# Introduction to Theoretical Computer Science, Fall 2024
# Assignment 7 Solutions

Q1. Let $A$ be a DFA. Consider the state diagram of $A$. It can be viewed as a directed graph $G$. We delete from $G$ the nodes that are not reachable from the initial sate and the nodes from which we cannot reach the final states. Then $L(A)$ is infinite if and only if $G$ has a cycle, the latter of which can be determined using DFS in linear time.

Q2. Let $M$ be a Turing machine. Let $w$ be a string. In the following, we shall construct a Turing machine $M'$ such that $M$ accepts $w$ if and only if $L(M')$ is infinite.

$M'$ = on input $x$:
1. run $M$ on $w$
2. if $M$ accepts $w$
3. accept $x$
4. else
5. reject $x$

The above is a reduction from $A_{\text{TM}}$ to $\text{INF}_{\text{TM}}$. Since $A_{\text{TM}}$ is not recursive, nor is $\text{INF}_{\text{TM}}$.

Q3. Let $A$ be a DFA. Let $B$ be the NFA with $L(B) = L^R(A)$. $B$ can be constructed from $A$ by reversing all the transitions of $A$. One can see that "$A$" $\in S$ if and only if $L(A) = L(B)$. Therefore, $S$ can be reduced to $\text{EQ}_{DFA}$. Since $\text{EQ}_{DFA}$ is recursive, so is $S$.

Q4. Let $A$ be the stated language in the question. Let $B = \{$"$M$" : $M$ is a TM that accepts $e\}$. We already know that $B$ is not recursive. To prove $A$ is not recursive, it suffices to reduce $B$ to $A$. Let $M$ be a TM. Let $M'$ be the following Turing machine.

$M'$ = on input $x$:
1. if $x == e$
2. accept $x$
3. else
4. reject $x$

It is easy to see that $L(M') = \{e\}$. Then we have that $M$ accepts $e$ if and only if $L(M) \cap L(M') \geq 1$. This finishes the reduction.