Intro to Linear Algebra I

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Overview

Why linear algebra?

Kerner et al. 2015. "Does it pay to be poor? Testing for systematically underreported GNI estimates"

2.5.2 Regression estimates

Our second empirical strategy to test whether the discontinuity in the historical GNI per capita data is really "there" is to estimate a series of regression models that assess whether being IDA eligible in the present "causes" countries to be aid-dependent in the past. The logic behind this test is again borrowed from diagnostic tests used in the regression discontinuity literature (see, for example, Lee (2008: 690–91) and Caughey and Sekhon (2011: 392–393)). To do so we estimate the following equation:

$$\text{(log)Aid per capital}_{it-10} = \alpha + \beta 1\text{Threshold\_Distance}_{it} + \beta 2\text{Threshold\_Distance}^2{}_{it} + \beta 3\text{IDA}_{it} + \beta 4\text{IDA}_{it} * \text{Threshold\_Distance}_{it} + \beta 5\text{IDA}_{it} * \text{Threshold\_Distance}^2{}_{it} \cdots + e_{it}$$

where the i subscript indicates country and the t subscript indicates year. IDA is a

Why linear algebra?

Nunn and Wantchekon, "The origins of mistrust in Africa"

III. Estimating Equations and Empirical Results

A. *OLS Estimates*

We begin by estimating the relationship between the number of slaves that were taken from an individual's ethnic group and the individual's current level of trust. Our baseline estimating equation is:

$$(1) \quad trust_{i,e,d,c} = \alpha_c + \beta\, slave\ exports_e + \mathbf{X}'_{i,e,d,c}\Gamma + \mathbf{X}'_{d,c}\Omega + \mathbf{X}'_{e}\Phi + \varepsilon_{i,e,d,c},$$

where i indexes individuals, e ethnic groups, d districts, and c countries. The variable $trust_{i,e,d,c}$ denotes one of our five measures of trust, which vary across individuals. α_c denotes country fixed effects, which are included to capture country-specific factors, such as government regulations, that may affect trust (e.g., Philippe Aghion et al. 2010; Aghion, Algan, and Cahuc 2008). $slave\ exports_e$ is a measure of the number of slaves taken from ethnic group e during the slave trade. (We discuss this variable in more detail below.) Our coefficient of interest is β, the estimated relationship between the slave exports of an individual's ethnic group and the individual's current level of trust.

Why linear algebra?

B. Basic Specification

Ignoring nonlinearities, we can write the economic relationship we are interested in identifying as

$$Y_c = \alpha \cdot F_c + \beta \cdot I_c + \mathbf{Z}_c' \cdot \boldsymbol{\gamma}_0 + \epsilon_c, \tag{1}$$

where Y_c is the outcome of interest for country c, F_c is a measure of contracting institutions, I_c is a measure of property rights institutions, and $\mathbf{Z}_c$ is a vector of other controls. The coefficients α and β are the parameters of interest, and $\boldsymbol{\gamma}_0$ is a vector capturing effects of the control variables in $\mathbf{Z}_c$.[5]

What are vectors?

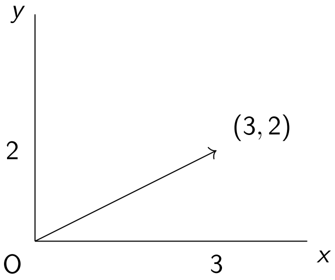
- Vector: A serial listing of numbers where the order matters (Gill, 83)

a row vector: $[1, 2, 3, 4]$

a column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

- Vector is written in bold type: $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

- Vector in a more general form: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$


y
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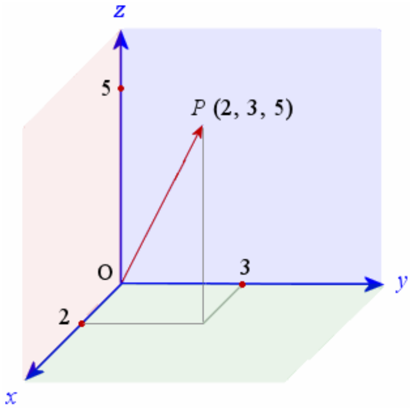
z
y
x
O
5
3
2
P (2, 3, 5)

Vector calculation

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

- **u+v**
- **u-v**

Vectors should be **conformable** to conduct calculation.

