1 R Tutorial

1.1 Loading Data

```r
# loading csv files
data <- read.table("whatever.csv", sep="", header=T)

# csv files can be stored with (almost) any kind of file ending, e.g.:
data <- read.table("whatever.dat", sep="", header=T)
data <- read.table("whatever.txt", sep="", header=T)
```

2 Probability And Statistics

2.1 Probability Models for Measurement Data

2.1.1 Random Variables

<table>
<tr><th colspan="2">Random Variables</th></tr>
<tr><td>Definition</td><td>$X : \Omega \longrightarrow W_{\mathrm{x}}$</td></tr>
<tr><td>Example</td><td>A Coin is thrown three times, head and tails is observed:
$\Omega = \{hhh, hht, htt, hth, ttt, tth, thh, tht\}$
Total number of heads $W_{\mathrm{x}} = \{0, 1, 2, 3\}$
Total number of tails $W_{\mathrm{x}} = \{0, 1, 2, 3\}$
Number of heads minus tails $W_{\mathrm{x}} = \{-3, -1, 1, 3\}$</td></tr>
<tr><th colspan="2">Probability Mass Function</th></tr>
<tr><td>Definition</td><td>The probability distribution of a discrete random variable:
$P(X = x)$</td></tr>
<tr><td>Example</td><td><table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>$P(X = x)$</td><td>$\frac{1}{8}$</td><td>$\frac{3}{8}$</td><td>$\frac{3}{8}$</td><td>$\frac{1}{8}$</td></tr></table></td></tr>
</table>

2.1.2 Probability Distributions

<table>
<tr><th colspan="2">Cumulative Density Function (cdf)</th></tr>
<tr><td>Definition</td><td>$F(x) = P(X \leqslant x)$</td></tr>
<tr><td>Properties</td><td>$P(a < X \leqslant b) = F(b) - F(a)$
$0 \leqslant F(x) \leqslant 1$
$P(X = a) = F(a) - F(a) = 0$</td></tr>
</table>

Probability Density Function (pdf)	
Definition	$f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$
Properties	$f(x) \geqslant 0$ $P(a < X \leqslant b) = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$ $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$

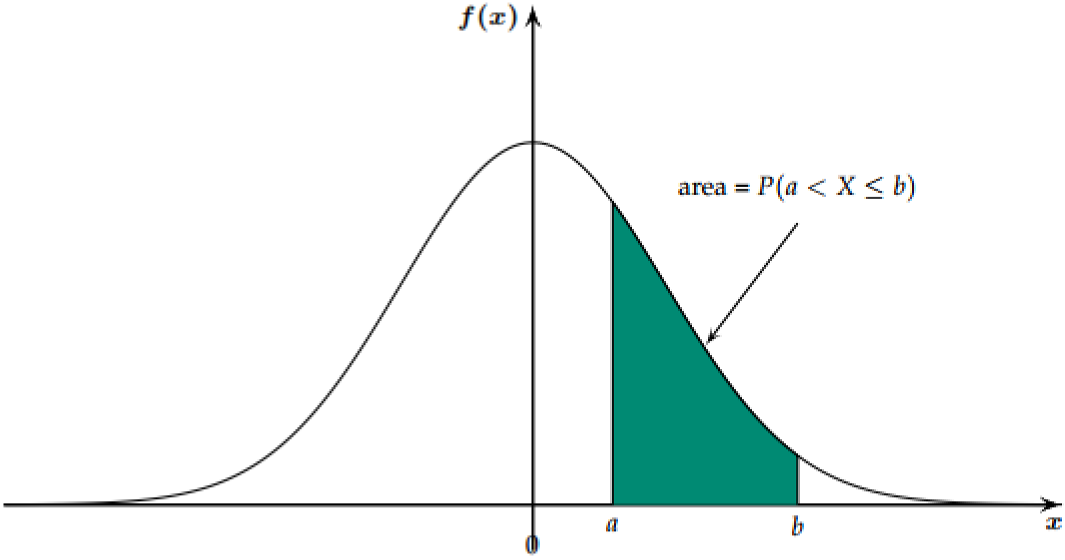


Figure 1: Probability density of a random variable and the probability of measuring a value from (a,b]

2.1.3 Summary Statistics of Continuous Distributions

Expected Value, Variance and Quantile	
Expected value	Discrete: $\mathrm{E}(X) = \sum_i x_i P(X = x_i)$ Continuous: $\mathrm{E}(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x$
Variance	$\mathrm{Var}(X) = \sigma_x^2 = \mathrm{E}((X - \mathrm{E}(X))^2) = \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2 \cdot f(x)\mathrm{d}x$
Quantile	$P(X \leqslant q(\alpha)) = \alpha$ $F(q(\alpha)) = \alpha \Leftrightarrow q(\alpha) = F^{-1}(\alpha)$ Note: When you're asked for the 50%-quantile, that means $\alpha = 50\%$, and you must find $q(0.5)$
Example Body Length	If α=0.75 and the corresponding quantile is $q(\alpha)$=182.5cm then 75% of the persons is shorter or equal 182.5cm.

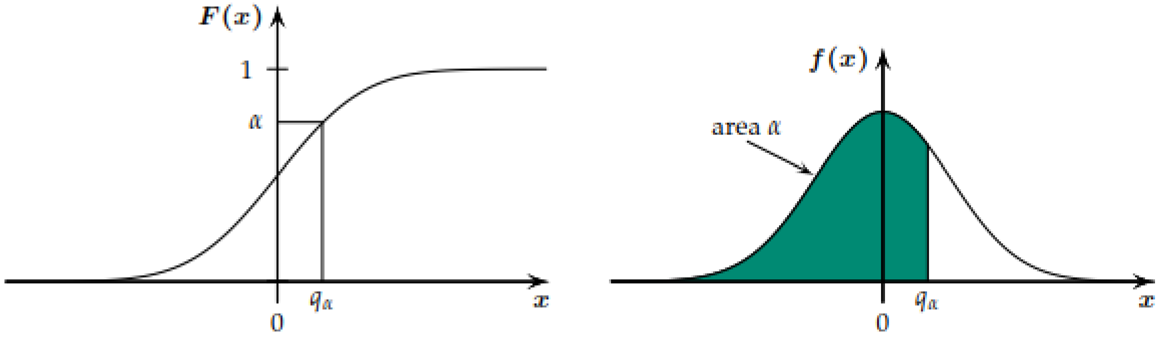


Figure 2: Quantiles

2.1.4 Important Distributions

2.1.4.1 Uniform Distribution

Theory

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & x > b \end{cases}$$

$$\mathrm{E}(x) = \frac{a+b}{2}$$

$$\mathrm{Var}(x) = \frac{(b-a)^2}{12}$$

$$\sigma_{\mathrm{x}} = \frac{b-a}{\sqrt{12}}$$

Code Example

```
# value of the probability density function
    Uniform([1, 10]) at the position x = 5
dunif(x=5, min=1, max=10)
[1] 0.1111111

# P(X <= 5)
punif(q=5, min=1, max=10)
[1] 0.4444444

# P(1.2 < X <= 4.8)
punif(4.8, 1, 10) - punif(1.2, 1, 10)
[1] 0.4

# 5 uniformly distributed random values in
    Uniform([1, 10])
runif(5,min=1,max=10)
[1] 1.061933 6.484813 5.928334 8.459887 8.852405

# TODO: ADD MORE HERE
```

2.1.4.2 Exponential Distribution

Theory

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 1 - \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}(x) = \frac{1}{\lambda}$$

$$\mathrm{Var}(x) = \frac{1}{\lambda^2}$$

$$\sigma_{\mathrm{x}} = \frac{1}{\lambda}$$

Code Example

```
# P(0 <= X <= 4) of X ~ Exp(3)
pexp(4, rate=3)
[1] 0.9999939

# TODO: ADD MORE HERE
```

2.1.4.3 Normal Distribution

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})}$$
$$F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}y$$
$$\mathrm{E}(x) = \mu$$
$$\mathrm{Var}(x) = \sigma^2$$
$$\sigma_{\mathrm{x}} = \sigma$$

</td><td>

```
# X~N(u, sigma^2) --> X~N(100,15^2)
# In R we compute P(X>130) as 1 - P(X<=130)
1-pnorm(130, mean=100, sd=15)
[1] 0.02275013

#P(85<=X<=115)
pnorm(115, mean=100, sd=15)-pnorm(85, mean=100,
    sd=15)
[1] 0.6826895

# TODO: ADD MORE HERE
```

</td></tr>
</table>

2.1.4.4 Linear Transformation of Random Variables

<table>
<tr><th colspan="2">Properties of Linear Transformation of a Random Variable</th></tr>
<tr><td>Definition</td><td>

