1 R Tutorial

1.1 Loading Data

```
# loading csv files
data <- read.table("whatever.csv", sep="", header=T)

# csv files can be stored with (almost) any kind of file ending, e.g.:
data <- read.table("whatever.dat", sep="", header=T)
data <- read.table("whatever.txt", sep="", header=T)
```

2 Probability And Statistics

2.1 Probability Models for Measurement Data

2.1.1 Random Variables

Random Variables	
Definition	$X : \Omega \longrightarrow W_{\mathrm{x}}$
Example	A Coin is thrown three times, head and tails is observed: $\Omega = \{hhh, hht, htt, hth, ttt, tth, thh, tht\}$ Total number of heads $W_{\mathrm{x}} = \{0, 1, 2, 3\}$ Total number of tails $W_{\mathrm{x}} = \{0, 1, 2, 3\}$ Number of heads minus tails $W_{\mathrm{x}} = \{-3, -1, 1, 3\}$
Probability Mass Function	
Definition	The probability distribution of a discrete random variable: $P(X = x)$
Example	x: 0, 1, 2, 3 $P(X = x)$: $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$

2.1.2 Probability Distributions

Cumulative Density Function (cdf)	
Definition	$F(x) = P(X \leqslant x)$
Properties	$P(a < X \leqslant b) = F(b) - F(a)$ $0 \leqslant F(x) \leqslant 1$ $P(X = a) = F(a) - F(a) = 0$

Probability Density Function (pdf)	
Definition	$f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$
Properties	$f(x) \geqslant 0$ $P(a < X \leqslant b) = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$ $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$

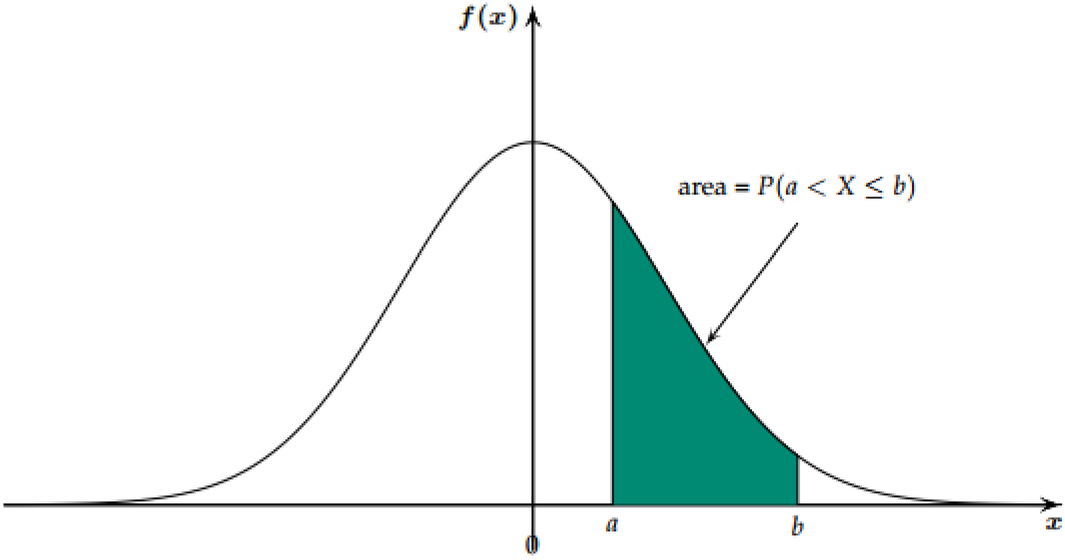

Figure 1: Probability density of a random variable and the probability of measuring a value from (a,b]

2.1.3 Summary Statistics of Continuous Distributions

Expected Value, Variance and Quantile	
Expected value	Discrete: $\mathrm{E}(X) = \sum_i x_i P(X = x_i)$ Continuous: $\mathrm{E}(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x$
Variance	$\mathrm{Var}(X) = \sigma_x^2 = \mathrm{E}((X - \mathrm{E}(X))^2) = \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2 \cdot f(x)\mathrm{d}x$
Quantile	$P(X \leqslant q(\alpha)) = \alpha$ $F(q(\alpha)) = \alpha \Leftrightarrow q(\alpha) = F^{-1}(\alpha)$ Note: When you're asked for the 50%-quantile, that means $\alpha = 50\%$, and you must find $q(0.5)$
Example Body Length	If α=0.75 and the corresponding quantile is $q(\alpha)$=182.5cm then 75% of the persons is shorter or equal 182.5cm.

