Price-Time Priority and Pro Rata Matching in an Order Book Model of Financial Markets

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Abstract. Using our recently introduced order book model of financial markets we analyzed two different matching principles for order allocation – price-time priority and pro rata matching. Price-time priority uses the submission timestamp which prioritizes orders in the book with the same price. The order which was entered earliest at a given price limit gets executed first. Pro rata matching is used for products with low intraday volatility of best bid and best ask price. Pro rata matching ensures constant access for orders of all sizes. We demonstrate how a multiagent-based model of financial market can be used to study microscopic aspects of order books.

1 Introduction

In recent years, econophysicists started to investigate and understand the price formation process in detail on a microscopic level. In this context, a statistical model of the continuous double auction [1, 2] was developed. Based on this model, we proposed an multiagent-based order book model recently. These Monte Carlo based simulations of financial markets' order books were introduced in [3] and studied in detail in [4]. Here we will provide simulation based evidence for two different order matching principles which can be found in order books of real exchanges[1].

The definition of the order book model and its main results are provided in Sect. 2. Sect. 3 will focus on Monte Carlo based simulations of the order book model using both price-time priority and pro rata matching. Sect. 4 summarizes our findings.

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Econophysics of Order-driven Markets.

2 Continuous Double Auction and the Definition of the Order Book Model

The order book model [3, 4] is based on microscopic structures which one can find at electronic financial markets. The function of an exchange based order book is to store buy orders and sell orders of all market participants. In our simulations, only one order book is used, in which one individual asset – e.g. a stock – is traded. There are various types of orders in financial markets' exchange systems. In our order book model, only the two most important types are implemented, namely limit orders and market orders. Limit orders are executed only at a specific price or a price, which is better for the trader, who placed this limit order in the book, whereas market orders are executed immediately against the best available limit order stored in the order book. Limit orders are added chronologically, which realizes a price time priority based matching algorithm. Thus, at a given price level with more than one limit order, the limit order, which was inserted first, has execution priority.

Limit and market orders are performed by agents in the model, in which we also distinguish between two types of market participants – liquidity providers and liquidity takers. These two groups of agents differ in the types of orders they are permitted to submit. On the one hand, N_A liquidity providers transmit only limit orders. In the case of a limit sell order, a liquidity provider offers an asset for sale at a set limit price or a higher price. Analogously, a limit buy order indicates a demand for buying an asset and the order is executed at a set limit price or any better price for the liquidity provider.

Let p_a be the best ask price, which is the lowest price level of all limit sell prices in the order book, and analogously p_b the best bid price, being the highest price level for which at least one limit buy order is stored in the order book. In the order book model, limit orders are placed around the midpoint $p_m = \frac{p_a+p_b}{2}$ with a rate α, i.e., $\alpha \cdot N_A$ new limit orders are submitted per time step. Let q_{provider} be the probability with which a limit order is a limit buy order. Thus, with probability $1 - q_{\text{provider}}$, the limit order to be placed is a limit sell order. The liquidity provider, which can be identified as market maker, supplies liquidity to the market in order to exploit the spread $s = p_a - p_b$: such market participants intend, e.g., to sell an asset at price p_a or higher and then to buy it back at price p_b or lower. Thus, they have earned at least the spread s. As seen in this example, short sales are allowed, i.e., it is allowed for agents to sell assets even if they do not possess them.

On the other hand, there are N_A liquidity takers, who transmit only market orders. These market orders are submitted with rate μ, i.e., $\mu \cdot N_A$ market orders are inserted per time step into the order book. A market order will be immediately executed after arrival. A market sell order is executed at price level p_b, a market buy order at price level p_a. A market order of a liquidity taker is a market buy order with probability q_{taker} and a market sell order with probability $1 - q_{\text{taker}}$. In this basic version of the order book model, the simple case $q_{\text{provider}} = q_{\text{taker}} = \frac{1}{2}$ is applied. Thus, all orders will be produced symmetrically around the midpoint. In practice, it is possible, that the limit price of a limit sell order is lower than the current best bid price and the limit price of a limit buy order is higher than the current best ask price. Such "crossing" limit orders degenerate to market orders and thus, they are executed immediately. In our order book model, only pure limit orders will be used.

