參考文章為:Modelling and control of actuated lower limb exoskeletons : a mathematical application using central pattern generators and nonlinear feedback control techniques

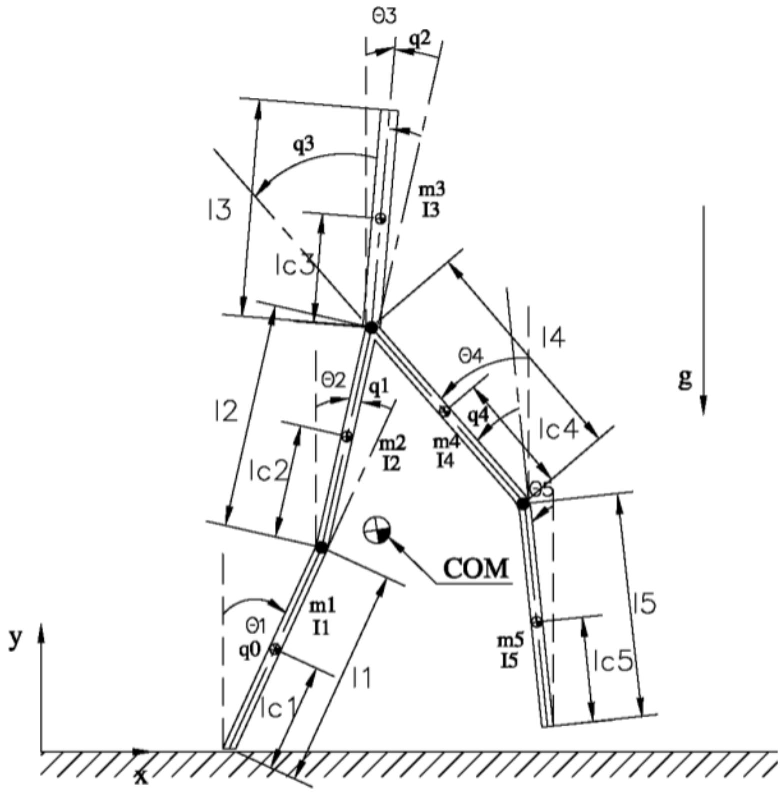
In [1]:
```
# 定義需要用到的函式
from sympy import symbols, Matrix, cos, sin, simplify, diff, expand
from sympy.physics.mechanics import dynamicsymbols
```

In [2]:
```
# th1 = theta1
# _d0 = 0次微分, _d1 = 1次微分, _d2 = 2次微分
th1_d0, th2_d0, th3_d0, th4_d0, th5_d0 = dynamicsymbols('theta1 theta2 theta3 theta4 theta5')
th1_d1, th2_d1, th3_d1, th4_d1, th5_d1 = dynamicsymbols('theta1 theta2 theta3 theta4 theta5',1)
th1_d2, th2_d2, th3_d2, th4_d2, th5_d2 = dynamicsymbols('theta1 theta2 theta3 theta4 theta5',2)
```

In [3]:
```
# define scalars
t, g = symbols('t g') # time, gravity
m1, m2, m3, m4, m5 = symbols('m_1 m_2 m_3 m_4 m_5')
l1, l2, l3, l4, l5 = symbols('l_1 l_2 l_3 l_4 l_5')
lc1, lc2, lc3, lc4, lc5 = symbols('lc_1 lc_2 lc_3 lc_4 lc_5')
I1, I2, I3, I4, I5 = symbols('I_1  I_2  I_3  I_4  I_5')
```

座標定義: 向右為+x, 向上為+z, 入射紙面為+y

$$\hat{z} \times \hat{x} = \hat{y}$$


Θ3
q2
q3
l3
m3
I3
lc3
l4
Θ4
g
Θ2
q1
l2
q4
lc4
m2
I2
m4
I4
lc2
Θ5
COM
l5
Θ1
m1
I1
y
q0
l1
m5
I5
lc5
lc1
x

