Example of product calculation of a Lie group from brackets of its Lie algebra

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1 Introduction:

So far, computing multiplication and exponential mapping on a Lie groups takes a lot of time, even in very small dimensions. But in this file we will use the Python library Sympy (**Py**thon **Sym**bols) which helps us to perform this formal calculation in a very fast and easy way from the Lie brackets of the Lie algebra of Lie group, we will take the group N_5 as an exampl. Let $\mathfrak{n}_5$ be the Lie algebra of 5×5 upper triangular matrices:

$$\mathbb{R}^{10} \ni (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) = \begin{pmatrix} 0 & x_1 & x_5 & x_8 & x_{10} \\ 0 & 0 & x_2 & x_6 & x_9 \\ 0 & 0 & 0 & x_3 & x_7 \\ 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\mathfrak{n}_5 = \mathbb{R}X_1+\mathbb{R}X_2+\mathbb{R}X_3+\mathbb{R}X_4+\mathbb{R}X_5+\mathbb{R}X_6+\mathbb{R}X_7+\mathbb{R}X_8+\mathbb{R}X_9+\mathbb{R}X_{10}$, where $X_1 = E_{1,2}, X_2 = E_{2,3}, X_3 = E_{3,4}, X_4 = E_{4,5}, X_5 = E_{1,3}, X_6 = E_{2,4}, X_7 = E_{3,5}, X_8 = E_{1,4}, X_9 = E_{2,5}, X_{10} = E_{1,5}$, equipped with the Lie brackets:

$$[X_1,X_2]=X_5;\ [X_1,X_6]=X_8;\ [X_1,X_9]=X_{10};\ [X_2,X_3]=X_6;\ [X_2,X_7]=X_9;\ [X_3,X_4]=X_7;$$
$$[X_3,X_5]=-X_8;\ [X_4,X_6]=-X_9;\ [X_4,X_8]=-X_{10};\ [X_5,X_7]=X_{10}.$$

[2]:
```
from sympy import*
```

2 Variables definition in python symbols:

[7]:
```
x1,x2,x3,x4,x5,x6,x7,x8,x9=symbols('x1 x2 x3 x4 x5 x6 x7 x8 x9')
x10,x11,x21,x31,x41,x51,x61=symbols('x10 x11 x21 x31 x41 x51 x61')
x71,x81,x91,x101,x12,x22,x32=symbols('x71 x81 x91 x101 x12 x22 x32')
x42,x52,x62,x72,x82,x92,x102=symbols('x42 x52 x62 x72 x82 x92 x102')
```

From the Lie brackets we see that $\mathfrak{h}_0 = \text{Span} < X_6,X_7,X_8,X_9,X_{10} >$ is an Abelian ideal, for which we can identify the Lie group H_0 and its Lie algebra $\mathfrak{h}_0$, obviously which the Lie algebra H_0 is equipped with the usual addition. Next, at each step we will add a vector X_i, $1 \le i \le 5$ to obtain the subgroup of N_5, each time we compute the multiplication [[1]] it gives, then we explicitly determine the exponential mapping and use Campbell- Baker-Hausdorff's formula which is also introduced in [[1]] on the subalgebra to get the The product on it.

2.1 Add the vector X_5:

We denote the subalgebra of $\mathfrak{n}_5$ by $\mathfrak{h}_1$, where $\mathfrak{h}_1 = \mathbb{R}X_5 \oplus \mathfrak{h}_0 = \text{Span} < X_5, X_6, X_7, X_8, X_9, X_{10} >$, and its Lie group is $H_1 = \mathbb{R} \ltimes H_0$. By the product formula:

$$(x_5, X).{}_{H_1}(x_5', X') = (x_5 + x_5', Ad(x_5'X_5)^{-1}X + X'), (x_5, X), (x_5', X') \in H_1$$

we have the following explicit multiplication on H_1:

[8]:
```
def prod1 (x5,x6,x7,x8,x9,x10,x51,x61,x71,x81,x91,x101):
    return(x5+x51,x6+x61,x7+x71,x8+x81,x9+x91,x10+x101-x51*x7)
```

So the Product on H_1 is given by:

$$\begin{aligned} & (x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_1}(x_5', x_6', x_7', x_8', x_9', x_{10}') \\ = \quad & (x_5 + x_5', x_6 + x_6', x_7 + x_7', x_8 + x_8', x_9 + x_9', x_{10} + x_{10}' - x_5'x_7), \\ & \forall,\ (x_5, x_6, x_7, x_8, x_9, x_{10}), (x_5', x_6', x_7', x_8', x_9', x_{10}') \in H_1. \end{aligned}$$

Now, we us the formula

$$\exp(x_5, X) = (x_5, \frac{\exp(-ad(x_5X_5, X)) - I}{-ad(x_5, X)})(X), (x_5, X) \in \mathfrak{h}_1,$$

to define the expression for the **exponential** mapping from $\mathfrak{h}_1$ to H_1 as follows:

[9]:
```
def exp1 (x5,x6,x7,x8,x9,x10):
    return(x5,x6,x7,x8,x9,x10-x5*x7/2)
```

Then,

$$\exp_1(x_5, x_6, x_7, x_8, x_9, x_{10}) = \left(x_5, x_6, x_7, x_8, x_9, x_{10} - \frac{x_5x7}{2}\right)$$

Note that the exponential map is a diffeomorphism from $\mathfrak{h}_1$ to H_1, and its inverse is a **logarithmic** map as follows:

[10]:
```
def log1 (x5,x6,x7,x8,x9,x10):
    return(x5,x6,x7,x8,x9,x10+x5*x7/2)
```

Then

$$\log_1(x_5, x_6, x_7, x_8, x_9, x_{10}) = \left(x_5, x_6, x_7, x_8, x_9, x_{10} + \frac{x_5x7}{2}\right)$$

Let us now use the Campbell-Baker-Hausdorff formula:

$$X *_{\mathfrak{h}_1} X' = \log_1(\exp_1(X).{}_{H_1} \exp_1(X')), X, X' \in \mathfrak{h}_1$$

to compute the multiplication on the vector space $\mathfrak{h}_1$.

$\exp_1(x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_1} \exp_1(x_5', x_6', x_7', x_8', x_9', x_{10}')$:

[11]:
```
prod1(x5,x6,x7,x8,x9,x10-x5*x7/2,x51,x61,x71,x81,x91,x101-
      x51*x71/2)
```

```
[11]: (x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x9 + x91,
 x10 + x101 - x5*x7/2 - x51*x7 - x51*x71/2)
```

$\log(\exp_1(x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_1} \exp_1(x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10})$:

```
[13]: log1 (x5+x51,x6+x61,x7+x71,x8+x81,x9+x91,x10+x101-x5*x7/2-x51*x7
      -x51*x71/2)
```

```
[13]: (x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x9 + x91,
 x10 + x101 - x5*x7/2 - x51*x7 - x51*x71/2 + (x5 + x51)*(x7 + x71)/2)
```

After simplification we obtain:

```
[14]: def prod11 (x5,x6,x7,x8,x9,x10,x51,x61,x71,x81,x91,x101):
          return(x5 + x51,
       x6 + x61,
       x7 + x71,
       x8 + x81,
       x9 + x91,
       x10 + x101 + x5*x71/2 - x51*x7/2)
```

The multiplication on the algebra $\mathfrak{h}_1$ is given by:

$$\begin{aligned} & (x_5, x_6, x_7, x_8, x_9, x_{10}) *_{\mathfrak{h}_1} (x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \\ = & (x_5 + x'_5, x_6 + x'_6, x_7 + x'_7, x_8 + x'_8, x_9 + x'_9, x_{10} + x'_{10} + \frac{x_5 x'_7}{2} - \frac{x'_5 x_7}{2}) \\ & \forall,\ (x_5, x_6, x_7, x_8, x_9, x_{10}), (x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \in \mathfrak{h}_1 \end{aligned}$$

2.2 Add the vector X_4:

In this step we define the subalgebra $\mathfrak{h}_2 := \mathbb{R}X_4 \oplus \mathfrak{h}_1 = \mathrm{Span} < X_4, X_5, X_6, X_7, X_8, X_9, X_{10} >$ and its Lie group $H_2 := \mathbb{R} \ltimes H_1$.