Limit orders, which are stored in the order book, can also expire or can be deleted. In the model, this canceling process is realized in the way that each stored order is deleted with probability δ per time unit. As there are overall $2N_A$ agents in the multiagent system, each Monte Carlo step (MCS) consists of $2N_A$ moves. In each move, one agent is randomly selected and can perform one action according to the probability rates. If the chosen agent is a liquidity provider, then a limit order with probability α is submitted by the chosen agent. On the other hand, if the selected agent is a liquidity taker, then a market order with probability μ is placed in the order book which will be immediately executed. Orders in our model have the constant order volume 1. Thus, it is possible only to buy or sell one asset unit with an individual order.

Based on this simple rules, first an unrealistic independent identically distributed order placement depth can be applied. This is realized in the way that limit buy orders are entered on each price level in the interval of $[p_a - 1 - p_{\text{int}}; p_a - 1]$ with the same probability, and accordingly, limit sell orders are transmitted uniformly distributed in the interval of $[p_b + 1; p_b + 1 + p_{\text{int}}]$. Already with this definition of the order book model, profits and losses of the agents can be analyzed. Using this setup, the averaged wealth value of liquidity takers and liquidity providers drifts apart linearly in time [3]. Comparing these results with real financial markets, it has to be stated that liquidity takers are systematically disadvantaged in comparison to liquidity providers. The distinction in our model between the two groups of market participants reflects the two types of orders. In general, market participants are not restricted to one order type in reality.

In the next step, a more realistic order placement depth will be integrated in the order book model. The order book depth of real financial markets can be described by a log-normal distribution [5]. And, to take this into account the independent identically distributed limit order placement is replaced by an exponentially distributed order placement depth. Thus, for placing a limit order, the limit price p_l is determined for a limit buy order through

$$p_l = p_a - 1 - \eta \tag{1}$$

and for a limit sell order according to

$$p_l = p_b + 1 + \eta \tag{2}$$

whereby η is an exponentially distributed integer random number created by $\eta = \lfloor -\lambda_0 \cdot \ln(x) \rfloor$ with x being a uniformly distributed random number in the interval $[0; 1)$ and $\lfloor z \rfloor$ denoting the integer part of z. With this construction, the submission process of limit orders has the tendency to reduce the gap between best bid price and best ask price. Also, crossing limit orders are avoided, as the price of a limit buy order cannot be equal or larger than the best ask price and the price of a limit sell order cannot be equal or lower than the best bid price.

As result of applying the exponential order placement rule, a log-normally distributed order book depth profile is obtained [3, 4]. This basic version is used in order to study both order matching algorithms. This simple variant is also able to reproduce the results of [1, 2]. The price time series of this basic version possesses an antipersistent price behavior on short time scales which is due to the order book structure. On medium and long time scales the Hurst exponent converges towards a diffusive regime. The price change distributions exhibit an almost Gaussian shape. The model, which is characterized by a symmetry created by identical buy and sell probabilities, describes a stationary market. However, when one additionally introduces a symmetry disturbance, the order book model is displaced from its stationary state. This extension is implemented by a temporal modulation of the buy probability q_{taker} of the liquidity takers or the buy probability q_{provider} of the liquidity providers [3]. Qualitatively identical results are achieved, if both probabilities are modulated independently of each other. Employing a feedback random walk to introduce micro market trends into the market, one additionally obtains a persistent price behavior on medium time scales. However, no fat tails can be reproduced with such a symmetry-breaking extension of the order book model. When one furthermore couples the characteristic order placement depth to the prevailing market trend, widened price change distributions are achieved, with so-called fat tails. Thus, with these extensions of our order book model, we could demonstrate that the generation of a nontrivial Hurst exponent is independent of the generation of fat tails [4]. This disproves the implication which can be often found in the literature that a persistent price behavior corresponds to non-Gaussian price changes. Furthermore, we are able to support the statement in [6, 7] that $H > 1/2$ implies not necessarily long time correlations.