For $Y = a + bX$ the following apply

(i) $E(Y) = a + bE(X)$

(ii) $Var(Y) = b^2 Var(X)$, $\sigma_Y = |b|\sigma_X$

(iii) $\alpha - Quantile\, of\, Y = q_Y(\alpha) = a + bq_X(\alpha)$

(iv) $f_Y(y) = \frac{1}{b} f_X(\frac{y-a}{b})$

</td></tr>
<tr><th colspan="2">Summary Statistics of S_n and $\bar{X}_n$</th></tr>
<tr><td>Summary Statistics of Sample Total S_n</td><td>

$E(S_n) = E(X_1 + X_2 + ... + X_n) = \sum_{i=1}^{n} E(X_i) = n\mu$

$Var(S_n) = \sum_{i=1}^{n} Var(X_i) = nVar(X_i)$

$\sigma(S_n) = \sqrt{n}\sigma_X$

</td></tr>
<tr><td>Summary Statistics of Sample Mean $\bar{X}_n$

Standard Error</td><td>

$E(\bar{X}_n) = E(\frac{X_1+X_2+...+X_n}{n}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} nE(X_i) = \mu$

$Var(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma_X^2 = \frac{\sigma_X^2}{n}$

$\sigma(\bar{X}_n) = \frac{\sigma_X}{\sqrt{n}}$

</td></tr>
</table>

2.1.4.5 Distributions of S_n and $\bar{X}_n$

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

1. For $X_i \in \{0, 1\}$, we have
$$S_n \sim \mathrm{Bin}(n, \pi) \text{ with } \pi = P(X_i = 1)$$
2. For $X_i \sim \mathrm{Pois}(\lambda)$, we have
$$S_n \sim \mathrm{Pois}(n\lambda)$$
3. For $X_i \sim N(\mu, \sigma^2)$
$$S_n \sim N(n\mu, n\sigma^2) \text{ and } \bar{X}_n \sim N(\mu, \frac{\sigma_X^2}{n})$$

</td><td>

```
What is the probability that among 10000 tosses
    of a fair coin, heads would appear in
    maximum 5100 cases?
#Approximated: X~N(5000,2500)
pnorm(5100,5000,sqrt(2500))
[1] 0.9772499

#"True Result": X~Bin(10000,0.5)
pbinom(5100,10000,0.5)
[1] 0.9777871
```

</td></tr>
</table>

2.2 Statistics for Measurement Data

bgg

2.2.1 Assess the Normal Distribution Assumption

2.2.1.1 Q-Q Plot

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

1. For

$$\alpha_k = \frac{k-0.5}{n} \text{ with } k = 1, ..., n$$

calculate the corresponding theoretical quantiles of the model distribution

$$q(\alpha_k) = F^{-1}(\alpha_k)$$

2. Determine the empirical α_k-quantiles,

$$x_{(1)} < x_{(2)} < ... < x_{(n)}$$

3. Plot the empirical quantiles x_k on the y-axis against the theoretical quantiles $q(\alpha_k)$ on the x-axis.

</td><td>

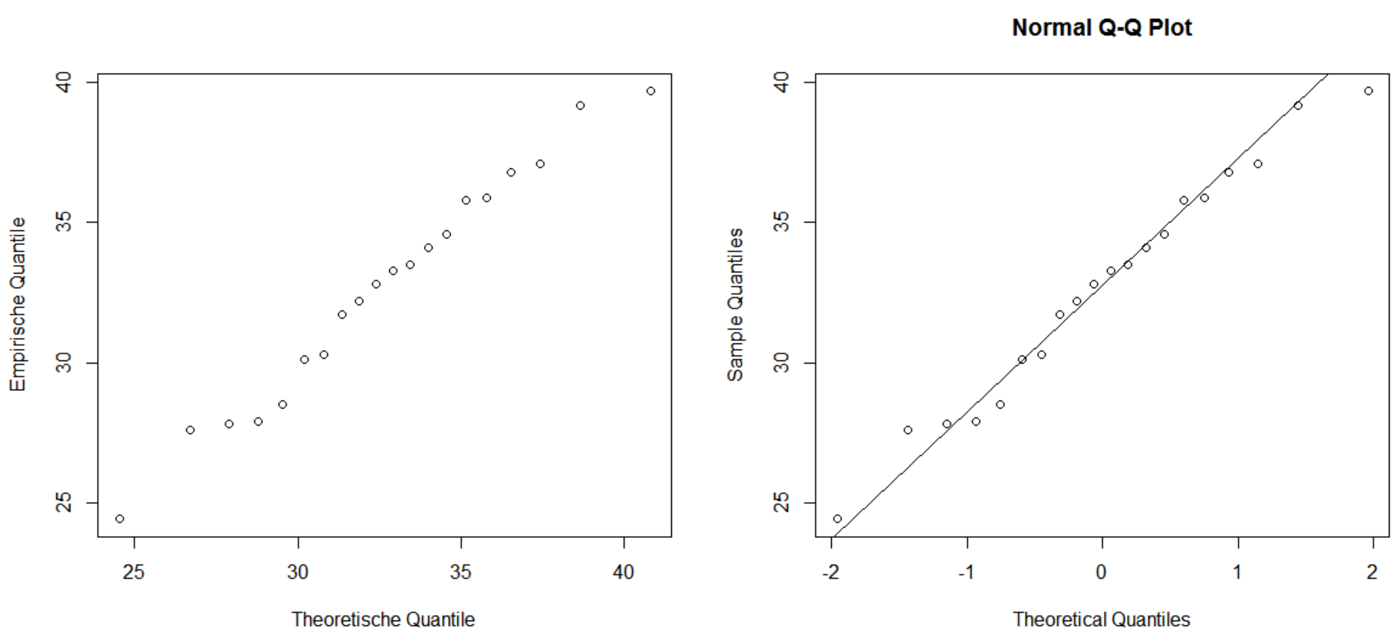
```
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3,
     31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6,
    35.8, 35.9, 36.8, 37.1, 39.2, 39.7)

alpha_k <- (seq(1, length(x), by=1)-0.5)/length(
    x)

quantile_th <- qnorm(alpha_k, mean=mean(x), sd=
    sd(x))

quantile_emp <- sort(x)
#image qqplot
qqplot(quantile_th, quantile_emp, xlab="
    Theoretische Quantile", ylab = "Empirische
    Quantile")
#image qqnorm;qqline
qqnorm(x);qqline(x)
```

</td></tr>
</table>



(a) qqplot()

(b) qqnorm();qqline()

k	$x_{(k)}$	$\alpha_k = (k-0.5)/n$	q_{α_k} for $\mathcal{N}(32.7, 4.15^2)$	$\Phi^{-1}(\alpha_k)$
1	24.4	0.0250	24.5	-1.96
2	27.6	0.075	26.7	-1.44
3	27.8	0.125	27.9	-1.15
4	27.9	0.175	28.8	-0.935
5	28.5	0.225	29.5	-0.755
6	30.1	0.275	30.2	-0.600
7	30.3	0.325	30.8	-0.453
8	31.7	0.375	31.3	-0.319
9	32.2	0.425	31.9	-0.189
10	32.8	0.475	32.4	-0.0627
11	33.3	0.525	32.9	0.0627
12	33.5	0.575	33.4	0.189
13	34.1	0.625	34.0	0.319
14	34.6	0.675	34.5	0.454
15	35.8	0.725	35.1	0.598
16	35.9	0.775	36.0	0.755
17	36.8	0.825	36.5	0.935
18	37.1	0.875	37.4	1.15
19	39.2	0.925	38.6	1.44
20	39.7	0.975	40.8	1.96

```r
#x(k) are the measured values N(u,sigma^2)
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3,
    31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6, 35.8,
     35.9, 36.8, 37.1, 39.2, 39.7)
mean(x)
[1] 32.665
sd(x)
[1] 4.149734
#N(32.7,4.15)
#a_k = (k-0.5)/n = qnorm(q_ak, 32.7, 4.15)
pnorm(24.5, 32.7, 4.15)
[1] 0.02408285
pnorm(32.4, 32.7, 4.15)
[1] 0.4711859
pnorm(35.8, 32.7, 4.15)
[1] 0.7724646
pnorm(40.8, 32.7, 4.15)
[1] 0.9745195
#q_ak for N(32.7,4.15) = qnorm(a_k, 32.7, 4.15)
qnorm(0.025, 32.7, 4.15)
[1] 24.56615
qnorm(0.475, 32.7, 4.15)
[1] 32.43977
qnorm(0.725, 32.7, 4.15)
[1] 35.1807
qnorm(0.975, 32.7, 4.15)
[1] 40.83385
#phi^{-1}(a_k)
qnorm(0.025)
[1] -1.959964
qnorm(0.475)
[1] -0.06270678
qnorm(0.725)
[1] 0.5977601
qnorm(0.975)
[1] 1.959964
```

2.2.2 Parameter Esitmation for Continuous Probability Distributions

Method of Moments (not unbiased)

1. We consider our data measurements $x_1, x_2, ..., x_n$ as realization of random variables $X_1, X_2, ..., X_n$ originating from the same known distribution.