Let's start by computing the pruduct on H_2 with the formule:

$$(x_4, X).{}_{H_2}(x'_4, X') = (x_4 + x'_4, Ad(x'_4 X_4)^{-1} X *_{\mathfrak{h}_1} X'), (x_4, X), (x'_4, X') \in H_2$$

```
[15]: prod11(x5,x6,x7,x8,x9+x41*x6,x10+x41*x8,x51,x61,x71,x81,x91,x101)
```

[15]:
```
(x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x41*x6 + x9 + x91,
 x10 + x101 + x41*x8 + x5*x71/2 - x51*x7/2)
```

[16]:
```
def prod2 (x4,x5,x6,x7,x8,x9,x10,x41,x51,x61,x71,x81,x91,x101):
    return(x4+x41,x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x41*x6 + x9 + x91,
 x10 + x101 + x41*x8 + x5*x71/2 - x51*x7/2)
```

So the Product on H_2 is given by:

$$\begin{aligned} & (x_4, x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_2}(x_4', x_5', x_6', x_7', x_8', x_9', x_{10}') \\ = \; & (x_4 + x_4', x_5 + x_5', x_6 + x_6', x_7 + x_7', x_8 + x_8', x_9 + x_9' + x_4' x_6, x_{10} + x_{10}' + x_4' x_8 + \frac{x_5 x_7'}{2} - \frac{x_5' x_7}{2}) \\ & \forall, \; (x_4, x_5, x_6, x_7, x_8, x_9, x_{10}), (x_4', x_5', x_6', x_7', x_8', x_9', x_{10}') \in H_2. \end{aligned}$$

We will determine in the following the expression of the **exponential** from $\mathfrak{h}_2$ to H_2:

[17]:
```
def exp2 (x4,x5,x6,x7,x8,x9,x10):
    return(x4,x5,x6,x7,x8,x9+x4*x6/2,x10+x4*x8/2)
```

$\exp_2(x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (x_4, x_5, x_6, x_7, x_8, x_9 + \frac{x_4 x_6}{2}, x_{10} + \frac{x_4 x_8}{2})$

Clearly the inverse map is the **logarithm** :

[18]:
```
def log2 (x4,x5,x6,x7,x8,x9,x10):
    return(x4,x5,x6,x7,x8,x9-x4*x6/2,x10-x4*x8/2)
```

$\log_2(x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (x_4, x_5, x_6, x_7, x_8, x_9 - \frac{x_4 x_6}{2}, x_{10} - \frac{x_4 x_8}{2})$

It remains now to complete the product on the algebra $\mathfrak{h}_2$ by the Campbell-Baker-Hausdroff formula:

$\exp_2(x_4, x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_2} \exp_2(x_4', x_5', x_6', x_7', x_8', x_9', x_{10}')$:

[20]:
```
prod2(x4,x5,x6,x7,x8,x9+x4*x6/2,x10+x4*x8/2,x41,x51,x61,x71,x81,
      x91+x41*x61/2,x101+x41*x81/2)
```

[20]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x4*x6/2 + x41*x6 + x41*x61/2 + x9 + x91,
 x10 + x101 + x4*x8/2 + x41*x8 + x41*x81/2 + x5*x71/2 - x51*x7/2)
```

$\log(\exp_2(x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) ._{H_2} \exp_2(x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) :$

[21]:
```
log2(x4+x41,x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x4*x6/2 + x41*x6 + x41*x61/2 + x9 + x91,
 x10 + x101 + x4*x8/2 + x41*x8 + x41*x81/2 + x5*x71/2 - x51*x7/2)
```

[21]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 x4*x6/2 + x41*x6 + x41*x61/2 + x9 + x91 - (x4 + x41)*(x6 + x61)/2,
 x10 + x101 + x4*x8/2 + x41*x8 + x41*x81/2 + x5*x71/2 - x51*x7/2 - (x4 +
x41)*(x8 + x81)/2)
```

To improve the result we use the function **Simplify**:

[22]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 simplify(x4*x6/2 + x41*x6 + x41*x61/2 + x9 + x91 -
          (x4 + x41)*(x6 + x61)/2),
 simplify(x10 + x101 + x4*x8/2 + x41*x8 + x41*x81/2 + x5*x71/2 -
          x51*x7/2 - (x4 + x41)*(x8 + x81)/2))
```

[22]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[23]:
```
def prod21 (x4,x5,x6,x7,x8,x9,x10,x41,x51,x61,x71,x81,x91,x101):
    return(x4 + x41,
 x5 + x51,
 x6 + x61,
 x7 + x71,
 x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

The multiplication on the algebra $\mathfrak{h}_2$ is given by:

$$
\begin{aligned}
& (x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) *_{\mathfrak{h}_2} (x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \\
= \; & \log_2[\exp_2(x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) ._{H_2} \exp_2(x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10})] \\
= \; & \Bigg(x_4 + x'_4, x_5 + x'_5, x_6 + x'_6, x_7 + x'_7, x_8 + x'_8, x_9 + x'_9 + \frac{x'_4 x_6}{2} - \frac{x_4 x'_6}{2}, x_{10} + x'_{10} + \frac{x_5 x'_7}{2} - \frac{x'_5 x_7}{2} \\
& + \frac{x'_4 x_8}{2} - \frac{x_4 x'_8}{2} \Bigg) \; \forall, \; (x_4, x_5, x_6, x_7, x_8, x_9, x_{10}), (x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \in \mathfrak{h}_2
\end{aligned}
$$

2.3 Add the vector X_3:

As in the previous steps, we add the vector X_3 to construct the algebra $\mathfrak{h}_3 := \mathbb{R}X_3 \oplus \mathfrak{h}_2 = \text{Span} < X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10} >$ and its Lie group $H_3 := \mathbb{R} \ltimes H_2$.

Let's start by computing the pruduct on H_3 with the formule:

$$(x_3, X)._{H_3}(x'_3, X') = (x_3 + x'_3, Ad(x'_3 X_3)^{-1} X *_{\mathfrak{h}_2} X'), (x_3, X), (x'_3, X') \in H_3$$

[24]:
```
prod21(x4,x5,x6,x7-x31*x4,x8+x31*x5,x9,x10,x41,x51,x61,x71,x81,
       x91,x101)
```

[24]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 -x31*x4 + x7 + x71,
 x31*x5 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x4*x81/2 + x41*(x31*x5 + x8)/2 + x5*x71/2 - x51*(-x31*x4 + x7)/2)
```

We need to simplify the result

[25]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 -x31*x4 + x7 + x71,
 x31*x5 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 simplify(x10 + x101 - x4*x81/2 + x41*(x31*x5 + x8)/2 + x5*x71/2
          - x51*(-x31*x4 + x7)/2))
```

[25]:
```
(x4 + x41,
 x5 + x51,
 x6 + x61,
 -x31*x4 + x7 + x71,
 x31*x5 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x4*x81/2 + x41*(x31*x5 + x8)/2 + x5*x71/2 + x51*(x31*x4 - x7)/2)
```

