

# Probability with Martingales

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March 15, 2021

## 1 Random Variables

### 1.1 Defs. Law, distribution function

Suppose that  $X$  is a random variable carried by some probability triple  $(\Omega, \mathcal{F}, \mathcal{P})$ . We have

$$\begin{aligned} \Omega &\xrightarrow{X} \mathbb{R} \\ [0, 1] &\xleftarrow{\mathcal{P}} \mathcal{F} \xleftarrow{X^{-1}} \mathcal{B} \\ \text{or indeed } [0, 1] &\xleftarrow{\mathcal{P}} \sigma(X) \xleftarrow{X^{-1}} \mathcal{B} \end{aligned}$$

Define the law  $\mathcal{L}_X$  of  $X$  by

$$\mathcal{L}_X := \mathcal{P} \circ X^{-1} : \mathcal{B} \rightarrow [0, 1]$$

Then  $\mathcal{L}_X$  is a probability measure on  $(\mathbb{R}, \mathcal{B})$ . Since  $\pi(\mathbb{R}) = \{(-\infty, c] : c \in \mathbb{R}\}$  is a  $\pi$ -system which generates  $\mathcal{B}$ , Uniqueness Lemma 1.6 shows that  $\mathcal{L}_X$  is determined by the function  $F_X : \mathbb{R} \rightarrow [0, 1]$  defined as follows:

$$F_X(c) := \mathcal{L}_X(-\infty, c] = P(X \leq c) = P\{w : X(w) \leq c\}$$

The function  $F_X$  is called the distribution function of  $X$ .