Classification with Linear Models

Losses for linear classification, logistic regression, multiclass classification

Machine Learning and Data Mining, 2021

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LAMBDA • HSE

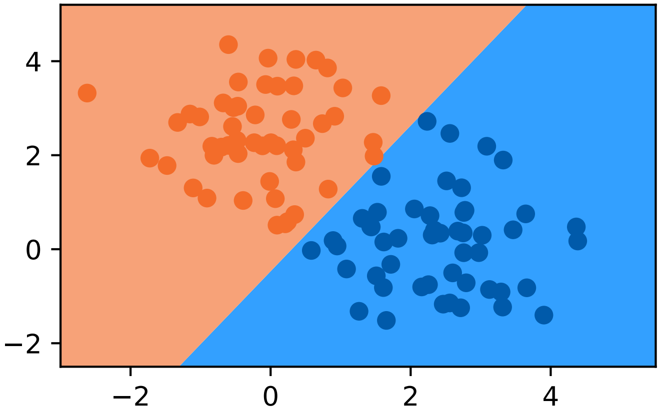
September 22, 2021

Can’t we just use linear regression for classification?

Classification with linear regression

Classification:

$$\hat{f}(x) = \text{sign}[\theta^{\mathrm{T}} x]$$



- For binary classification task, assign:
  - $y = +1$ for **positive** class
  - $y = -1$ for **negative** class

Classification with linear regression

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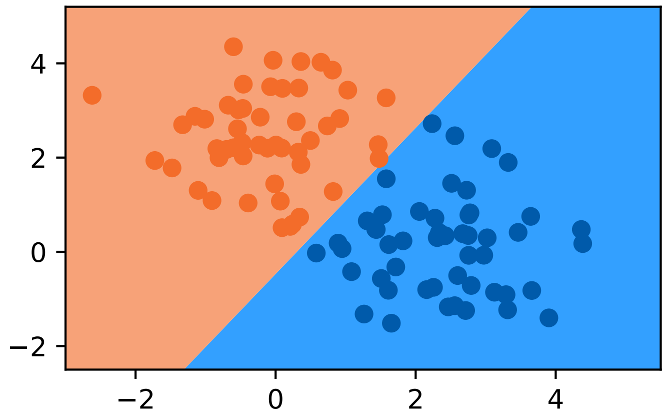

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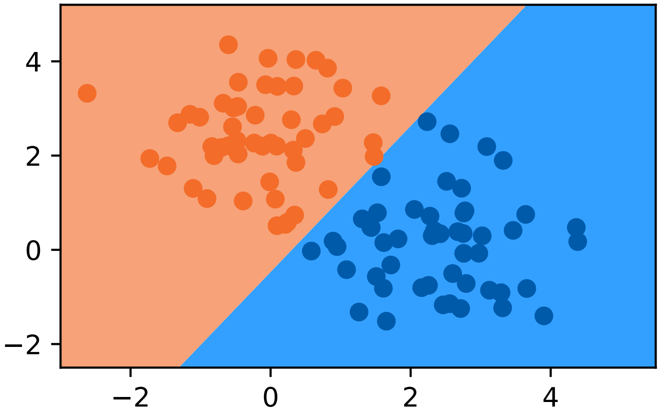

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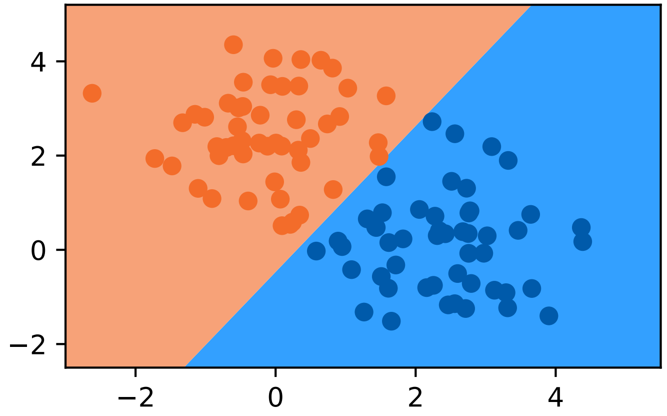

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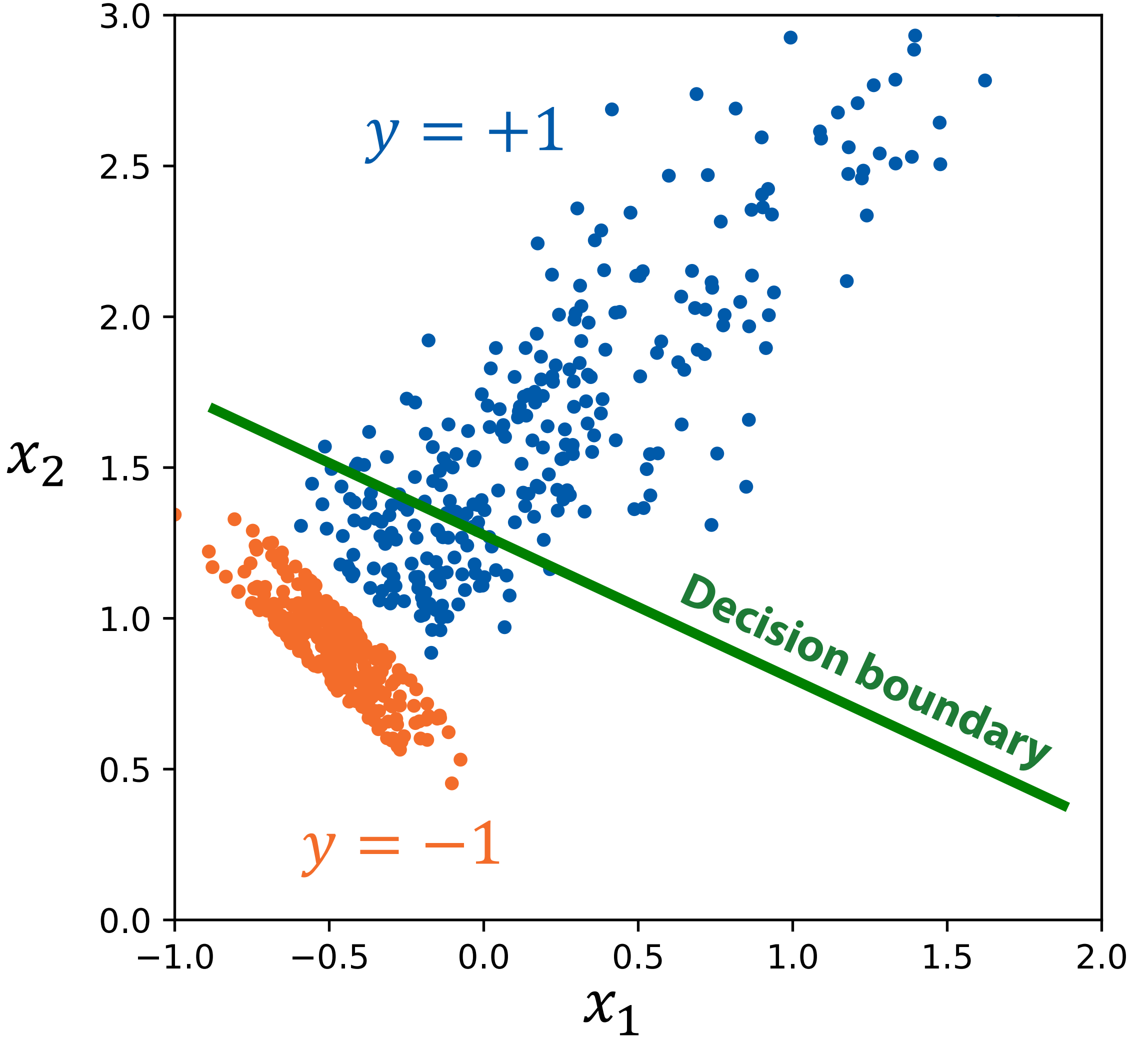
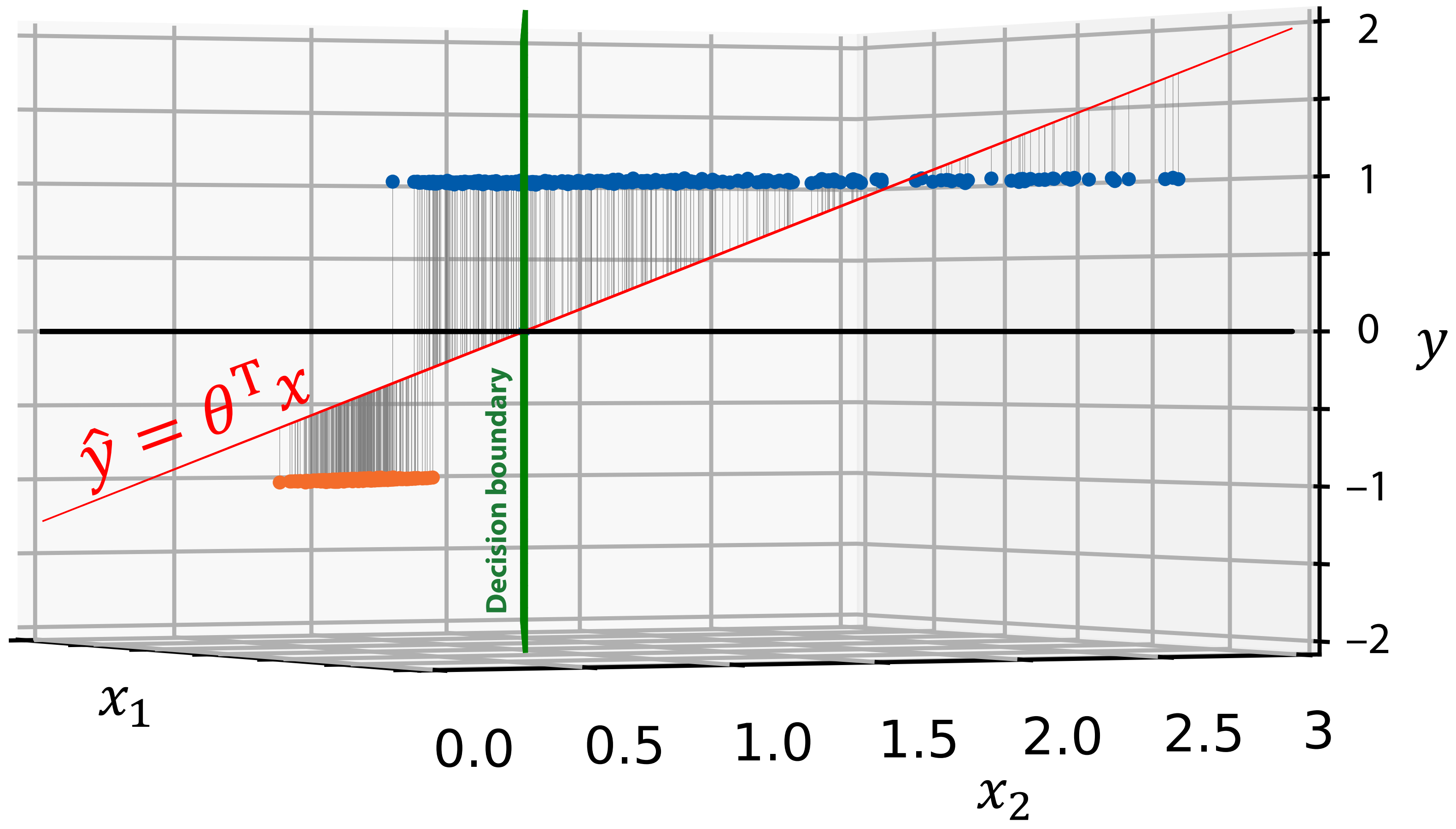
- Any problems with this approach?