In [4]:
```python
xc1 = lc1*sin(th1_d0)
zc1 = lc1*cos(th1_d0)

xc2 = l1*sin(th1_d0) + lc2*sin(th2_d0)
zc2 = l1*cos(th1_d0) + lc2*cos(th2_d0)

xc3 = l1*sin(th1_d0) + l2*sin(th2_d0) + lc3*sin(th3_d0)
zc3 = l1*cos(th1_d0) + l2*cos(th2_d0) + lc3*cos(th3_d0)

xc4 = l1*sin(th1_d0) + l2*sin(th2_d0) + (l4-lc4)*sin(th4_d0)
zc4 = l1*cos(th1_d0) + l2*cos(th2_d0) - (l4-lc4)*cos(th4_d0)

xc5 = l1*sin(th1_d0) + l2*sin(th2_d0) + l4*sin(th4_d0) + (l5-lc5)*sin(th5_d0)
zc5 = l1*cos(th1_d0) + l2*cos(th2_d0) - l4*cos(th4_d0) - (l5-lc5)*cos(th5_d0)
```

In [5]:
```python
vc1 = diff(Matrix([[xc1],[0],[zc1]]),t)
vc2 = diff(Matrix([[xc2],[0],[zc2]]),t)
vc3 = diff(Matrix([[xc3],[0],[zc3]]),t)
vc4 = diff(Matrix([[xc4],[0],[zc4]]),t)
vc5 = diff(Matrix([[xc5],[0],[zc5]]),t)
```

Kinetic Energy K

In [6]:
```python
# Translational kinetic energy
K_stand_shank = m1*vc1.T.dot(vc1)/2
K_stand_thigh = m2*vc2.T.dot(vc2)/2
K_body        = m3*vc3.T.dot(vc3)/2
K_swing_thigh = m4*vc4.T.dot(vc4)/2
K_swing_shank = m5*vc5.T.dot(vc5)/2

# Rotational kinetic energy
K_rotation = I1*th1_d1**2/2 + I2*th2_d1**2/2 + I3*th3_d1**2/2 + I4*th4_d1**2/2 + I5*th5_d1**2/2

# Total kinetic energy
K_total = K_stand_shank+K_stand_thigh+K_body+K_swing_thigh+K_swing_shank+K_rotation
```

Potential Energy U

In [7]:
```
U_stand_shank = m1*g*zc1
U_stand_thigh = m2*g*zc2
U_body        = m3*g*zc3
U_swing_thigh = m4*g*zc4
U_swing_shank = m5*g*zc5

U_total = U_stand_shank+U_stand_thigh+U_body+U_swing_thigh+U_swing_shank
```

Lagrangian

In [8]:
```
system_L = K_total - U_total
```

Calculate EoM

In [9]:
```
from sympy.physics.mechanics import LagrangesMethod, Lagrangian

system_LM = LagrangesMethod(system_L, [th1_d0, th2_d0, th3_d0, th4_d0, th5_d0])
system_EoM = system_LM.form_lagranges_equations()
```

Check equations

首先檢查mass matrix, 論文中的mass matrix依序為

$$M_{11} = I_1 + l_{1c}^2 m_1 + l_1^2 m_2 + l_1^2 m_3 + l_1^2 m_4 + l_1^2 m_5$$

$$M_{12} = l_1 l_{c2} m_2 cos(\theta_1 - \theta_2) + l_1 l_2 m_3 cos(\theta_1 - \theta_2) + l_1 l_2 m_4 cos(\theta_1 - \theta_2) + l_1 l_2 m_5 cos(\theta_1 - \theta_2)$$

$$M_{13} = l_1 l_{c3} m_3 cos(\theta_1 - \theta_3)$$

$$M_{14} = l_1 l_4 m_4 cos(\theta_1 + \theta_4) + l_1 l_4 m_5 cos(\theta_1 + \theta_4) - l_1 l_{c4} m_4 cos(\theta_1 + \theta_4)$$

$$M_{15} = l_1 l_5 m_5 cos(\theta_1 + \theta_5) - l_1 l_{c5} m_5 cos(\theta_1 + \theta_5)$$

