MA 202 Project Report

Probabilistic Methods

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Abstract

This report discusses the probabilistic method for problem-solving. In this paper, probabilistic methods are used to prove theorems based on graph theory. Some examples of problems are addressed.

1 Introduction

Probabilistic Methods: If a random object in a set satisfies some property with positive probability, then there exists an object in that set that satisfies that property. The method used to prove such problems are called Probabilistic Method. For example - The Ramsey-number $R(k,l)$ is the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or there is a blue K_l.

Proposition 1.1 If $\begin{pmatrix} n \\ k \end{pmatrix} \cdot 2^{1-\begin{pmatrix} k \\ 2 \end{pmatrix}} < 1$ then $R(k,k) > n$. Thus $R(k,k) > \lfloor 2^{k/2} \rfloor$ for all $k \geq 3$

Proof. : We want to show that:

$$\mathbb{P}\left(\nexists R \subseteq V\left(K_R\right) \text{ with } |R| = k \text{ such that } A_R\right) > 0$$

Let A_R denote the event that $K_n[R]$ is monochromatic for $R \subseteq V\left(K_n\right)$

$$\mathbb{P}\left(A_R\right) = \frac{2}{2^{\left(\frac{k}{2}\right)}} = 2^{1-\left(\frac{k}{2}\right)}$$

$$\mathbb{P}\left(\nexists R \subseteq V\left(K_R\right) \text{ with } |R| = k \text{ such that } A_R\right)$$

$$= 1 \text{ - P}(\exists R \subseteq V\left(K_n\right) \text{ with } |R| = k \text{ such that } A_R)$$
$$= 1 \text{ - P}(\underbrace{U}_{R:R \subseteq V(K_n)} A_R)$$
$$\geq 1 - \textstyle\sum_{R:R\subseteq V(K_n)} \mathbb{P}\left(A_R\right)$$
$$= 1 \text{ - } \begin{pmatrix} n \\ k \end{pmatrix} \cdot 2^{1-\begin{pmatrix} k \\ 2 \end{pmatrix}}$$

Now$further$ $\begin{pmatrix} n \\ k \end{pmatrix} \cdot 2^{1-\begin{pmatrix} k \\ 2 \end{pmatrix}} \leqslant \left(\frac{2^{k^2/2}}{2^{k^2/2}} \right) \cdot \left(\frac{2^{1+k/2}}{k} \right)$ for n = $\lfloor 2^{k/2} \rfloor$

the second term $\frac{2^{1+k/2}}{k} \leqslant 1 \; for \; k \geq 3$

$Hence$ $\begin{pmatrix} n \\ k \end{pmatrix} \cdot 2^{1-\begin{pmatrix} k \\ 2 \end{pmatrix}}$ <1

For a large value of k, *a random coloring of* K_n is not likely to contain a monochromatic subgraph K_k. This probabilistic, nonconstructive method provides an effective probabilistic algorithm.

Let us take an example, find the value of $R(3,3)$. Here $k = 3$, $n = \lfloor 2^{3/2} \rfloor$. Hence $R(3,3) \geq 6$

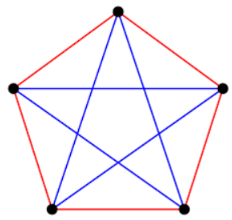


Figure 1: Graph with 5 vertices

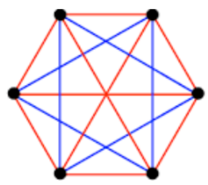


Figure 2: $R(3,3) = 6$

$\square$

2 Definitions

Tournament: Tournament on n vertices is a directed graph where for every pair of vertices x,y either (x, y) or (y, x) belongs to the edge set.
S_k**:** A graph with n vertices has property, S_k if for every set of k players there is one who beats them all.