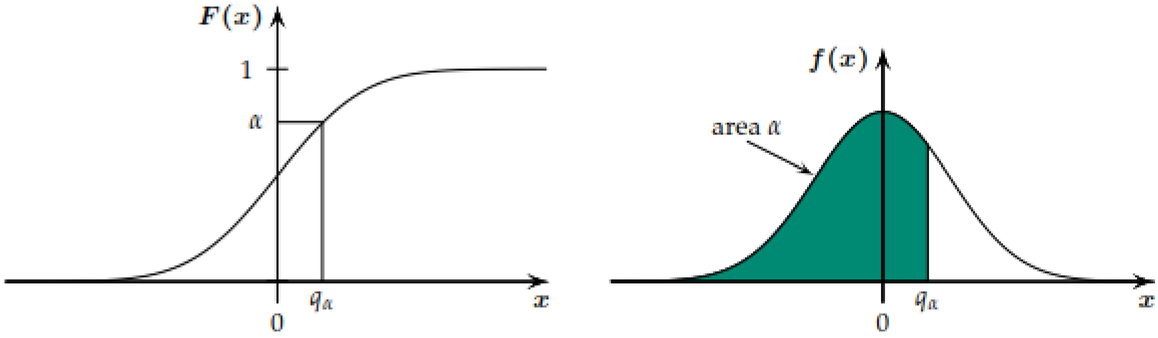

Figure 2: Quantiles

2.1.4 Important Distributions

2.1.4.1 Uniform Distribution

Theory	Code Example

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$ $$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & x > b \end{cases}$$ $$\mathrm{E}(x) = \frac{a+b}{2}$$ $$\mathrm{Var}(x) = \frac{(b-a)^2}{12}$$ $$\sigma_{\mathrm{x}} = \frac{b-a}{\sqrt{12}}$$	`# value of the probability density function Uniform([1, 10]) at the position x = 5`
	`dunif(x=5, min=1, max=10)`
	`[1] 0.1111111`
	
	`# P(X <= 5)`
	`punif(q=5, min=1, max=10)`
	`[1] 0.4444444`
	
	`# P(1.2 < X <= 4.8)`
	`punif(4.8, 1, 10) - punif(1.2, 1, 10)`
	`[1] 0.4`
	
	`# 5 uniformly distributed random values in Uniform([1, 10])`
	`runif(5,min=1,max=10)`
	`[1] 1.061933 6.484813 5.928334 8.459887 8.852405`
	
	`# TODO: ADD MORE HERE`

2.1.4.2 Exponential Distribution

Theory	Code Example

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$ $$F(x) = \begin{cases} 1 - \lambda \cdot e^{-\lambda \cdot x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$ $$\mathrm{E}(x) = \frac{1}{\lambda}$$ $$\mathrm{Var}(x) = \frac{1}{\lambda^2}$$ $$\sigma_{\mathrm{x}} = \frac{1}{\lambda}$$	`# P(0 <= X <= 4) of X ~ Exp(3)`
	`pexp(4, rate=3)`
	`[1] 0.9999939`
	
	`# TODO: ADD MORE HERE`

2.1.4.3 Normal Distribution

Theory	Code Example

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})}$
$F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}y$
$\mathrm{E}(x) = \mu$
$\mathrm{Var}(x) = \sigma^2$
$\sigma_{\mathrm{x}} = \sigma$
	`# X~N(u, sigma^2) --> X~N(100,15^2)`
	`# In R we compute P(X>130) as 1 - P(X<=130)`
	`1-pnorm(130, mean=100, sd=15)`
	`[1] 0.02275013`
	
	`#P(85<=X<=115)`
	`pnorm(115, mean=100, sd=15)-pnorm(85, mean=100, sd=15)`
	`[1] 0.6826895`
	
	`# TODO: ADD MORE HERE`

2.1.4.4 Linear Transformation of Random Variables

Properties of Linear Transformation of a Random Variable	
Definition	For $Y = a + bX$ the following apply (i) $E(Y) = a + bE(X)$ (ii) $Var(Y) = b^2 Var(X)$, $\sigma_Y = \lvert b\rvert\sigma_X$ (iii) $\alpha - Quantile\,of\,Y = q_Y(\alpha) = a + bq_X(\alpha)$ (iv) $f_Y(y) = \frac{1}{b} f_X(\frac{y-a}{b})$
Summary Statistics of S_n and $\bar{X}_n$	
Summary Statistics of Sample Total S_n	$E(S_n) = E(X_1 + X_2 + ... + X_n) = \sum_{i=1}^{n} E(X_i) = n\mu$ $Var(S_n) = \sum_{i=1}^{n} Var(X_i) = nVar(X_i)$ $\sigma(S_n) = \sqrt{n}\sigma_X$
Summary Statistics of Sample Mean $\bar{X}_n$ *Standard Error*	$E(\bar{X}_n) = E(\frac{X_1+X_2+...+X_n}{n}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} nE(X_i) = \mu$ $Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma_X^2 = \frac{\sigma_X^2}{n}$ $\sigma(\bar{X}_n) = \frac{\sigma_X}{\sqrt{n}}$

