1 R Tutorial

1.1 Loading Data

```r
# loading csv files
data <- read.table("whatever.csv", sep="", header=T)

# csv files can be stored with (almost) any kind of file ending, e.g.:
data <- read.table("whatever.dat", sep="", header=T)
data <- read.table("whatever.txt", sep="", header=T)
```

2 Probability And Statistics

2.1 Probability Models for Measurement Data

2.1.1 Random Variables

Random Variables	
Definition	$X : \Omega \longrightarrow W_{\mathrm{x}}$
Example	A Coin is thrown three times, head and tails is observed: $\Omega = \{hhh, hht, htt, hth, ttt, tth, thh, tht\}$ Total number of heads $W_{\mathrm{x}} = \{0, 1, 2, 3\}$ Total number of tails $W_{\mathrm{x}} = \{0, 1, 2, 3\}$ Number of heads minus tails $W_{\mathrm{x}} = \{-3, -1, 1, 3\}$
Probability Mass Function	
Definition	The probability distribution of a discrete random variable: $P(X = x)$
Example	$\begin{array}{c\|cccc} \mathrm{x} & 0 & 1 & 2 & 3 \\ \hline P(X=x) & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$

2.1.2 Probability Distributions

Cumulative Density Function (cdf)	
Definition	$F(x) = P(X \leqslant x)$
Properties	$P(a < X \leqslant b) = F(b) - F(a)$ $0 \leqslant F(x) \leqslant 1$ $P(X = a) = F(a) - F(a) = 0$

Probability Density Function (pdf)	
Definition	$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$
Properties	$f(x) \geqslant 0$ $P(a < X \leqslant b) = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$ $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$

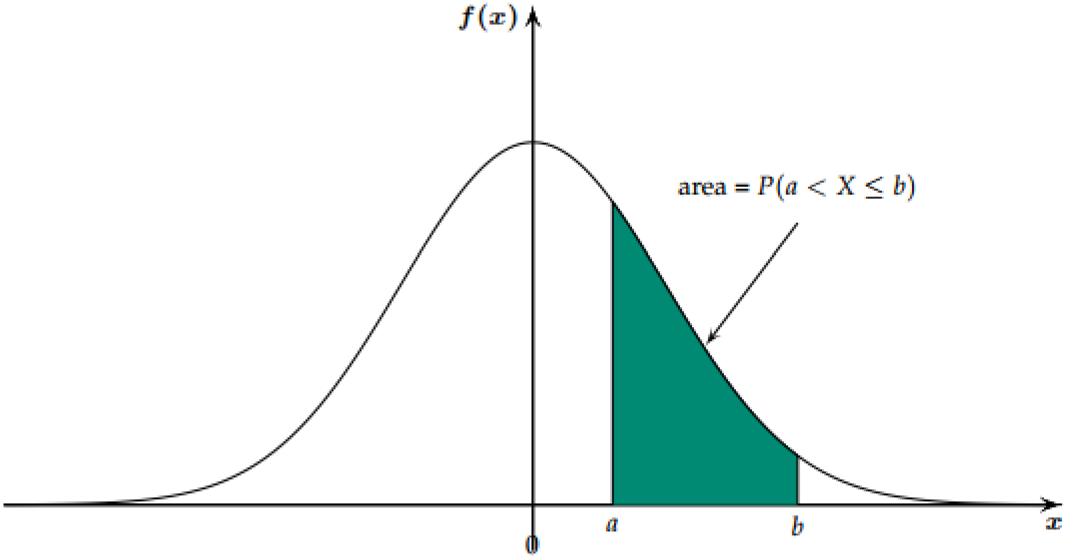


Figure 1: Probability density of a random variable and the probability of measuring a value from (a,b]

2.1.3 Summary Statistics of Continuous Distributions

Expected Value, Variance and Quantile	
Expected value	Discrete: $\mathrm{E}(X) = \sum_i x_i P(X = x_i)$ Continuous: $\mathrm{E}(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x$
Variance	$\mathrm{Var}(X) = \sigma_x^2 = \mathrm{E}((X - \mathrm{E}(X))^2) = \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2 \cdot f(x)\mathrm{d}x$
Quantile	$P(X \leqslant q(\alpha)) = \alpha$ $F(q(\alpha)) = \alpha \Leftrightarrow q(\alpha) = F^{-1}(\alpha)$ Note: When you're asked for the 50%-quantile, that means $\alpha = 50\%$, and you must find $q(0.5)$
Example Body Length	If α=0.75 and the corresponding quantile is $q(\alpha)$=182.5cm then 75% of the persons is shorter or equal 182.5cm.