Classification with linear regression



- May face problems when classes are unbalanced or have different spread

Classification with linear regression



MSE loss makes the model **avoid high residuals**

at a price of **pushing the decision boundary** towards the class with higher spread

Can we find a better loss function?

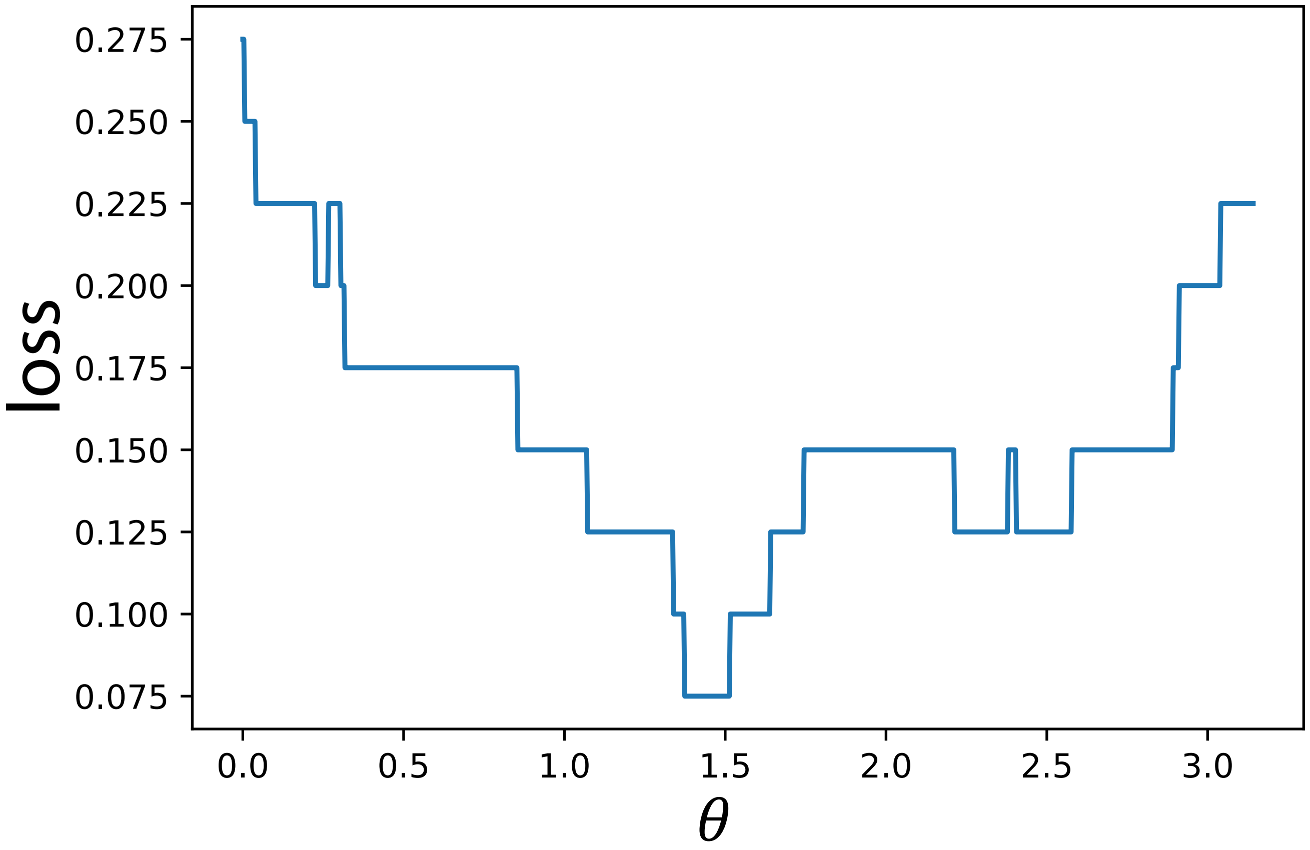
Classification loss functions

0-1 Loss

- Probability of an error

$$\mathcal{L}_{0-1} = \frac{1}{N} \sum_{i=1\ldots N} \mathbb{I}\left(\theta^{\mathrm{T}} x_i \cdot y_i < 0\right)$$

$$y_i \in \{-1, +1\}$$



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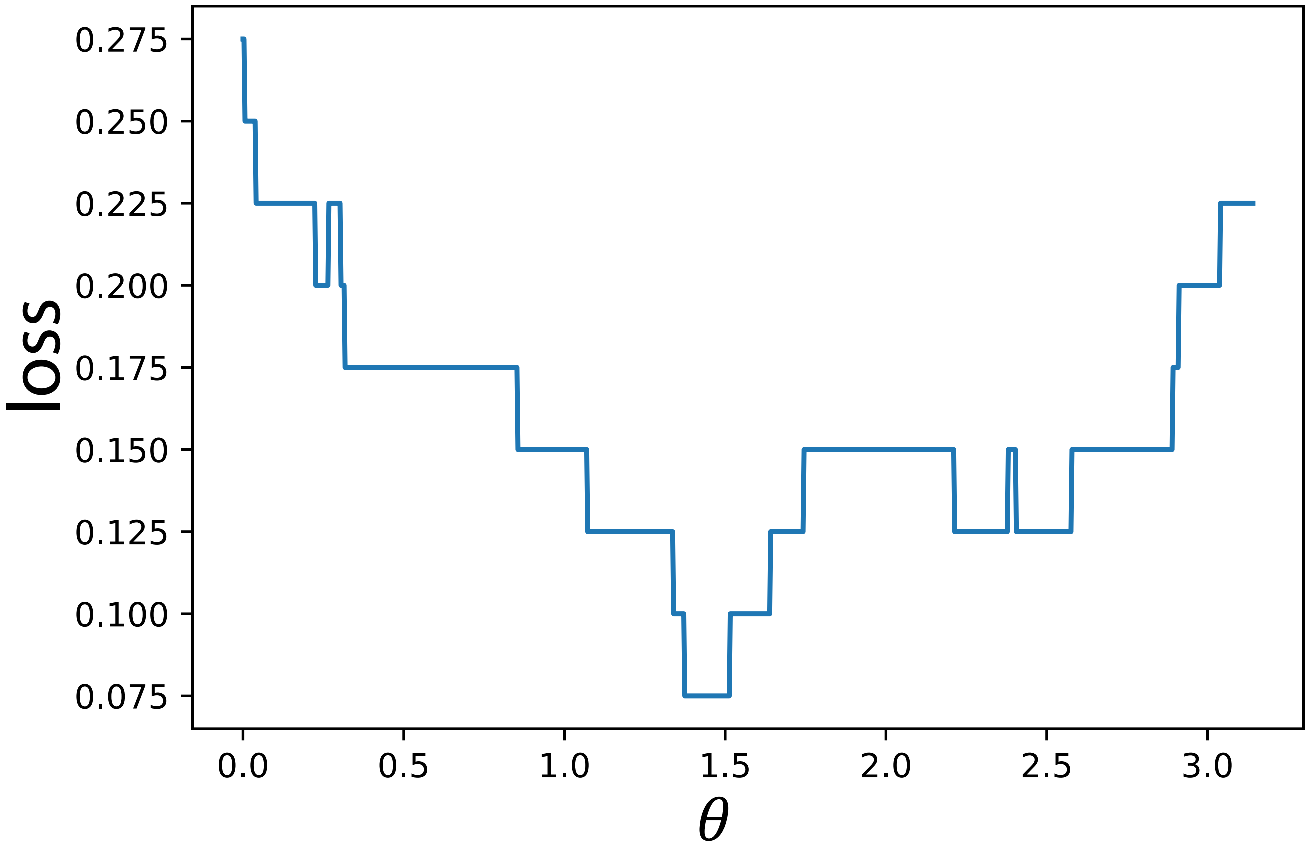


- Can't optimize **piecewise constant** function with gradient-based methods*

*other techniques exist (still quite limited)

