotation. Submit appropriate screenshots with your answers. Explain quantitatively (using diagrams) how the operation moves the state on the Bloch Sphere.

Answer. Individually rotating the computational state $|0\rangle$ and $|1\rangle$ by $\frac{\pi}{2}$ around the $(1,1,0)$-axis yields

$$\begin{aligned} R_{(1,1,0)}\left(\frac{\pi}{2}\right)|0\rangle &= \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \end{bmatrix} \\ R_{(1,1,0)}\left(\frac{\pi}{2}\right)|1\rangle &= \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix} \end{aligned}$$

Thus, we have

$$r_1 = \frac{1}{\sqrt{2}},\ \ r_2 = \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}),\ \ r_3 = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}),\ \ r_4 = \frac{1}{\sqrt{2}}$$

and the rotation operator $R_{(1,1,0)}\left(\frac{\pi}{2}\right)$ is given by

$$R_{(1,1,0)}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & 1 \end{bmatrix}$$

The action of $R_{(1,1,0)}\left(\frac{\pi}{2}\right)$ on a state results in a $\frac{\pi}{2}$ rotation around the axis $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$. Figure 3 provides a visual representation of this rotation on the Bloch sphere.

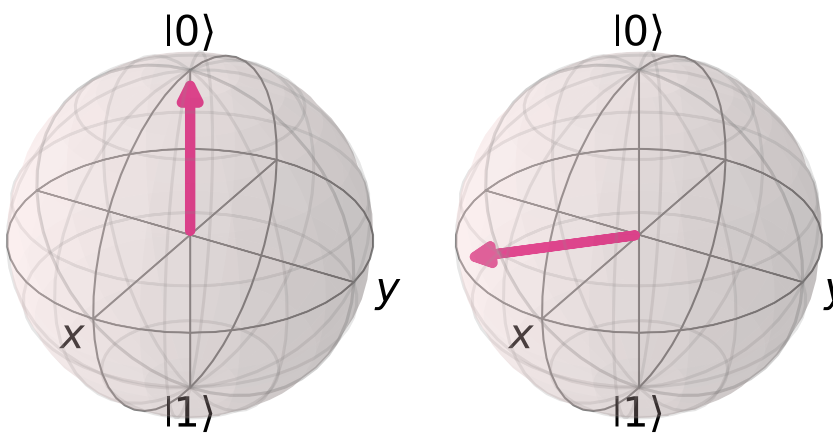

Figure 2: Visual representation of applying $R_{(1,1,0)}\left(\frac{\pi}{2}\right)$ to computational state $|0\rangle$ on the Bloch sphere. Left: original state $|0\rangle$. Right: resulting state after rotation $R_{(1,1,0)}\left(\frac{\pi}{2}\right)|0\rangle$.

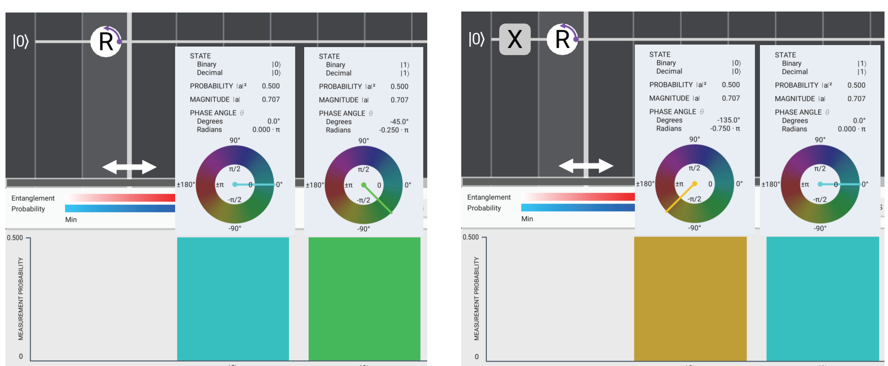

Figure 3: Screenshots from Quantum User Interface (QUI), applying $R_{(1,1,0)}\left(\frac{\pi}{2}\right)$ to computational states $|0\rangle$ (left) and $|1\rangle$ (right).

Problem 2a: Actions on single qubit. Write down the action of H, T and $T^\dagger$ on an arbitrary single qubit state in ket notation.

Answer. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ represent an arbitrary single qubit, where $|\alpha|^2 + |\beta|^2 = 1$.