3 Matching Principles

When orders are entered into the electronic order book, they are sorted by type, price, and submission time. Market orders are always given the highest priority for matching purposes. Orders at a given price level are aggregated, although the number of orders remains unknown. Market participants only see the specific details of their own limit orders[2]. Most exchange traded products – e.g. equity index derivatives at the European Exchange (EUREX) in Germany – follow the matching principle which is known as price-time priority. This is not the case for money market products which show typically smaller intraday fluctuations. These products follow pro rata matching.

Price-time priority can be described as follows. When an order is entered into the order book, it is assigned a timestamp with a resolution of milliseconds. This timestamp is used to prioritize orders in the book with the same price. The order which was entered earliest at a given price limit gets executed first.

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[2] More information can be found, e.g., on www.eurexchange.com.

Pro rata matching is used for products with low intraday volatility of best bid or best ask price. Pro rata matching ensures constant access for orders of all sizes. Otherwise, using price-time priority a large order may prevent smaller orders from participating in the order matching process. When matching existing orders in the electronic order book against an incoming order, the pro rata matching algorithm takes into account every book order at best bid or best ask price according to its percentage of the overall volume bid or offered at the price. Its timestamp is neglected. Thus, pro rata principles avoid conflicts in priority between orders with small and large volumes [8].

In this section, we study the consequences of both matching algorithms using our order book model. The price-time priority based matching algorithm was already implemented in the basic version of the order book model [3, 4]. Without loss of generality, we study in both versions the simple case that the order volume is set to 1 – for the price-time priority and for the pro rata matching. However, in this case it is not possible to observe the situation that a large order prevent smaller orders from participating in the matching process. Thus, one can not distinguish orders based on volume. Additionally, pro rata matching ignores timestamps. In our framework with constant order volume the matching process can be realized by executing a randomly chosen limit order at the best available price (best bid or best ask) if there are more than one limit orders at this price level.

Before we compare both order matching algorithms, we have to think about an appropriate macroscopic variable for that purpose. As the change of matching algorithms does not effect the times and sales records, it is not useful to analyze the price time series. Only variables referring to execution times of individual orders are useful to study. Thus, we will analyze the time-to-fill distributions of limit orders. The time-to-fill quantity T_i of an individual limit order i is given by the time interval which starts with the submission of the limit order to the central order book at time t_i^l and which ends with its execution at time t_i^e. The time-to-fill T_i is given by

$$T_i = t_i^e - t_i^l . \tag{3}$$

As the order volume is set to 1 one has not to handle the special cases of partial executions. Deleted limit orders do not contribute to the time-to-fill distributions. In order to measure the time-to-fill distributions, we have to store for each limit order in the order book the submission timestamp t_i^l. Thus, we can determine the time difference T_i. Please note that also market orders are neglected for the calculation of this distribution as they have an execution time interval of 0 MCS by definition – of course, this is not true if the order book is empty.

We choose the same parameters for the simulation of the order book model as used in [3, 4]: $\alpha = 0,15$, $\mu = 0,025$, $\delta = 0,025$, $\lambda = 100$, and $N_A = 250$. In Fig. 1a, the time-to-fill distribution of limit orders is shown for this parameter set for both order allocation methods. The calculation of the time-to-fill distributions is based on simulations lasting 10^5 MCS. Results are averaged over 10 runs. In Fig. 1a, one can clearly see that the price-time priority matching algorithm has a larger probability in comparison to the pro rata based allocation only for $T \in [10; 50]$. This