In [10]:
```
system_MM = simplify(system_LM.mass_matrix)
```

In [11]:
```
system_MM[0,0]
```

Out[11]: $I_1 + l_1^2 m_2 + l_1^2 m_3 + l_1^2 m_4 + l_1^2 m_5 + lc_1^2 m_1$

In [12]:
```
system_MM[0,1]
```

Out[12]: $l_1 \left(l_2 m_3 + l_2 m_4 + l_2 m_5 + lc_2 m_2\right) \cos\left(\theta_1(t) - \theta_2(t)\right)$

In [13]:
```
system_MM[0,2]
```

Out[13]: $l_1 lc_3 m_3 \cos\left(\theta_1(t) - \theta_3(t)\right)$

In [14]:
```
system_MM[0,3]
```

Out[14]: $l_1 \left(l_4 m_5 + m_4 \left(l_4 - lc_4\right)\right) \cos\left(\theta_1(t) + \theta_4(t)\right)$

In [15]:
```
system_MM[0,4]
```

Out[15]: $l_1 m_5 \left(l_5 - lc_5\right) \cos\left(\theta_1(t) + \theta_5(t)\right)$

$$M_{21} = l_1 l_{c2} m_2 cos(\theta_1 - \theta_2) + l_1 l_2 m_3 cos(\theta_1 - \theta_2) + l_1 l_2 m_4 cos(\theta_1 - \theta_2) + l_1 l_2 m_5 cos(\theta_1 - \theta_2)$$

$$M_{22} = I_2 + l_{c2}^2 m_2 + l_2^2 m_3 + l_2^2 m_4 + l_2^2 m_5$$

$$M_{23} = l_2 l_{c3} m_3 cos(\theta_2 - \theta_3)$$

$$M_{24} = l_2 l_4 m_4 cos(\theta_2 + \theta_4) - l_2 l_{c4} m_4 cos(\theta_2 + \theta_4) + l_2 l_4 m_5 cos(\theta_2 + \theta_4)$$

$$M_{25} = l_2 l_5 m_5 cos(\theta_2 + \theta_5) - l_2 l_{c5} m_5 cos(\theta_2 + \theta_5)$$

In [16]:
```
system_MM[1,0]
```

Out[16]: $l_1 \left(l_2 m_3 + l_2 m_4 + l_2 m_5 + lc_2 m_2\right) \cos\left(\theta_1(t) - \theta_2(t)\right)$

In [17]:
```
system_MM[1,1]
```

Out[17]: $I_2 + l_2^2 m_3 + l_2^2 m_4 + l_2^2 m_5 + lc_2^2 m_2$

In [18]:
```
system_MM[1,2]
```

Out[18]: $l_2 lc_3 m_3 \cos(\theta_2(t) - \theta_3(t))$

In [19]:
```
system_MM[1,3]
```

Out[19]: $l_2 (l_4 m_5 + m_4 (l_4 - lc_4)) \cos(\theta_2(t) + \theta_4(t))$

In [20]:
```
system_MM[1,4]
```

Out[20]: $l_2 m_5 (l_5 - lc_5) \cos(\theta_2(t) + \theta_5(t))$

$$M_{31} = l_1 l_{c3} m_3 cos(\theta_1 - \theta_3)$$

$$M_{32} = l_2 l_{c3} m_3 cos(\theta_2 - \theta_3)$$

$$M_{33} = I_3 + l_{c3}^2 m_3$$

$$M_{34} = 0$$

$$M_{35} = 0$$

In [21]:
```
system_MM[2,0]
```

Out[21]: $l_1 lc_3 m_3 \cos(\theta_1(t) - \theta_3(t))$

In [22]:
```
system_MM[2,1]
```

Out[22]: $l_2 lc_3 m_3 \cos(\theta_2(t) - \theta_3(t))$

In [23]:
```
system_MM[2,2]
```

Out[23]: $I_3 + lc_3^2 m_3$

In [24]:
```
system_MM[2,3]
```

Out[24]: 0

In [25]:
```
system_MM[2,4]
```

Out[25]: 0

$$M_{41} = l_1 l_4 m_4 cos(\theta_1 + \theta_4) - l_1 l_{c4} m_4 cos(\theta_1 + \theta_4) + l_1 l_4 m_5 cos(\theta_1 + \theta_4)$$