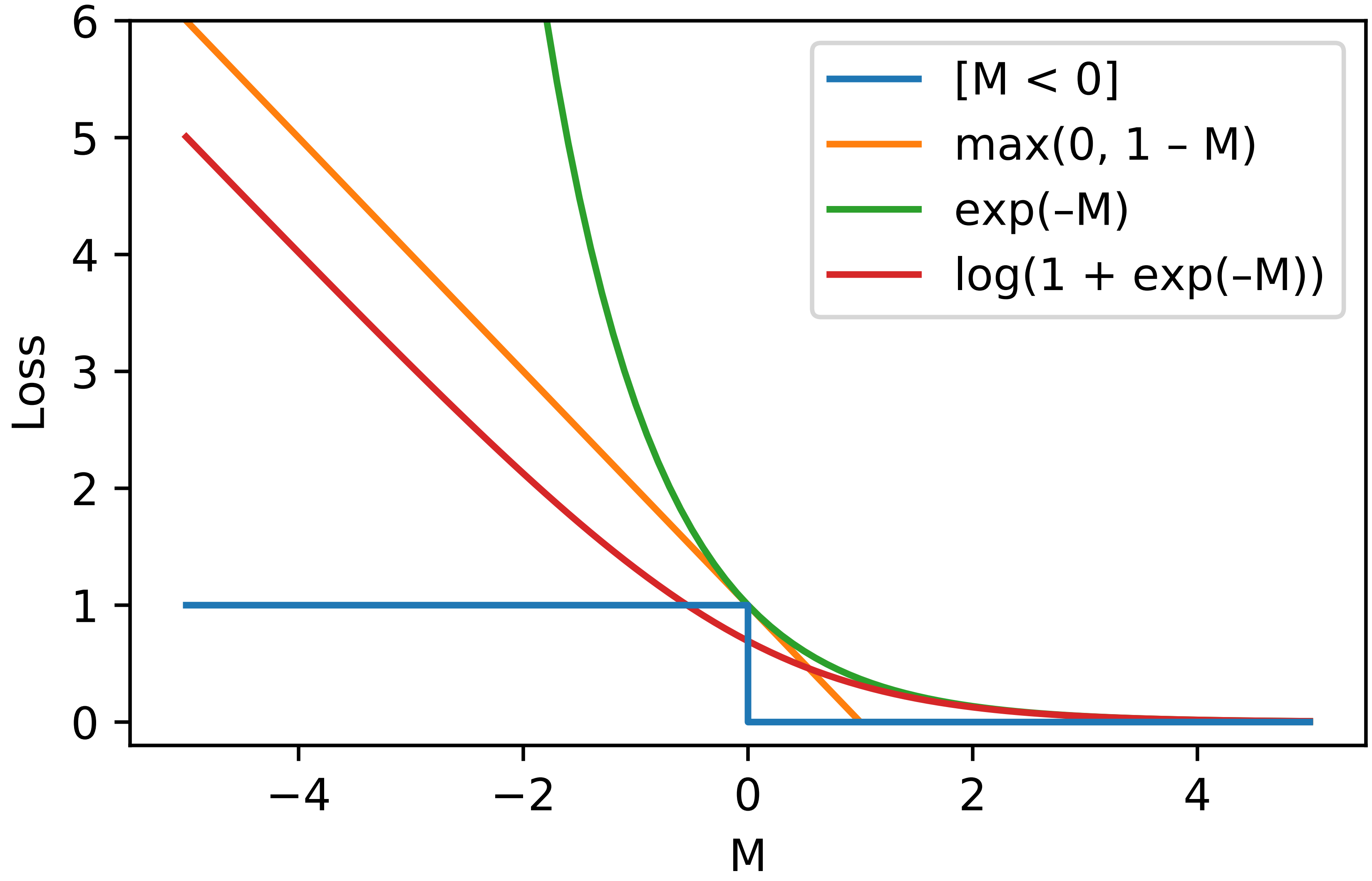
Margin

$$M = \theta^{\mathrm{T}} x \cdot y$$

$$\mathcal{L}_{0-1} = \frac{1}{N} \sum_{i=1 \ldots N} \mathbb{I}\big(\underbrace{\theta^{\mathrm{T}} x_i \cdot y_i}_{\text{margin}} < 0\big)$$

$M > 0$ – correct classification
$M < 0$ – incorrect classification

Upper bounds on 0-1 loss



Instead of optimizing the 0-1 loss we can optimize a **differentiable upper bound**

Logistic Regression

Idea

- Let's model the **class probabilities**

$$P(y = +1|x) = \widehat{f_\theta}(x)$$
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$$= \prod_{i=1\ldots N} \left[\mathbb{I}[y_i = +1] \cdot \widehat{f_\theta}(x_i) + \mathbb{I}[y_i = -1] \cdot \left(1 - \widehat{f_\theta}(x_i)\right)\right]$$

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- Fit with **maximum (log) likelihood**

$$\theta=\underset{\theta}{\operatorname{argmax}}\sum_{i=1\ldots N}\left[\mathbb{I}[y_i=+1]\cdot\log\widehat{f_\theta}(x_i)+\mathbb{I}[y_i=-1]\cdot\log\left(1-\widehat{f_\theta}(x_i)\right)\right]$$

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- Predict the class with **highest probability***

*more generally: find a probability threshold suitable for your problem

Linear probability model

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- Common choice – **sigmoid function**:

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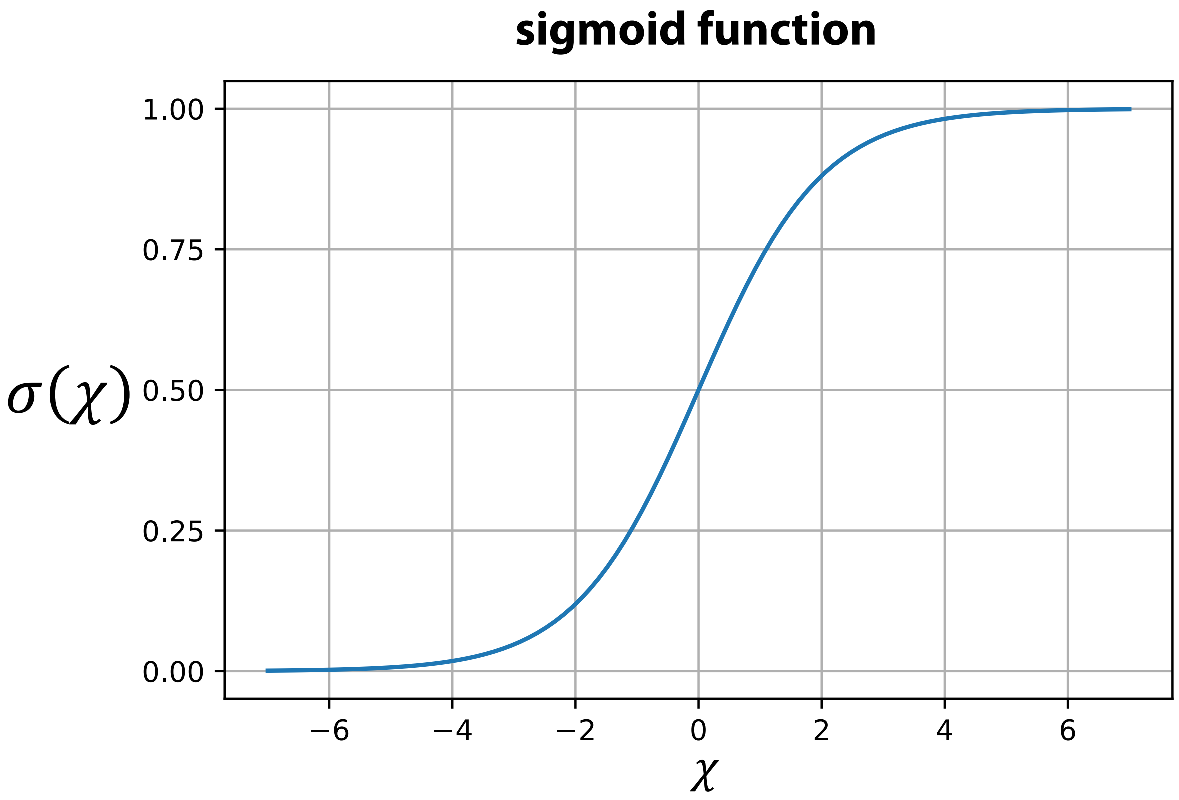

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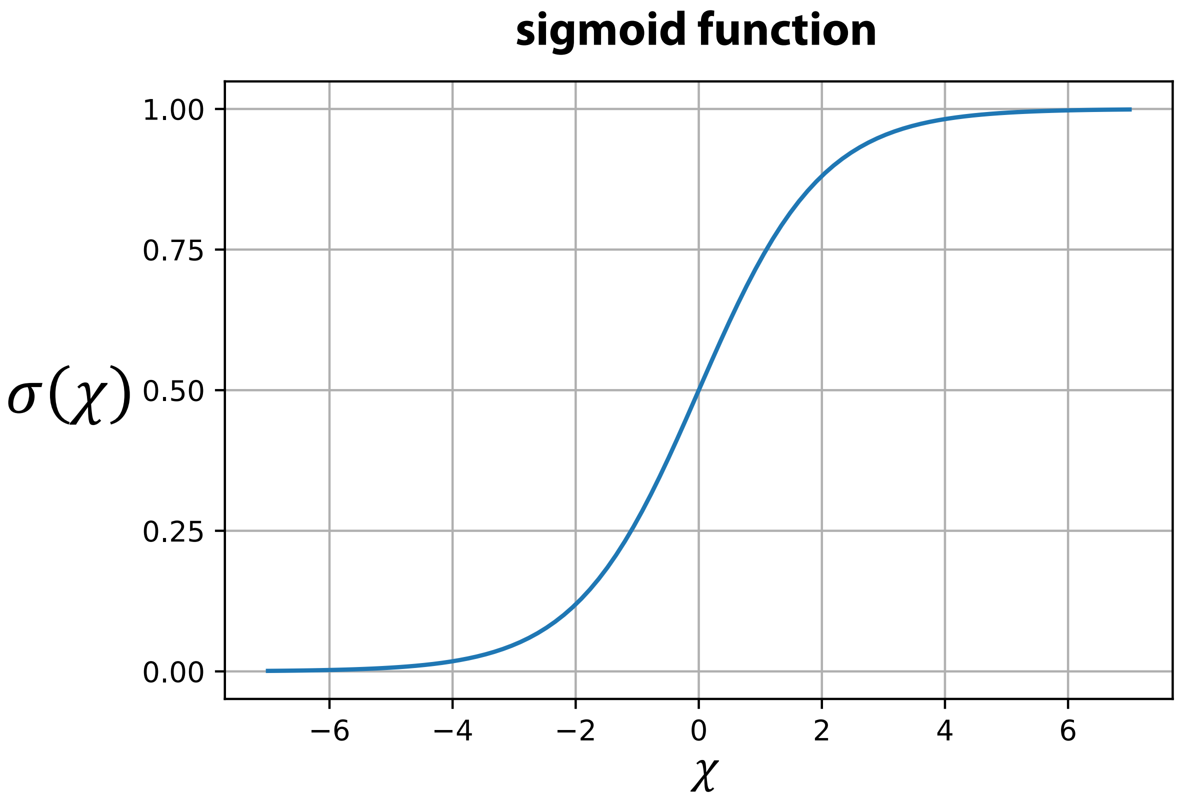