[26]:
```
def prod3(x3,x4,x5,x6,x7,x8,x9,x10,x31,x41,x51,x61,x71,
          x81,x91,x101):
    return(x3+x31,x4 + x41,
 x5 + x51,
 x6 + x61,
 -x31*x4 + x7 + x71,
 x31*x5 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x4*x81/2 + x41*(x31*x5 + x8)/2 + x5*x71/2 +
           x51*(x31*x4 - x7)/2)
```

Then we get the multiplication on H_3 :

$$\begin{aligned}
&(x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}).{}_{H_3}(x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}')\\
=&\Big(x_3+x_3',x_4+x_4',x_5+x_5',x_6+x_6',x_7+x_7'-x_3'x_4,x_8+x_8'+x_3'x_5,x_9+x_9'+\frac{x_4'x_6}{2}-\frac{x_4x_6'}{2},x_{10}\\
&+x_{10}'+\frac{x_4'x_8}{2}-\frac{x_4x_8'}{2}+\frac{x_5x_7'}{2}-\frac{x_5'x_7}{2}+\frac{x_3'x_4'x_5}{2}+\frac{x_3'x_4x_5'}{2}\Big)\\
&\forall,\ (x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}),(x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}')\in H_3
\end{aligned}$$

Now consider the exponential expression from $\mathfrak{h}_3$ to H_3 as follows:

[27]:
```
def exp3(x3,x4,x5,x6,x7,x8,x9,x10):
    return (x3,x4,x5,x6,x7-x3*x4/2,x8+x3*x5/2,x9,x10+x3*x4*x5/6)
```

$\exp_3(x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})=(x_3,x_4,x_5,x_6,x_7-\frac{x_3x_4}{2},x_8+\frac{x_3x_5}{2},x_9,x_{10}+\frac{x_3x_4x_5}{6})$

Obviously the reciprocal logarithm map is expressed as:

[28]:
```
def log3 (x3,x4,x5,x6,x7,x8,x9,x10):
    return(x3,x4,x5,x6,x7+x3*x4/2,x8-x3*x5/2,x9,x10-x3*x4*x5/6)
```

$\log_3(x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})=(x_3,x_4,x_5,x_6,x_7+\frac{x_3x_4}{2},x_8-\frac{x_3x_5}{2},x_9,x_{10}-\frac{x_3x_4x_5}{6})$

Next we will calculate:

$\exp_3(x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}).{}_{H_3}\exp_3(x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}')$:

[29]:
```
prod3(x3,x4,x5,x6,x7-x3*x4/2,x8+x3*x5/2,x9,x10+x3*x4*x5/6,x31,
      x41,x51,x61,x71-x31*x41/2,x81+x31*x51/2,x91,
      x101+x31*x41*x51/6)
```

[29]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 -x3*x4/2 - x31*x4 - x31*x41/2 + x7 + x71,
 x3*x5/2 + x31*x5 + x31*x51/2 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x5/6 + x31*x41*x51/6 - x4*(x31*x51/2 + x81)/2 + x41*(x3*x5/2
```

```
+ x31*x5 + x8)/2 + x5*(-x31*x41/2 + x71)/2 + x51*(x3*x4/2 + x31*x4 - x7)/2)
```

$\log_3[\exp_3(x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}).{}_{H_3} \exp_3(x'_3, x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10})]$:

[30]:
```
log3(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 -x3*x4/2 - x31*x4 - x31*x41/2 + x7 + x71,
 x3*x5/2 + x31*x5 + x31*x51/2 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x5/6 + x31*x41*x51/6 - x4*(x31*x51/2 + x81)/2
     + x41*(x3*x5/2 + x31*x5 + x8)/2 + x5*(-x31*x41/2 + x71)/2 +
     x51*(x3*x4/2 + x31*x4 - x7)/2)
```

[30]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 -x3*x4/2 - x31*x4 - x31*x41/2 + x7 + x71 + (x3 + x31)*(x4 + x41)/2,
 x3*x5/2 + x31*x5 + x31*x51/2 + x8 + x81 - (x3 + x31)*(x5 + x51)/2,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x5/6 + x31*x41*x51/6 - x4*(x31*x51/2 + x81)/2 + x41*(x3*x5/2
+ x31*x5 + x8)/2 + x5*(-x31*x41/2 + x71)/2 + x51*(x3*x4/2 + x31*x4 - x7)/2 - (x3
+ x31)*(x4 + x41)*(x5 + x51)/6)
```

[32]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 simplify(-x3*x4/2 - x31*x4 - x31*x41/2 + x7 + x71 +
          (x3 + x31)*(x4 + x41)/2),
 simplify(x3*x5/2 + x31*x5 + x31*x51/2 + x8 + x81 -
          (x3 + x31)*(x5 + x51)/2),
 simplify(-x4*x61/2 + x41*x6/2 + x9 + x91),
 simplify(x10 + x101 + x3*x4*x5/6 + x31*x41*x51/6 - x4*(x31*x51/2
        + x81)/2 + x41*(x3*x5/2 + x31*x5 + x8)/2 +
          x5*(-x31*x41/2 + x71)/2 + x51*(x3*x4/2 + x31*x4 - x7)/2
          - (x3 + x31)*(x4 + x41)*(x5 + x51)/6))
```

[32]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
```

```
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[33]:
```
def prod31 (x3,x4,x5,x6,x7,x8,x9,x10,x31,x41,x51,x61,x71,x81,x91,
            x101):
    return(x3 + x31,
 x4 + x41,
 x5 + x51,
 x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 -
        x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 +
           x41*x8/2 + x5*x71/2 - x51*x7/2)
```

So after simplifying we have that:

$$\begin{aligned}
& (x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) *_{\mathfrak{h}_3} (x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \\
= & \Bigg(x_3+x_3', x_4+x_4', x_5+x_5', x_6+x_6', x_7+x_7'+\frac{x_3x_4'}{2}-\frac{x_3'x_4}{2}, x_8+x_8'+\frac{x_3'x_5}{2}-\frac{x_3x_5'}{2}, x_9+x_9' \\
& +\frac{x_4'x_6}{2}-\frac{x_4x_6'}{2}, x_{10}+x_{10}'+\frac{x_5x_7'}{2}-\frac{x_5'x_7}{2}+\frac{x_4'x_8}{2}-\frac{x_4x_8'}{2}-\frac{x_3'x_4x_5}{6}-\frac{x_3x_4'x_5'}{6}+\frac{x_3'x_4'x_5}{12} \\
& +\frac{x_3'x_4x_5'}{12}+\frac{x_3x_4'x_5}{12}+\frac{x_3x_4x_5'}{12}\Bigg) \\
& \forall,\ (x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}),(x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \in \mathfrak{h}_3
\end{aligned}$$

2.4 Add the vector X_2:

Now we can define the Lie algebra $\mathfrak{h}_4 := \mathbb{R}X_4 \oplus \mathfrak{h}_3 = \text{Span} < X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10} >$, also, its Lie group $H_4 := \mathbb{R} \ltimes H_4$.

We start as an evidence by computing the pruduct on H_4 by the formule:

$$(x_2, X)._{H_4}(x_2', X') = (x_2 + x_2', Ad(x_2'X_2)^{-1}X *_{\mathfrak{h}_3} X'), (x_2, X), (x_2', X') \in H_4$$

[34]:
```
prod31 (x3,x4,x5,x6-x21*x3,x7,x8,x9-x21*x7,x10,x31,x41,x51,x61,
        x71,x81,x91,x101)
```

[34]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 -x21*x3 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x21*x7 - x4*x61/2 + x41*(-x21*x3 + x6)/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[35]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 -x21*x3 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
simplify( -x21*x7 - x4*x61/2 + x41*(-x21*x3 + x6)/2 + x9 + x91),
simplify( x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6
         - x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2
         + x41*x8/2 + x5*x71/2 - x51*x7/2))
```