$$M_{42} = l_2 l_4 m_4 cos(\theta_2 + \theta_4) - l_2 l_{c4} m_4 cos(\theta_2 + \theta_4) + l_2 l_4 m_5 cos(\theta_2 + \theta_4)$$

$$M_{43} = 0$$

$$M_{44} = I_4 + l_4^2 m_4 + l_4^2 m_5 + l_{c4}^2 m_4 - 2 l_4 l_{c4} m_4$$

$$M_{45} = l_4 l_5 m_5 cos(\theta_4 - \theta_5) - l_4 l_{c5} m_5 cos(\theta_4 - \theta_5)$$

In [26]:
```
system_MM[3,0]
```

Out[26]: $l_1 \left(l_4 m_5 + m_4 \left(l_4 - lc_4\right)\right) \cos\left(\theta_1(t) + \theta_4(t)\right)$

In [27]:
```
system_MM[3,1]
```

Out[27]: $l_2 \left(l_4 m_5 + m_4 \left(l_4 - lc_4\right)\right) \cos\left(\theta_2(t) + \theta_4(t)\right)$

In [28]:
```
system_MM[3,2]
```

Out[28]: 0

In [29]:
```
system_MM[3,3]
```
Out[29]: $I_4 + l_4^2 m_5 + m_4(l_4 - lc_4)^2$

In [30]:
```
system_MM[3,4]
```
Out[30]: $l_4 m_5 (l_5 - lc_5) \cos(\theta_4(t) - \theta_5(t))$

$$M_{51} = l_1 l_5 m_5 cos(\theta_1 + \theta_5) - l_1 l_{c5} m_5 cos(\theta_1 + \theta_5)$$
$$M_{52} = l_2 l_5 m_5 cos(\theta_2 + \theta_5) - l_2 l_{c5} m_5 cos(\theta_2 + \theta_5)$$
$$M_{53} = 0$$
$$M_{54} = l_4 l_5 m_5 cos(\theta_4 - \theta_5) - l_4 l_{c5} m_5 cos(\theta_4 - \theta_5)$$
$$M_{55} = I_5 + l_5^2 m_5 + l_{c5}^2 m_5 - 2 l_5 l_{c5} m_5$$

In [31]:
```
system_MM[4,0]
```
Out[31]: $l_1 m_5 (l_5 - lc_5) \cos(\theta_1(t) + \theta_5(t))$

In [32]:
```
system_MM[4,1]
```
Out[32]: $l_2 m_5 (l_5 - lc_5) \cos(\theta_2(t) + \theta_5(t))$

In [33]:
```
system_MM[4,2]
```
Out[33]: 0

In [34]:
```
system_MM[4,3]
```
Out[34]: $l_4 m_5 (l_5 - lc_5) \cos(\theta_4(t) - \theta_5(t))$

In [35]:
```python
system_MM[4,4]
```
Out[35]: $I_5 + m_5(l_5 - lc_5)^2$

Force Matrix 計算

In [36]:
```python
system_LM_forcing = -system_LM.forcing
system_ForceMatrix = expand(system_LM_forcing - diff(system_LM_forcing, g)*g)
```