1. $\underset{1\times 4}{\mathbf{u}} + \underset{1\times 4}{\mathbf{v}}$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 0 \\ 5 \end{bmatrix}$

2. $\underset{1\times 4}{\mathbf{u}} + \underset{1\times 3}{\mathbf{v}}$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \\ -3 \end{bmatrix} = \cancel{\begin{bmatrix} 8 \\ 5 \\ 0 \\ 4 \end{bmatrix}}$

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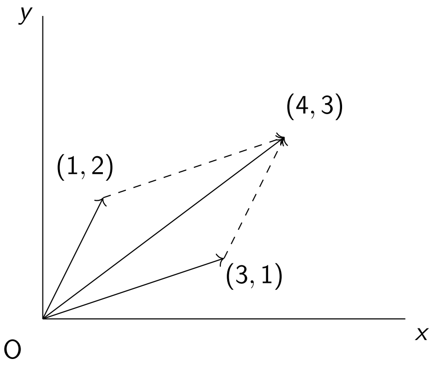
Vector calculation visualization

$$\underset{1\times 2}{\mathbf{u}} + \underset{1\times 2}{\mathbf{v}}: \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$



Vector multiplication and product

1. Vector scalar multiplication

$$3\mathbf{v} = 3 \cdot \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \\ -9 \\ 3 \end{bmatrix}$$

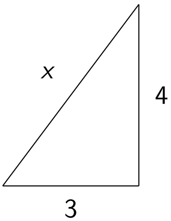
2. Vector inner product

$$\mathbf{u} \cdot \mathbf{v} = [u_1 v_1 + u_2 v_2 + \cdots + u_k v_k] = \sum u_i v_i$$

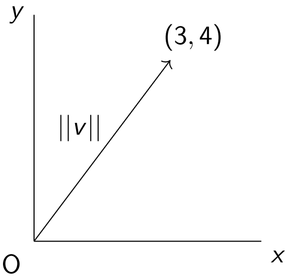
$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 7 + 2 \cdot 3 + 3 \cdot -3 + 4 \cdot 1 = 8$$

Vector distance visualization



$$x = \sqrt{3^2 + 4^2}$$



$$||v|| = \sqrt{v_1^2 + v_2^2}$$

Distance of vector: vector norm $||\mathbf{v}||$

- For example, with vector $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
- The length of the diagonal line is: $\sqrt{4^2 + 2^2} = \sqrt{20}$

Vector norm

$$||v|| = (v_1^2 + v_2^2 + \dots v_n^2)^{\frac{1}{2}} = (\mathbf{v}'\mathbf{v})^{\frac{1}{2}}$$

- Practice: calculate vector norms

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$$

- Which vector is longer? $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$

Matrix

What are matrices?

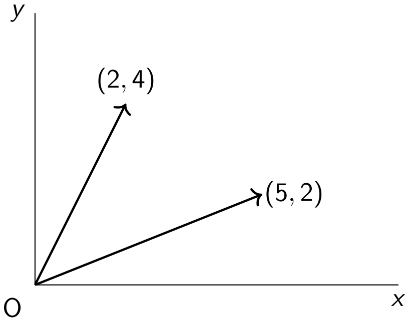
- two dimensions: rows and columns $\underset{2\times 2}{\mathbf{X}} = \begin{bmatrix} 1 & 3 \\ 7 & 3 \end{bmatrix}$
- Examples: dimensions?

$$\begin{bmatrix} 1 & 3 & 4 \\ 7 & 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 9 \\ 8 & 10 \\ 9 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 9 & 2 & 7 \\ 8 & 10 & 2 & 1 \end{bmatrix}$$

$\underset{i\times j}{\mathbf{X}}$: i for rows, j for columns.