[35]:
```
(x3 + x31,
 x4 + x41,
 x5 + x51,
 -x21*x3 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x21*x7 - x4*x61/2 - x41*(x21*x3 - x6)/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[36]:
```
def prod4 (x2,x3,x4,x5,x6,x7,x8,x9,x10,x21,x31,x41,x51,x61,x71,
           x81,x91,x101):
    return(x2+x21,x3 + x31,
 x4 + x41,
 x5 + x51,
 -x21*x3 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x21*x7 - x4*x61/2 - x41*(x21*x3 - x6)/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 -
    x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 +
           x41*x8/2 + x5*x71/2 - x51*x7/2)
```

Thus the multiplication on H_4 is :

$$\begin{aligned}
& (x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}).{}_{H_4}(x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \\
= & \Bigg(x_2+x_2', x_3+x_3', x_4+x_4', x_5+x_5', x_6+x_6'-x_2'x_3, x_7+x_7'-\frac{x_3'x_4}{2}+\frac{x_3x_4'}{2}, x_8+x_8' \\
& +\frac{x_3'x_5}{2}-\frac{x_3x_5'}{2}, x_9+x_9'-x_2'x_7+\frac{x_4'x_6}{2}-\frac{x_4x_6'}{2}-\frac{x_2'x_3x_4'}{2}, x_{10}+x_{10}'+\frac{x_4'x_8}{2}-\frac{x_4x_8'}{2}+\frac{x_5x_7'}{2} \\
& -\frac{x_5'x_7}{2}-\frac{x_3x_4'x_5'}{6}-\frac{x_3'x_4x_5}{6}+\frac{x_3'x_4'x_5}{12}+\frac{x_3'x_4x_5'}{12}+\frac{x_3x_4'x_5}{12}+\frac{x_3x_4x_5'}{12}\Bigg) \\
& \forall,\ (x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}),(x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \in H_4
\end{aligned}$$

Now let us define the **exponential** function of $\mathfrak{h}_4$ in H_4:

[37]:
```
def exp4(x2,x3,x4,x5,x6,x7,x8,x9,x10):
    return (x2,x3,x4,x5,x6-x2*x3/2,x7,x8,x9-x2*x7/2-x2*x3*x4/12,
            x10)
```

Explicitly the formula is written as:

$\exp_4(x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) = (x_2,x_3,x_4,x_5,x_6 - \frac{x_2x_3}{2}, x_7,x_8,x_9 - \frac{x_2x_7}{2} - \frac{x_2x_3x_4}{12}, x_{10})$

[38]:
```
def log4 (x2,x3,x4,x5,x6,x7,x8,x9,x10):
    return(x2,x3,x4,x5,x6+x2*x3/2,x7,x8,x9+x2*x7/2+x2*x3*x4/12,
           x10)
```

Inversely:

$$\log_4(x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) = (x_2,x_3,x_4,x_5,x_6 + \frac{x_2x_3}{2}, x_7,x_8,x_9 + \frac{x_2x_7}{2} + \frac{x_2x_3x_4}{12}, x_{10})$$

we apply now the Campbell Baker-Hausdorff-formula:

$\exp_4(x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) ._{H_4} \exp_4(x'_2,x'_3,x'_4,x'_5,x'_6,x'_7,x'_8,x'_9,x'_{10})$:

[39]:
```
prod4(x2,x3,x4,x5,x6-x2*x3/2,x7,x8,x9-x2*x7/2-x2*x3*x4/12,x10,x21
,x31,x41,x51,x61-x21*x31/2,x71,x81,x91-x21*x71/2-x21*x31*x41/12,
      x101)
```

[39]:
```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 -x2*x3/2 - x21*x3 - x21*x31/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x2*x3*x4/12 - x2*x7/2 - x21*x31*x41/12 - x21*x7 - x21*x71/2 - x4*(-x21*x31/2 +
x61)/2 - x41*(x2*x3/2 + x21*x3 - x6)/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

$\log_4[\exp_4(x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) ._{H_4} \exp_4(x'_2,x'_3,x'_4,x'_5,x'_6,x'_7,x'_8,x'_9,x'_{10})]$:

[40]:
```
log4(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 -x2*x3/2 - x21*x3 - x21*x31/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x2*x3*x4/12 - x2*x7/2 - x21*x31*x41/12 - x21*x7 - x21*x71/2 -
     x4*(-x21*x31/2 + x61)/2 - x41*(x2*x3/2 + x21*x3 - x6)/2 + x9
     + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 -
```

```
    x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 +
    x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[40]:

```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 -x2*x3/2 - x21*x3 - x21*x31/2 + x6 + x61 + (x2 + x21)*(x3 + x31)/2,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 -x2*x3*x4/12 - x2*x7/2 - x21*x31*x41/12 - x21*x7 - x21*x71/2 - x4*(-x21*x31/2 +
x61)/2 - x41*(x2*x3/2 + x21*x3 - x6)/2 + x9 + x91 + (x2 + x21)*(x3 + x31)*(x4 +
x41)/12 + (x2 + x21)*(x3*x41/2 - x31*x4/2 + x7 + x71)/2,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[41]:

```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 simplify(-x2*x3/2 - x21*x3 - x21*x31/2 + x6 + x61 +
          (x2 + x21)*(x3 + x31)/2),
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 simplify(-x2*x3*x4/12 - x2*x7/2 - x21*x31*x41/12 - x21*x7 -
        x21*x71/2 - x4*(-x21*x31/2 + x61)/2 -
          x41*(x2*x3/2 + x21*x3 - x6)/2 + x9 + x91 +
          (x2 + x21)*(x3 + x31)*(x4 + x41)/12 +
          (x2 + x21)*(x3*x41/2 - x31*x4/2 + x7 + x71)/2),
 simplify(x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6
          - x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 -
          x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2))
```

[41]:

```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 + x21*x3*x4/12 -
x21*x3*x41/6 + x21*x31*x4/12 - x21*x7/2 - x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 - x31*x4*x5/6 +
x31*x4*x51/12 + x31*x41*x5/12 - x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

[42]:

```
def prod41 (x2,x3,x4,x5,x6,x7,x8,x9,x10,x21,x31,x41,x51,x61,x71,
             x81,x91,x101):
```