$\mu = E(X) \Rightarrow \hat{\mu} = \bar{x}_n = \frac{x_1+x_2+...+x_n}{n} = \frac{653.3}{20} = 32.7$

$\sigma^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

2. We calculate the expected value $E(X)$ and solve the equation for the unknown parameter that we intend to estimate.

$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2$

$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2}{n}$

3. We replace the expected value with its counterpart, the empirical mean value and obtain an estimate of the unknown parameter. A method of moments estimate of the standard deviation is the empirical standard deviation.

$\hat{\sigma}^2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2} = \sqrt{\frac{1}{20} \sum_{i=1}^{20} (x_i - 32.7)^2} = 4.04$

Method of Maximum Likelihood

We have n observations that are i.i.d.

$X_1 = x_1, X_2 = x_2, ..., X_n = x_n$

For a discrete probability distribution: probability that these n observations (events) actually have occurred can be expressed as follows

$P[(X_1 = x_1) \cap (X_2 = x_2) \cap ... \cap (X_n = x_n)] =$
$P[X_1 = x_1] \cdot P[X_2 = x_2] \cdot ... \cdot P[X_n = x_n] = \prod_{i=1}^{n} P[X_i = x_i]$

Probability that the n independent random variables $x_1, x_2, ..., x_n$ are observed, depends on parameter θ, which we wish to estimate. Therefore the Likelihood function is given by $L(\theta)$ where $P[X_i = x_i\|\theta]$ denotes probability mass function that value x_i has been observed, given the parameter value θ.	$L(\theta) = P[X_1 = x_1\|\theta] \cdot P[X_2 = x_2\|\theta] \cdot ... \cdot P[X_n = x_n\|\theta] = \prod_{i=1}^{n} P[X_i = x_i\|\theta]$
Idea of Maximum Likelihood : estimate the parameter θ in such a way that the likelihood is maximized, that is, that it makes the observed data most likely or most probable.	
Continuous probability distributions : with probability density function $f(x;\theta)$. Probability, that each observation x_i falls into its corresponding interval $[x_i, x_i + dx_i]$:	$\prod_{i=1}^{n} f(x_i;\theta)dx_i$
Infinitesimal intervals dx_i do not depend on the parameter value θ : we omit them in the likelihood function	$\prod_{i=1}^{n} f(x_i;\theta)$
If assumed probability density function $f(x_i;\theta)$ and parameter value of θ are correct, we expect a high probability for the actually observed data to occur : maximization of $L(\theta)$	

Example: Maximum Likelihood for Exponential Distribution	
Let $X_1, X_2..., X_n$ i.i.d. $\sim \text{Exp}(\lambda)$, that is	$f(x_i;\lambda) = \lambda e^{-\lambda x_i}$
Likelihood function for a given data set $x_1, x_2, ..., x_n$ is given by	$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$
Log likelihood function is	$\log(L(\lambda)) = n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i$
If we calculate the derivative of the log likelihood function with respect to λ and set it equal to 0, then we obtain	$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \overset{!}{=} 0$
The maximum likelihood estimate $\hat{\lambda}$ thus corresponds to the solution of the previous equation	$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

2.2.3 Statistical Tests and Confidence Interval for Normally Distributed Data

z-Test (σ_x known)	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X known
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0$ H_A: $\mu \neq \mu_0$ (or $<$ or $>$)
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{(\bar{X}_n - \mu_0)}{\sigma_{\bar{X}_n}} = \frac{(\bar{X}_n - \mu_0)}{\sigma_{X_n}/\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{observed - expected}{standard\,error}$ $Z \sim N(0,1)$
4. Significance level:	α
5. Rejection region for the test statistic: where	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A: \mu \neq \mu_0$, $K = (-\infty, z_\alpha]$ with $H_A: \mu < \mu_0$, $K = [z_{1-\alpha}, \infty)$ with $H_A: \mu > \mu_0$ $z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2)$

6. Test decision:	Check whether the observed value of the test statistic falls into the rejection region.

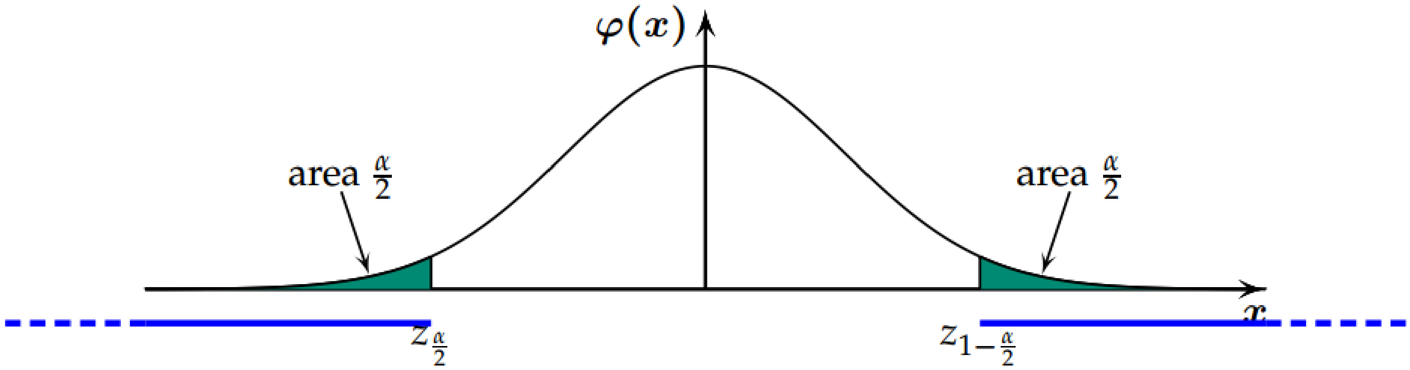


Figure 4: z-Test: Rejection Region

z-Test (σ_x known): Example	
Measurement of fusion heat:	The empirical mean value of $n = 13$ measurements is 80.02. From previous measurements the standard deviation is $\sigma_X = 0.01$. Is a fusion heat of exactly $80.00\frac{g}{cal}$ plausible?
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, $\sigma_X = 0.01$ known, $n = 13$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0 = 80.00$ H_A: $\mu \neq \mu_0$
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{\sqrt{n}\bar{X}_n - \mu_0)}{\sigma_{X_n}}$ $Z \sim N(0,1)$
4. Significance level:	$\alpha = 0.05$ (commonly used α-level)

5. Rejection region for the test statistic:
 	
Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.
 	
The following rejection region for the test statistic results
	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$
	
	`qnorm(0.025)`
	`[1] -1.959964`
	
	$z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2) = \Phi^{-1}(0.025) = -1.96$
	$K = (-\infty, -1.96] \cup [1.96, \infty)$

6. Test decision: **Remarks:** Standardizing is in principle unnecessary because of technical aid of computer software.	Hence the value for the statistics is $z = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{\sqrt{13}(80.02-80.00)}{0.01} = 7.211$ Therefore the observed value falls into the rejection region.
3. Test statistic: (not standardized)	The mean value of the measurements T: $\bar{X}_n$

Null distribution (assuming H_0 is true):	$T \sim N(\mu_0, \frac{\sigma_X^2}{n}) = N(80, \frac{0.01^2}{13})$

5. Rejection region for the test statistic: (not standardized)
 	
Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.
	$K = (-\infty, c_u] \cup [c_o, \infty)$ with $H_A : \mu \neq \mu_0$
	
	`qnorm(0.025, 80.0, 0.01/sqrt(13))`
	`[1] 79.99456`
	`qnorm(0.975, 80.0, 0.01/sqrt(13))`
	`[1] 80.00544`

In this way, we obtain the rejection region tor the test statistic:	$K = (-\infty, 79.99] \cup [80.01, \infty)$

***t*-Test (σ_x unknown)**	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X is estimated by $\hat{\sigma}_X$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0$ H_A: $\mu \neq \mu_0$ (or $<$ or $>$)
3. Test statistic: Null distribution (assuming H_0 is true):	$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = \frac{observed - expected}{estimated\ standard\ error}$ $T \sim t_{n-1}$
4. Significance level:	α
5. Rejection region for the test statistic:	$K = (-\infty, t_{n-1;\frac{\alpha}{2}}] \cup [t_{n-1;1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$, $K = (-\infty, t_{n-1;\alpha}]$ with $H_A : \mu < \mu_0$, $K = [t_{n-1;1-\alpha}, \infty)$ with $H_A : \mu > \mu_0$
6. Test decision:	Check whether the observed value of the test statistic falls into the rejection region.
Example	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X is estimated, $\hat{\sigma}_X = 0.024$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0 = 80.00$ H_A: $\mu \neq \mu_0$
3. Test statistic: Null distribution (assuming H_0 is true):	$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X}$ $T \sim t_{n-1}$
4. Significance level:	$\alpha = 0.05$

5. Rejection region for the test statistic:
 	
We determine the value
 	
by means of R, where $\alpha = 0.05$ and $n = 13$.
 	