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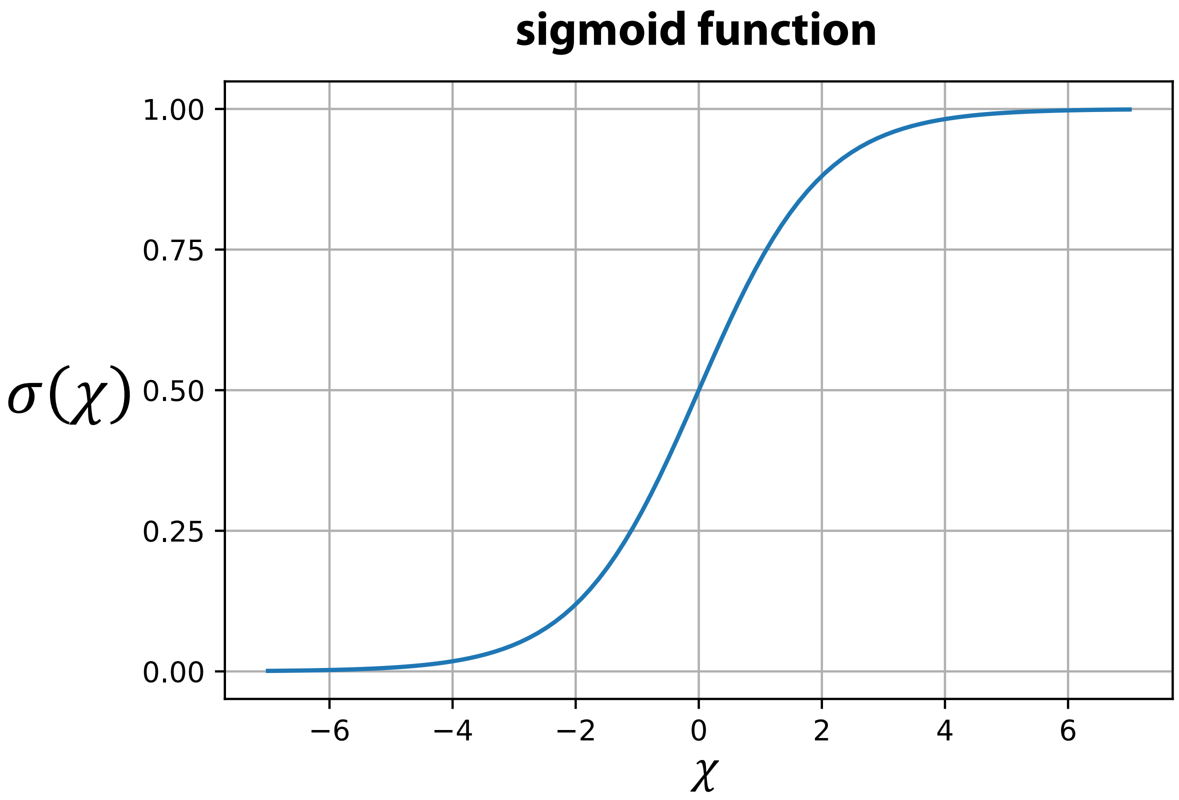

$$\log \frac{P(y = +1|x)}{P(y = -1|x)} = \log\left(\frac{1}{1 + e^{-\theta^{\mathrm{T}} x}} \cdot \frac{1 + e^{-\theta^{\mathrm{T}} x}}{e^{-\theta^{\mathrm{T}} x}}\right) = \theta^{\mathrm{T}} x$$

Bringing it all together

- Use negative log likelihood as our loss function:

$$\mathcal{L} = -\sum_{i=1\dots N}\left[\mathbb{I}[y_i = +1] \cdot \log \widehat{f_\theta}(x_i) + \mathbb{I}[y_i = -1] \cdot \log\left(1 - \widehat{f_\theta}(x_i)\right)\right]$$

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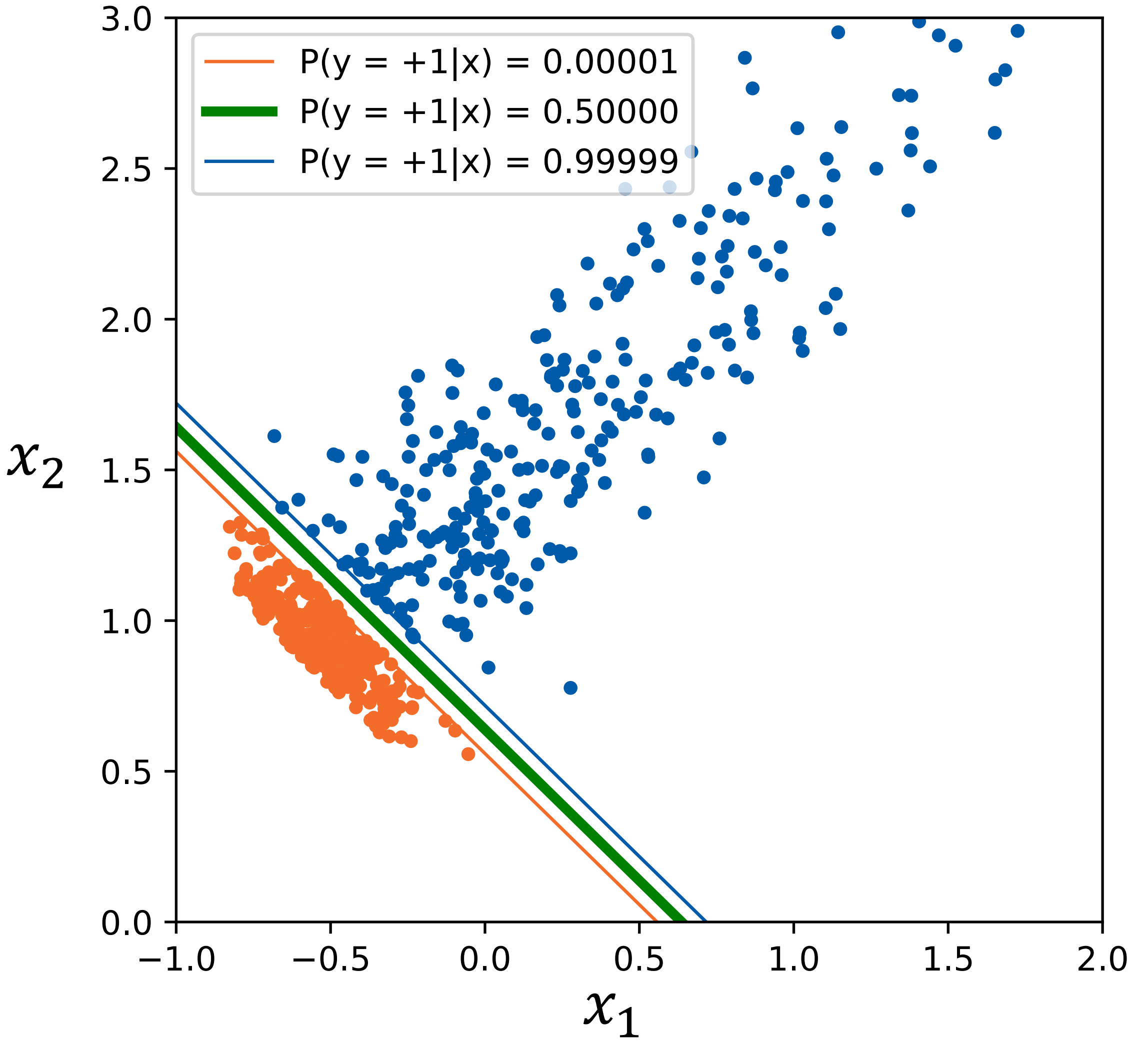
- This can be optimized **numerically**

Example



- Now the boundary is at the right place

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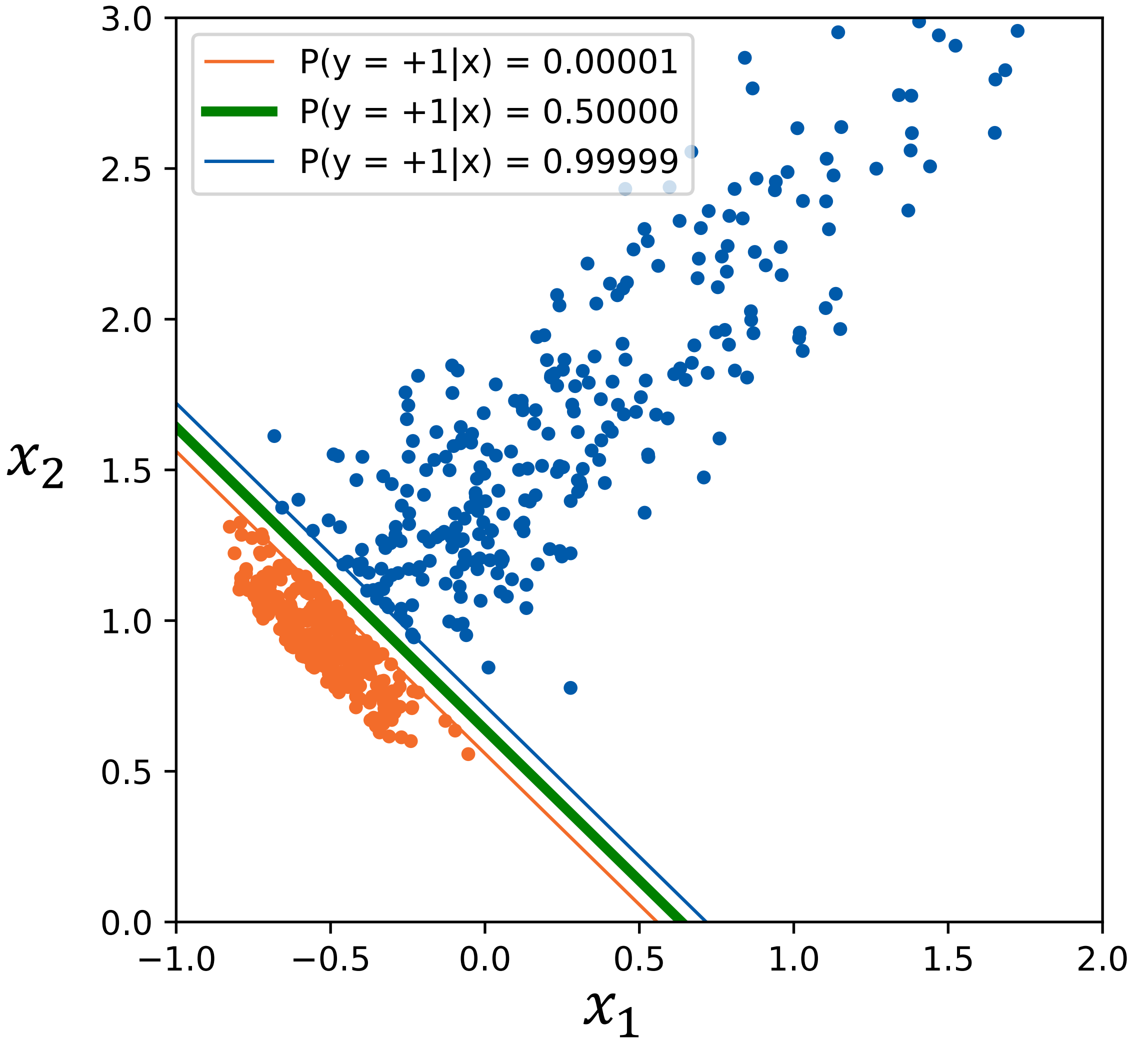