在使用global angle θ 定義的情況下，可能是因為角度沒有耦聯(couple)，force matrix的項次都"係數*一次微分項^2"，因此使用兩次偏微分提出需要的項次

$$C_{11} = 0$$
$$C_{12} = \dot{\theta}_2 l_1 l_{c2} m_2 sin(\theta_1 - \theta_2) + \dot{\theta}_2 l_1 l_2 m_3 sin(\theta_1 - \theta_2) + \dot{\theta}_2 l_1 l_2 m_4 sin(\theta_1 - \theta_2) + \dot{\theta}_2 l_1 l_2 m_5 sin(\theta_1 - \theta_2)$$
$$C_{13} = \dot{\theta}_3 l_1 l_{c3} m_3 sin(\theta_1 - \theta_3)$$
$$C_{14} = \dot{\theta}_4 l_1 l_{c4} m_4 sin(\theta_1 + \theta_4) - \dot{\theta}_4 l_1 l_4 m_4 sin(\theta_1 + \theta_4) - \dot{\theta}_4 l_1 l_4 m_5 sin(\theta_1 + \theta_4)$$
$$C_{15} = -\dot{\theta}_5 l_1 l_5 m_5 sin(\theta_1 + \theta_5) + \dot{\theta}_5 l_1 l_{c5} m_5 sin(\theta_1 + \theta_5)$$

In [37]:
```python
C11 = simplify(diff(system_ForceMatrix[0], th1_d1, 2)/2)*th1_d1; C11
```
Out[37]: 0

In [38]:
```python
C12 = simplify(diff(system_ForceMatrix[0], th2_d1, 2)/2)*th2_d1; C12
```
Out[38]: $l_1\left(l_2 m_3 + l_2 m_4 + l_2 m_5 + lc_2 m_2\right)\sin\left(\theta_1(t) - \theta_2(t)\right)\frac{d}{dt}\theta_2(t)$

In [39]:
```python
C13 = simplify(diff(system_ForceMatrix[0], th3_d1, 2)/2)*th3_d1; C13
```
Out[39]: $l_1 lc_3 m_3 \sin\left(\theta_1(t) - \theta_3(t)\right)\frac{d}{dt}\theta_3(t)$

In [40]:
```
C14 = simplify(diff(system_ForceMatrix[0], th4_d1, 2)/2)*th4_d1; C14
```

Out[40]:

$$l_1 \left(-l_4 m_4 - l_4 m_5 + lc_4 m_4\right) \sin\left(\theta_1(t) + \theta_4(t)\right) \frac{d}{dt} \theta_4(t)$$

In [41]:
```
C15 = simplify(diff(system_ForceMatrix[0], th5_d1, 2)/2)*th5_d1; C15
```

Out[41]:

$$l_1 m_5 \left(-l_5 + lc_5\right) \sin\left(\theta_1(t) + \theta_5(t)\right) \frac{d}{dt} \theta_5(t)$$

$$C_{21} = -\dot{\theta}_1 l_1 l_{c2} m_2 sin(\theta_1 - \theta_2) - \dot{\theta}_1 l_1 l_2 m_3 sin(\theta_1 - \theta_2) - \dot{\theta}_1 l_1 l_2 m_4 sin(\theta_1 - \theta_2) - \dot{\theta}_1 l_1 l_2 m_5 sin(\theta_1 - \theta_2)$$

$$C_{22} = 0$$

$$C_{23} = \dot{\theta}_3 l_2 l_{c3} m_3 sin(\theta_2 - \theta_3)$$

$$C_{24} = -\dot{\theta}_4 l_2 l_4 m_4 sin(\theta_2 + \theta_4) - \dot{\theta}_4 l_2 l_4 m_5 sin(\theta_2 + \theta_4) + \dot{\theta}_4 l_2 l_{c4} m_4 sin(\theta_2 + \theta_4)$$

$$C_{25} = -\dot{\theta}_5 l_2 l_5 m_5 sin(\theta_2 + \theta_5) + \dot{\theta}_5 l_2 l_{c5} m_5 sin(\theta_2 + \theta_5)$$

In [42]:
```
C21 = simplify(diff(system_ForceMatrix[1], th1_d1, 2)/2)*th1_d1; C21
```

Out[42]:

$$-l_1 \left(l_2 m_3 + l_2 m_4 + l_2 m_5 + lc_2 m_2\right) \sin\left(\theta_1(t) - \theta_2(t)\right) \frac{d}{dt} \theta_1(t)$$

In [43]:
```
C22 = simplify(diff(system_ForceMatrix[1], th2_d1, 2)/2)*th2_d1; C22
```