```
    return(x2 + x21,
 x3 + x31,
 x4 + x41,
 x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x3*x51/2 + x31*x5/2 + x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 +
           x21*x3*x4/12 - x21*x3*x41/6 + x21*x31*x4/12 - x21*x7/2
           - x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 + x3*x4*x51/12 + x3*x41*x5/12 - x3*x41*x51/6 -
           x31*x4*x5/6 + x31*x4*x51/12 + x31*x41*x5/12 -
           x4*x81/2 + x41*x8/2 + x5*x71/2 - x51*x7/2)
```

So after simplifying we have that:

$$\begin{aligned}
&(x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) *_{\mathfrak{h}_4} (x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \\
= &\Big(x_2+x_2', x_3+x_3', x_4+x_4', x_5+x_5', x_6+x_6'+\frac{x_2x_3'}{2}-\frac{x_2'x_3}{2}, x_7+x_7'+\frac{x_3x_4'}{2}-\frac{x_3'x_4}{2}, x_8+x_8' \\
&+\frac{x_3'x_5}{2}-\frac{x_3x_5'}{2}, x_9+x_9'+\frac{x_4'x_6}{2}-\frac{x_4x_6'}{2}+\frac{x_2x_7'}{2}-\frac{x_2'x_7}{2}-\frac{x_2x_3'x_4}{6}-\frac{x_2'x_3x_4'}{6}+\frac{x_2x_3x_4'}{12}+\frac{x_2x_3'x_4'}{12} \\
&+\frac{x_2'x_3x_4}{12}+\frac{x_2x_3'x_4}{12}, x_{10}+x_{10}'+\frac{x_5x_7'}{2}-\frac{x_5'x_7}{2}+\frac{x_4'x_8}{2}-\frac{x_4x_8'}{2}-\frac{x_3'x_4x_5}{6}-\frac{x_3x_4'x_5'}{6}+\frac{x_3'x_4'x_5}{12} \\
&+\frac{x_3'x_4x_5'}{12}+\frac{x_3x_4'x_5}{12}+\frac{x_3x_4x_5'}{12}\Big) \\
&\forall,\ (x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}),(x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \in \mathfrak{h}_4
\end{aligned}$$

2.5 Add the vector X_1:

We go directly to the last step in which we add the vector X_1, and by this we obtain the algebra $\mathfrak{n}_5$ and its N_5 we use of course

$$(x_1,X)._{N_5}(x_1',X') = (x_1+x_1', Ad(x_1'X_1)^{-1}X *_{\mathfrak{h}_4} X'), (x_1,X),(x_1',X') \in N_5$$

[43]:
```
prod41(x2,x3,x4,x5-x11*x2,x6,x7,x8-x11*x6,x9,x10-x11*x9,x21,x31,
       x41,x51,x61,x71,x81,x91,x101)
```

[43]:
```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 -x11*x2 + x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x11*x6 - x3*x51/2 + x31*(-x11*x2 + x5)/2 + x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 + x21*x3*x4/12 -
x21*x3*x41/6 + x21*x31*x4/12 - x21*x7/2 - x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x11*x9 + x3*x4*x51/12 - x3*x41*x51/6 + x3*x41*(-x11*x2 + x5)/12 +
```

```
x31*x4*x51/12 - x31*x4*(-x11*x2 + x5)/6 + x31*x41*(-x11*x2 + x5)/12 - x4*x81/2 +
x41*(-x11*x6 + x8)/2 - x51*x7/2 + x71*(-x11*x2 + x5)/2)
```

[45]:
```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 -x11*x2 + x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 simplify(-x11*x6 - x3*x51/2 + x31*(-x11*x2 + x5)/2 + x8 + x81),
 simplify(x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 +
          x21*x3*x4/12 - x21*x3*x41/6 + x21*x31*x4/12 - x21*x7/2
          - x4*x61/2 + x41*x6/2 + x9 + x91),
 simplify(x10 + x101 - x11*x9 + x3*x4*x51/12 - x3*x41*x51/6 +
          x3*x41*(-x11*x2 + x5)/12 + x31*x4*x51/12 -
          x31*x4*(-x11*x2 + x5)/6 + x31*x41*(-x11*x2 + x5)/12 -
          x4*x81/2 + x41*(-x11*x6 + x8)/2 - x51*x7/2 +
          x71*(-x11*x2 + x5)/2))
```

[45]:
```
(x2 + x21,
 x3 + x31,
 x4 + x41,
 -x11*x2 + x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x11*x6 - x3*x51/2 - x31*(x11*x2 - x5)/2 + x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 + x21*x3*x4/12 -
x21*x3*x41/6 + x21*x31*x4/12 - x21*x7/2 - x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x11*x9 + x3*x4*x51/12 - x3*x41*x51/6 - x3*x41*(x11*x2 - x5)/12 +
x31*x4*x51/12 + x31*x4*(x11*x2 - x5)/6 - x31*x41*(x11*x2 - x5)/12 - x4*x81/2 -
x41*(x11*x6 - x8)/2 - x51*x7/2 - x71*(x11*x2 - x5)/2)
```

[46]:
```
def prod5 (x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x21,x31,x41,x51,x61,
           x71,x81,x91,x101):
    return(x1+x11,x2 + x21,
 x3 + x31,
 x4 + x41,
 -x11*x2 + x5 + x51,
 x2*x31/2 - x21*x3/2 + x6 + x61,
 x3*x41/2 - x31*x4/2 + x7 + x71,
 -x11*x6 - x3*x51/2 - x31*(x11*x2 - x5)/2 + x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 + x2*x31*x41/12 + x2*x71/2 +
           x21*x3*x4/12 - x21*x3*x41/6 + x21*x31*x4/12 -
           x21*x7/2 - x4*x61/2 + x41*x6/2 + x9 + x91,
 x10 + x101 - x11*x9 + x3*x4*x51/12 - x3*x41*x51/6 -
x3*x41*(x11*x2 - x5)/12 + x31*x4*x51/12 + x31*x4*(x11*x2 - x5)/6
           - x31*x41*(x11*x2 - x5)/12 - x4*x81/2 -
```

```
        x41*(x11*x6 - x8)/2 - x51*x7/2 - x71*(x11*x2 - x5)/2)
```

Now we can equipped the N_5 with the followin group low:

$$
\begin{aligned}
& (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}).{}_{N_5}(x_1',x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \\
= & \Big(x_1+x_1', x_2+x_2', x_3+x_3', x_4+x_4', x_5+x_5'-x_1'x_2, x_6+x_6'-\frac{x_2'x_3}{2}+\frac{x_2x_3'}{2}, x_7+x_7'-\frac{x_3'x_4}{2} \\
& +\frac{x_3x_4'}{2}, x_8+x_8'-x_1'x_6+\frac{x_3'x_5}{2}-\frac{x_3x_5'}{2}-\frac{x_1'x_2x_3'}{2}, x_9+x_9'-\frac{x_2'x_7}{2}+\frac{x_2x_7'}{2}+\frac{x_4'x_6}{2}-\frac{x_4x_6'}{2}- \\
& \frac{x_2'x_3x_4'}{6}-\frac{x_2x_3'x_4}{6}+\frac{x_2'x_3'x'4}{12}+\frac{x_2'x_3x_4}{12}+\frac{x_2x_3x_4'}{12}+\frac{x_2x_3'x_4'}{12}, x_{10}+x_{10}'-x_1'x_9+\frac{x_4'x_8}{2}-\frac{x_4x_8'}{2} \\
& +\frac{x_5x_7'}{2}-\frac{x_5'x_7}{2}-\frac{x_1'x_4'x_6}{2}-\frac{x_1'x_2x_7'}{2}-\frac{x_3x_4'x_5'}{6}-\frac{x_3'x_4x_5}{6}+\frac{x_3'x_4x_1'x_2}{6}+\frac{x_3'x_4'x_5}{12}+\frac{x_3'x_4x_5'}{12}+\frac{x_3x_4'x_5}{12} \\
& +\frac{x_3x_4x_5'}{12}-\frac{x_1'x_2x_3x_4'}{12}-\frac{x_1'x_2x_3'x_4'}{12}\Big) \\
& \forall,\ (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}),(x_1,x_2',x_3',x_4',x_5',x_6',x_7',x_8',x_9',x_{10}') \in N_5
\end{aligned}
$$

We define the reverse of an element by ${}_{N_5}$ as follows:

[47]:
```
def inverse (x1,x2,x3,x4,x5,x6,x7,x8,x9,x10):
    return (-x1,-x2,-x3,-x4,-x5-x1*x2,-x6,-x7,-x8-x1*x6,-x9,
            -x10-x1*x9)
```

Its inverse is determined by:

$$
\begin{aligned}
& (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})^{-1} \\
= & (-x_1,-x_2,-x_3,-x_4,-x_5-x_1x_2,-x_6,-x_7,-x_8-x_1x_6,-x_9,-x_{10}-x_1x_9)
\end{aligned}
$$

Exponential and **logratime** are defined as:

[48]:
```
def exp5 (x1,x2,x3,x4,x5,x6,x7,x8,x9,x10):
    return(x1,x2,x3,x4,x5-x1*x2/2,x6,x7,x8-x1*x6/2-x1*x2*x3/12,
           x9,x10-x1*x9/2-x1*x2*x7/12-x1*x4*x6/12)
```

$$
\begin{aligned}
& \exp_5(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) \\
= & \Big(x_1,x_2,x_3,x_4,x_5-\frac{x_1x_2}{2},x_6,x_7,x_8-\frac{x_1x_6}{2}+\frac{x_1x_2x_3}{12},x_9,x_{10}-\frac{x_1x_9}{2}-\frac{x_1x_2x_7}{12}-\frac{x_1x_4x_6}{12}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& \log_5(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) \\
= & \Big(x_1,x_2,x_3,x_4,x_5+\frac{x_1x_2}{2},x_6,x_7,x_8+\frac{x_1x_6}{2}+\frac{x_1x_2x_3}{12},x_9,x_{10}+\frac{x_1x_9}{2}+\frac{x_1x_2x_7}{12}+\frac{x_1x_4x_6}{12}\Big)
\end{aligned}
$$

[49]:
```
def log5 (x1,x2,x3,x4,x5,x6,x7,x8,x9,x10):
    return(x1,x2,x3,x4,x5+x1*x2/2,x6,x7,x8+x1*x6/2+x1*x2*x3/12,
```