- Now the boundary is at the right place
- Note: when classes are linearly separable for any correct decision boundary

$$\theta \longrightarrow C \cdot \theta, \text{for some } C > 1 \in \mathbb{R}$$

keeps the boundary at the same place, yet improves the loss

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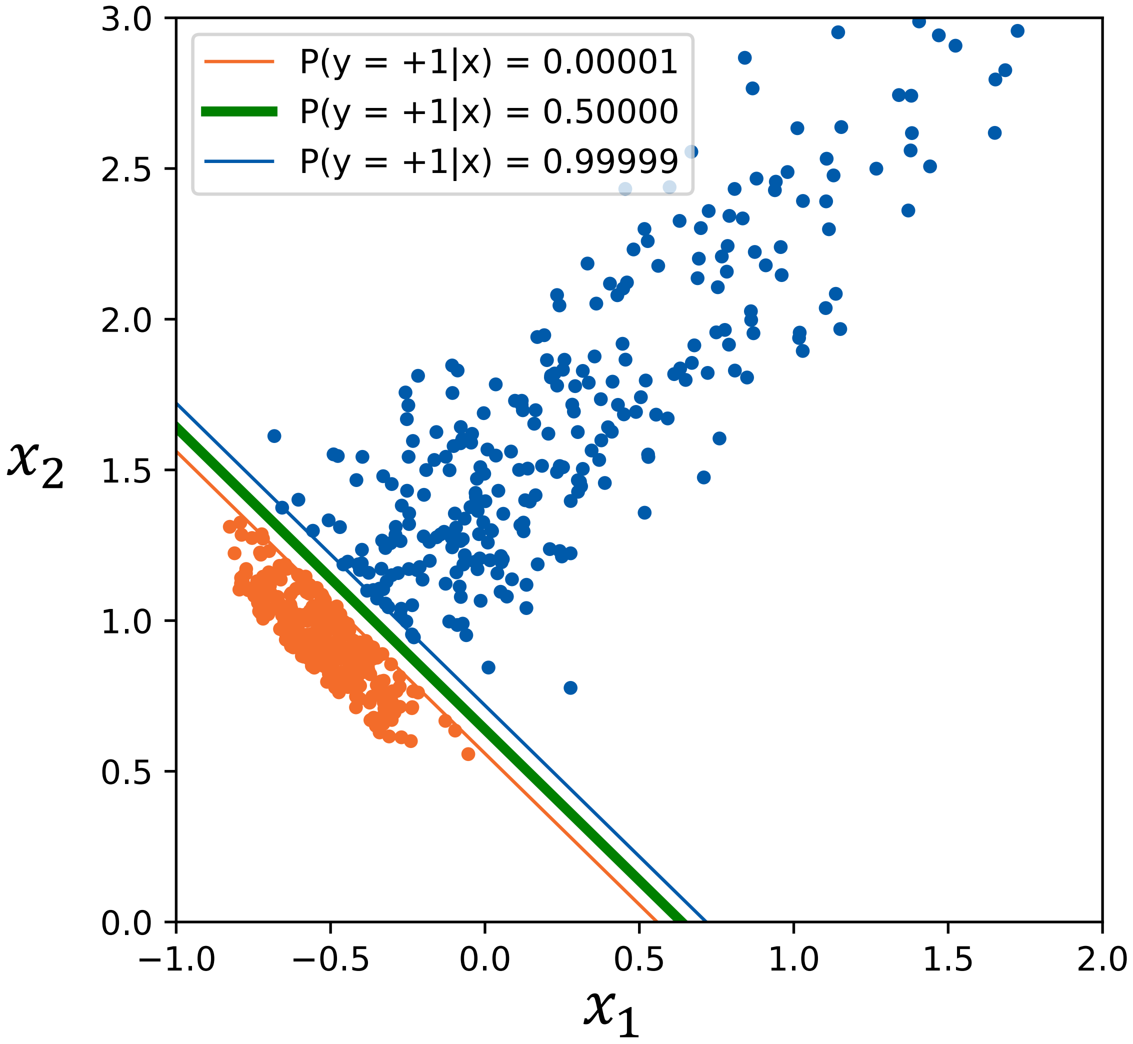

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- ideal fit when sigmoid turns into a step function (at infinitely large θ)

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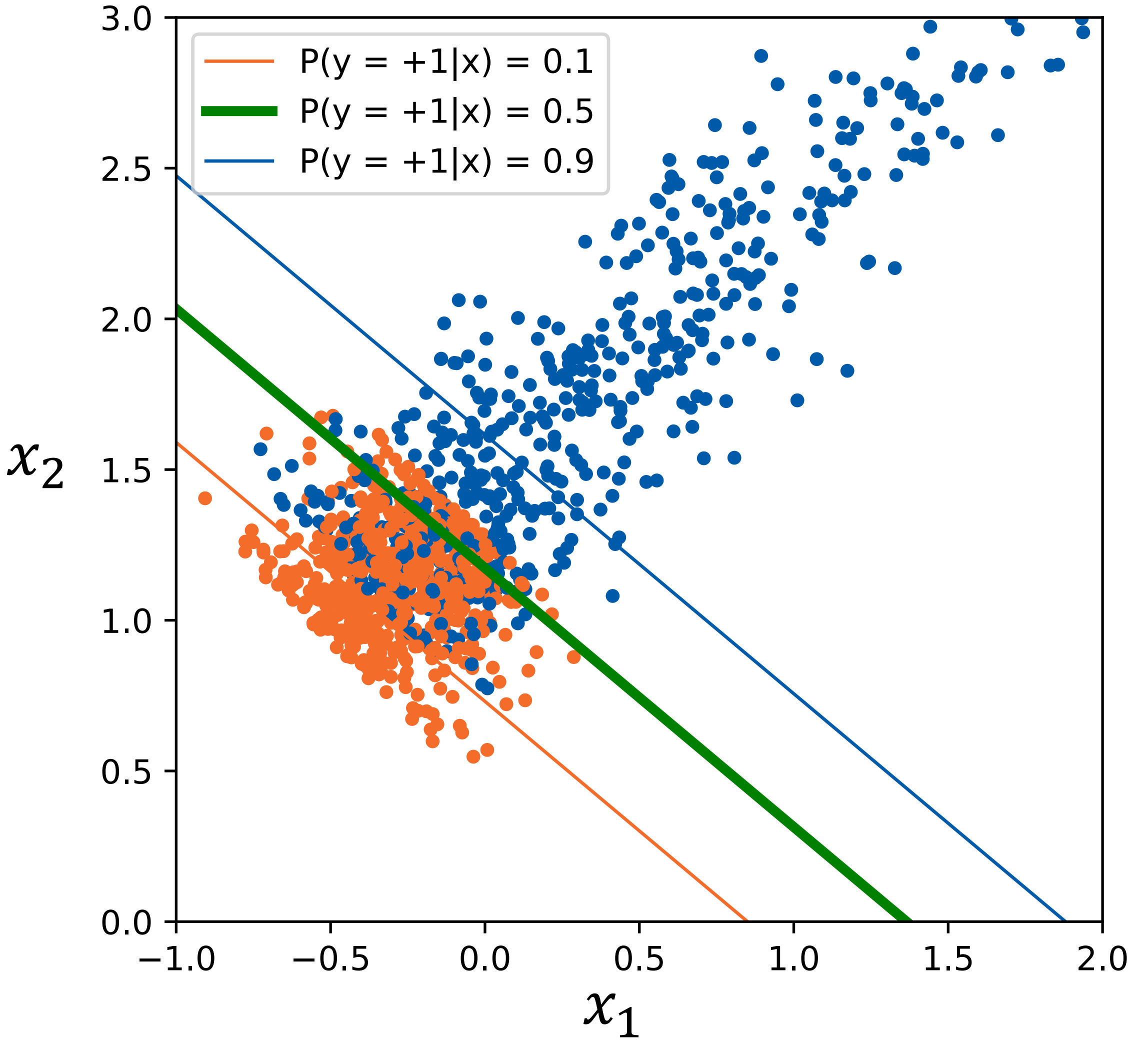

- When classes overlap the loss has a finite minimum
- Predicted class probability changes smoothly

Multiclass Logistic Regression

Multinomial Logistic Regression

- Similarly to the binary case, we'll model the class probabilities
- Let's model **unnormalized** class probabilities like this:

$$\tilde{P}(y = k|x) = \exp \theta_k^{\mathrm{T}} x$$

Note: now we have K parameter vectors

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- Then, the **normalized** probabilities are:

$$P(y = k|x) = \frac{\tilde{P}(y = k|x)}{\sum_{k'=1\ldots K} \tilde{P}(y = k'|x)} = \frac{\exp \theta_k^{\mathrm{T}} x}{\sum_{k'=1\ldots K} \exp \theta_{k'}^{\mathrm{T}} x}$$