The action of a Hadamard gate on an arbitrary qubit $|\psi\rangle$ is given by

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|0\rangle + \alpha|1\rangle - \beta ket1) \\ &= \frac{1}{\sqrt{2}}\Big(\alpha(|0\rangle + |1\rangle) + \beta(|0\rangle - |1\rangle)\Big) \\ &= \alpha|+\rangle + \beta|-\rangle \end{aligned}$$

The action of a T gate on an arbitrary qubit $|\psi\rangle$ is given by

$$\begin{aligned} T|\psi\rangle &= \alpha|0\rangle + \beta e^{i\frac{\pi}{4}}|1\rangle \\ &= \alpha|0\rangle + \frac{\beta}{\sqrt{2}}(1+i)|1\rangle \end{aligned}$$

The action of a $T^\dagger$ gate on an arbitrary qubit $|\psi\rangle$ is given by

$$\begin{aligned} T^\dagger|\psi\rangle &= \alpha|0\rangle + \beta e^{-i\frac{\pi}{4}}|1\rangle \\ &= \alpha|0\rangle + \frac{\beta}{\sqrt{2}}(1-i)|1\rangle \end{aligned}$$

Problem 2b: Toffoli gate decomposition. Starting from the initial 3-qubit state $|111\rangle$ (ordering: top-middle-lower), and staying in ket notation, trace through each step of the circuit to verify the output produced.

Answer. The circuit commences with quantum state $|\psi\rangle_0 = |1\rangle \otimes |1\rangle \otimes |1\rangle$. Step 1: apply a Hadamard gate to the third qubit

$$\begin{aligned} |\psi_1\rangle &= H_3|\psi_0\rangle \\ &= |1\rangle \otimes |1\rangle \otimes |-\rangle \end{aligned}$$

Step 2: apply a CNOT gate with the second qubit as the control, and the third qubit as the target

$$\begin{aligned} |\psi_2\rangle &= CNOT_{2,3}|\psi_1\rangle \\ &= |1\rangle \otimes |1\rangle \otimes - |-\rangle \end{aligned}$$

Step 3: apply a $T^\dagger$ gate to qubit 3

$$\begin{aligned} |\psi_3\rangle &= T_3^\dagger|\psi_2\rangle \\ &= |1\rangle \otimes |1\rangle \otimes -\frac{1}{\sqrt{2}}(|0\rangle - e^{-i\pi/4}|1\rangle) \end{aligned}$$

Step 4: apply a CNOT gate with the first qubit as the control, and the third qubit as the target

$$\begin{aligned} |\psi_4\rangle &= CNOT_{1,3}|\psi_3\rangle \\ &= |1\rangle \otimes |1\rangle \otimes \frac{1}{\sqrt{2}}(e^{-i\pi/4}|0\rangle - |1\rangle) \end{aligned}$$

Step 5: apply a T gate to qubit 3

$$\begin{aligned}
|\psi_5\rangle &= T_3 |\psi_4\rangle \\
&= |1\rangle \otimes |1\rangle \otimes \frac{1}{\sqrt{2}}(e^{-i\pi/4}|0\rangle - e^{i\pi/4}|1\rangle)
\end{aligned}$$

Step 6: apply a CNOT gate with the second qubit as the control, and the third qubit as the target

$$\begin{aligned}
|\psi_6\rangle &= CNOT_{2,3} |\psi_5\rangle \\
&= |1\rangle \otimes |1\rangle \otimes -\frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle - e^{-i\pi/4}|1\rangle)
\end{aligned}$$

Step 7: apply a $T^\dagger$ gate to qubit 3

$$\begin{aligned}
|\psi_7\rangle &= T_3^\dagger |\psi_6\rangle \\
&= |1\rangle \otimes |1\rangle \otimes -\frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle - e^{-i\pi/2}|1\rangle)
\end{aligned}$$

Step 8: apply a CNOT gate with the first qubit as the control, and the third qubit as the target

$$\begin{aligned}
|\psi_8\rangle &= CNOT_{1,3} |\psi_7\rangle \\
&= |1\rangle \otimes |1\rangle \otimes \frac{1}{\sqrt{2}}(e^{-i\pi/2}|0\rangle - e^{i\pi/4}|1\rangle)
\end{aligned}$$

Step 9: apply T gates to qubits 2 and 3

$$\begin{aligned}
|\psi_9\rangle &= T_2 T_3 |\psi_8\rangle \\
&= |1\rangle \otimes e^{i\pi/4}|1\rangle \otimes \frac{1}{\sqrt{2}}(e^{-i\pi/2}|0\rangle - e^{i\pi/2}|1\rangle) \\
&= |1\rangle \otimes e^{i\pi/4}|1\rangle \otimes \frac{e^{-i\pi/2}}{\sqrt{2}}(|0\rangle - e^{i\pi}|1\rangle) \\
&= |1\rangle \otimes e^{i\pi/4}|1\rangle \otimes e^{-i\pi/2}|+\rangle
\end{aligned}$$