2.1.4.5 Distributions of S_n and $\bar{X}_n$

Theory	Code Example

1. For $X_i \in \{0,1\}$, we have
$S_n \sim \mathrm{Bin}(n,\pi)$ with $\pi = P(X_i = 1)$
2. For $X_i \sim \mathrm{Pois}(\lambda)$, we have
$S_n \sim \mathrm{Pois}(n\lambda)$
3. For $X_i \sim N(\mu, \sigma^2)$
$S_n \sim N(n\mu, n\sigma^2)$ and $\bar{X}_n \sim N(\mu, \frac{\sigma_X^2}{n})$
	`What is the probability that among 10000 tosses of a fair coin, heads would appear in maximum 5100 cases?`
	`#Approximated: X~N(5000,2500)`
	`pnorm(5100,5000,sqrt(2500))`
	`[1] 0.9772499`
	
	`#"True Result": X~Bin(10000,0.5)`
	`pbinom(5100,10000,0.5)`
	`[1] 0.9777871`

2.2 Statistics for Measurement Data

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2.2.1 Assess the Normal Distribution Assumption

2.2.1.1 Q-Q Plot

Theory

1. For

$$\alpha_k = \frac{k-0.5}{n} \text{ with } k = 1, ..., n$$

calculate the corresponding theoretical quantiles of the model distribution

$$q(\alpha_k) = F^{-1}(\alpha_k)$$

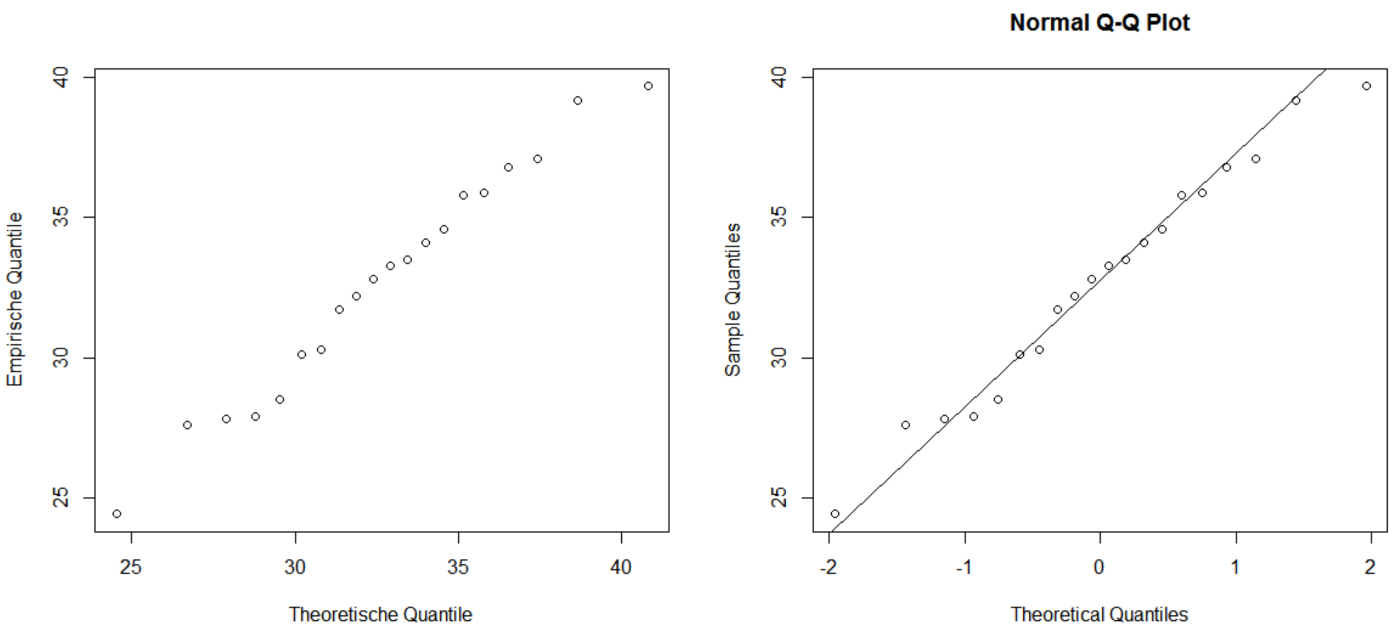
2. Determine the empirical α_k-quantiles,