```
            x9,x10+x1*x9/2+x1*x2*x7/12+x1*x4*x6/12)
```

The action $\begin{array}{rccc} Ad: & N_5 \times \mathfrak{n}_5 & \longrightarrow & N_5 \\ & (X, X') & \longmapsto & (Ad(X))(X') \end{array}$ is colled the adjoint action of N_5 and it satisfies the formula $Ad(X^{-1})(X') = \log(X \exp(X')X^{-1})$ such that $X \in N_5$ and $X' \in \mathfrak{n}_5$.

So let's start by calculate $\exp(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \cdot_{\mathfrak{n}_5} (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})^{-1}$:

[50]:
```
prod5(x11,x21,x31,x41,x51-x11*x21/2,x61,x71,x81-x11*x61/2-
      x11*x21*x31/12,x91,x101-x11*x91/2-x11*x21*x71/12-
      x11*x41*x61/12,-x1,-x2,-x3,-x4,-x5-x1*x2,-x6,-x7,
      -x8-x1*x6,-x9,-x10-x1*x9)
```

[50]:
```
(-x1 + x11,
 -x2 + x21,
 -x3 + x31,
 -x4 + x41,
 -x1*x2 + x1*x21 - x11*x21/2 - x5 + x51,
 x2*x31/2 - x21*x3/2 - x6 + x61,
 x3*x41/2 - x31*x4/2 - x7 + x71,
 -x1*x6 + x1*x61 - x11*x21*x31/12 - x11*x61/2 + x3*(-x1*x21 + x11*x21/2 - x51)/2
- x31*(-x1*x2 - x5)/2 - x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 - x2*x31*x41/12 + x2*x71/2 + x21*x3*x4/12 +
x21*x3*x41/6 - x21*x31*x4/12 - x21*x7/2 - x4*x61/2 + x41*x6/2 - x9 + x91,
 -x1*x9 + x1*x91 - x10 + x101 - x11*x21*x71/12 - x11*x41*x61/12 - x11*x91/2 -
x3*x4*(-x1*x21 + x11*x21/2 - x51)/12 - x3*x41*(-x1*x2 - x5)/12 - x3*x41*(-x1*x21
+ x11*x21/2 - x51)/6 + x31*x4*(-x1*x2 - x5)/6 + x31*x4*(-x1*x21 + x11*x21/2 -
x51)/12 + x31*x41*(-x1*x2 - x5)/12 + x4*(-x1*x61 + x11*x21*x31/12 + x11*x61/2 -
x81)/2 - x41*(-x1*x6 - x8)/2 + x7*(-x1*x21 + x11*x21/2 - x51)/2 - x71*(-x1*x2 -
x5)/2)
```

Now we compute

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \cdot_{\mathfrak{n}_5} [\exp(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}) \\ \cdot_{\mathfrak{n}_5} (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})^{-1}]$$

[51]:
```
prod5(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,-x1 + x11,
 -x2 + x21,
 -x3 + x31,
 -x4 + x41,
 -x1*x2 + x1*x21 - x11*x21/2 - x5 + x51,
 x2*x31/2 - x21*x3/2 - x6 + x61,
 x3*x41/2 - x31*x4/2 - x7 + x71,
 -x1*x6 + x1*x61 - x11*x21*x31/12 - x11*x61/2 +
x3*(-x1*x21 + x11*x21/2 - x51)/2 - x31*(-x1*x2 - x5)/2 -
      x8 + x81,
 x2*x3*x41/12 - x2*x31*x4/6 - x2*x31*x41/12 + x2*x71/2 +
    x21*x3*x4/12 + x21*x3*x41/6 - x21*x31*x4/12 - x21*x7/2 -
```

```
      x4*x61/2 + x41*x6/2 - x9 + x91,
 -x1*x9 + x1*x91 - x10 + x101 - x11*x21*x71/12 - x11*x41*x61/12 -
x11*x91/2 - x3*x4*(-x1*x21 + x11*x21/2 - x51)/12 -
x3*x41*(-x1*x2 - x5)/12 - x3*x41*(-x1*x21 + x11*x21/2 - x51)/6
+ x31*x4*(-x1*x2 - x5)/6 + x31*x4*(-x1*x21 + x11*x21/2 - x51)/12
+ x31*x41*(-x1*x2 - x5)/12 + x4*(-x1*x61 + x11*x21*x31/12 +
x11*x61/2 - x81)/2 - x41*(-x1*x6 - x8)/2 +
      x7*(-x1*x21 + x11*x21/2 - x51)/2 - x71*(-x1*x2 - x5)/2)
```

[51]:
```
(x11,
 x21,
 x31,
 x41,
 -x1*x2 + x1*x21 - x11*x21/2 + x2*(x1 - x11) + x51,
 x2*x31/2 + x2*(-x3 + x31)/2 - x21*x3/2 - x3*(-x2 + x21)/2 + x61,
 x3*x41/2 + x3*(-x4 + x41)/2 - x31*x4/2 - x4*(-x3 + x31)/2 + x71,
 -x1*x6 + x1*x61 - x11*x21*x31/12 - x11*x61/2 + x3*(-x1*x21 + x11*x21/2 - x51)/2
- x3*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/2 - x31*(-x1*x2 - x5)/2 + x6*(x1 -
x11) + x81 - (-x3 + x31)*(x2*(-x1 + x11) - x5)/2,
 x2*x3*x41/12 + x2*x3*(-x4 + x41)/12 - x2*x31*x4/6 - x2*x31*x41/12 - x2*x4*(-x3
+ x31)/6 + x2*x71/2 + x2*(-x3 + x31)*(-x4 + x41)/12 + x2*(x3*x41/2 - x31*x4/2 -
x7 + x71)/2 + x21*x3*x4/12 + x21*x3*x41/6 - x21*x31*x4/12 - x21*x7/2 +
x3*x4*(-x2 + x21)/12 - x3*(-x2 + x21)*(-x4 + x41)/6 - x4*x61/2 + x4*(-x2 +
x21)*(-x3 + x31)/12 - x4*(x2*x31/2 - x21*x3/2 - x6 + x61)/2 + x41*x6/2 + x6*(-x4
+ x41)/2 - x7*(-x2 + x21)/2 + x91,
 -x1*x9 + x1*x91 + x101 - x11*x21*x71/12 - x11*x41*x61/12 - x11*x91/2 -
x3*x4*(-x1*x21 + x11*x21/2 - x51)/12 + x3*x4*(-x1*x2 + x1*x21 - x11*x21/2 - x5 +
x51)/12 - x3*x41*(-x1*x2 - x5)/12 - x3*x41*(-x1*x21 + x11*x21/2 - x51)/6 -
x3*(-x4 + x41)*(x2*(-x1 + x11) - x5)/12 - x3*(-x4 + x41)*(-x1*x2 + x1*x21 -
x11*x21/2 - x5 + x51)/6 + x31*x4*(-x1*x2 - x5)/6 + x31*x4*(-x1*x21 + x11*x21/2 -
x51)/12 + x31*x41*(-x1*x2 - x5)/12 + x4*(-x3 + x31)*(x2*(-x1 + x11) - x5)/6 +
x4*(-x3 + x31)*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/12 + x4*(-x1*x61 +
x11*x21*x31/12 + x11*x61/2 - x81)/2 - x4*(-x1*x6 + x1*x61 - x11*x21*x31/12 -
x11*x61/2 + x3*(-x1*x21 + x11*x21/2 - x51)/2 - x31*(-x1*x2 - x5)/2 - x8 + x81)/2
- x41*(-x1*x6 - x8)/2 + x7*(-x1*x21 + x11*x21/2 - x51)/2 - x7*(-x1*x2 + x1*x21 -
x11*x21/2 - x5 + x51)/2 - x71*(-x1*x2 - x5)/2 - x9*(-x1 + x11) - (-x3 +
x31)*(-x4 + x41)*(x2*(-x1 + x11) - x5)/12 - (-x4 + x41)*(x6*(-x1 + x11) - x8)/2
- (x2*(-x1 + x11) - x5)*(x3*x41/2 - x31*x4/2 - x7 + x71)/2)
```

The result is simplified as follows:

[52]:
```
(x11,
 x21,
 x31,
 x41,
 simplify(-x1*x2 + x1*x21 - x11*x21/2 + x2*(x1 - x11) + x51),
 simplify(x2*x31/2 + x2*(-x3 + x31)/2 - x21*x3/2 -
```