– This function is called **softmax** and is commonly used in neural networks

Multinomial Logistic Regression

- Note that transforming all $\theta_k \longrightarrow \theta_k + v$ by some constant vector v does not affect the normalized probability

$$\tilde{P}(y = k|x) = e^{\theta_k^{\mathrm{T}} x} \longrightarrow e^{v^{\mathrm{T}} x} \cdot e^{\theta_k^{\mathrm{T}} x} = e^{v^{\mathrm{T}} x} \cdot \tilde{P}(y = k|x)$$

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- This means we can **set one of the vectors $\boldsymbol{\theta_k}$ to 0**, e.g. the last one:

$$\theta_K = 0$$

Multinomial Logistic Regression

- We now have $K - 1$ parameter vectors
- Individual linear outputs $\theta_k^{\mathrm{T}} x$ now have the meaning of **log odds ratio** between the classes k and K:

$$\log \frac{P(y = k|x)}{P(y = K|x)} = \log \frac{\tilde{P}(y = k|x)}{\tilde{P}(y = K|x)} = \log \frac{e^{\theta_k^{\mathrm{T}} x}}{e^0} = \theta_k^{\mathrm{T}} x$$

Multinomial Logistic Regression

- Plugging everything into the negative log likelihood we get our loss function:

$$\mathcal{L} = -\sum_{i=1\dots N} \log \frac{\exp \theta_{y_i}^{\mathrm{T}} x_i}{1 + \sum_{k'=1\dots K-1} \exp \theta_{k'}^{\mathrm{T}} x_i}$$

$$(\theta_K = 0)$$

- Again, this can be optimized **numerically**

Multiclass classification: general approach

General idea

For a problem with K classes introduce K predictors:

$$\widehat{f}_k(x) \colon \mathcal{X} \to \mathbb{R}, \text{for } \mathrm{k} = 1, \ldots, K$$

each of which outputs a corresponding **class score**.

Predict the class with the **highest score**:

$$\hat{y}_i = \underset{k}{\operatorname{argmax}} \widehat{f}_k(x_i)$$

Example: binary $\longrightarrow$ multiclass

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- For each class k train a binary model $\widehat{f_k}(x) = \theta_{(k)}^{\mathrm{T}} x$ separating the given class from all others, $\hat{y}_{(k)}^{\mathrm{1-vs-rest}} = \mathrm{sign}\big[\widehat{f_k}(x)\big]$

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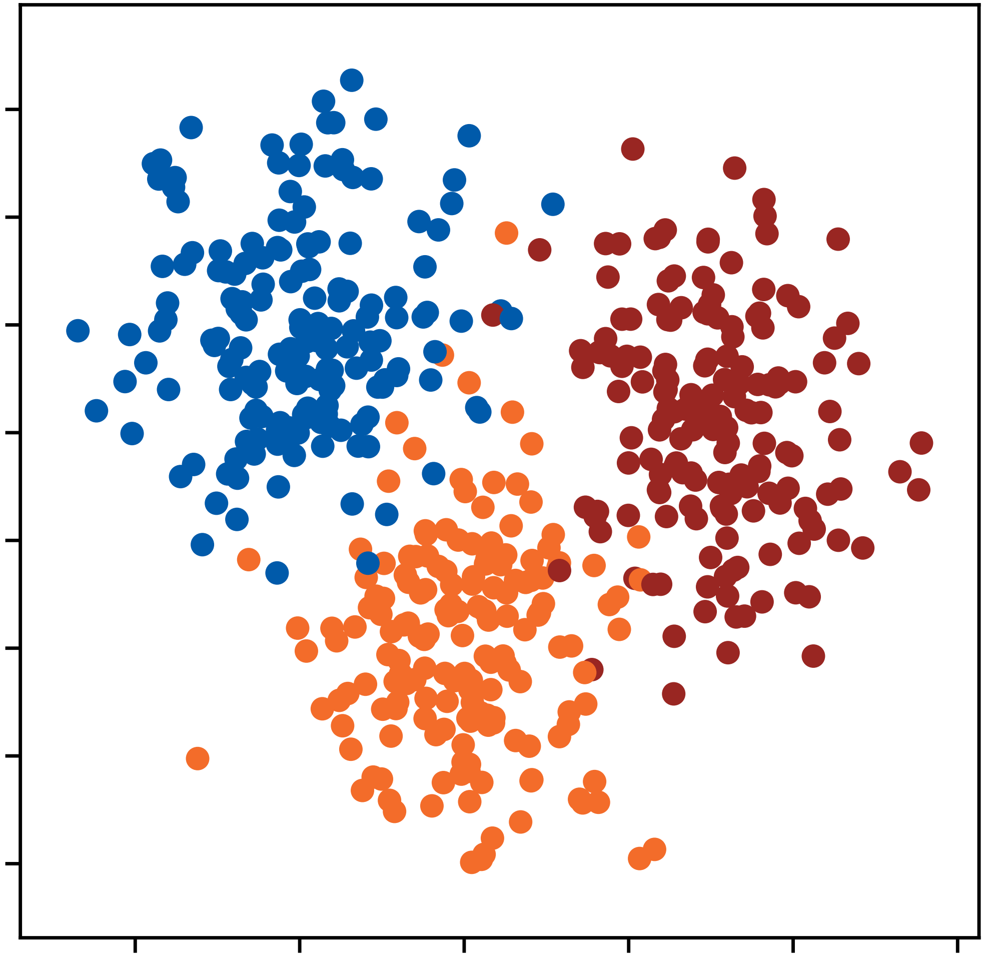
- Use the outputs of $\widehat{f}_k$ as class scores for multiclass classification:

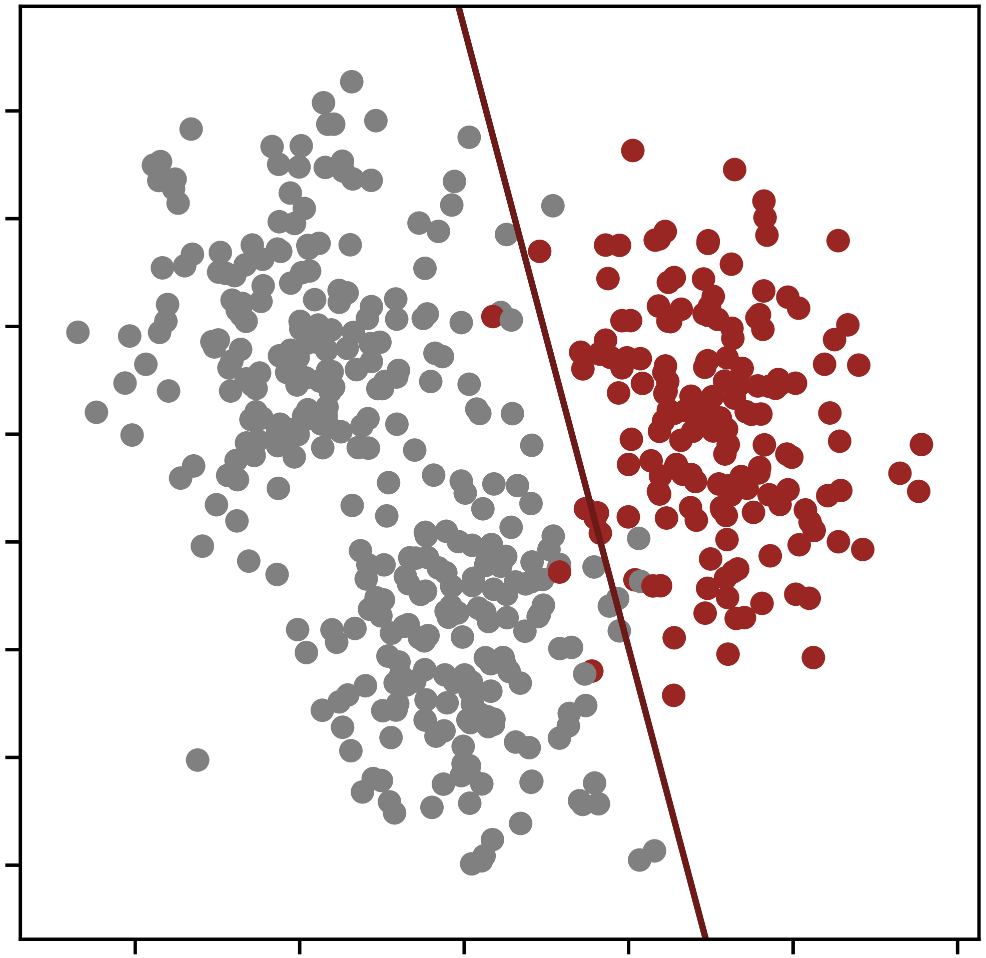
$$\hat{y}_i = \underset{k}{\mathrm{argmax}}\, \widehat{f}_k(x_i)$$