$$x_{(1)} < x_{(2)} < ... < x_{(n)}$$

3. Plot the empirical quantiles x_k on the y-axis against the theoretical quantiles $q(\alpha_k)$ on the x-axis.

Code Example

```r
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3,
     31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6,
    35.8, 35.9, 36.8, 37.1, 39.2, 39.7)

alpha_k <- (seq(1, length(x), by=1)-0.5)/length(
    x)

quantile_th <- qnorm(alpha_k, mean=mean(x), sd=
    sd(x))

quantile_emp <- sort(x)
#image qqplot
qqplot(quantile_th, quantile_emp, xlab="
    Theoretische Quantile", ylab = "Empirische
    Quantile")
#image qqnorm;qqline
qqnorm(x);qqline(x)
```



(a) qqplot()

(b) qqnorm();qqline()

k	$x_{(k)}$	$\alpha_k = (k-0.5)/n$	q_{α_k} for $\mathcal{N}(32.7, 4.15^2)$	$\Phi^{-1}(\alpha_k)$
1	24.4	0.0250	24.5	-1.96
2	27.6	0.075	26.7	-1.44
3	27.8	0.125	27.9	-1.15
4	27.9	0.175	28.8	-0.935
5	28.5	0.225	29.5	-0.755
6	30.1	0.275	30.2	-0.600
7	30.3	0.325	30.8	-0.453
8	31.7	0.375	31.3	-0.319
9	32.2	0.425	31.9	-0.189
10	32.8	0.475	32.4	-0.0627
11	33.3	0.525	32.9	0.0627
12	33.5	0.575	33.4	0.189
13	34.1	0.625	34.0	0.319
14	34.6	0.675	34.5	0.454
15	35.8	0.725	35.1	0.598
16	35.9	0.775	36.0	0.755
17	36.8	0.825	36.5	0.935
18	37.1	0.875	37.4	1.15
19	39.2	0.925	38.6	1.44
20	39.7	0.975	40.8	1.96

```r
#x(k) are the measured values N(u,sigma^2)
x <- c(24.4, 27.6, 27.8, 27.9, 28.5, 30.1, 30.3, 31.7, 32.2, 32.8, 33.3, 33.5, 34.1, 34.6, 35.8, 35.9, 36.8, 37.1, 39.2, 39.7)
mean(x)
[1] 32.665
sd(x)
[1] 4.149734
#N(32.7,4.15)
#a_k = (k-0.5)/n = qnorm(q_ak, 32.7, 4.15)
pnorm(24.5, 32.7, 4.15)
[1] 0.02408285
pnorm(32.4, 32.7, 4.15)
[1] 0.4711859
pnorm(35.8, 32.7, 4.15)
[1] 0.7724646
pnorm(40.8, 32.7, 4.15)
[1] 0.9745195
#q_ak for N(32.7,4.15) = qnorm(a_k, 32.7, 4.15)
qnorm(0.025, 32.7, 4.15)
[1] 24.56615
qnorm(0.475, 32.7, 4.15)
[1] 32.43977
qnorm(0.725, 32.7, 4.15)
[1] 35.1807
qnorm(0.975, 32.7, 4.15)
[1] 40.83385
#phi^{-1}(a_k)
qnorm(0.025)
[1] -1.959964
qnorm(0.475)
[1] -0.06270678
qnorm(0.725)
[1] 0.5977601
qnorm(0.975)
[1] 1.959964
```

2.2.2 Parameter Esitmation for Continuous Probability Distributions

Method of Moments (not unbiased)

1. We consider our data measurements $x_1, x_2, ..., x_n$ as realization of random variables $X_1, X_2, ..., X_n$ originating from the same known distribution.

2. We calculate the expected value $E(X)$ and solve the equation for the unknown parameter that we intend to estimate.

3. We replace the expected value with its counterpart, the empirical mean value and obtain an estimate of the unknown parameter. A method of moments estimate of the standard deviation is the empirical standard deviation.

$$\mu = E(X) \Rightarrow \hat{\mu} = \bar{x}_n = \frac{x_1+x_2+...+x_n}{n} = \frac{653.3}{20} = 32.7$$

$$\sigma^2 = E(X^2) - E(X)^2 = E(X^2) - \mu^2$$

$$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}{n}$$

$$\hat{\sigma}^2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2} = \sqrt{\frac{1}{20}\sum_{i=1}^{20}(x_i - 32.7)^2} = 4.04$$

Method of Maximum Likelihood

We have n observations that are i.i.d.