The rejection region of the test statistic thus is given by
	$K = (-\infty, t_{n-1;\frac{\alpha}{2}}] \cup [t_{n-1;1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$,
	
	$t_{n-1;1-\frac{\alpha}{2}} = t_{12;0.975} = 2.179$
	
	`qt(0.975,12)`
	`[1] 2.178813`
	
	$K = (-\infty, -2.179] \cup [2.179, \infty)$

6. Test decision:	On the basis of $n = 13$ measurements, we find

$\bar{x} = 80.02$ and $\hat{\sigma}_X = 0.024$

Hence, the realized value of the test statistic is

$t = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = \frac{\sqrt{13}(80.02-80.00)}{0.024} = 3.00$

The observed value falls into the rejection region. Therefore, the null hypothesis is rejected at 5% level.

The t-test directly performed in R using the function t.test()

```
x <- c(79.98, 80.04, 80.02, 80.04, 80.03,
80.03, 80.04, 79.97, 80.05, 80.03,
80.02, 80.00, 80.02)

t.test(x, alternative = "two.sided",
mu = 80.00, conf.level = 0.95)

##
## One Sample t-test
##
## data:  x
## t = 3.1246, df = 12, p-value = 0.008779
## alternative hypothesis: true mean is not
    equal to 80
## 95 percent confidence interval:
## 80.00629 80.03525
## sample estimates:
## mean of x
## 80.02077
```

Remarks:
(i) The observed value of the test statistic is 3.12. Assuming the null hypothesis is true, then the test statistic follows a t-distribution with $df = 12$ degrees of freedom.

(ii) The observed mean value of the data is 80.02. A 95% confidence interval for the true mean is $[80.006, 80.035]$.

(iii) The R functions qt(p,df) calculates the quantile from the probability density and the degrees of freedom and pt(q,df) calculates the probabilty density from the quantile and the degrees of freedom.

(iv) The **confidence interval** for measurement data consists of the values µ, for which the corresponding statistical test does not reject the null hypothesis.

```
qt(0.975,12)
[1] 2.178813
pt(2.178813,12)
[1] 0.975
qt(0.5,12)
[1] 0.0
pt(0.0,12)
[1] 0.5
```

P-Value

The p-value is the probability that the test statistic will take on a value that is at least as extreme (with respect to the alternative hypothesis) as the observed value of the statistic when the null hypothesis H_0 is true.

For the one-sided alternative hypothesis H_A: $\mu > \mu_0$, the p-value can be calculated as follows - the observed value of the statistics is $t = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = 3.1246$:

p-value= $P(T > t) = P(T > 3.1246) = 0.00439$ For the two-sided alternative hypothesis H_A: $\mu \neq \mu_0$, the p-value can be calculated as follows (the observed value of the test statistics is $t = \frac{\sqrt{n}|\bar{X}_n - \mu_0|}{\hat{\sigma}_X}$):

p-value= $2 \cdot P(T > |t|)$

In R we compute the one-sided and the two sided p-value as follows:

These p-values are evidence against the null hypothesis at 5% level. Whereas the two-sided value is statistically significant at the 5% value.

```
#one-sided p-value
1-pt(3.1246, df=12)
[1] 0.004389739
#two-sided p-value
2*(1-pt(3.1246, df=12))
[1] 0.008779477
```


p-value and Statistical Test 1. Reject H_0 if p-value$\leqslant \alpha$ 2. Retain H_0 if p-value$> \alpha$ *The p-value is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.*	The p-value quantifies how significant an alternative is: p-value ≈ 0.05 : weakly significant, "." p-value ≈ 0.01 : weakly significant, "*" p-value ≈ 0.001 : weakly significant, "**" p-value $\leqslant 10^{-4}$: weakly significant, "***"

2.3 Joint Distributions

2.3.1 Joint, Marginal and Conditional Distributions

Discrete Joint Probability Distribution	
The **Joint Probability Distribution** of X and Y is defined by the following distributions:	$P(X = x, Y = y)$, $x \in W_x, y \in W_y$
Marginal Distributions are single distributions $P(X = x)$ of X and $P(Y = y)$ of Y. They can be calculated based on their joint distribution: Joint distribution of (X, Y) starting from the marginal distribution of X and Y is only possible for **independent** X and Y. Then it holds:	$P(X = x) = \sum\limits_{y \in W_y} P(X = x, Y = y)$, $x \in W_x$ $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$, $x \in W_x, y \in W_y$
Conditional probability of X given $Y = y$ is defined as: The **marginal distributions** then can be expressed as follows:	$P(X = x \mid Y = y) = \frac{P(X=x,Y=y)}{P(Y=y}$ $P(X = x) = \sum\limits_{y \in W_y} P(X = x \mid Y = y)P(Y = y)$, $x \in W_x$
Conditional Expected Value of Y given $X = x$ is defined as:	$E[Y \mid X = x] = \sum\limits_{y \in W_y} y \cdot P(Y = y \mid X = x)$

Example

X\ Y	1	2	3	4	$\sum$
1	0.080	0.015	0.003	0.002	0.100
2	0.050	0.350	0.050	0.050	0.500
3	0.030	0.060	0.180	0.030	0.300
4	0.001	0.002	0.007	0.090	0.100
$\sum$	0.161	0.427	0.240	0.172	1

$P(X = 3, Y = 4) = 0.030$ or $P(X = 3 \cup Y = 4) = 0.030$

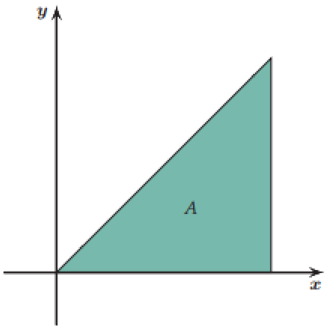
$P(X = 3) = P(X = 3, Y = 1) + P(X = 3, Y = 2)+$
$P(X = 3, Y = 3) + P(X = 3, Y = 4) =$
$0.030 + 0.060 + 0.180 + 0.030 = 0.300$

$P(Y = 2 \mid X = 4) = \frac{P(Y=2,X=4)}{P(X=4)} = \frac{0.002}{0.1} = 0.02$

$P(X = Y) = P(X = 1, Y = 1) + P(X = 2, Y = 2)+$
$P(X = 3, Y = 3) + P(X = 4, Y = 4) = 0.700$

If random variables are independent it must hold that
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$
From the marginal distribution follows
$P(X = 1) \cdot P(Y = 2) = 0.100 \cdot 0.427 = 0.043$
and this is not equal to
$P(X = 1, Y = 2) = 0.15$
X and Y are **not independent**

Joint Density Function	
The **probability** that the **joint random variable** (X,Y) lies in a two-dimensional region A, i.e., $A \subset \mathbb{R}^2$, is given by	$P((X, Y) \in A) = \iint\limits_A f_{X,Y}(x, y)dx\,dy$

The (bivariate) **joint density function** needs to satisfy	$\iint\limits_{\mathbb{R}} f_{X,Y}(x,y)dx\,dy = 1$
X and Y are only **independent** if	$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y),\ x, y \in \mathbb{R}$
Marginal Density	$f_X(x) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dy,\ f_Y(y) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dx$
Conditional Probability	$f_{Y\|X=x}(y) = f_Y(y\|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
X and Y are only independent if the following apply:	$f_{Y\|X=x}(y) = f_Y(y)$ resp. $f_{X\|Y=y}(x) = f_X(x)$
Conditional Expected Value of a continuous random variable Y given $X = x$	$E[Y\|X=x] = \int\limits_{-\infty}^{\infty} y \cdot f_{Y\|X=x}(y)dy$
Example	
Two machines with exponentially distributed life expectancy $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$, where X and Y are independent. $f_X(x) = \lambda_1 e^{-\lambda_1 x}$ and $f_Y(y) = \lambda_2 e^{-\lambda_2 y}$  	Due to independence: $_{X,Y}(x,y) = \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}$ $P(Y < X) = \int\limits_0^{\infty}(\int\limits_0^{x}\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}dy)dx$ $P(Y < X) = \int\limits_0^{\infty}\lambda_1 e^{-\lambda_1 x}(1 - e^{-\lambda_2 y})dx = \frac{\lambda_2}{\lambda_1+\lambda_2}$