Example



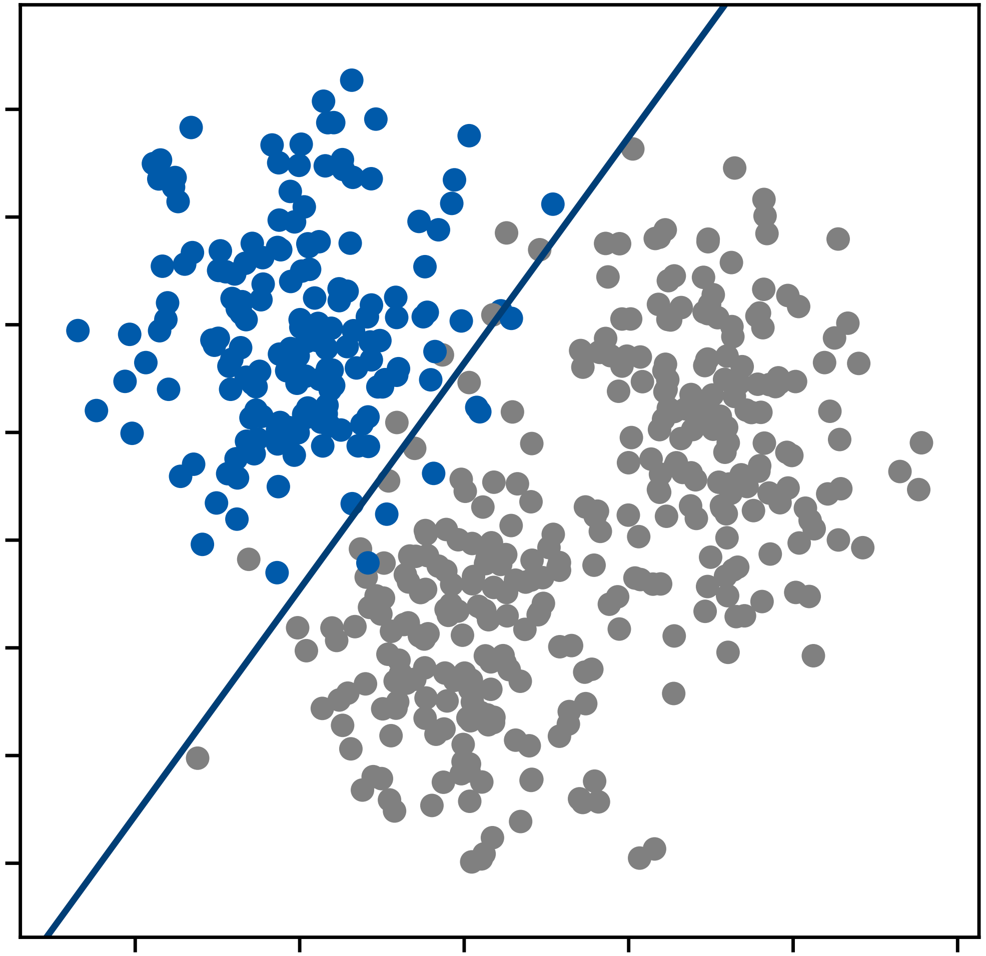
- Consider the following 3 class problem

Example



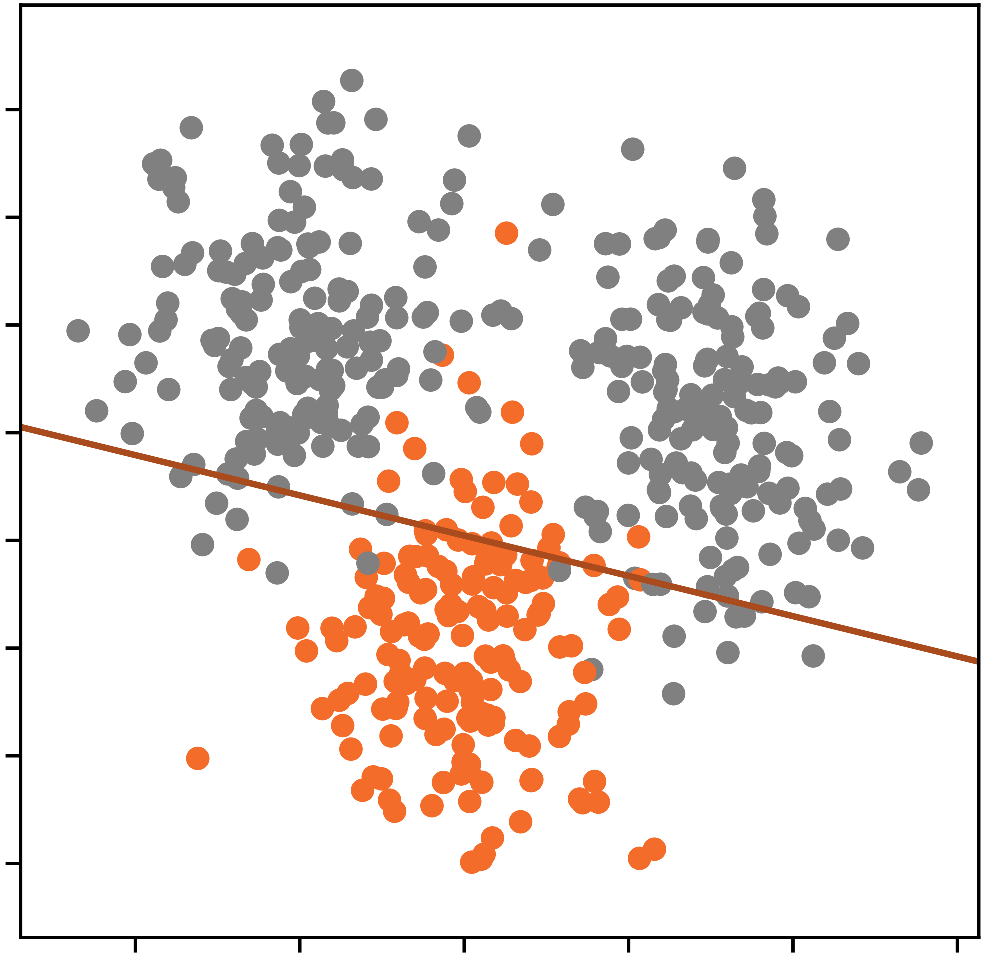
- “Class-1 VS rest” binary classifier

Example



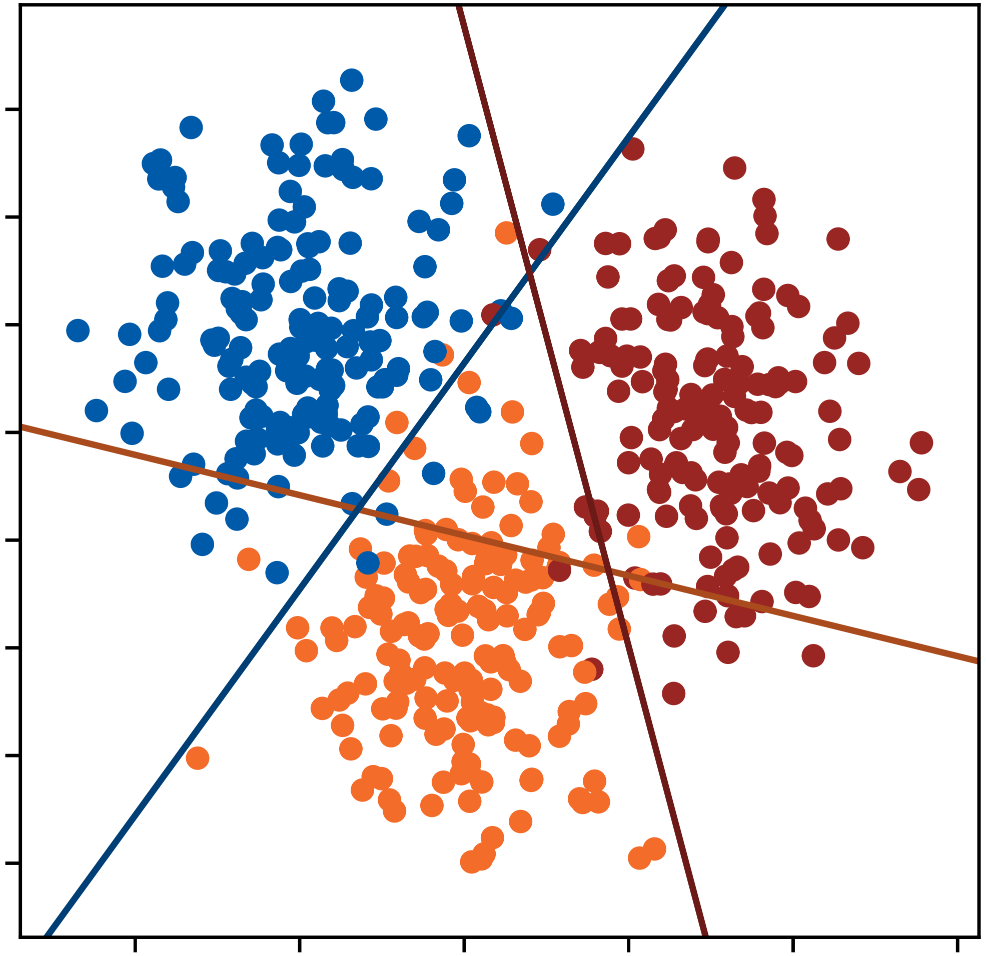
- “Class-2 VS rest” binary classifier

Example



- “Class-3 VS rest” binary classifier

Example



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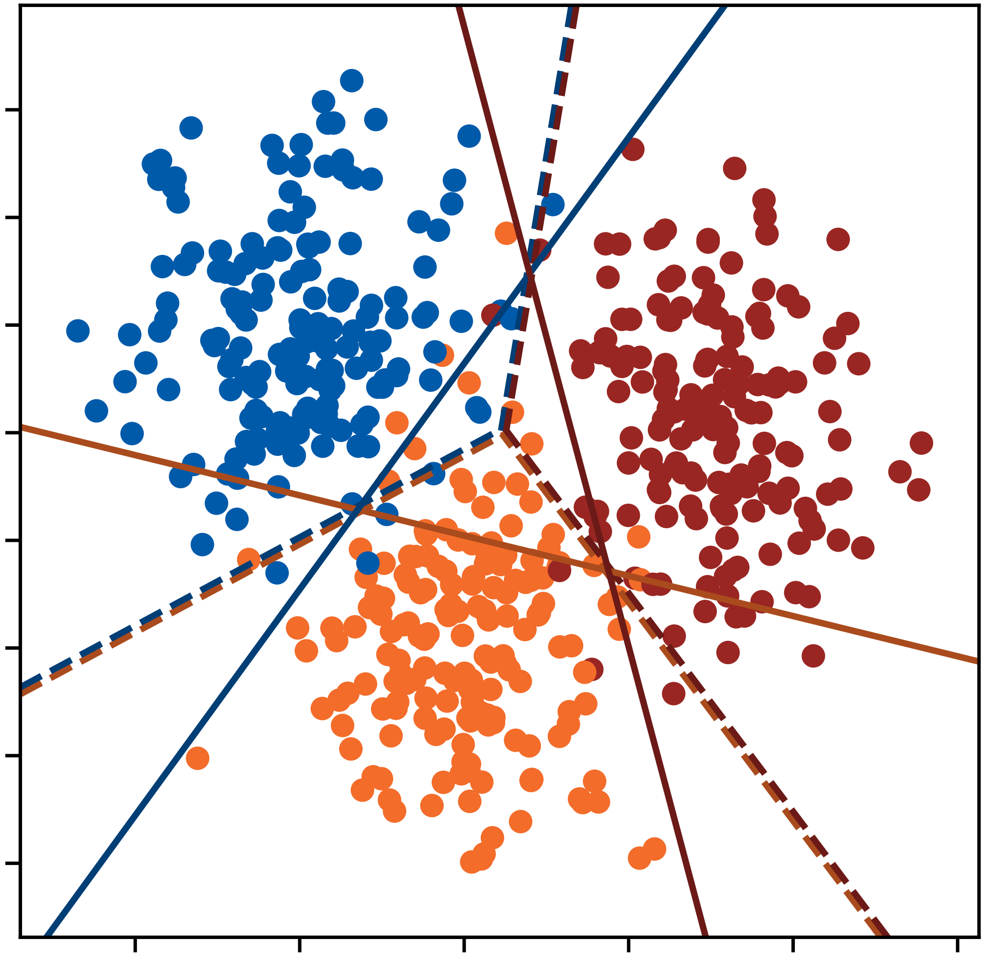