```
          x3*(-x2 + x21)/2 + x61),
 simplify(x3*x41/2 + x3*(-x4 + x41)/2 - x31*x4/2 -
          x4*(-x3 + x31)/2 + x71),
 simplify(-x1*x6 + x1*x61 - x11*x21*x31/12 - x11*x61/2 +
x3*(-x1*x21 + x11*x21/2 - x51)/2 -
x3*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/2
    - x31*(-x1*x2 - x5)/2 + x6*(x1 - x11) + x81 -
          (-x3 + x31)*(x2*(-x1 + x11) - x5)/2),
 simplify( x2*x3*x41/12 + x2*x3*(-x4 + x41)/12 - x2*x31*x4/6 -
    x2*x31*x41/12 - x2*x4*(-x3 + x31)/6 + x2*x71/2 +
    x2*(-x3 + x31)*(-x4 + x41)/12 + x2*(x3*x41/2 - x31*x4/2 -
    x7 + x71)/2 + x21*x3*x4/12 + x21*x3*x41/6 - x21*x31*x4/12 -
    x21*x7/2 + x3*x4*(-x2 + x21)/12 -
    x3*(-x2 + x21)*(-x4 + x41)/6 - x4*x61/2 +
    x4*(-x2 + x21)*(-x3 + x31)/12 - x4*(x2*x31/2 - x21*x3/2 - x6
    + x61)/2 + x41*x6/2 + x6*(-x4 + x41)/2 -
          x7*(-x2 + x21)/2 + x91),
 simplify( -x1*x9 + x1*x91 + x101 - x11*x21*x71/12 -
x11*x41*x61/12 - x11*x91/2 - x3*x4*(-x1*x21 + x11*x21/2 - x51)/12
+ x3*x4*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/12 -
x3*x41*(-x1*x2 - x5)/12 - x3*x41*(-x1*x21 + x11*x21/2 - x51)/6
- x3*(-x4 + x41)*(x2*(-x1 + x11) - x5)/12 -
x3*(-x4 + x41)*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/6
+ x31*x4*(-x1*x2 - x5)/6 + x31*x4*(-x1*x21 + x11*x21/2 - x51)/12
+ x31*x41*(-x1*x2 - x5)/12 +
x4*(-x3 + x31)*(x2*(-x1 + x11) - x5)/6 +
x4*(-x3 + x31)*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/12 +
x4*(-x1*x61 + x11*x21*x31/12 + x11*x61/2 - x81)/2 -
x4*(-x1*x6 + x1*x61 - x11*x21*x31/12 - x11*x61/2 +
x3*(-x1*x21 + x11*x21/2 - x51)/2 - x31*(-x1*x2 - x5)/2 - x8 +
x81)/2 - x41*(-x1*x6 - x8)/2 +
x7*(-x1*x21 + x11*x21/2 - x51)/2 -
x7*(-x1*x2 + x1*x21 - x11*x21/2 - x5 + x51)/2 -
x71*(-x1*x2 - x5)/2 - x9*(-x1 + x11) -
(-x3 + x31)*(-x4 + x41)*(x2*(-x1 + x11) - x5)/12 -
(-x4 + x41)*(x6*(-x1 + x11) - x8)/2 -
(x2*(-x1 + x11) - x5)*(x3*x41/2 - x31*x4/2 - x7 + x71)/2))
```

[52]:

```
(x11,
 x21,
 x31,
 x41,
 x1*x21 - x11*x2 - x11*x21/2 + x51,
 x2*x31 - x21*x3 + x61,
 x3*x41 - x31*x4 + x71,
 x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x2*x31/2 + x11*x21*x3/2 -
x11*x21*x31/12 - x11*x6 - x11*x61/2 - x3*x51 + x31*x5 + x81,
```

```
 x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61 + x41*x6 +
x91,
 x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 - x1*x21*x7 -
x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 - x11*x2*x3*x4/6 - x11*x2*x3*x41/4 +
x11*x2*x31*x4/2 - x11*x2*x31*x41/12 + x11*x2*x7/2 - x11*x2*x71/2 -
x11*x21*x3*x4/4 + x11*x21*x31*x4/12 + x11*x21*x7/2 - x11*x21*x71/12 +
x11*x4*x6/2 + x11*x4*x61/2 - x11*x41*x6/2 - x11*x41*x61/12 - x11*x9 - x11*x91/2
+ x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8 + x5*x71 - x51*x7)
```

Latest we have :

$$\log[(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})\cdot_{\mathfrak{n}_5}[\exp(x'_1,x'_2,x'_3,x'_4,x'_5,x'_6,x'_7,x'_8,x'_9,x'_{10})$$
$$\cdot_{\mathfrak{n}_5}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})^{-1}]]$$

[57]:
```
log5(x11,
 x21,
 x31,
 x41,
 x1*x21 - x11*x2 - x11*x21/2 + x51,
 x2*x31 - x21*x3 + x61,
 x3*x41 - x31*x4 + x71,
 x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x2*x31/2 +
     x11*x21*x3/2 - x11*x21*x31/12 - x11*x6 - x11*x61/2 -
     x3*x51 + x31*x5 + x81,
 x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61
     + x41*x6 + x91,
 x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 -
     x1*x21*x7 - x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 -
     x11*x2*x3*x4/6 - x11*x2*x3*x41/4 + x11*x2*x31*x4/2 -
     x11*x2*x31*x41/12 + x11*x2*x7/2 - x11*x2*x71/2 -
     x11*x21*x3*x4/4 + x11*x21*x31*x4/12 + x11*x21*x7/2 -
     x11*x21*x71/12 + x11*x4*x6/2 + x11*x4*x61/2 - x11*x41*x6/2
     - x11*x41*x61/12 - x11*x9 - x11*x91/2 + x3*x4*x51/2 +
     x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8 + x5*x71 - x51*x7)
```

[57]:
```
(x11,
 x21,
 x31,
 x41,
 x1*x21 - x11*x2 + x51,
 x2*x31 - x21*x3 + x61,
 x3*x41 - x31*x4 + x71,
 x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x2*x31/2 + x11*x21*x3/2 -
x11*x6 - x11*x61/2 + x11*(x2*x31 - x21*x3 + x61)/2 - x3*x51 + x31*x5 + x81,
 x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61 + x41*x6 +
x91,
 x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 - x1*x21*x7 -
```