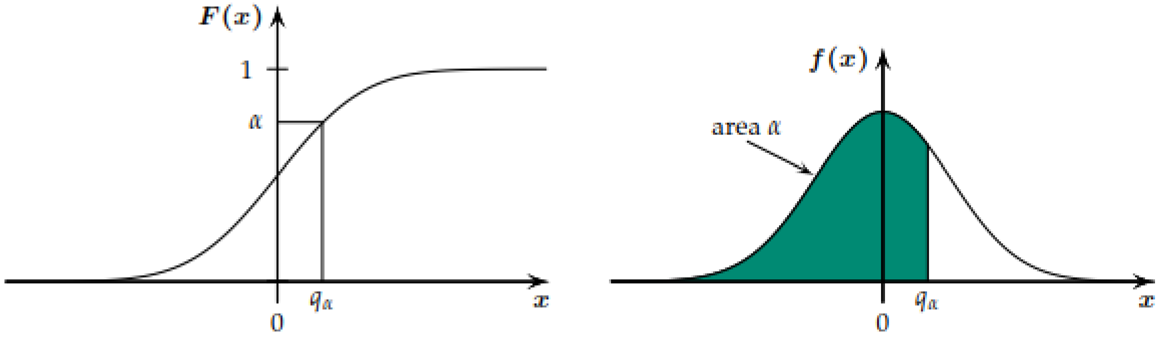


Figure 2: Quantiles

2.1.4 Important Distributions

2.1.4.1 Uniform Distribution

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & x > b \end{cases}$$

$$\mathrm{E}(x) = \frac{a+b}{2}$$

$$\mathrm{Var}(x) = \frac{(b-a)^2}{12}$$

$$\sigma_{\mathrm{x}} = \frac{b-a}{\sqrt{12}}$$

</td><td>

```r
# value of the probability density function
    Uniform([1, 10]) at the position x = 5
dunif(x=5, min=1, max=10)
[1] 0.1111111

# P(X <= 5)
punif(q=5, min=1, max=10)
[1] 0.4444444

# P(1.2 < X <= 4.8)
punif(4.8, 1, 10) - punif(1.2, 1, 10)
[1] 0.4

# 5 uniformly distributed random values in
    Uniform([1, 10])
runif(5,min=1,max=10)
[1] 1.061933 6.484813 5.928334 8.459887 8.852405

# TODO: ADD MORE HERE
```

</td></tr>
</table>

2.1.4.2 Exponential Distribution

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 1 - \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}(x) = \frac{1}{\lambda}$$

$$\mathrm{Var}(x) = \frac{1}{\lambda^2}$$

$$\sigma_{\mathrm{x}} = \frac{1}{\lambda}$$

</td><td>

```r
# P(0 <= X <= 4) of X ~ Exp(3)
pexp(4, rate=3)
[1] 0.9999939

# TODO: ADD MORE HERE
```

</td></tr>
</table>

2.1.4.3 Normal Distribution

Theory	Code Example

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})}$
$F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}y$
$\mathrm{E}(x) = \mu$
$\mathrm{Var}(x) = \sigma^2$
$\sigma_{\mathrm{x}} = \sigma$
	`# X~N(u, sigma^2) --> X~N(100,15^2)`
	`# In R we compute P(X>130) as 1 - P(X<=130)`
	`1-pnorm(130, mean=100, sd=15)`
	`[1] 0.02275013`
	
	`#P(85<=X<=115)`
	`pnorm(115, mean=100, sd=15)-pnorm(85, mean=100, sd=15)`
	`[1] 0.6826895`
	
	`# TODO: ADD MORE HERE`

2.1.4.4 Linear Transformation of Random Variables

Properties of Linear Transformation of a Random Variable	
Definition	For $Y = a + bX$ the following apply (i) $E(Y) = a + bE(X)$ (ii) $Var(Y) = b^2 Var(X)$, $\sigma_Y = \lvert b \rvert \sigma_X$ (iii) $\alpha - Quantile\, of\, Y = q_Y(\alpha) = a + bq_X(\alpha)$ (iv) $f_Y(y) = \frac{1}{b} f_X(\frac{y-a}{b})$
Summary Statistics of S_n and $\bar{X}_n$	
Summary Statistics of Sample Total S_n	$E(S_n) = E(X_1 + X_2 + ... + X_n) = \sum_{i=1}^{n} E(X_i) = n\mu$ $Var(S_n) = \sum_{i=1}^{n} Var(X_i) = nVar(X_i)$ $\sigma(S_n) = \sqrt{n}\sigma_X$
Summary Statistics of Sample Mean $\bar{X}_n$	$E(\bar{X}_n) = E(\frac{X_1+X_2+...+X_n}{n}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} nE(X_i) = \mu$ $Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma_X^2 = \frac{\sigma_X^2}{n}$
Standard Error	$\sigma(\bar{X}_n) = \frac{\sigma_X}{\sqrt{n}}$

2.1.4.5 Distributions of S_n and $\bar{X}_n$

Theory	Code Example

1. For $X_i \in \{0, 1\}$, we have
$S_n \sim \mathrm{Bin}(n, \pi)$ with $\pi = P(X_i = 1)$
2. For $X_i \sim \mathrm{Pois}(\lambda)$, we have
$S_n \sim \mathrm{Pois}(n\lambda)$
3. For $X_i \sim N(\mu, \sigma^2)$
$S_n \sim N(n\mu, n\sigma^2)$ and $\bar{X}_n \sim N(\mu, \frac{\sigma_X^2}{n})$
	`What is the probability that among 10000 tosses of a fair coin, heads would appear in maximum 5100 cases?`
	`#Approximated: X~N(5000,2500)`
	`pnorm(5100,5000,sqrt(2500))`
	`[1] 0.9772499`
	