- $\widehat{f}_k(x) = 0$ lines (binary decision boundaries)
- Adding decision boundaries for $\hat{y} = \underset{k}{\operatorname{argmax}} \widehat{f}_k(x)$

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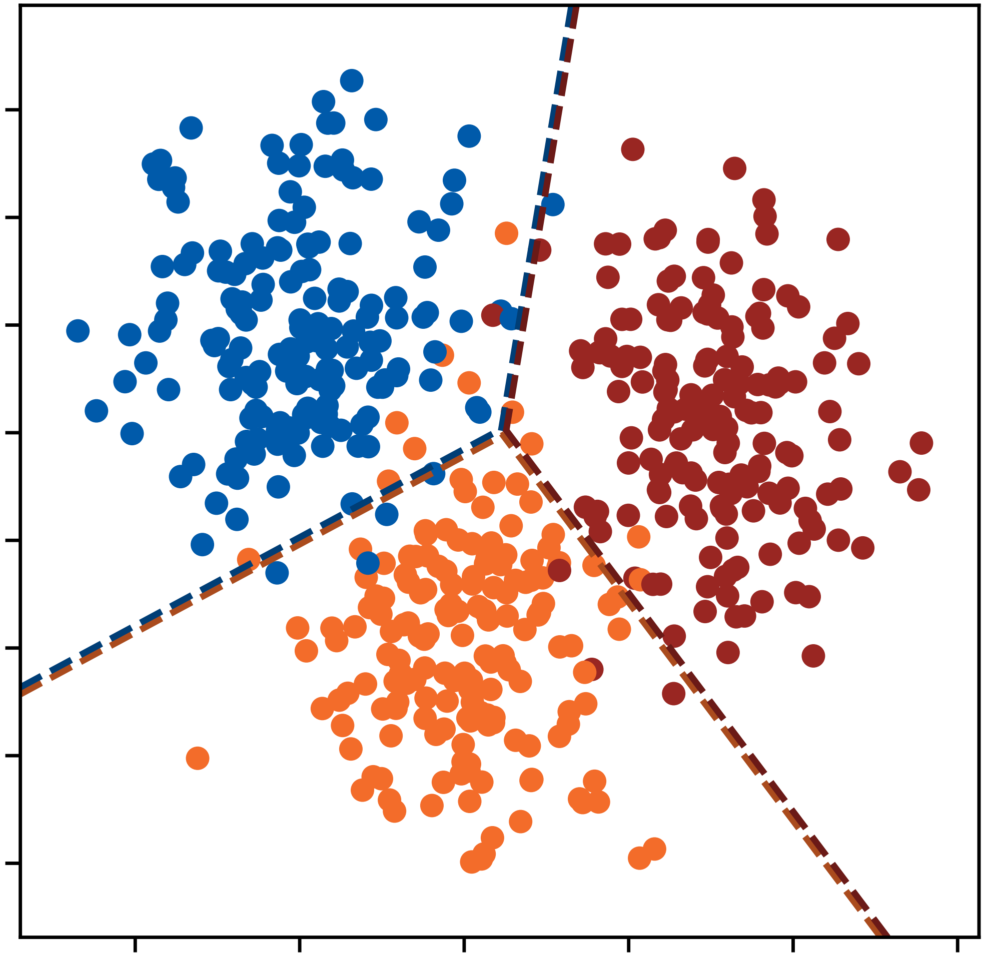

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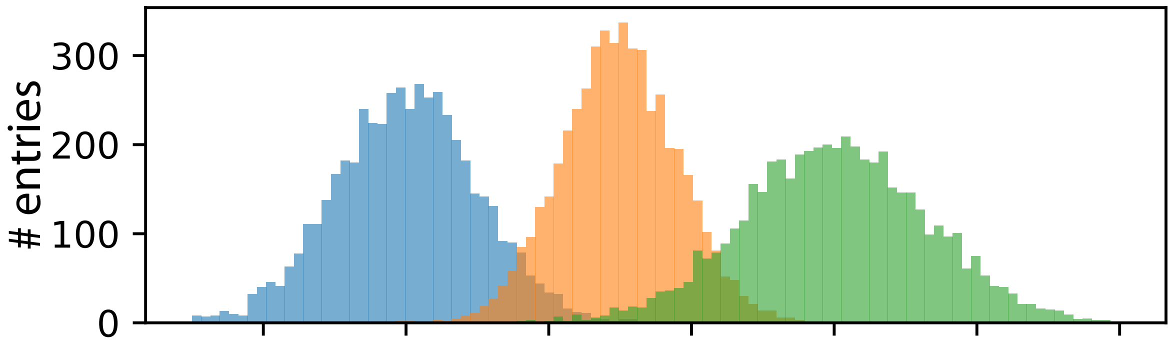
Logistic regression: multinomial or one-vs-rest?



Some of the binary classification tasks not linearly solvable

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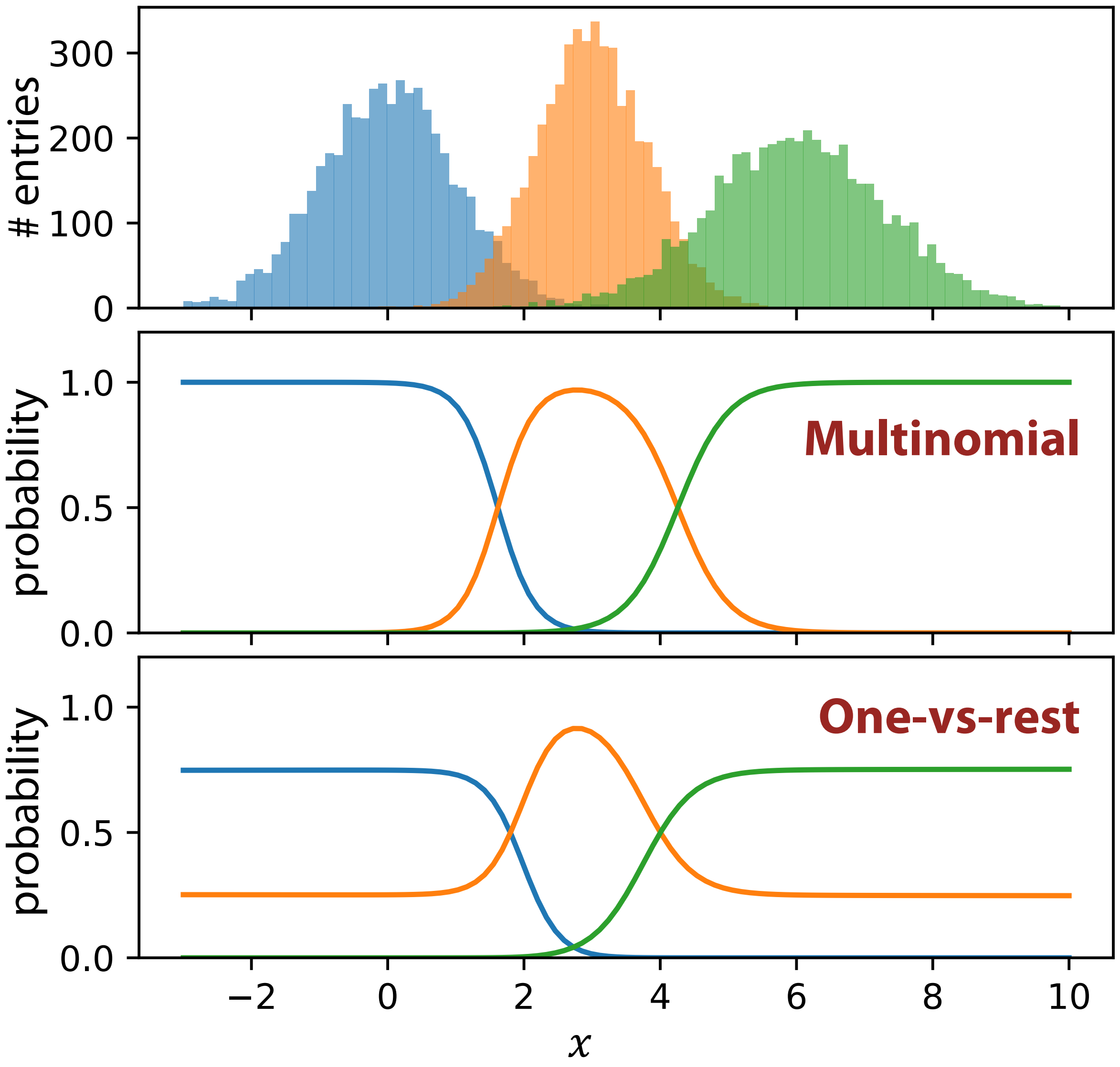

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⇒ one-vs-rest results in biased class probabilities

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- 0-1 loss function is better, but is **hard to optimize** directly
- Various **differentiable upper bounds** on 0-1 loss may be used instead
- Logistic Regression combines such an upper bound with a **probabilistic model** using the **sigmoid function**

Summary

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- Food for thought: how can you mitigate the biased probability problems when using one-vs-rest strategy (as discussed on the previous slide)?

Thank you!

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