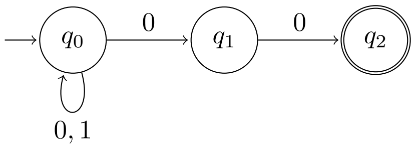
Quiz 1 Solutions

Problem 1
Construct an NFA that accepts the following language. Your NFA should have no more than 3 states.

$$L = \{w \in \{0,1\}^* : w \text{ ends with } 00\}$$

Solution:
A possible answer is shown below.



Problem 2
Let A and B be two regular languages over some alphabet Σ. Prove that the following language is also regular:

$$L = \{a_1 b_1 a_2 b_2 \cdots a_k b_k : (a_i, b_i \in \Sigma^*) \wedge (a_1 a_2 \cdots a_k \in A) \wedge (b_1 b_2 \cdots b_k \in B) \wedge (k \geq 0)\}$$

Solution:
Since A and B are both regular language, so A and B can be accepted by some DFA respectively, denoted by $M_A = (Q_A, \Sigma, \delta_A, s_{0,A}, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, s_{0,B}, F_B)$. Then we can construct a new NFA $M = (Q, \Sigma, s_0, \Delta, F)$:

$$\begin{cases} Q = Q_A \times Q_B \\ s_0 = (s_{0,A}, s_{0,B}) \\ \Delta = \{((q_A, q_B), a, (\delta_A(q_A, a), q_B))\} \cup \{((q_A, q_B), a, (q_A, \delta_B(q_B, a)))\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (a \in \Sigma, q_A \in Q_A, q_B \in Q_B) \\ F = F_A \times F_B \end{cases}$$

It can be proved that $L(M) = L$.

Problem 3
Let M be a DFA with k states. Prove the following statements. You should prove from scratch. That is, you should not use any theorem that has been proven in class.

(a) If $L(M)$ is not empty, then $L(M)$ must contains a string of length at most k.

(b) If there is a string $w \in L(M)$ with $|w| \geq k$, then $L(M)$ contains an infinite number of strings.

Solution:

(a) Consider the state diagram of M. Since $L(M)$ is non-empty, there must be a path from the initial state to the final state. Take the shortest one among all these paths, and denote it as P. The length of P is at most k, and it corresponds to a string $w \in L(M)$ with $|w| \leq k - 1$.

(b) Suppose that $w = a_1 a_2 \dots a_n$ with $n \geq k$. Consider all the configurations of M when we run it on w. They must be as follows.

$$(q_0, a_1 \dots a_n) \vdash_M (q_1, a_2 \dots, a_n) \vdash_M \dots \vdash_M (q_{n-1}, a_n) \vdash_M (q_n, e)$$

Since there are only k distinct states and $n \geq k$, it must be that $q_i = q_j$ for some $i < j$. Let $x = a_1 \dots a_i$. Let $y = a_{i+1} \dots a_j$. Let $z = a_{j+1} \dots a_n$. We have that $xy^k z \in L(M)$ for any $k \geq 0$. Moreover, $|y| > 0$ as $j > i$. Therefore, $L(M)$ contains an infinite number of strings.