	`#"True Result": X~Bin(10000,0.5)`
	`pbinom(5100,10000,0.5)`
	`[1] 0.9777871`

2.2 Statistics for Measurement Data

TO DO: Chapter 2

2.2.1 Assess the Normal Distribution Assumption

2.2.1.1 Q-Q Plot

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

1. For

$$\alpha_k = \frac{k-0.5}{n} \text{ with } k = 1, ..., n$$

calculate the corresponding theoretical quantiles of the model distribution

$$q(\alpha_k) = F^{-1}(\alpha_k)$$

2. Determine the empirical α_k-quantiles,

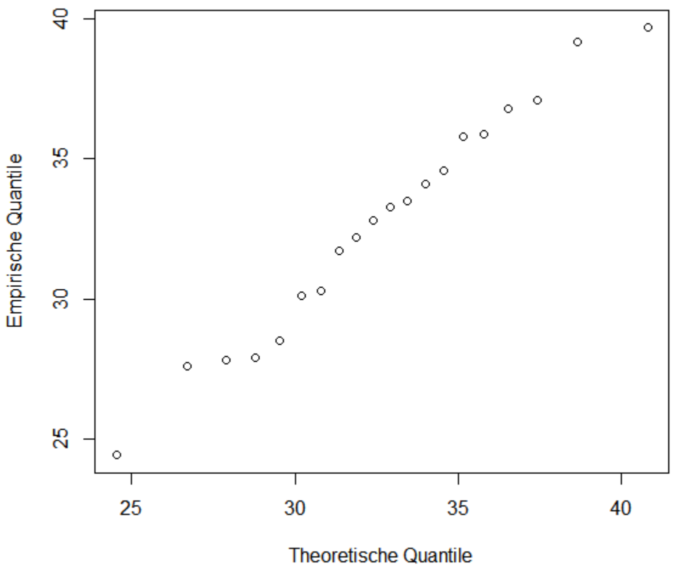
$$x_{(1)} < x_{(2)} < ... < x_{(n)}$$

3. Plot the empirical quantiles x_k on the y-axis against the theoretical quantiles $q(\alpha_k)$ on the x-axis.

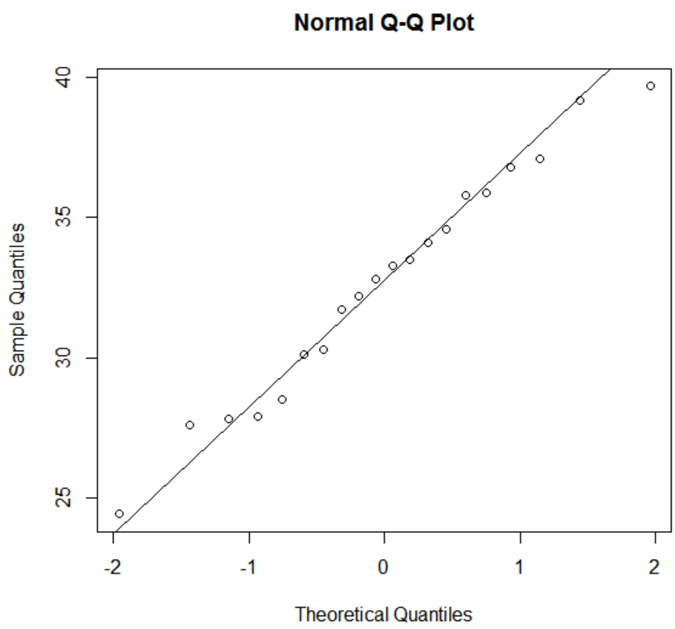
</td><td>

```
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3,
    31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6,
    35.8, 35.9, 36.8, 37.1, 39.2, 39.7)

alpha_k <- (seq(1, length(x), by=1)-0.5)/length(
    x)

quantile_th <- qnorm(alpha_k, mean=mean(x), sd=
    sd(x))

quantile_emp <- sort(x)
#image qqplot
qqplot(quantile_th, quantile_emp, xlab="
    Theoretische Quantile", ylab = "Empirische
    Quantile")
#image qqnorm;qqline
qqnorm(x);qqline(x)
```

</td></tr>
</table>



(a) qqplot()



(b) qqnorm();qqline()