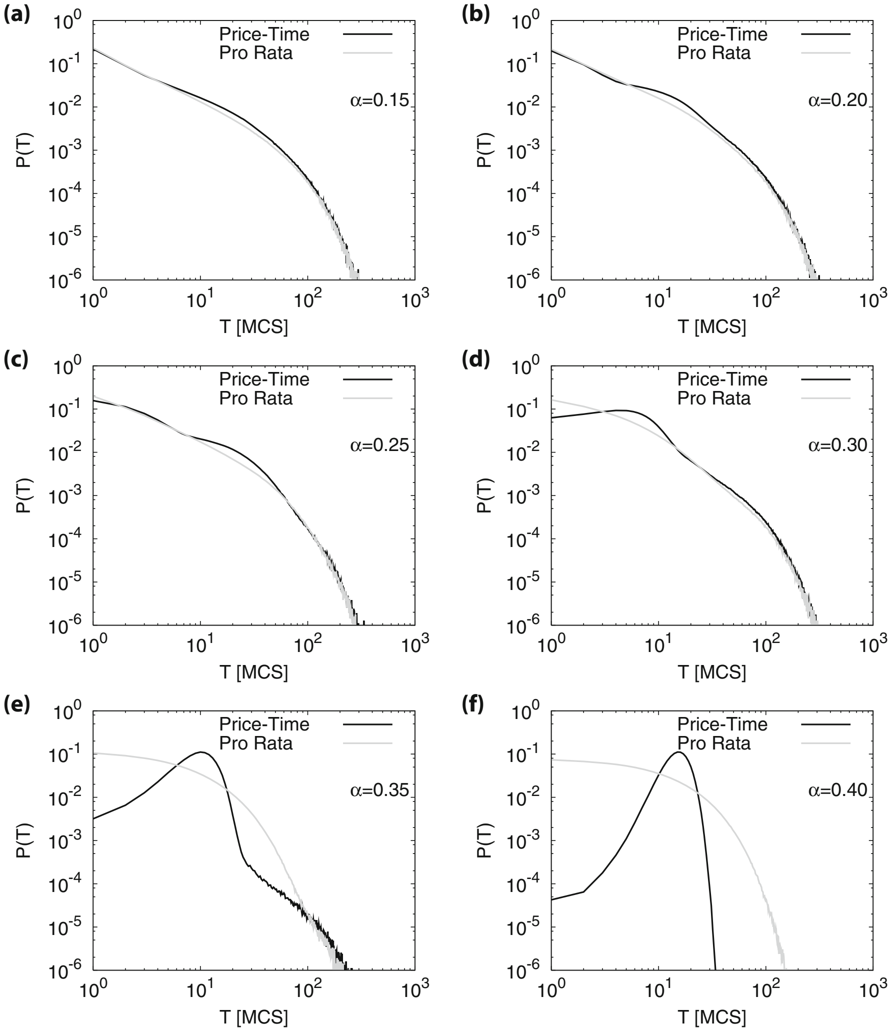

Fig. 1 Time-to-fill distributions of a price-time priority and pro rata based order matching for (a) $\alpha = 0, 15$, (b) $\alpha = 0, 20$, (c) $\alpha = 0, 25$, (d) $\alpha = 0, 30$, (e) $\alpha = 0, 35$, (f) $\alpha = 0, 40$

effect will be analyzed in more detail when we start to "freeze" the order book stepwise, i.e., when we reduce the volatility of best bid and best ask price.

Based on the results which were obtained by the parameter space analysis [4], freezing can be realized increasing the limit order rate α. If the market order rate μ is constant and we increase the limit order rate, then more and more limit orders are placed around around the midpoint. In the end of this process, α is so large that the number of market orders per time unit is too less in order to change the best bid price and the best ask price. Thus, the last traded price is jumping from the constant best