2.3.2 Covariance and Correlation

Covariance and Correlation	
Covariance	$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$
X, Y independent	$E[XY] = E[X]E[Y]$
	$Cov(X,X) = E[(X-\mu_X)(X-\mu_X)] = E[(X-\mu_X)^2] = Var(X)$
Sum of Variances	$Var(\sum\limits_{i=1}^{n} X_i) = Cov(\sum\limits_{i=1}^{n} X_i, \sum\limits_{i=1}^{n} X_i) = \sum\limits_{i=1}^{n} Var(X_i) + 2\sum\limits_{i<j}^{n} Cov(X_i, X_j)$
2 Random Variables	$Var(X+Y) = Cov(X+Y, X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$
If all X_i are independent	$Var(X_1 + X_2 + ... + X_n) = Var(X_1) + ... + Var(X_n)$
Correlation	$Cor(X,Y) = \rho_{XY} = \frac{Cov(X,Y}{\rho_X \rho_Y}$ where $-1 \leqslant Cor(X,Y) \leqslant 1$
Measure for strength and direction of the *linear dependency* between X and Y.	$Cor(X,Y) = +1$ if $Y = a + bX$ for $a \in \mathbb{R}$ and $b > 0$ $Cor(X,Y) = -1$ if $Y = a + bX$ for $a \in \mathbb{R}$ and $b < 0$

	$\lvert Cor(X,Y)\rvert = 1$ means perfect linear relationship between X and Y. $Cor(X,Y) = 0$ means X and Y are uncorrelated.
X and Y **linear independent**	$Cor(X,Y) = 0$ (and thus $Cov(X,Y) = 0$)

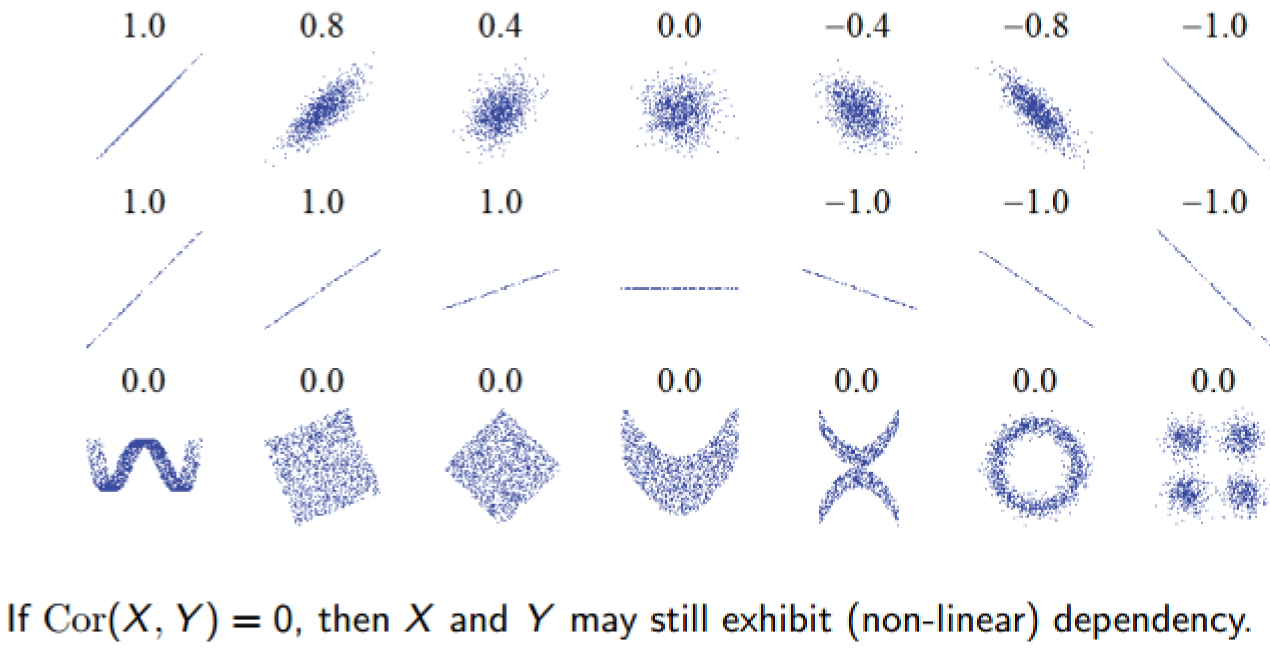


Figure 5: Correlations

2.3.3 Bivariate Normal Distribution

Bivariate Normal Distribution	
Expected values and variances of the marginal distribution Covariance between X and Y	μ_X, σ_X^2 and μ_Y, σ_Y^2 $Cov(X,Y) = \rho_{XY}\sigma_X\sigma_Y$
Joint Density	$f_{X,Y}(x,y) =$ $\frac{1}{2\pi\sqrt{det(\sum)}} exp\Big(-\frac{1}{2}(x-\mu_X, y-\mu_Y)\sum^{-1}\begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}\Big)$
Covariance Matrix	$\sum = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$

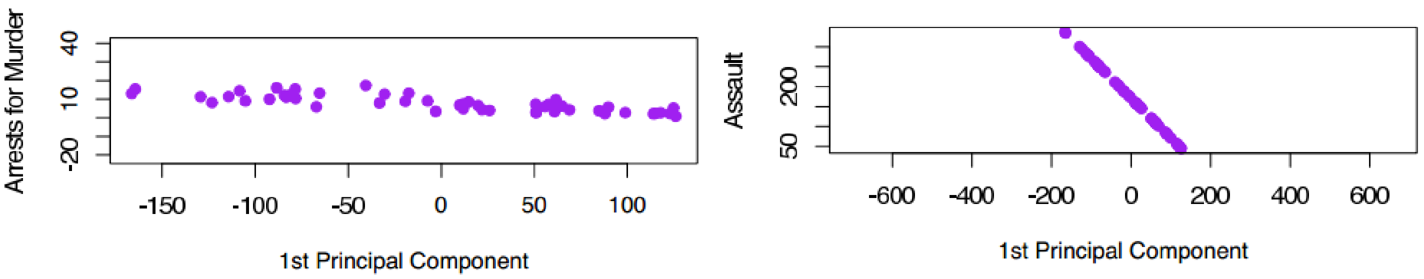
2.3.4 Principal Component Analysis (PCA)

PCA is a popular approach for deriving a low-dimensional set of features from a large set of variables. PCA is a technique for reducing the dimension of a n x p data matrix X where n corresponds to the number of observations and p to the number of variables.

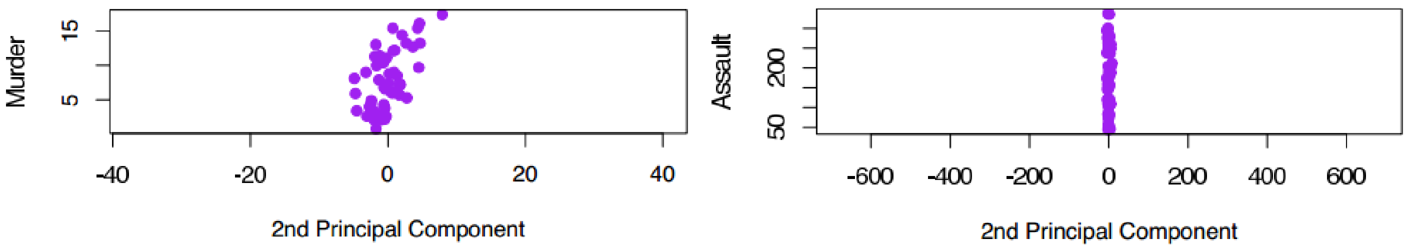
2.3.4.1 Example: USArrests

(a) The data vary the most along the first principal component (b) Counter Clockwise Rotation that 1. PC coincides with x-axis

Figure 6: 1st and 2nd Principal Component



(a) z_{i1} versus Murder (b) z_{i1} versus Assault



(a) z_{i2} versus Murder (b) z_{i2} versus Assault

Figure 8: The fact that the 2nd principal component scores are much closer to zero indicates that this component captures far less information as the 1st principal component.

Theory

$Z_1 = -0.0419126(\text{Murder - }\overline{\text{Murder}}) - 0.9991213(\text{Assault-}\overline{\text{Assault}})$

$\phi_{11} = -0.00419126$ and $\phi_{21} = -0.9991213$ are the *principal component loadings*

The idea is that every that out of every linear combination of Murder and Assault sucht that

$$\phi_{11}^2 + \phi_{21}^2 = 1$$

and

$$Var(\phi_{11})(\text{Murder} - \overline{\text{Murder}}) + \phi_{21}(\text{Assault} - \overline{\text{Assault}})$$

is maximized.

prcomp() centers the variables to have mean zero. This corresponds to how the first principal component is defined.

$z_{i1} = -0.0419126(\text{Murder - }\overline{\text{Murder}}) - 0.9991213(\text{Assault-}\overline{\text{Assault}})$

The values of $z_{i1}, ..., z_{n1}$ are known as *principal component scores*, seen in the right-hand panel in Figure 6. $z_{i1} > 0$ indicates a state with below-average arrests for murder and below average for assault. A negative score suggests the opposite.