```
x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 - x11*x2*x3*x4/6 - x11*x2*x3*x41/4 +
x11*x2*x31*x4/2 - x11*x2*x31*x41/12 + x11*x2*x7/2 - x11*x2*x71/2 -
x11*x21*x3*x4/4 + x11*x21*x31*x4/12 + x11*x21*x7/2 - x11*x21*x71/12 +
x11*x21*(x3*x41 - x31*x4 + x71)/12 + x11*x4*x6/2 + x11*x4*x61/2 - x11*x41*x6/2 -
x11*x41*x61/12 + x11*x41*(x2*x31 - x21*x3 + x61)/12 - x11*x9 - x11*x91/2 +
x11*(x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61 + x41*x6 +
x91)/2 + x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8 + x5*x71 -
x51*x7)
```

[58]:
```
(x11,
 x21,
 x31,
 x41,
 x1*x21 - x11*x2 + x51,
 x2*x31 - x21*x3 + x61,
 x3*x41 - x31*x4 + x71,
 simplify(x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 -
          x11*x2*x31/2 + x11*x21*x3/2 - x11*x6 - x11*x61/2 +
          x11*(x2*x31 - x21*x3 + x61)/2 - x3*x51 + x31*x5 + x81),
 simplify(x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7
          - x4*x61 + x41*x6 + x91),
 simplify(x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 +
          x1*x21*x3*x4/2 - x1*x21*x7 - x1*x4*x61 + x1*x41*x6 +
          x1*x91 + x101 - x11*x2*x3*x4/6 - x11*x2*x3*x41/4 +
          x11*x2*x31*x4/2 - x11*x2*x31*x41/12 + x11*x2*x7/2 -
          x11*x2*x71/2 - x11*x21*x3*x4/4 + x11*x21*x31*x4/12 +
          x11*x21*x7/2 - x11*x21*x71/12 +
          x11*x21*(x3*x41 - x31*x4 + x71)/12 + x11*x4*x6/2 +
          x11*x4*x61/2 - x11*x41*x6/2 - x11*x41*x61/12 +
          x11*x41*(x2*x31 - x21*x3 + x61)/12 - x11*x9 - x11*x91/2
          + x11*(x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2
                 - x21*x7 - x4*x61 + x41*x6 + x91)/2 +
          x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 +
          x41*x8 + x5*x71 - x51*x7))
```

[58]:
```
(x11,
 x21,
 x31,
 x41,
 x1*x21 - x11*x2 + x51,
 x2*x31 - x21*x3 + x61,
 x3*x41 - x31*x4 + x71,
 x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x6 - x3*x51 + x31*x5 + x81,
 x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61 + x41*x6 +
x91,
 x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 - x1*x21*x7 -
x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 - x11*x2*x3*x4/6 + x11*x2*x7/2 +
```

```
x11*x4*x6/2 - x11*x9 + x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8 +
x5*x71 - x51*x7)
```

we need to define the natural scalar product:

```
[59]: def prod(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x21,x31,x41,x51,x61,
              x71,x81,x91,x101):
          return (x1+x11,x2+x21,x3+x31,x4+x41,x5+x51,x6+x61,x7+x71,
                  x8+x81,x9+x91,x10+x101)
```

Now we can use the scalar product to identify $\mathfrak{n}_5$ by its dual space $\mathfrak{n}_5^*$ and we obtain the following expression of Ad^*, where $Ad^*(X)(X")(X') =< X'', Ad(X^{-1})(X') >, X \in N_5, X' \in \mathfrak{n}_5, X" \in \mathfrak{n}_5^*$

```
[56]: prod(x11,
       x21,
       x31,
       x41,
       x1*x21 - x11*x2 + x51,
       x2*x31 - x21*x3 + x61,
       x3*x41 - x31*x4 + x71,
       x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x6 - x3*x51
           + x31*x5 + x81,
       x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61
           + x41*x6 + x91,
       x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 -
           x1*x21*x7 - x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 -
           x11*x2*x3*x4/6 + x11*x2*x7/2 + x11*x4*x6/2 - x11*x9 +
           x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8
           + x5*x71 - x51*x7,x12,x22,x32,x42,x52,x62,x72,x82,x92,x102)
```

```
[56]: (x11 + x12,
       x21 + x22,
       x31 + x32,
       x41 + x42,
       x1*x21 - x11*x2 + x51 + x52,
       x2*x31 - x21*x3 + x61 + x62,
       x3*x41 - x31*x4 + x71 + x72,
       x1*x2*x31 - x1*x21*x3 + x1*x61 + x11*x2*x3/2 - x11*x6 - x3*x51 + x31*x5 + x81 +
      x82,
       x2*x3*x41/2 - x2*x31*x4 + x2*x71 + x21*x3*x4/2 - x21*x7 - x4*x61 + x41*x6 + x91
      + x92,
       x1*x2*x3*x41/2 - x1*x2*x31*x4 + x1*x2*x71 + x1*x21*x3*x4/2 - x1*x21*x7 -
      x1*x4*x61 + x1*x41*x6 + x1*x91 + x101 + x102 - x11*x2*x3*x4/6 + x11*x2*x7/2 +
      x11*x4*x6/2 - x11*x9 + x3*x4*x51/2 + x3*x41*x5/2 - x31*x4*x5 - x4*x81 + x41*x8 +
      x5*x71 - x51*x7)
```

So, For all $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \in N_5$ and $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}) \in \mathfrak{n}_5^*$ we have that:

$$
\begin{aligned}
& Ad^*(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9,f_{10}) \\
= & \Big(f_1 - f_5x_2 - f_8x_6 + f_8\frac{x_2x_3}{2} - f_{10}x_9 - f_{10}\frac{x_2x_3x_4}{6} + f_{10}\frac{x_2x_7}{2} + f_{10}\frac{x_4x_6}{2}, f_2 + f_5x_1 - f_6x_3 - f_8x_1x_3 \\
& + f_9\frac{x_3x_4}{2} - f_9x_7 + f_{10}\frac{x_1x_3x_4}{2} - f_{10}x_1x_7, f_3 + f_6x_2 - f_7x_4 + f_8x_1x_2 + f_8x_5 - f_9x_2x_4 - f_{10}x_1x_2x_4 \\
& - f_{10}x_4x_5, f_4 + f_7x_3 + f_9\frac{x_2x_3}{2} + f_9x_6 + f_{10}\frac{x_1x_2x_3}{2} + f_{10}x_1x_6 + f_{10}\frac{x_3x_5}{2} + f_{10}x_8, f_5 - f_8x_3 \\
& + f_{10}\frac{x_3x_4}{2} - f_{10}x_7, f_6 + f_8x_1 - f_9x_4 - f_{10}x_1x_4, f_7 + f_9x_2 + f_{10}x_1x_2 + f_{10}x_5, f_8 - f_{10}x_4, f_9 + f_{10}x_1, f_{10}\Big).
\end{aligned}
$$

References

[1] Corwin,L.J, Greenleaf,F.P. Representations of nilpotent Lie groups and their application. Part I. Basic theory and examples. Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, 1990. 142 pp.