Probability with Martingales

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1 Random Variables

1.1 Defs. Law, distribution function

Suppose that X is a random variable carried by some probability triple $(\Omega, \mathcal{F}, \mathcal{P})$. We have

$$\Omega \xrightarrow{X} \mathbb{R}$$
$$[0,1] \xleftarrow{\mathcal{P}} \mathcal{F} \xleftarrow{X^{-1}} \mathcal{B}$$
$$\textit{or indeed } [0,1] \xleftarrow{\mathcal{P}} \sigma(X) \xleftarrow{X^{-1}} \mathcal{B}$$

Define the law $\mathcal{L}_X$ of X by

$$\mathcal{L}_X := \mathcal{P} \circ X^{-1} \mathcal{L}_X : \mathcal{B} \to [0,1]$$

Then $\mathcal{L}_X$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. Since $\pi(\mathbb{R}) = \{(-\infty, c] : c \in \mathbb{R}\}$ is a π-system which generates $\mathcal{B}$, Uniqueness Lemma 1.6 shows that $\mathcal{L}_X$ is determined by the function $F_X : \mathbb{R} \to [0,1]$ defined as follows:

$$F_X(c) := \mathcal{L}_X(-\infty, c] = P(X \leq c) = P\{w : X(w) \leq c\}$$

The funtion F_X is called the distribution function of X.