$Z_2 = 0.9991213(\text{Murder - }\overline{\text{Murder}}) - 0.0419126(\text{Assault-}\overline{\text{Assault}})$

Code Example

```
#First principal component
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
pr.out$rotation[,1]

## Murder   Assault
## -0.0419126  -0.9991213

#principal component scores (z_i1 to z_in)
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
head(pr.out$x)

##      PC1(z_i1)  PC2(z_i2)
## Alabama    -65.40950  2.6728663
## Alaska   -92.25166 -1.6559620
## Arizona    -123.14478  -4.8535831
## Arkansas  -19.26551  0.2047123
## California  -105.19832  -3.1999471
## Colorado  -33.21549  -1.2812733

#Second principal component
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
pr.out$rotation[,2]

## Murder   Assault
## 0.9991213  -0.0419126
```

With two-dimensional data, such as in our USArrests example, we can construct at most two principal components. However, if we had other variables, such as Rape, then additional components could be constructed.

2.3.4.2 PCA and Covariance Matrix

The covariance matrix of two random variables X and Y is defined as

$$\sum = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(Y,X) & \sigma_Y^2 \end{pmatrix}$$

Since Z_1 and Z_2 are required to be uncorrelated, this implies for their covariance matrix $\sum$ to have vanishing off diagonal elements. Therefore the covariance has to be diagonalized. This can be done with by a rotation matrix Φ so that

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = (X - \mu_X, Y - \mu_Y) \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} \phi_{11}(X - \mu_X) & \phi_{12}(Y - \mu_Y) \\ \phi_{21}(X - \mu_X) & \phi_{22}(Y - \mu_Y) \end{pmatrix}$$

and

$$\begin{pmatrix} Cov(Z_1,Z_1) & Cov(Z_1,Z_2) \\ Cov(Z_2,Z_1) & Cov(Z_2,Z_2) \end{pmatrix} = \begin{pmatrix} \sigma_{Z_1}^2 & 0 \\ 0 & \sigma_{Z_2}^2 \end{pmatrix}$$

The rotation matrix Φ needs to satisfy the condition $\phi_{11}^2 + \phi_{21}^2 = 1$ and $\phi_{12}^2 + \phi_{22}^2 = 1$ It is straightforward to generalize the case of $p = 2$ to an arbitrary p.

2.3.4.3 Proportion of Variance Explained by Principal Components

Theory	Code Example

There is an information loss of the given data by projecting the observations onto the first few principal components. Therefore we want to know the *proportion of variance explained (PVE)*. The *total variance* is defined as $$\sum_{j=1}^{p} Var(X_j) = \sum_{j=1}^{p} \frac{1}{n} \sum_{i=1}^{n} x_{ij}^2$$ and the variance of the mth principal component is $$\frac{1}{n} \sum_{i=1}^{n} z_{im}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{p} \phi_{jm} x_{ij} \right)^2$$ Therefore the PVE by the mth principal component is given by $$\frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{p} \phi_{jm} x_{ij} \right)^2}{\sum_{j=1}^{p} \sum_{i=1}^{n} x_{ij}^2}$$	`pr.out <- prcomp(USArrests[,c("Murder","Assault")],scale=FALSE)`
	`pr.var <- pr.out$sdev^2`
	`pve <- pr.var/sum(pr.var)`
	`pve`
	
	`## [1] 0.9990292327 0.0009707673`
	
	`## Most of the information of the data about the arrests for Murder and Assault is contained in the first principal component.`

3 Regression Analysis

3.1 Simple Linear Regression

TO DO: Chapter 5

...

3.1.1 Estimating the Coefficients

Theory	Code Example
Estimation of response variable Y based on a predictor variable X. $$Y \simeq \beta_0 + \beta_1 X$$	`lm(Y ~ X, data=someData)`

Source code:	Output:

<pre>advertising <- read.csv("../Data/Advertising.csv
    ")
model <- lm(sales ~ TV, data=advertising)
summary(model)</pre>
	<pre>##
	## Call:
	## lm(formula = sales ~ TV, data = Advertising)
	##
	## Residuals:
	## Min 1Q Median 3Q Max
	## -8.3860 -1.9545 -0.1913 2.0671 7.2124
	##
	## Coefficients:
	## Estimate Std. Error t value Pr(>|t|)
	## (Intercept) 7.032594 0.457843 15.36 <2e-16 **
	    *
	## TV 0.047537 0.002691 17.67 <2e-16 ***
	## ---
	## Signif. codes:
	## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
	    1
	##
	## Residual standard error: 3.259 on 198 degrees
	    of freedom
	## Multiple R-squared: 0.6119,Adjusted R-squared
	    : 0.6099
	## F-statistic: 312.1 on 1 and 198 DF, p-value:
	    < 2.2e-16</pre>

Interpretation of output:	
TO DO: interpretation here	

3.2 Residual Analysis

TO DO: Chapter 6

3.3 Multiple Linear Regression

TO DO: Chapter 7

3.4 Linear Model Selection

TO DO: Chapter 8

4 Classification

4.1 Logistic Regression

TO DO: Chapter 10

4.2 Decision Trees

Decision trees are applied to both, classification and regression. **TO DO: Chapter 11**

4.2.1 Classification Trees

4.2.1.1 Binary Splitting

In binary splitting, a training set is used to split up the predictor domain into regions which contain data for which the response variable belongs to the same class. By **binary** it is meant that a region is split into **two** subregions (i.e. "is a predictor less or greater than a threshold value?" → yes/no).

Theory

Algorithm:

1. Initialise the set of regions $\mathcal{R} = R$ by the predictor domain R
2. Choose the optimal region R in $\mathcal{R}$ and the optimal predictor X_i such that a binary split of R with respect to X

$$R_1 = \{\vec{x} \in R | x_i > t\} \quad \text{and} \quad R_2 = \{\vec{x} \in R | x_i \leqslant t\}$$

   gives the highest gain in purity (for some threshold t).

3. Replace R in $\mathcal{R}$ with R_1 and R_2 and return to 2.

The iteration is stopped if the current splitting fulfils a pre-defined stopping criterion.

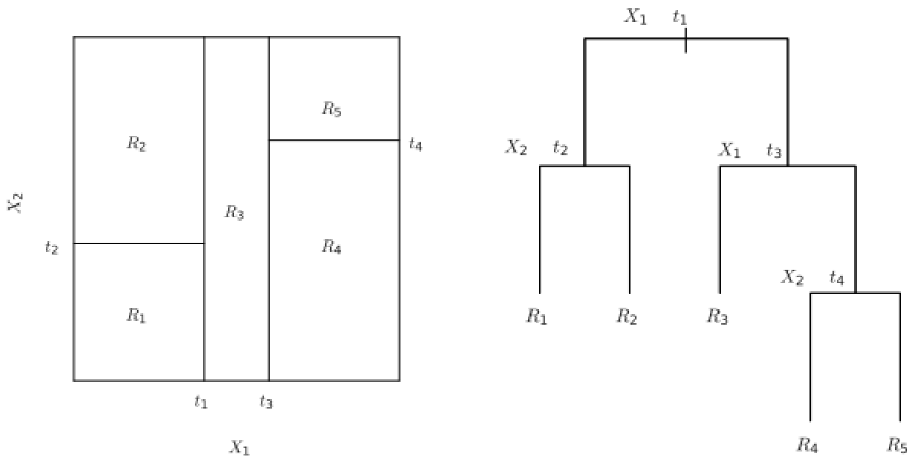
Code Example

```
require(tree)
#default controls
tc = tree.control(nobs = 303, mincut = 5,
    minsize = 10, mindev = 0.01)

#grow tree
tree.model = tree(AHD~MaxHR+Age, data = heart,
    control = tc)

#plot tree and label splits
plot(tree.model)
text(tree.model, cex=0.8)

#plot partition (only for two predictor case)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label
    ], pch=20)
```



(a) Example regions resulting from binary splitting

(b) Example decision tree resulting from binary splitting

4.2.1.2 Node Purity

Notation:

Variable	Description
Y	Response variable
K	Levels (categories) of the response variable
T	The decision tree
M	Amount of terminal nodes
$\hat{p}_{mk}$	proportion of the training data in region m from level k

Purity Measures:

Classification error rate	$E_m(T) = 1 - \max_k(\hat{p}_{mk})$

Gini index	$G_m(T) = \sum_{k=1}^{K} \hat{p}_{mk} \cdot (1 - \hat{p}_{mk})$
Cross-entropy	$D_m(T) = -\sum_{k=1}^{K} \hat{p}_{mk} \cdot \log(\hat{p}_{mk})$

Code example: Cross Entropy and Gini measures in R

```
require(tree)
# deviance or cross entropy
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "deviance")
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)

# Gini index
tc = tree.control(303, mincut = 5, minsize = 60, mindev = 0.01)
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "gini", control = tc)
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)
```

4.3 Random Forests

TO DO: Chapter 12

5 Time Series Analysis

5.1 Introduction to Time Series

Models are not always *independent of the order* of the training data. Many real life measuring and data recording processes result in data sets that are *serially correlated.* For example *machine monitoring, stock, environmental observations or federal statistics.* These kind of data is called *time series data.* Usually there are several goals that one wants to achieve in time series data.