k	$x_{(k)}$	$\alpha_k = (k-0.5)/n$	q_{α_k} for $\mathcal{N}(32.7, 4.15^2)$	$\Phi^{-1}(\alpha_k)$
1	24.4	0.0250	24.5	-1.96
2	27.6	0.075	26.7	-1.44
3	27.8	0.125	27.9	-1.15
4	27.9	0.175	28.8	-0.935
5	28.5	0.225	29.5	-0.755
6	30.1	0.275	30.2	-0.600
7	30.3	0.325	30.8	-0.453
8	31.7	0.375	31.3	-0.319
9	32.2	0.425	31.9	-0.189
10	32.8	0.475	32.4	-0.0627
11	33.3	0.525	32.9	0.0627
12	33.5	0.575	33.4	0.189
13	34.1	0.625	34.0	0.319
14	34.6	0.675	34.5	0.454
15	35.8	0.725	35.1	0.598
16	35.9	0.775	36.0	0.755
17	36.8	0.825	36.5	0.935
18	37.1	0.875	37.4	1.15
19	39.2	0.925	38.6	1.44
20	39.7	0.975	40.8	1.96

```r
#x(k) are the measured values N(u,sigma^2)
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3,
    31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6, 35.8,
    35.9, 36.8, 37.1, 39.2, 39.7)
mean(x)
[1] 32.665
sd(x)
[1] 4.149734
#N(32.7,4.15)
#a_k = (k-0.5)/n = qnorm(q_ak, 32.7, 4.15)
pnorm(24.5, 32.7, 4.15)
[1] 0.02408285
pnorm(32.4, 32.7, 4.15)
[1] 0.4711859
pnorm(35.8, 32.7, 4.15)
[1] 0.7724646
pnorm(40.8, 32.7, 4.15)
[1] 0.9745195
#q_ak for N(32.7,4.15) = qnorm(a_k, 32.7, 4.15)
qnorm(0.025, 32.7, 4.15)
[1] 24.56615
qnorm(0.475, 32.7, 4.15)
[1] 32.43977
qnorm(0.725, 32.7, 4.15)
[1] 35.1807
qnorm(0.975, 32.7, 4.15)
[1] 40.83385
#phi^{-1}(a_k)
qnorm(0.025)
[1] -1.959964
qnorm(0.475)
[1] -0.06270678
qnorm(0.725)
[1] 0.5977601
qnorm(0.975)
[1] 1.959964
```

2.2.2 Parameter Esitmation for Continuous Probability Distributions

Method of Moments (not unbiased)

1. We consider our data measurements $x_1, x_2, ..., x_n$ as realization of random variables $X_1, X_2, ..., X_n$ originating from the same known distribution.	$\mu = E(X) \Rightarrow \hat{\mu} = \bar{x}_n = \frac{x_1+x_2+...+x_n}{n} = \frac{653.3}{20} = 32.7$ $\sigma^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2$
2. We calculate the expected value $E(X)$ and solve the equation for the unknown parameter that we intend to estimate.	$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{n}\sum\limits_{i=1}^{n} x_i^2$ $\hat{\sigma}^2 = \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x}_n)^2}{n}$
3. We replace the expected value with its counterpart, the empirical mean value and obtain an estimate of the unknown parameter. A method of moments estimate of the standard deviation is the empirical standard deviation.	$\hat{\sigma}^2 = \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i - \hat{\mu})^2} = \sqrt{\frac{1}{20}\sum\limits_{i=1}^{20}(x_i - 32.7)^2} = 4.04$

Method of Maximum Likelihood

We have n observations that are i.i.d.	$X_1 = x_1, X_2 = x_2, ..., X_n = x_n$
For a discrete probability distribution: probability that these n observations (events) actually have occurred can be expressed as follows	$P[(X_1 = x_1) \cap (X_2 = x_2) \cap ... \cap (X_n = x_n)] =$ $P[X_1 = x_1] \cdot P[X_2 = x_2] \cdot ... \cdot P[X_n = x_n] = \prod\limits_{i=1}^{n} P[X_i = x_i]$

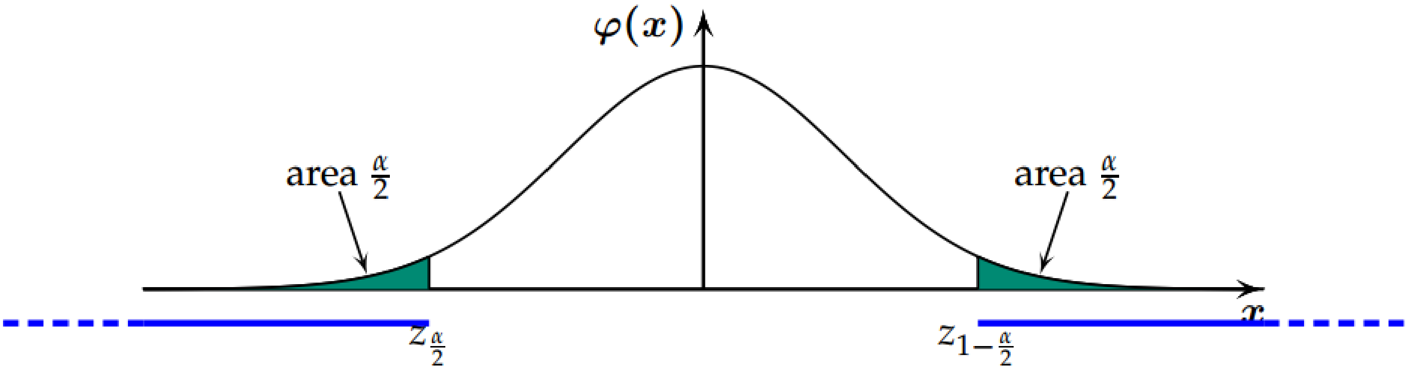
Probability that the n independent random variables $x_1, x_2, ..., x_n$ are observed, depends on parameter θ, which we wish to estimate. Therefore the Likelihood function is given by $L(\theta)$ where $P[X_i = x_i \| \theta]$ denotes probability mass function that value x_i has been observed, given the parameter value θ.	$L(\theta) = P[X_1 = x_1 \mid \theta] \cdot P[X_2 = x_2 \mid \theta] \cdot ... \cdot P[X_n = x_n \mid \theta] = \prod_{i=1}^{n} P[X_i = x_i \mid \theta]$
Idea of Maximum Likelihood : estimate the parameter θ in such a way that the likelihood is maximized, that is, that it makes the observed data most likely or most probable.	
Continuous probability distributions : with probability density function $f(x;\theta)$. Probability, that each observation x_i falls into its corresponding interval $[x_i, x_i + dx_i]$:	$\prod_{i=1}^{n} f(x_i;\theta) dx_i$
Infinitesimal intervals dx_i do not depend on the parameter value θ : we omit them in the likelihood function	$\prod_{i=1}^{n} f(x_i;\theta)$
If assumed probability density function $f(x_i;\theta)$ and parameter value of θ are correct, we expect a high probability for the actually observed data to occur : maximization of $L(\theta)$	