Step 10: apply a CNOT gate with the first qubit as the control, and the second qubit as the target, and apply a Hadamard gate to qubit 3

$$\begin{aligned}
|\psi_{10}\rangle &= CNOT_{1,2} H_3 |\psi_9\rangle \\
&= |1\rangle \otimes e^{i\pi/4}|0\rangle \otimes e^{-i\pi/2}|0\rangle
\end{aligned}$$

Step 11: apply a T gate to qubit 1, and a $T^\dagger$ gate to qubit 2

$$\begin{aligned}
|\psi_{11}\rangle &= T_1 T_2^\dagger |\psi_{10}\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{i\pi/4}|0\rangle \otimes e^{-i\pi/2}|0\rangle
\end{aligned}$$

Step 12: apply a CNOT gate with the first qubit as the control, and the second qubit as the target

$$\begin{aligned}
|\psi_{12}\rangle &= CNOT_{1,2}\,|\psi_{11}\rangle \\
&= e^{i\pi/4}\,|1\rangle \otimes e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/2}\,|0\rangle \\
&= e^{i\pi/4}e^{i\pi/4}e^{-i\pi/2}\,|110\rangle \\
&= |110\rangle
\end{aligned}$$

Problem 3a: Oracle construction. Construct an oracle with five qubits, which identifies (i.e. applies a phase to) numbers in the set $(1, 3, 7, 22, 25)$. You may use multiply controlled operations. Optimise your circuit to use as few operations as possible. Briefly describe your implementation of this circuit including any optimisations which you have made.

Answer. [Insert Answer]

Problem 3b: Grover's algorithm (single iteration). Use the oracle which you constructed in part (a) to implement a single iteration of Grover's algorithm, starting from an equal superposition, and calculate the probability of measuring a number which is in the given set of numbers at the output. Show your working.

Answer. [Insert Answer]

Problem 3c: Grover's algorithm (optimal iterations). How many iterations of Grover's algorithm are required to give the highest probability of success using your oracle? Use the geometric picture and show your working.

Answer. [Insert Answer]

Problem 4: Four-qubit GHZ state. Using QUI, create a circuit which constructs the four qubit GHZ-state, using only single qubit and two-qubit operations. Optimise the circuit as much as possible. Write down the state at each time-step through the circuit.

Answer. [Insert Answer]

Problem 5a: CHSH inequality (classical states). Suppose that Q, R, S, and T take values $\{\pm 1\}$. Consider the quantity composed of combined operators: $QS+RS+RT-QT$. Note that there is an implied tensor product in these combined operators. In this case, what is the upper bound on the expectation (mean value) of the quantity $QS+RS+RT-QT$? This is called Bell's inequality.

Answer. [Insert Answer]

Problem 5b: CHSH inequality (quantum states). Next we consider measuring the expectation value of the quantity $QS + RS + RT - QT$ when Alice and Bob have access to quantum states. What are the eigenvalues for each of Q, R, S, and T? What is the expectation value $\langle QS + RS + RT - QT\rangle$ for the product state $|\psi_{\text{Alice}}\rangle \otimes |\psi_{\text{Bob}}\rangle$? What is the max possible value?

Answer. [Insert Answer]

Problem 5c: CHSH inequality (shared qubits). Now consider the case where Alice and Bob share the state (Alice qubit first, Bob qubit second): $\psi_{AB} = \cos(\phi)\,|01\rangle - \sin(\phi)\,|10\rangle$.

For what values of ψ is the quantity $\langle QS+RS+RT-QT\rangle$ maximum (given the observables in part (b))? What is the significance of such ψ for the state $|\psi_{AB}\rangle$? How does this value for $\langle QS+RS+RT-QT\rangle$ compare to the value in part (a)? This is the basis of the CHSH inequality.

Answer. [Insert Answer]

Problem 6a: Quantum circuit (3-qubit). Design a quantum circuit that implements $f(x)$ for x represented by $n = 3$ qubits (in ket notation, define the left-most qubit corresponding to the most significant bit). Draw the quantum circuit using one-qubit gates, two-qubit gates and as few Toffoli gates as possible. Explain its working. You may use ancilla qubits.

Answer. [Insert Answer]

Problem 6b: Quantum circuit (5-qubit). Work out a 5-bit quantum implementation of $f(x)$ – i.e. if there are more 1's than 0's in the input, your function should return 1, otherwise returns 0. Draw a schematic quantum circuit which demonstrates the working of this function. You do not need to draw the full quantum circuit, a clear strategy that demonstrates the working for 5-qubit inputs is sufficient. You must use only one, two and three-qubit gates. Hint: You may use your 3-bit implementation from part (a) as a black box.

Answer. [Insert Answer]