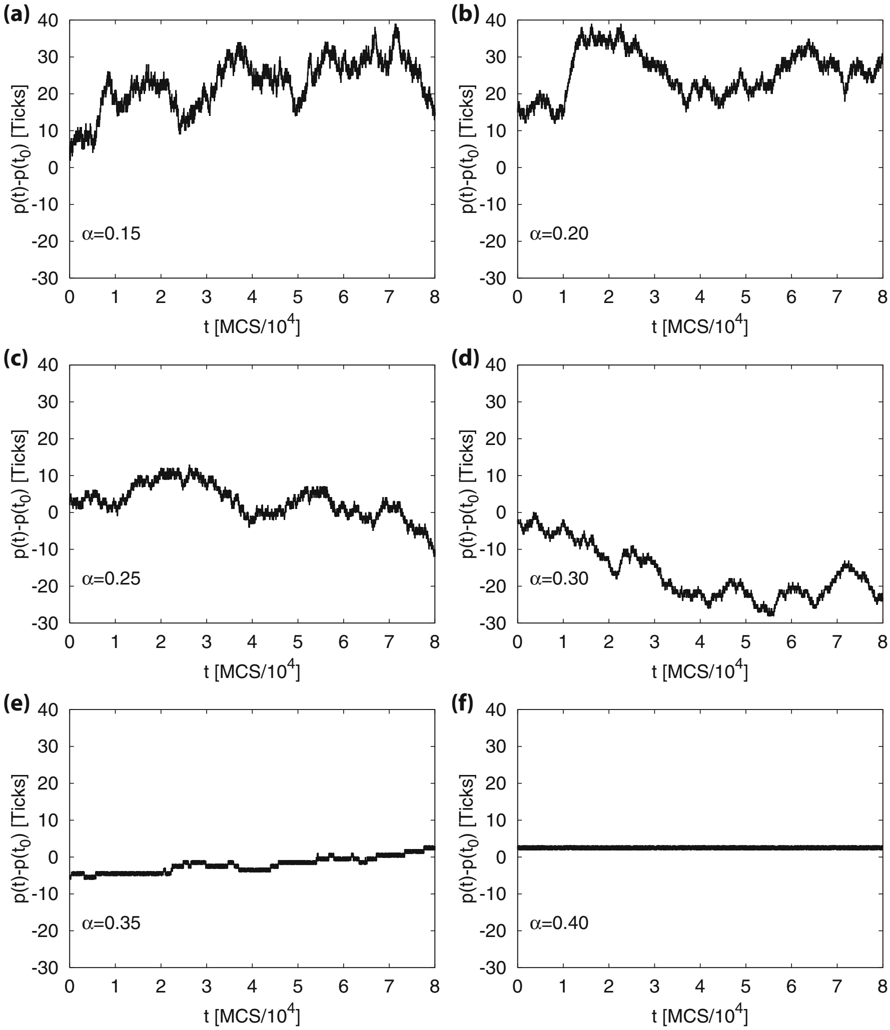

Fig. 2 Freezing of the price time series: Subsets comprising 8×10^4 MCS for (a) $\alpha = 0,15$, (b) $\alpha = 0,20$, (c) $\alpha = 0,25$, (d) $\alpha = 0,30$, (e) $\alpha = 0,35$, (f) $\alpha = 0,40$

bid price to the constant best ask price and vice versa. Fig. 2 shows such a stepwise freezing of the price time series for various values of α. The corresponding time-to-fill distributions are shown in Fig. 1. The larger α the larger is the qualitative change of the distribution shape. Using a pro rata allocation method, we find in all cases a strictly monotonic decreasing distribution. However, if we apply a price-time priority allocation method in our order book, we end up with a distribution which has a distinct maximum located at $T > 1$. This supports in an impressive way why exchange operators are using pro rata matching algorithms for low volatility products as it guarantees more fairness. On the other hand, the question arises whether price-time priority should be replaced by pro rata matching in general as the time-to-fill profile is not changing when we use the pro rata approach. This result supports the tendency in the USA in recent years. There, a larger number of products obeys pro rata matching.

In case of a completely frozen order book and price-time priority (see Fig. 1f and Fig. 2f), a limit buy or limit sell order has to pass the whole queue at best bid or best ask before it can be matched with an arriving market order. If we use pro rata matching the limit orders are randomly selected. Thus, strictly monotonic decreasing shape of the distribution persists also for a frozen order book.

4 Conclusions

Based on our recently introduced order book model of financial markets [3, 4] we analyzed two different matching principles for order allocation – price-time priority and pro rata matching. Price-time priority uses the submission timestamp which prioritizes orders in the book with the same price. The order which was entered earliest at a given price limit gets executed first. Pro rata matching is used for products with low intraday volatility of best bid and best ask price. Pro rata matching ensures constant access for orders of all sizes. The results obtained from simulations of the order book model show that the larger the limit order rate the larger is the qualitative change of the time-to-fill distribution shape. Using a pro rata allocation method, we find in all cases a strictly monotonic decreasing distribution. However, if we apply a price-time priority allocation method in our order book, we end up with a distribution which has a distinct maximum located at $T > 1$.

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