Example: Maximum Likelihood for Exponential Distribution	
Let $X_1, X_2 ..., X_n$ i.i.d. $\sim$ Exp(λ), that is	$f(x_i;\lambda) = \lambda e^{-\lambda x_i}$
Likelihood function for a given data set $x_1, x_2, ..., x_n$ is given by	$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$
Log likelihood function is	$\log(L(\lambda)) = n \log(\lambda) - \lambda \sum_{i=1}^{n} x_i$
If we calculate the derivative of the log likelihood function with respect to λ and set it equal to 0, then we obtain	$\frac{d \log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \stackrel{!}{=} 0$
The maximum likelihood estimate $\hat{\lambda}$ thus corresponds to the solution of the previous equation	$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

2.2.3 Statistical Tests and Confidence Interval for Normally Distributed Data

z-Test (σ_x known)	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X known
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0$ H_A: $\mu \neq \mu_0$ (or $<$ or $>$)
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{(\bar{X}_n - \mu_0)}{\sigma_{\bar{X}_n}} = \frac{(\bar{X}_n - \mu_0)}{\sigma_{X_n}/\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{observed - expected}{standard\,error}$ $Z \sim N(0,1)$
4. Significance level:	α
5. Rejection region for the test statistic: where	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$, $K = (-\infty, z_{\frac{\alpha}{2}}]$ with $H_A : \mu < \mu_0$, $K = [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu > \mu_0$ $z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2)$

6. Test decision:	Check whether the observed value of the test statistic falls into the rejection region.



z-Test (σ_x known): Example	
Measurement of fusion heat:	The empirical mean value of $n = 13$ measurements is 80.02. From previous measurements the standard deviation is $\sigma_X = 0.01$. Is a fusion heat of exactly $80.00\frac{g}{cal}$ plausible?
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, $\sigma_X = 0.01$ known, $n = 13$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0 = 80.00$ H_A: $\mu \neq \mu_0$
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{\sqrt{n}\bar{X}_n - \mu_0)}{\sigma_{X_n}}$ $Z \sim N(0,1)$
4. Significance level:	$\alpha = 0.05$ (commonly used α-level)

5. Rejection region for the test statistic:
 	
Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.
 	
The following rejection region for the test statistic results
	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$
	
	`qnorm(0.025)`
	`[1] -1.959964`
	
	$z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2) = \Phi^{-1}(0.025) = -1.96$
	$K = (-\infty, -1.96] \cup [1.96, \infty)$

6. Test decision: **Remarks:** Standardizing is in principle unnecessary because of technical aid of computer software.	Hence the value for the statistics is $z = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{\sqrt{13}(80.02-80.00)}{0.01} = 7.211$ Therefore the observed value falls into the rejection region.
3. Test statistic: (not standardized) Null distribution (assuming H_0 is true):	The mean value of the measurements T: $\bar{X}_n$ $T \sim N(\mu_0, \frac{\sigma_X^2}{n}) = N(80, \frac{0.01^2}{13})$

5. Rejection region for the test statistic: (not standardized)	$K = (-\infty, c_u] \cup [c_o, \infty)$ with $H_A : \mu \neq \mu_0$

Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.	`qnorm(0.025, 80.0, 0.01/sqrt(13))`
	`[1] 79.99456`
	`qnorm(0.975, 80.0, 0.01/sqrt(13))`
	`[1] 80.00544`

In this way, we obtain the rejection region tor the test statistic	$K = (-\infty, 79.99] \cup [80.01, \infty)$

2.3 Joint Distributions

TO DO: Chapter 3

3 Regression Analysis

3.1 Simple Linear Regression

TO DO: Chapter 5

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3.1.1 Estimating the Coefficients

Theory	Code Example
Estimation of response variable Y based on a predictor variable X. $$Y \simeq \beta_0 + \beta_1 X$$	`lm(Y ~ X, data=someData)`

Source code:

```
advertising <- read.csv("../Data/Advertising.csv")
model <- lm(sales ~ TV, data=advertising)
summary(model)
```

Output:

```
##
## Call:
## lm(formula = sales ~ TV, data = Advertising)
##
## Residuals:
## Min 1Q Median 3Q Max
## -8.3860 -1.9545 -0.1913 2.0671 7.2124
##
## Coefficients:
## Estimate Std. Error t value Pr(>|t|)
## (Intercept) 7.032594 0.457843 15.36 <2e-16 ***
## TV 0.047537 0.002691 17.67 <2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.259 on 198 degrees of freedom
## Multiple R-squared: 0.6119,Adjusted R-squared: 0.6099
## F-statistic: 312.1 on 1 and 198 DF, p-value: < 2.2e-16
```