$$X_1 = x_1, X_2 = x_2, ..., X_n = x_n$$

For a discrete probability distribution: probability that these n observations (events) actually have occurred can be expressed as follows

$$P[(X_1 = x_1) \cap (X_2 = x_2) \cap ... \cap (X_n = x_n)] =$$
$$P[X_1 = x_1] \cdot P[X_2 = x_2] \cdot ... \cdot P[X_n = x_n] = \prod_{i=1}^{n} P[X_i = x_i]$$

Probability that the n independent random variables $x_1, x_2, ..., x_n$ are observed, depends on parameter θ, which we wish to estimate. Therefore the Likelihood function is given by $L(\theta)$ where $P[X_i = x_i\|\theta]$ denotes probability mass function that value x_i has been observed, given the parameter value θ.	$L(\theta) = P[X_1 = x_1\|\theta] \cdot P[X_2 = x_2\|\theta] \cdot ... \cdot P[X_n = x_n\|\theta] = \prod_{i=1}^{n} P[X_i = x_i\|\theta]$
Idea of Maximum Likelihood : estimate the parameter θ in such a way that the likelihood is maximized, that is, that it makes the observed data most likely or most probable.	
Continuous probability distributions : with probability density function $f(x;\theta)$. Probability, that each observation x_i falls into its corresponding interval $[x_i, x_i + dx_i]$:	$\prod_{i=1}^{n} f(x_i;\theta)dx_i$
Infinitesimal intervals dx_i do not depend on the parameter value θ : we omit them in the likelihood function	$\prod_{i=1}^{n} f(x_i;\theta)$
If assumed probability density function $f(x_i;\theta)$ and parameter value of θ are correct, we expect a high probability for the actually observed data to occur : maximization of $L(\theta)$	

Example: Maximum Likelihood for Exponential Distribution	
Let $X_1, X_2..., X_n$ i.i.d. $\sim$ Exp(λ), that is	$f(x_i;\lambda) = \lambda e^{-\lambda x_i}$
Likelihood function for a given data set $x_1, x_2, ..., x_n$ is given by	$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$
Log likelihood function is	$\log(L(\lambda)) = n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i$
If we calculate the derivative of the log likelihood function with respect to λ and set it equal to 0, then we obtain	$\frac{d\log(L(\lambda))}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \overset{!}{=} 0$
The maximum likelihood estimate $\hat{\lambda}$ thus corresponds to the solution of the previous equation	$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

2.2.3 Statistical Tests and Confidence Interval for Normally Distributed Data

z-Test (σ_x known)	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X known
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0$ H_A: $\mu \neq \mu_0$ (or $<$ or $>$)
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{(\bar{X}_n - \mu_0)}{\sigma_{\bar{X}_n}} = \frac{(\bar{X}_n - \mu_0)}{\sigma_{X_n}/\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{observed - expected}{standard\, error}$ $Z \sim N(0,1)$
4. Significance level:	α
5. Rejection region for the test statistic: where	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$, $K = (-\infty, z_\alpha]$ with $H_A : \mu < \mu_0$, $K = [z_{1-\alpha}, \infty)$ with $H_A : \mu > \mu_0$ $z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2)$