Out[43]: 0

In [44]:
```
C23 = simplify(diff(system_ForceMatrix[1], th3_d1, 2)/2)*th3_d1; C23
```

Out[44]:

$$l_2 lc_3 m_3 \sin\left(\theta_2(t) - \theta_3(t)\right) \frac{d}{dt} \theta_3(t)$$

In [45]:
```
1 C24 = simplify(diff(system_ForceMatrix[1], th4_d1, 2)/2)*th4_d1; C24
```

Out[45]:

$$l_2\left(-l_4 m_4 - l_4 m_5 + lc_4 m_4\right)\sin\left(\theta_2(t) + \theta_4(t)\right)\frac{d}{dt}\theta_4(t)$$

In [46]:
```
1 C25 = simplify(diff(system_ForceMatrix[1], th5_d1, 2)/2)*th5_d1; C25
```

Out[46]:

$$l_2 m_5\left(-l_5 + lc_5\right)\sin\left(\theta_2(t) + \theta_5(t)\right)\frac{d}{dt}\theta_5(t)$$

$$C_{31} = -\dot{\theta}_1 l_1 l_{c3} m_3 sin(\theta_1 - \theta_3)$$

$$C_{32} = -\dot{\theta}_2 l_2 l_{c3} m_3 sin(\theta_2 - \theta_3)$$

$$C_{33} = 0$$

$$C_{34} = 0$$

$$C_{35} = 0$$

In [47]:
```
1 C31 = simplify(diff(system_ForceMatrix[2], th1_d1, 2)/2)*th1_d1; C31
```

Out[47]:

$$-l_1 lc_3 m_3 \sin\left(\theta_1(t) - \theta_3(t)\right)\frac{d}{dt}\theta_1(t)$$

In [48]:
```
1 C32 = simplify(diff(system_ForceMatrix[2], th2_d1, 2)/2)*th2_d1; C32
```

Out[48]:

$$-l_2 lc_3 m_3 \sin\left(\theta_2(t) - \theta_3(t)\right)\frac{d}{dt}\theta_2(t)$$

In [49]:
```
1 C33 = simplify(diff(system_ForceMatrix[2], th3_d1, 2)/2)*th3_d1; C33
```

Out[49]: 0

In [50]:
```
1 C34 = simplify(diff(system_ForceMatrix[2], th4_d1, 2)/2)*th4_d1; C34
```

Out[50]: 0

In [51]:
```
C35 = simplify(diff(system_ForceMatrix[2], th5_d1, 2)/2)*th5_d1; C35
```
Out[51]: 0

$$C_{41} = \dot{\theta}_1 l_1 l_{c4} m_4 sin(\theta_1 + \theta_4) - \dot{\theta}_1 l_1 l_4 m_4 sin(\theta_1 + \theta_4) - \dot{\theta}_1 l_1 l_4 m_5 sin(\theta_1 + \theta_4)$$
$$C_{42} = \dot{\theta}_2 l_2 l_{c4} m_4 sin(\theta_2 + \theta_4) - \dot{\theta}_2 l_2 l_4 m_4 sin(\theta_2 + \theta_4) - \dot{\theta}_2 l_2 l_4 m_5 sin(\theta_2 + \theta_4)$$
$$C_{43} = 0$$
$$C_{44} = 0$$
$$C_{45} = \dot{\theta}_5 l_4 l_5 m_5 sin(\theta_4 - \theta_5) - \dot{\theta}_5 l_4 l_{c5} m_5 sin(\theta_4 - \theta_5);$$

In [52]:
```
C41 = simplify(diff(system_ForceMatrix[3], th1_d1, 2)/2)*th1_d1; C41
```
Out[52]: $$l_1 \left(-l_4 m_4 - l_4 m_5 + lc_4 m_4\right) \sin\left(\theta_1(t) + \theta_4(t)\right) \frac{d}{dt} \theta_1(t)$$