Interpretation of output:

Source code:	Output:
TO DO: interpretation here	

3.2 Residual Analysis

TO DO: Chapter 6

3.3 Multiple Linear Regression

TO DO: Chapter 7

3.4 Linear Model Selection

TO DO: Chapter 8

4 Classification

4.1 Logistic Regression

TO DO: Chapter 10

4.2 Decision Trees

Decision trees are applied to both, classification and regression. **TO DO: Chapter 11**

4.2.1 Classification Trees

4.2.1.1 Binary Splitting

In binary splitting, a training set is used to split up the predictor domain into regions which contain data for which the response variable belongs to the same class. By **binary** it is meant that a region is split into **two** subregions (i.e. "is a predictor less or greater than a threshold value?" $\rightarrow$ yes/no).

<table>
<tr><th>Theory</th><th>Code Example</th></tr>
<tr><td>

Algorithm:

1. Initialise the set of regions $\mathcal{R} = R$ by the predictor domain R
2. Choose the optimal region R in $\mathcal{R}$ and the optimal predictor X_i such that a binary split of R with respect to X

   $$R_1 = \{\vec{x} \in R | x_i > t\} \quad \text{and} \quad R_2 = \{\vec{x} \in R | x_i \leqslant t\}$$

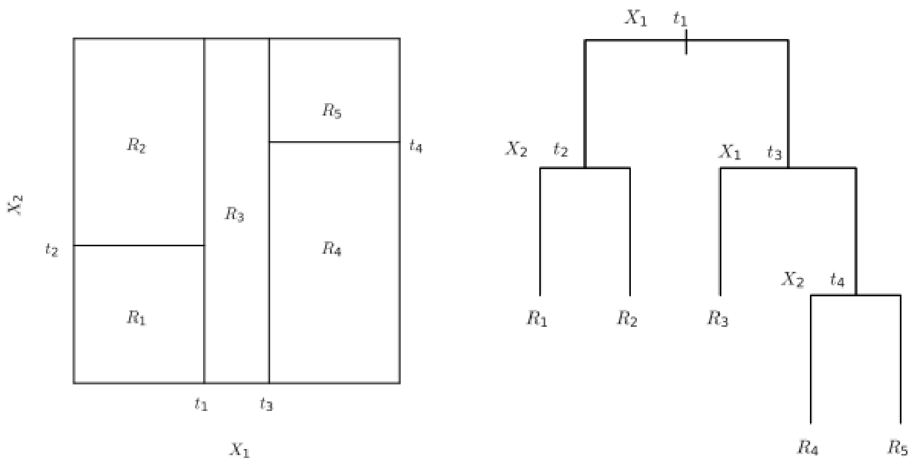
   gives the highest gain in purity (for some threshold t).
3. Replace R in $\mathcal{R}$ with R_1 and R_2 and return to 2.

The iteration is stopped if the current splitting fulfils a pre-defined stopping criterion.

</td><td>

```
require(tree)
#default controls
tc = tree.control(nobs = 303, mincut = 5,
    minsize = 10, mindev = 0.01)

#grow tree
tree.model = tree(AHD~MaxHR+Age, data = heart,
    control = tc)

#plot tree and label splits
plot(tree.model)
text(tree.model, cex=0.8)

#plot partition (only for two predictor case)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label
    ], pch=20)
```

</td></tr>
</table>



(a) Example regions resulting from binary splitting

(b) Example decision tree resulting from binary splitting

4.2.1.2 Node Purity

Notation:

Variable	Description
Y	Response variable
K	Levels (categories) of the response variable
T	The decision tree
M	Amount of terminal nodes
$\hat{p}_{mk}$	proportion of the training data in region m from level k

Purity Measures:

Classification error rate	$E_m(T) = 1 - \max_k(\hat{p}_{mk})$
Gini index	$G_m(T) = \sum_{k=1}^{K} \hat{p}_{mk} \cdot (1 - \hat{p}_{mk})$
Cross-entropy	$D_m(T) = -\sum_{k=1}^{K} \hat{p}_{mk} \cdot \log(\hat{p}_{mk})$

Code example: Cross Entropy and Gini measures in R

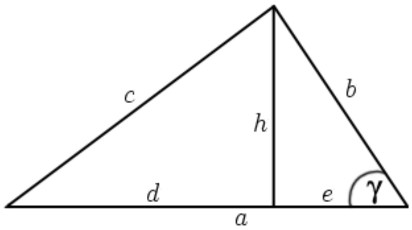
```
require(tree)
# deviance or cross entropy
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "deviance")
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)

# Gini index
tc = tree.control(303, mincut = 5, minsize = 60, mindev = 0.01)
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "gini", control = tc)
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)
```