6. Test decision:	Check whether the observed value of the test statistic falls into the rejection region.

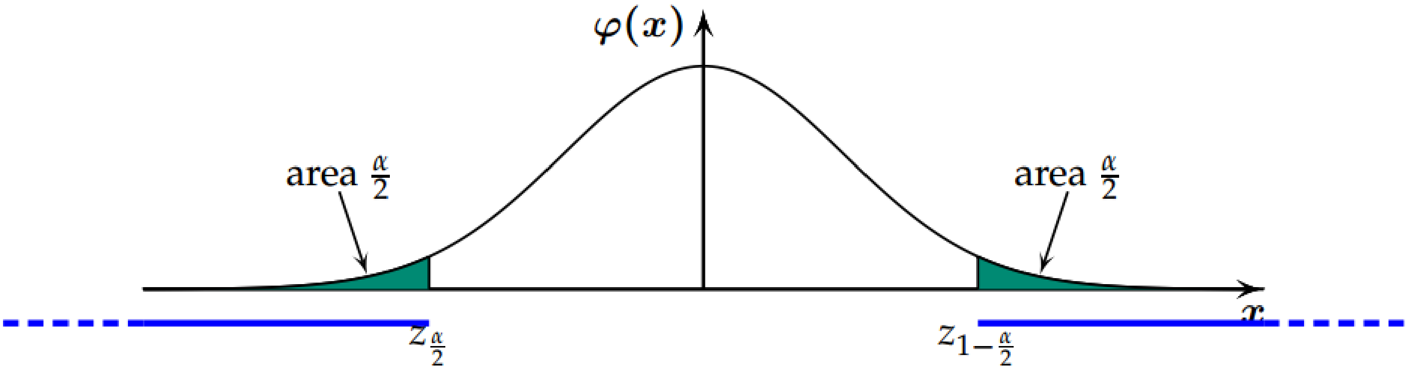


Figure 4: z-Test: Rejection Region

z-Test (σ_x known): Example	
Measurement of fusion heat:	The empirical mean value of $n = 13$ measurements is 80.02. From previous measurements the standard deviation is $\sigma_X = 0.01$. Is a fusion heat of exactly $80.00\frac{g}{cal}$ plausible?
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, $\sigma_X = 0.01$ known, $n = 13$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0 = 80.00$ H_A: $\mu \neq \mu_0$
3. Test statistic: Null distribution (assuming H_0 is true):	$Z = \frac{\sqrt{n}\bar{X}_n - \mu_0)}{\sigma_{X_n}}$ $Z \sim N(0, 1)$
4. Significance level:	$\alpha = 0.05$ (commonly used α-level)

5. Rejection region for the test statistic:
 	
Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.
 	
The following rejection region for the test statistic results
	$K = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$
	
	`qnorm(0.025)`
	`[1] -1.959964`
	
	$z_{\frac{\alpha}{2}} = \Phi^{-1}(\alpha/2) = \Phi^{-1}(0.025) = -1.96$
	$K = (-\infty, -1.96] \cup [1.96, \infty)$

6. Test decision: **Remarks:** Standardizing is in principle unnecessary because of technical aid of computer software.	Hence the value for the statistics is $z = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_{X_n}} = \frac{\sqrt{13}(80.02 - 80.00)}{0.01} = 7.211$ Therefore the observed value falls into the rejection region.
3. Test statistic: (not standardized)	The mean value of the measurements T: $\bar{X}_n$

Null distribution (assuming H_0 is true):	$T \sim N(\mu_0, \frac{\sigma_X^2}{n}) = N(80, \frac{0.01^2}{13})$

5. Rejection region for the test statistic: (not standardized)
 	
Given $\alpha = 0.05$, R yields the following 2.5% quantile of the standard normal distribution.
 	
In this way, we obtain the rejection region tor the test statistic:
	$K = (-\infty, c_u] \cup [c_o, \infty)$ with $H_A : \mu \neq \mu_0$
	
	`qnorm(0.025, 80.0, 0.01/sqrt(13))`
	`[1] 79.99456`
	`qnorm(0.975, 80.0, 0.01/sqrt(13))`
	`[1] 80.00544`
	
	$K = (-\infty, 79.99] \cup [80.01, \infty)$

t-Test (σ_x unknown)	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X is estimated by $\hat{\sigma}_X$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0$ H_A: $\mu \neq \mu_0$ (or $<$ or $>$)
3. Test statistic: Null distribution (assuming H_0 is true):	$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = \frac{observed - expected}{estimated\ standard\ error}$ $T \sim t_{n-1}$
4. Significance level:	α
5. Rejection region for the test statistic:	$K = (-\infty, t_{n-1;\frac{\alpha}{2}}] \cup [t_{n-1;1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$, $K = (-\infty, t_{n-1;\alpha}]$ with $H_A : \mu < \mu_0$, $K = [t_{n-1;1-\alpha}, \infty)$ with $H_A : \mu > \mu_0$
6. Test decision:	Check whether the observed value of the test statistic falls into the rejection region.
Example	
1. Model:	$X_1, ..., X_n$ i.i.d. $\sim N(\mu, \sigma_X^2)$, σ_X is estimated, $\hat{\sigma}_X = 0.024$
2. Null hypothesis: Alternative:	H_0: $\mu = \mu_0 = 80.00$ H_A: $\mu \neq \mu_0$
3. Test statistic: Null distribution (assuming H_0 is true):	$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X}$ $T \sim t_{n-1}$
4. Significance level:	$\alpha = 0.05$

5. Rejection region for the test statistic:
 	
We determine the value
 	
by means of R, where $\alpha = 0.05$ and $n = 13$.
 	
The rejection region of the test statistic thus is given by
	$K = (-\infty, t_{n-1;\frac{\alpha}{2}}] \cup [t_{n-1;1-\frac{\alpha}{2}}, \infty)$ with $H_A : \mu \neq \mu_0$,
	
	$t_{n-1;1-\frac{\alpha}{2}} = t_{12;0.975} = 2.179$
	
	`qt(0.975,12)`
	`[1] 2.178813`
	
	$K = (-\infty, -2.179] \cup [2.179, \infty)$

6. Test decision:	On the basis of $n = 13$ measurements, we find

$\bar{x} = 80.02$ and $\hat{\sigma}_X = 0.024$

Hence, the realized value of the test statistic is

$t = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = \frac{\sqrt{13}(80.02 - 80.00)}{0.024} = 3.00$

The observed value falls into the rejection region. Therefore, the null hypothesis is rejected at 5% level.