- **Descriptive Analysis**
- **Modelling and Interpretation**
- **Decomposition**
- **Predection**
- **Regression**

TO DO: Chapter 13

5.1.1 Time Series with R

Theory	Code Example

All data in R are stored in objects, which provide a range of methods. The class of an object can be found using the class function. For example, we have already encountered the data.frame class. It has a series of methods, such as names or nrow:
(The data set iris contains 50 samples of three types of Iris flowers, measured along four variables.)
	`class(iris); names(iris); nrow(iris)`
	
	`## [1] "data.frame"`
	`## [1] "Sepal.Length" "Sepal.Width" "Petal.Length" "Petal.Width"`
	`## [5] "Species"`
	`## [1] 150`

5.1.1.1 The ts Class

Theory	Code Example

Basic properties:
The AirPassengers-data is a built in set of class ts. Most important methods for ts class are:

1. start() returns the start time of the series.
2. end() returns the end time of the series.
3. frequency() returns the number of samples per unit time.
4. plot() displays the time series as a function over the time axis. plot function calls plot.ts which is tailored for time series. See Figure AirPassengers.

```r
class(AirPassengers)
## [1] "ts"

start(AirPassengers); end(AirPassengers);
    frequency(AirPassengers)
## [1] 1949 1
## [1] 1960 12
## [1] 12

#1/frequency = 1/12 = 0.0833
deltat(AirPassengers)
## [1] 0.0833

#output in figure AirPassengers.
plot(AirPassengers, main = "Passengers", ylab="
    Number (in 1000s)")
grid()
```

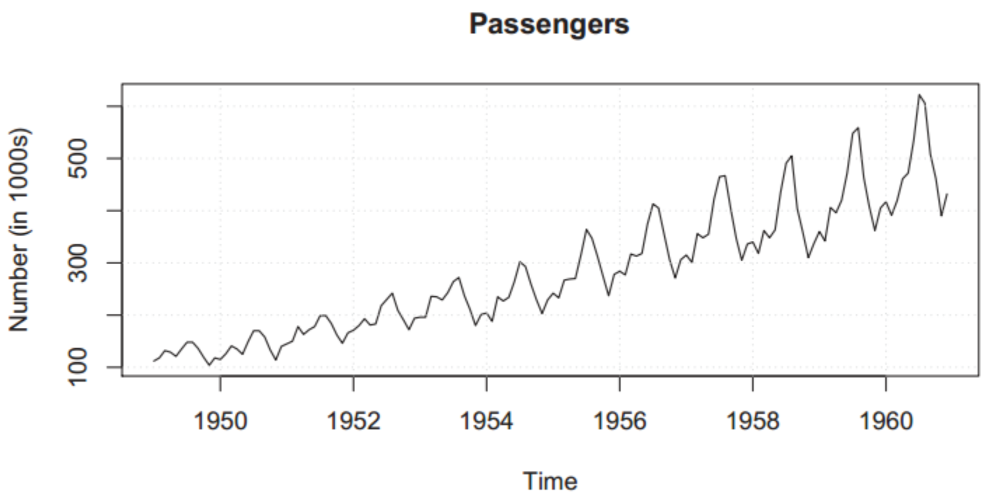


Figure 10: AirPassengers

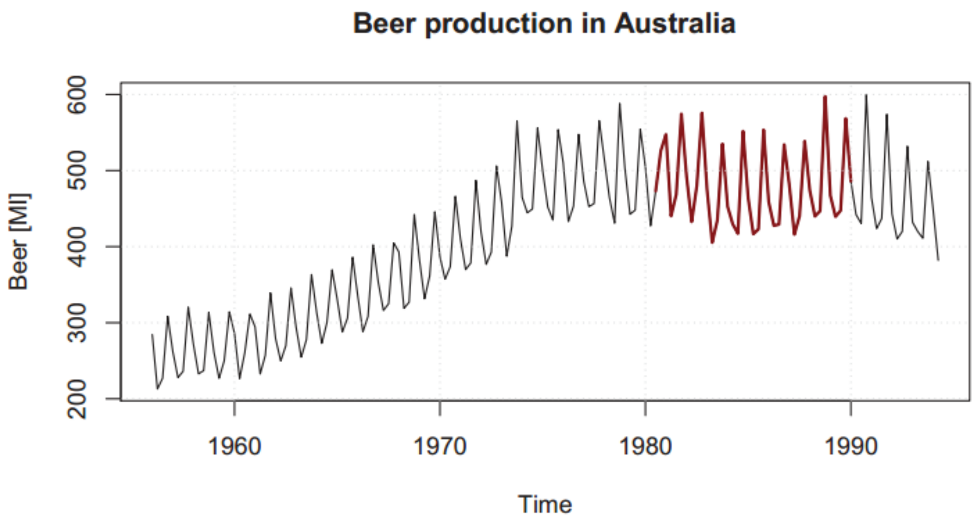


Figure 11: Subset of a Time Series (seasonal behaviour)

Theory	Code Example
Defining a ts class If data is not in a time series form we can make a ts object by using the ts function. This is not necessary for AirPassengers, therefore the example AustralianBeer is used. 1. summary() gives the five-number summary as well as the mean of the time series.This function shows the minimum, the first quartile, the median, the second quartile and the maximum of the time series. This is called the *five-number-summary* of a data set. Additionally the mean is also computed. 2. window() returns a subset of the time series defined by a start and an end time.	see code below

```r
X.beer = read.table("../Daten/AustralianBeer.csv
    ", sep=";", header = T)

X.beer.ts = ts(X.beer[,2], start = c(1956,1),
    end = c(1994, 2),frequency = 4)
summary(X.beer.ts)
## Min. 1st Qu. Median Mean 3rd Qu. Max.
## 213   325   427   408   467   600

#Figure Subset of Time Series
plot(X.beer.ts, ylab="Beer [Ml]", main="Beer
    production in Australia")
X.ts.w = window(X.beer.ts, start = c(1980,3),
    end = c(1990, 1))
summary(X.ts.w)
## Min. 1st Qu. Median   Mean   3rd Qu. Max.
## 405   437   467   478     530      598
lines(X.ts.w, col = "darkred", lwd=2)
grid()
```

5.2 Mathematical Models for Time Series

TO DO: Chapter 14

5.3 Forecasting ime Series

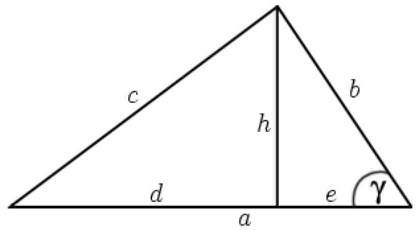
TO DO: Chapter 15

6 Idiotenseite

6.1 Dreiecksformeln

Cosinussatz

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos\gamma$$



Sinussatz

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r = \frac{u}{\pi}$$

Pythagoras beim Sinus

$$\sin^2(b) + \cos^2(b) = 1 \qquad \tan(b) = \frac{\sin(b)}{\cos(b)}$$

$$\sin\beta = \frac{b}{a} = \frac{\text{Gegenkathete}}{\text{Hypotenuse}} \qquad \tan\beta = \frac{c}{b} = \frac{\text{Gegenkathete}}{\text{Ankathete}}$$
$$\cos\beta = \frac{c}{a} = \frac{\text{Ankathete}}{\text{Hypotenuse}} \qquad \cot\beta = \frac{c}{b} = \frac{\text{Ankathete}}{\text{Gegenkathete}}$$

6.2 Funktionswerte für Winkelargumente

deg	rad	sin	cos	tan
0 °	0	0	1	0
30 °	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45 °	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60 °	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

deg	rad	sin	cos
90 °	$\frac{\pi}{2}$	1	0
120 °	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
135 °	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
150 °	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

deg	rad	sin	cos
180 °	π	0	-1
210 °	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
225 °	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
240 °	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

deg	rad	sin	cos
270 °	$\frac{3\pi}{2}$	-1	0
300 °	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
315 °	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
330 °	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