5 Idiotenseite

5.1 Dreiecksformeln

Cosinussatz

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos\gamma$$



Sinussatz

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r = \frac{u}{\pi}$$

Pythagoras beim Sinus

$$\sin^2(b) + \cos^2(b) = 1 \qquad \tan(b) = \frac{\sin(b)}{\cos(b)}$$

$$\sin\beta = \tfrac{b}{a} = \tfrac{\text{Gegenkathete}}{\text{Hypotenuse}} \qquad \tan\beta = \tfrac{c}{b} = \tfrac{\text{Gegenkathete}}{\text{Ankathete}}$$
$$\cos\beta = \tfrac{c}{a} = \tfrac{\text{Ankathete}}{\text{Hypotenuse}} \qquad \cot\beta = \tfrac{c}{b} = \tfrac{\text{Ankathete}}{\text{Gegenkathete}}$$

5.2 Funktionswerte für Winkelargumente

deg	rad	sin	cos	tan	deg	rad	sin	cos	deg	rad	sin	cos	deg	rad	sin	cos
0°	0	0	1	0	90°	$\frac{\pi}{2}$	1	0	180°	π	0	-1	270°	$\frac{3\pi}{2}$	-1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

5.3 Periodizität

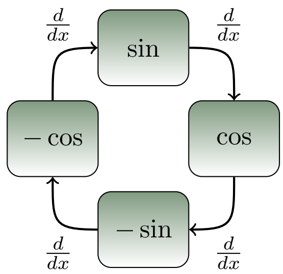
$\cos(a + k \cdot 2\pi) = \cos(a) \qquad \sin(a + k \cdot 2\pi) = \sin(a) \qquad (k \in \mathbb{Z})$

5.4 Quadrantenbeziehungen

$\sin(-a) = -\sin(a)$
$\sin(\pi - a) = \sin(a)$
$\sin(\pi + a) = -\sin(a)$
$\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right) = \cos(a)$

$\cos(-a) = \cos(a)$
$\cos(\pi - a) = -\cos(a)$
$\cos(\pi + a) = -\cos(a)$
$\cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\frac{\pi}{2} + a\right) = \sin(a)$

5.5 Ableitungen



5.6 Additionstheoreme

$\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b)$
$\cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$
$\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)}$

5.7 Doppel- und Halbwinkel

$\sin(2a) = 2\sin(a)\cos(a)$
$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\cos^2\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{2} \qquad \sin^2\left(\dfrac{a}{2}\right) = \frac{1-\cos(a)}{2}$

5.8 Produkte

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
$\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

5.9 Euler-Formeln

$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \qquad \cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$
$e^{x+jy} = e^x \cdot e^{jy} = e^x \cdot (\cos(y) + j\sin(y))$
$e^{j\pi} = e^{-j\pi} = -1$

5.10 Summe und Differenz

$\sin(a) + \sin(b) = 2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\sin(a) - \sin(b) = 2 \cdot \sin\left(\frac{a-b}{2}\right) \cdot \cos\left(\frac{a+b}{2}\right)$

$\cos(a) + \cos(b) = 2 \cdot \cos\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\cos(a) - \cos(b) = -2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right)$
$\tan(a) \pm \tan(b) = \dfrac{\sin(a \pm b)}{\cos(a)\cos(b)}$

5.12 Ableitungen elementarer Funktionen S436

Funktion	**Ableitung**	**Funktion**	**Ableitung**
C (Konstante)	0	$\sec x$	$\frac{\sin x}{\cos^2 x}$
x	1	$\sec^{-1} x$	$\frac{-\cos x}{\sin^2 x}$
$x^n \ (n \in \mathbb{R})$	nx^{n-1}	$\arcsin x \quad (\|x\| < 1)$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\arccos x \quad (\|x\| < 1)$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$	$\arctan x$	$\frac{1}{1+x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\mathrm{arccot}\ x$	$-\frac{1}{1+x^2}$
$\sqrt[n]{x} \quad (n \in \mathbb{R}, n \neq 0, x > 0)$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$	$\mathrm{arcsec}\ x$	$\frac{1}{x\sqrt{x^2-1}}$
e^x	e^x	$\mathrm{arcossec}\ x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\mathrm{e}^{bx} \quad (b \in \mathbb{R})$	$b\mathrm{e}^{bx}$	$\sinh x$	$\cosh x$
$a^x \quad (a > 0)$	$a^x \ln a$	$\cosh x$	$\sinh x$
$a^{bx} \quad (b \in \mathbb{R}, a > 0)$	$ba^{bx} \ln a$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\ln x$	$\frac{1}{x}$	$\coth x \quad (x \neq 0)$	$-\frac{1}{\sinh^2 x}$
$\log_a x \quad (a > 0, a \neq 1, x > 0)$	$\frac{1}{x}\log_a \mathrm{e} = \frac{1}{x \ln a}$	$\mathrm{Arsinh}\ x$	$\frac{1}{\sqrt{1+x^2}}$
$\lg x \quad (x > 0)$	$\frac{1}{x}\lg \mathrm{e} \approx \frac{0.4343}{x}$	$\mathrm{Arcosh}\ x \quad (x > 1)$	$\frac{1}{\sqrt{x^2-1}}$
$\sin x$	$\cos x$	$\mathrm{Artanh}\ x \quad (\|x\| < 1)$	$\frac{1}{1-x^2}$
$\cos x$	$-\sin x$	$\mathrm{Arcoth}\ x \quad (\|x\| > 1)$	$-\frac{1}{x^2-1}$
$\tan x \quad (x \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z})$	$\frac{1}{\cos^2 x} = \sec^2 x$	$[f(x)]^n \quad (n \in \mathbb{R})$	$n[f(x)]^{n-1}f'(x)$
$\cot x \quad (x \neq k\pi, k \in \mathbb{Z})$	$\frac{-1}{\sin^2 x} = -cosec^2 x$	$\ln f(x) \quad (f(x) > 0)$	$\frac{f'(x)}{f(x)}$