The t-test directly performed in R using the function t.test()

```
x <- c(79.98, 80.04, 80.02, 80.04, 80.03,
80.03, 80.04, 79.97, 80.05, 80.03,
80.02, 80.00, 80.02)

t.test(x, alternative = "two.sided",
mu = 80.00, conf.level = 0.95)

##
## One Sample t-test
##
## data:  x
## t = 3.1246, df = 12, p-value = 0.008779
## alternative hypothesis: true mean is not
    equal to 80
## 95 percent confidence interval:
## 80.00629 80.03525
## sample estimates:
## mean of x
## 80.02077
```

Remarks:
(i) The observed value of the test statistic is 3.12. Assuming the null hypothesis is true, then the test statistic follows a t-distribution with $df = 12$ degrees of freedom.

(ii) The observed mean value of the data is 80.02. A 95% confidence interval for the true mean is $[80.006, 80.035]$.

(iii) The R functions qt(p,df) calculates the quantile from the probability density and the degrees of freedom and pt(q,df) calculates the probabilty density from the quantile and the degrees of freedom.

(iv) The **confidence interval** for measurement data consists of the values µ, for which the corresponding statistical test does not reject the null hypothesis.

```
qt(0.975,12)
[1] 2.178813
pt(2.178813,12)
[1] 0.975
qt(0.5,12)
[1] 0.0
pt(0.0,12)
[1] 0.5
```

P-Value

The p-value is the probability that the test statistic will take on a value that is at least as extreme (with respect to the alternative hypothesis) as the observed value of the statistic when the null hypothesis H_0 is true.

For the one-sided alternative hypothesis H_A: $\mu > \mu_0$, the p-value can be calculated as follows - the observed value of the statistics is $t = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_X} = 3.1246$:

p-value$= P(T > t) = P(T > 3.1246) = 0.00439$ For the two-sided alternative hypothesis H_A: $\mu \neq \mu_0$, the p-value can be calculated as follows (the observed value of the test statistics is $t = \frac{\sqrt{n}|\bar{X}_n - \mu_0|}{\hat{\sigma}_X}$):

p-value$= 2 \cdot P(T > |t|)$

In R we compute the one-sided and the two sided p-value as follows:

```
#one-sided p-value
1-pt(3.1246, df=12)
[1] 0.004389739
#two-sided p-value
2*(1-pt(3.1246, df=12))
[1] 0.008779477
```

These p-values are evidence against the null hypothesis at 5% level. Whereas the two-sided value is statistically significant at the 5% value.

p-value and Statistical Test 1. Reject H_0 if p-value$\leqslant \alpha$ 2. Retain H_0 if p-value$> \alpha$ *The p-value is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.*	The p-value quantifies how significant an alternative is: p-value ≈ 0.05 : weakly significant, "." p-value ≈ 0.01 : weakly significant, "*" p-value ≈ 0.001 : weakly significant, "**" p-value $\leqslant 10^{-4}$: weakly significant, "***"

2.3 Joint Distributions

2.3.1 Joint, Marginal and Conditional Distributions

Discrete Joint Probability Distribution	
The **Joint Probability Distribution** of X and Y is defined by the following distributions:	$P(X = x, Y = y),\ x \in W_x, y \in W_y$
Marginal Distributions are single distributions $P(X = x)$ of X and $P(Y = y)$ of Y. They can be calculated based on their joint distribution:	$P(X = x) = \sum_{y \in W_y} P(X = x, Y = y),\ x \in W_x$
Joint distribution of (X, Y) starting from the marginal distribution of X and Y is only possible for **independent** X and Y. Then it holds:	$P(X = x, Y = y) = P(X = x) \cdot P(Y = y),\ x \in W_x, y \in W_y$
Conditional probability of X given $Y = y$ is defined as:	$P(X = x \mid Y = y) = \frac{P(X=x,Y=y)}{P(Y=y}$
The **marginal distributions** then can be expressed as follows:	$P(X = x) = \sum_{y \in W_y} P(X = x \mid Y = y) P(Y = y),\ x \in W_x$
Conditional Expected Value of Y given $X = x$ is defined as:	$E[Y \mid X = x] = \sum_{y \in W_y} y \cdot P(Y = y \mid X = x)$

Example

X\ Y	1	2	3	4	$\sum$
1	0.080	0.015	0.003	0.002	0.100
2	0.050	0.350	0.050	0.050	0.500
3	0.030	0.060	0.180	0.030	0.300
4	0.001	0.002	0.007	0.090	0.100
$\sum$	0.161	0.427	0.240	0.172	1