In [53]:
```
C42 = simplify(diff(system_ForceMatrix[3], th2_d1, 2)/2)*th2_d1; C42
```
Out[53]: $$l_2 \left(-l_4 m_4 - l_4 m_5 + lc_4 m_4\right) \sin\left(\theta_2(t) + \theta_4(t)\right) \frac{d}{dt} \theta_2(t)$$

In [54]:
```
C43 = simplify(diff(system_ForceMatrix[3], th3_d1, 2)/2)*th3_d1; C43
```
Out[54]: 0

In [55]:
```
C44 = simplify(diff(system_ForceMatrix[3], th4_d1, 2)/2)*th4_d1; C44
```
Out[55]: 0

In [56]:
```
C45 = simplify(diff(system_ForceMatrix[3], th5_d1, 2)/2)*th5_d1; C45
```
Out[56]: $$l_4 m_5 \left(l_5 - lc_5\right) \sin\left(\theta_4(t) - \theta_5(t)\right) \frac{d}{dt} \theta_5(t)$$

$$C_{51} = -\dot{\theta}_1 l_1 l_5 m_5 sin(\theta_1 + \theta_5) + \dot{\theta}_1 l_1 l_{c5} m_5 sin(\theta_1 + \theta_5)$$

$$C_{52} = -\dot{\theta}_2 l_2 l_5 m_5 sin(\theta_2 + \theta_5) + \dot{\theta}_2 l_2 l_{c5} m_5 sin(\theta_2 + \theta_5)$$

$$C_{53} = 0$$

$$C_{54} = -\dot{\theta}_4 l_4 l_5 m_5 sin(\theta_4 - \theta_5) + \dot{\theta}_4 l_4 l_{c5} m_5 sin(\theta_4 - \theta_5)$$

$$C_{55} = 0$$

In [57]:
```
C51 = simplify(diff(system_ForceMatrix[4], th1_d1, 2)/2)*th1_d1; C51
```

Out[57]:

$$l_1 m_5 (-l_5 + lc_5) \sin(\theta_1(t) + \theta_5(t)) \frac{d}{dt}\theta_1(t)$$

In [58]:
```
C52 = simplify(diff(system_ForceMatrix[4], th2_d1, 2)/2)*th2_d1; C52
```

Out[58]:

$$l_2 m_5 (-l_5 + lc_5) \sin(\theta_2(t) + \theta_5(t)) \frac{d}{dt}\theta_2(t)$$

In [59]:
```
C53 = simplify(diff(system_ForceMatrix[4], th3_d1, 2)/2)*th3_d1; C53
```

Out[59]: 0

In [60]:
```
C54 = simplify(diff(system_ForceMatrix[4], th4_d1, 2)/2)*th4_d1; C54
```

Out[60]:

$$l_4 m_5 (-l_5 + lc_5) \sin(\theta_4(t) - \theta_5(t)) \frac{d}{dt}\theta_4(t)$$

In [61]:
```
C55 = simplify(diff(system_ForceMatrix[4], th5_d1, 2)/2)*th5_d1; C55
```

Out[61]: 0

Gravity Matrix 計算

In [62]:
```
system_GravityMatrix = simplify(diff(system_LM_forcing, g)*g)
system_GravityMatrix
```

Out[62]:

$$\left[\begin{matrix}- g \left(l_{1} m_{2} + l_{1} m_{3} + l_{1} m_{4} + l_{1} m_{5} + lc_{1} m_{1}\right) \sin{\left(\theta_{1}(t) \right)}\\- g \left(l_{2} m_{3} + l_{2} m_{4} + l_{2} m_{5} + lc_{2} m_{2}\right) \sin{\left(\theta_{2}(t) \right)}\\- g lc_{3} m_{3} \sin{\left(\theta_{3}(t) \right)}\\g \left(l_{4} m_{5} + m_{4} \left(l_{4} - lc_{4}\right)\right) \sin{\left(\theta_{4}(t) \right)}\\g m_{5} \left(l_{5} - lc_{5}\right) \sin{\left(\theta_{5}(t) \right)}\end{matrix}\right]$$

計算結果與Ajayi論文中一致