5.11 Einige unbestimmte Integrale S1074

$\int dx = x + C$	$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$
$\int \frac{1}{x} dx = \ln\|x\| + C,\ x \neq 0$	$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C,\ a\epsilon\mathbb{R}^+\backslash\{1\}$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C,\ x \neq k\pi$ mit $k\epsilon\mathbb{Z}$
$\int \frac{dx}{\cos^2 x} = \tan x + C,\ x \neq \frac{\pi}{2} + k\pi$ mit$k\epsilon\mathbb{Z}$	$\int \sinh x dx = \cosh x + C$
$\int \cosh x dx = \sinh x + C$	$\int \frac{dx}{\sinh^2 x} = -\coth x + C,\ x \neq 0$
$\int \frac{dx}{\cosh^2 x} = \tanh x + C$	$\int \frac{dx}{ax+b} = \frac{1}{a} \ln\|ax + b\| + C,\ a \neq 0, x \neq -\frac{b}{a}$
$\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab} \arctan \frac{a}{b}x + C,\ a \neq 0,\ b \neq 0$	$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab} \ln\left\|\frac{ax-b}{ax+b}\right\| + C,\ a \neq 0,\ b \neq 0,\ x \neq \frac{b}{a},\ x \neq -\frac{b}{a}$
$\int \sqrt{a^2x^2 + b^2} dx = \frac{x}{2}\sqrt{a^2x^2 + b^2} + \frac{b^2}{2a} \ln(ax + \sqrt{a^2x^2 + b^2}) + C,\ a \neq 0,\ b \neq 0$	$\int \sqrt{a^2x^2 - b^2} dx = \frac{x}{2}\sqrt{a^2x^2 - b^2} - \frac{b^2}{2a} \ln\left\|ax + \sqrt{a^2x^2 - b^2}\right\| + C,\ a \neq 0,\ b \neq 0, a^2x^2 \geqq b^2$
$\int \sqrt{b^2 - a^2x^2} dx = \frac{x}{2}\sqrt{b^2 - a^2x^2} + \frac{b^2}{2a} \arcsin \frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 \leqq b^2$	$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \frac{1}{a} \ln(ax + \sqrt{a^2x^2 + b^2}) + C,\ a \neq 0,\ b \neq 0$
$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \frac{1}{a} \ln\left\|ax + \sqrt{a^2x^2 - b^2}\right\| + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 > b^2$	$\int \frac{dx}{\sqrt{b^2-a^2x^2}} = \frac{1}{a} \arcsin \frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 < b^2$
Die Integrale $\int \frac{dx}{X}, \int \sqrt{X} dx, \int \frac{dx}{\sqrt{X}}$ mit $X = ax^2 + 2bx + c,\ a \neq 0$ werden durch die Umformung $X = a(x + \frac{b}{a})^2 + (c - \frac{b^2}{a})$ und die Substitution $t = x + \frac{b}{a}$ in die oberen 4 Zeilen transformiert.	$\int \frac{xdx}{X} = \frac{1}{2a} \ln\|X\| - \frac{b}{a} \int \frac{dx}{X},\ a \neq 0,\ X = ax^2 + 2bx + c$
$\int \sin^2 axdx = \frac{x}{2} - \frac{1}{4a} \cdot \sin 2ax + C,\ a \neq 0$	$\int \cos^2 axdx = \frac{x}{2} + \frac{1}{4a} \cdot \sin 2ax + C,\ a \neq 0$
$\int \sin^n axdx = -\frac{sin^{n-1} ax \cdot \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} axdx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int \cos^n axdx = \frac{\cos^{n-1} ax \cdot \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} axdx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int \frac{dx}{\sin ax} = \frac{1}{a} \ln\left\|\tan \frac{ax}{2}\right\| + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$	$\int \frac{dx}{\cos ax} = \frac{1}{a} \ln\left\|\tan(\frac{ax}{2} + \frac{\pi}{4})\right\| + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$
$\int \tan axdx = -\frac{1}{a} \ln\|\cos ax\| + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$mit $k\epsilon\mathbb{Z}$	$\int \cot axdx = \frac{1}{a} \ln\|\sin ax\| + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}mitk\epsilon\mathbb{Z}$
$\int x^n \sin axdx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos axdx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int x^n \cos axdx = \frac{x^n}{a} \sin ax - \frac{n}{a} \int x^{n-1} \sin axdx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C,\ a \neq 0,\ b \neq 0$
$\int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C,\ a \neq 0,\ b \neq 0$	$\int \ln xdx = x(\ln x - 1) + C,\ x\epsilon\mathbb{R}^+$
$\int x^\alpha \cdot \ln xdx = \frac{x^{\alpha+1}}{(\alpha+1)^2}[(\alpha + 1) \ln x - 1] + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$	