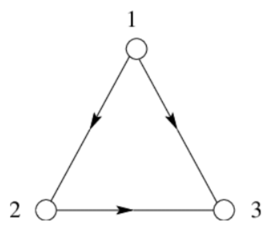


Figure 3: Tournament on 3 vertices with S_1 property

3 Body of the Work

Theorem 3.1. *If* $\begin{pmatrix} n \\ k \end{pmatrix}\left(1-2^{-k}\right)^{n-k}<1$ *then there is a tournament on* n *vertices that has the property* S_k*.*

Proof. A$_R$: Event such that $\exists$ set of k players R, $\nexists$ a player who beats every player in R $\mathbb{P}\left(A_R\right)=\left(1-2^{-k}\right)^{n-k}$

$\mathbb{P}(\forall$ set of k players $R, \exists$ a player who beats every player in $R)$
$=(1-\mathrm{P}(\exists$ set of k players R, $\nexists$ a player who beats every player in $R)$
$=1-\mathbb{P}\left(\bigcup_R A_R\right) \geq 1-\sum_R \mathbb{P}\left(A_R\right)$
$=1$ - $\begin{pmatrix} n \\ k \end{pmatrix}\left(1-2^{-k}\right)^{n-k}>0$

Therefore with positive probability there is a tournament on n vertices with property S_k.

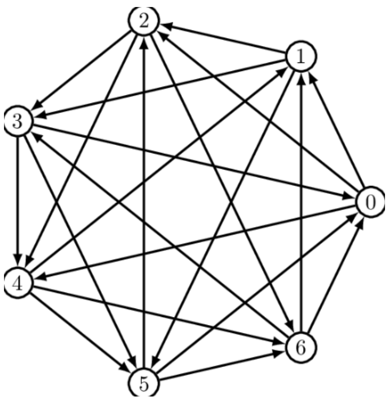


Figure 4: Tournament on 7 vertices with S_2 property

The Above theorem implies that the minimum possible number of vertices of tournament that has property S_k is given by $f(k) \leq k^2 \cdot 2^k \cdot(\ln 2)(1+o(1))$ and as proved by Szekeres f(k) $\geq c1 \cdot k \cdot 2^k$. we can verify for S_2 the value of $f(2)$ ≥ 5.33 which can be easily checked from Figure 4 that the values of n comes out to be 7 .

Theorem 3.2. *Let $G(V, E)$ be a graph with n vertices with minimum degree $\delta > 1$. Then G has a dominating set of at most $n[1+\ln(\delta+1)]/(\delta+1)$ vertices.*

Proof. Let X be a random subset of vertices where each vertex is included with probability P. Let Y_x be the set of vertices not in X who do not have a neighbor in X. Define $U_x = X \mathbb{U} Y_x$. Observe that U_x is a dominating set.

$$\mathbb{E}\left(|U_x|\right) = \mathbb{E}(|X|) + \mathbb{E}(|Y_x\ |)$$
$$= np + \sum_{v \in v} \mathbb{P}\left(v \& x \text{ and } \forall v' \in N(x) v' \& x\right)$$

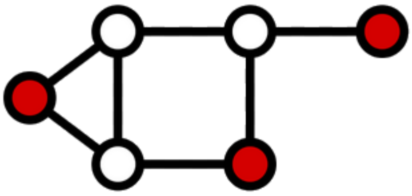
Now $\mathbb{P}\left(v \& x \text{ and } \forall v' \in N(x) v' \& x\right) = (1-p)^{|+|N(x)|} \leqslant (1-p)^{\delta+1}$

$$= \ \ n * p + (1-p)^{\delta+1}$$

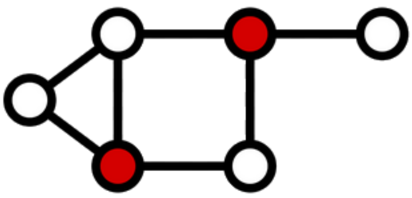
To minimize the above expected value of set U_x we substitute p $= \frac{\ln(\delta)+1}{\delta+1}$

$$= n[1+\ln(\delta+1)]/(\delta+1)$$

Hence required result is obtained. □



We can see that for the above graph the dominating set has 3 vertices and from our result the dominating set must contain at the most 5 vertices if we select vertices randomly.
But if we select the number of vertices for the dominating set wisely it will contain 2 vertices as shown in below figure



We used the property-Linerity of expectation in the above proof

Let $X_1, \ldots, X_n$ be random variables, X=c$_1 X_1 + \ldots + c_n X_n$. Linearity of Expectation states that

$$E[X] = c_1 E\left[X_1\right] + \ldots + c_n E\left[X_n\right]$$

4 Conclusions

Probabilistic methods provides an effective alternative for the exhaustive search approach whose runtime maybe more then polynomial time. We showed how probabilistic method can be applied on non-deterministic tournament problems to determine the bound for which the property satisfies. We showed using basic methods of probabilistic method to express the power of random selection which obtains satisfactory results for the existence of a property.

References

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