Matrix visualization

With a matrix of $\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$,



Matrix calculations

1. Addition, subtraction

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 1-2 & 2+2 \\ 3+0 & 4+1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 1+2 & 2-2 \\ 3-0 & 4-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

2. Scalar Multiplication

$$s = 5$$

$$s\mathbf{X} = \begin{bmatrix} 5 \times 1 & 5 \times 2 \\ 5 \times 3 & 5 \times 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

Matrix Multiplication

$$\underset{(k\times n)}{\mathbf{X}}\underset{(n\times p)}{\mathbf{Y}} = \underset{(k\times p)}{\mathbf{XY}}$$

$$\begin{aligned}\mathbf{XY} &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\ &= \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}\end{aligned}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}\mathbf{XY} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2+0 & 2+2 \\ -6+0 & 6+4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -6 & 10 \end{bmatrix}\end{aligned}$$

Matrix properties

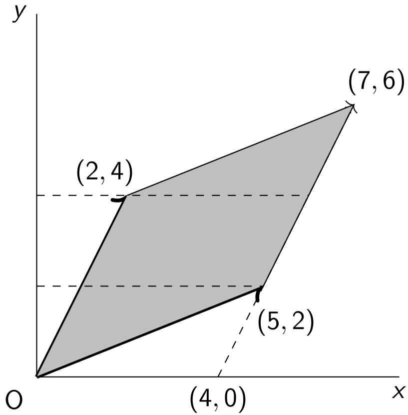
Matrix inversion $XX^{-1} = X^{-1}X = I$

- When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

  $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

  $|A| = ad - bc$, where $|A|$ is the **determinant** of matrix A.

Determinant



- Size of the shaded area:
  $4 \times 4 = 16$
- In a matrix form, we have
  $\begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$
- Determinant of the matrix:
  $ad - bc = 5 \cdot 4 - 2 \cdot 2 = 16$
- $|A|$ = shaded area of the matrix

Inverse

Show that when $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $A \cdot A^{-1} = A^{-1} \cdot A = I$

From equations to matrices

Vectors, Matrices are useful when it comes to dealing with complex algebraic calculations, especially when we delve into linear regressions. For example, in the next session, James will talk more about what it means to solve the following equation:

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

But, for now, it's the best to understand that linear regression is one of different equations that we want to find a solution to. So, as our final step, let's look at how we can solve equations with linear algebra.

Consider the following system of equations,

$$\begin{aligned} 2x_1 - 3x_2 &= 4 \\ 5x_1 + 5x_2 &= 3, \end{aligned}$$

we can rewrite them into matrix form,

$$\begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

x_1 and x_2 can be derived by multiplying the inverse of the 2×2 matrix on both sides

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.16 \\ -0.56 \end{bmatrix}$$

The above system of equations can be simplified as:

$$\mathbf{Ax} = \mathbf{b}$$

Linear Systems of Equations

For the following expression on the relationship between political blame and regional political variables, simplify it in matrix algebra form.

$$
\begin{aligned}
Y_i = \beta_0 &+ \beta_1 CHANGELIV + \beta_2 BLAMECOMM + \beta_3 INCOME \\
&+ \beta_4 FARMER + \beta_5 OWNER + \beta_6 BLUESTATE \\
&+ \beta_7 WHITESTATE + \beta_8 FORMMCOMM + \beta_9 AGE \\
&+ \beta_{10} SQAGE + \beta_{11} SEX + \beta_{12} SIZEPLACE \\
&+ \beta_{13} EDUC + \beta_{14} FINHS + \beta_{15} ED * HS \\
&+ \beta_{16} RELIG + \beta_{17} NATION + E_i, \text{for } i{=}1 \text{ to } n
\end{aligned}
$$

$\Rightarrow$

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$