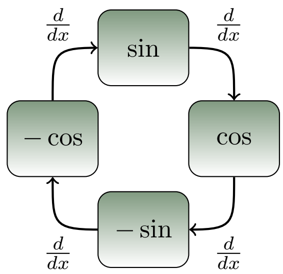
6.3 Periodizität

$\cos(a + k \cdot 2\pi) = \cos(a) \qquad \sin(a + k \cdot 2\pi) = \sin(a) \qquad (k \in \mathbb{Z})$

6.4 Quadrantenbeziehungen

$\sin(-a) = -\sin(a) \qquad \cos(-a) = \cos(a)$
$\sin(\pi - a) = \sin(a) \qquad \cos(\pi - a) = -\cos(a)$
$\sin(\pi + a) = -\sin(a) \qquad \cos(\pi + a) = -\cos(a)$
$\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right) = \cos(a) \qquad \cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\frac{\pi}{2} + a\right) = \sin(a)$

6.5 Ableitungen



6.6 Additionstheoreme

$\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b)$
$\cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$
$\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)}$

6.7 Doppel- und Halbwinkel

$\sin(2a) = 2\sin(a)\cos(a)$
$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\cos^2\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{2} \qquad \sin^2\left(\dfrac{a}{2}\right) = \frac{1-\cos(a)}{2}$

6.8 Produkte

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
$\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

6.9 Euler-Formeln

$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \qquad \cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$
$e^{x+jy} = e^x \cdot e^{jy} = e^x \cdot (\cos(y) + j\sin(y))$
$e^{j\pi} = e^{-j\pi} = -1$

6.10 Summe und Differenz

$\sin(a) + \sin(b) = 2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\sin(a) - \sin(b) = 2 \cdot \sin\left(\frac{a-b}{2}\right) \cdot \cos\left(\frac{a+b}{2}\right)$

$\cos(a) + \cos(b) = 2 \cdot \cos\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\cos(a) - \cos(b) = -2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right)$
$\tan(a) \pm \tan(b) = \dfrac{\sin(a \pm b)}{\cos(a)\cos(b)}$

6.12 Ableitungen elementarer FunktionenS436

Funktion	**Ableitung**	**Funktion**	**Ableitung**
C (Konstante)	0	$\sec x$	$\frac{\sin x}{\cos^2 x}$
x	1	$\sec^{-1} x$	$\frac{-\cos x}{\sin^2 x}$
$x^n \ (n \in \mathbb{R})$	nx^{n-1}	$\arcsin x \quad (\lvert x\rvert < 1)$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\arccos x \quad (\lvert x\rvert < 1)$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$	$\arctan x$	$\frac{1}{1+x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$
$\sqrt[n]{x} \quad (n \in \mathbb{R}, n \neq 0, x > 0)$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$	$\operatorname{arcsec} x$	$\frac{1}{x\sqrt{x^2-1}}$
e^x	e^x	$\operatorname{arcossec} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$e^{bx} \quad (b \in \mathbb{R})$	be^{bx}	$\sinh x$	$\cosh x$
$a^x \quad (a > 0)$	$a^x \ln a$	$\cosh x$	$\sinh x$
$a^{bx} \quad (b \in \mathbb{R}, a > 0)$	$ba^{bx} \ln a$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\ln x$	$\frac{1}{x}$	$\coth x \quad (x \neq 0)$	$-\frac{1}{\sinh^2 x}$
$\log_a x \quad (a > 0, a \neq 1, x > 0)$	$\frac{1}{x}\log_a e = \frac{1}{x \ln a}$	$\operatorname{Arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\lg x \quad (x > 0)$	$\frac{1}{x}\lg e \approx \frac{0.4343}{x}$	$\operatorname{Arcosh} x \quad (x > 1)$	$\frac{1}{\sqrt{x^2-1}}$
$\sin x$	$\cos x$	$\operatorname{Artanh} x \quad (\lvert x\rvert < 1)$	$\frac{1}{1-x^2}$
$\cos x$	$-\sin x$	$\operatorname{Arcoth} x \quad (\lvert x\rvert > 1)$	$-\frac{1}{x^2-1}$
$\tan x \quad (x \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z})$	$\frac{1}{\cos^2 x} = \sec^2 x$	$[f(x)]^n \quad (n \in \mathbb{R})$	$n[f(x)]^{n-1} f'(x)$
$\cot x \quad (x \neq k\pi, k \in \mathbb{Z})$	$\frac{-1}{\sin^2 x} = -cosec^2 x$	$\ln f(x) \quad (f(x) > 0)$	$\frac{f'(x)}{f(x)}$

6.11 Einige unbestimmte Integrale S1074

$\int dx = x + C$	$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C,\ x \neq 0$	$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C,\ a\epsilon\mathbb{R}^+\backslash\{1\}$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C,\ x \neq k\pi$ mit $k\epsilon\mathbb{Z}$
$\int \frac{dx}{\cos^2 x} = \tan x + C,\ x \neq \frac{\pi}{2} + k\pi$ mit$k\epsilon\mathbb{Z}$	$\int \sinh x dx = \cosh x + C$
$\int \cosh x dx = \sinh x + C$	$\int \frac{dx}{\sinh^2 x} = -\coth x + C,\ x \neq 0$
$\int \frac{dx}{\cosh^2 x} = \tanh x + C$	$\int \frac{dx}{ax+b} = \frac{1}{a}\ln\lvert ax+b\rvert + C,\ a \neq 0, x \neq -\frac{b}{a}$
$\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\arctan\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0$	$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left\lvert\frac{ax-b}{ax+b}\right\rvert + C,\ a \neq 0,\ b \neq 0,\ x \neq \frac{b}{a},\ x \neq -\frac{b}{a}$
$\int \sqrt{a^2x^2+b^2}dx = \frac{x}{2}\sqrt{a^2x^2+b^2} + \frac{b^2}{2a}\ln(ax + \sqrt{a^2x^2+b^2}) + C,\ a \neq 0,\ b \neq 0$	$\int \sqrt{a^2x^2-b^2}dx = \frac{x}{2}\sqrt{a^2x^2-b^2} - \frac{b^2}{2a}\ln\left\lvert ax + \sqrt{a^2x^2-b^2}\right\rvert + C,\ a \neq 0,\ b \neq 0, a^2x^2 \geqq b^2$
$\int \sqrt{b^2-a^2x^2}dx = \frac{x}{2}\sqrt{b^2-a^2x^2} + \frac{b^2}{2a}\arcsin\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 \leqq b^2$	$\int \frac{dx}{\sqrt{a^2x^2+b^2}} = \frac{1}{a}\ln(ax + \sqrt{a^2x^2+b^2}) + C,\ a \neq 0,\ b \neq 0$
$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \frac{1}{a}\ln\left\lvert ax + \sqrt{a^2x^2-b^2}\right\rvert + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 > b^2$	$\int \frac{dx}{\sqrt{b^2-a^2x^2}} = \frac{1}{a}\arcsin\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 < b^2$
Die Integrale $\int \frac{dx}{X}, \int \sqrt{X}dx, \int \frac{dx}{\sqrt{X}}$ mit $X = ax^2 + 2bx + c,\ a \neq 0$ werden durch die Umformung $X = a(x + \frac{b}{a})^2 + (c - \frac{b^2}{a})$ und die Substitution $t = x + \frac{b}{a}$ in die oberen 4 Zeilen transformiert.	$\int \frac{xdx}{X} = \frac{1}{2a}\ln\lvert X\rvert - \frac{b}{a}\int\frac{dx}{X},\ a \neq 0,\ X = ax^2 + 2bx + c$
$\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a}\cdot\sin 2ax + C,\ a \neq 0$	$\int \cos^2 ax dx = \frac{x}{2} + \frac{1}{4a}\cdot\sin 2ax + C,\ a \neq 0$
$\int \sin^n ax dx = -\frac{sin^{n-1}ax\cdot\cos ax}{na} + \frac{n-1}{n}\int\sin^{n-2}ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int \cos^n ax dx = \frac{\cos^{n-1}ax\cdot\sin ax}{na} + \frac{n-1}{n}\int\cos^{n-2}ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln\left\lvert\tan\frac{ax}{2}\right\rvert + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$	$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left\lvert\tan(\frac{ax}{2} + \frac{\pi}{4})\right\rvert + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$
$\int \tan ax dx = -\frac{1}{a}\ln\lvert\cos ax\rvert + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$mit $k\epsilon\mathbb{Z}$	$\int \cot ax dx = \frac{1}{a}\ln\lvert\sin ax\rvert + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}mitk\epsilon\mathbb{Z}$
$\int x^n \sin ax dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int x^n \cos ax dx = \frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C,\ a \neq 0,\ b \neq 0$
$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C,\ a \neq 0,\ b \neq 0$	$\int \ln x dx = x(\ln x - 1) + C,\ x\epsilon\mathbb{R}^+$
$\int x^\alpha\cdot\ln x dx = \frac{x^{\alpha+1}}{(\alpha+1)^2}[(\alpha+1)\ln x - 1] + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$	