$P(X = 3, Y = 4) = 0.030$ or $P(X = 3 \cup Y = 4) = 0.030$

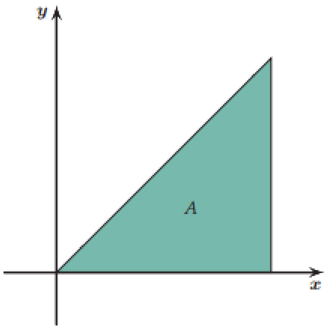
$P(X = 3) = P(X = 3, Y = 1) + P(X = 3, Y = 2) +$
$P(X = 3, Y = 3) + P(X = 3, Y = 4) =$
$0.030 + 0.060 + 0.180 + 0.030 = 0.300$

$P(Y = 2 \mid X = 4) = \frac{P(Y=2,X=4)}{P(X=4)} = \frac{0.002}{0.1} = 0.02$

$P(X = Y) = P(X = 1, Y = 1) + P(X = 2, Y = 2) +$
$P(X = 3, Y = 3) + P(X = 4, Y = 4) = 0.700$

If random variables are independent it must hold that
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$
From the marginal distribution follows
$P(X = 1) \cdot P(Y = 2) = 0.100 \cdot 0.427 = 0.043$
and this is not equal to
$P(X = 1, Y = 2) = 0.15$
X and Y are **not independent**

Joint Density Function	
The **probability** that the **joint random variable** (X,Y) lies in a two-dimensional region A, i.e., $A \subset \mathbb{R}^2$, is given by	$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) dx\, dy$

The (bivariate) **joint density function** needs to satisfy X and Y are only **independent** if	$\iint\limits_{\mathbb{R}} f_{X,Y}(x,y)dx\,dy = 1$ $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y),\ x, y \in \mathbb{R}$
Marginal Density	$f_X(x) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dy,\ f_Y(y) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dx$
Conditional Probability X and Y are only independent if the following apply:	$f_{Y\|X=x}(y) = f_Y(y\|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ $f_{Y\|X=x}(y) = f_Y(y)$ resp. $f_{X\|Y=y}(x) = f_X(x)$
Conditional Expected Value of a continuous random variable Y given $X = x$	$E[Y\|X=x] = \int\limits_{-\infty}^{\infty} y \cdot f_{Y\|X=x}(y)dy$
Example	
Two machines with exponentially distributed life expectancy $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$, where X and Y are independent. $f_X(x) = \lambda_1 e^{-\lambda_1 x}$ and $f_Y(y) = \lambda_2 e^{-\lambda_2 y}$ 	Due to independence: $_{X,Y}(x,y) = \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}$ $P(Y < X) = \int\limits_0^{\infty}(\int\limits_0^{x} \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}dy)dx$ $P(Y < X) = \int\limits_0^{\infty} \lambda_1 e^{-\lambda_1 x}(1 - e^{-\lambda_2 y})dx = \frac{\lambda_2}{\lambda_1+\lambda_2}$

2.3.2 Covariance and Correlation

Covariance and Correlation	
Covariance X, Y independent Sum of Variances 2 Random Variables If all X_i are independent	$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$ $E[XY] = E[X]E[Y]$ $Cov(X,X) = E[(X-\mu_X)(X-\mu_X)] = E[(X-\mu_X)^2] = Var(X)$ $Var(\sum\limits_{i=1}^{n} X_i) = Cov(\sum\limits_{i=1}^{n} X_i, \sum\limits_{i=1}^{n} X_i) = \sum\limits_{i=1}^{n} Var(X_i) + 2\sum\limits_{i<j}^{n} Cov(X_i, X_j)$ $Var(X+Y) = Cov(X+Y, X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ $Var(X_1 + X_2 + ... + X_n) = Var(X_1) + ... + Var(X_n)$
Correlation Measure for strength and direction of the *linear dependency* between X and Y.	$Cor(X,Y) = \rho_{XY} = \frac{Cov(X,Y}{\rho_X \rho_Y}$ where $-1 \leqslant Cor(X,Y) \leqslant 1$ $Cor(X,Y) = +1$ if $Y = a + bX$ for $a \in \mathbb{R}$ and $b > 0$ $Cor(X,Y) = -1$ if $Y = a + bX$ for $a \in \mathbb{R}$ and $b < 0$

	$\lvert Cor(X,Y)\rvert = 1$ means perfect linear relationship between X and Y. $Cor(X,Y) = 0$ means X and Y are uncorrelated.
X and Y **linear independent**	$Cor(X,Y) = 0$ (and thus $Cov(X,Y) = 0$)

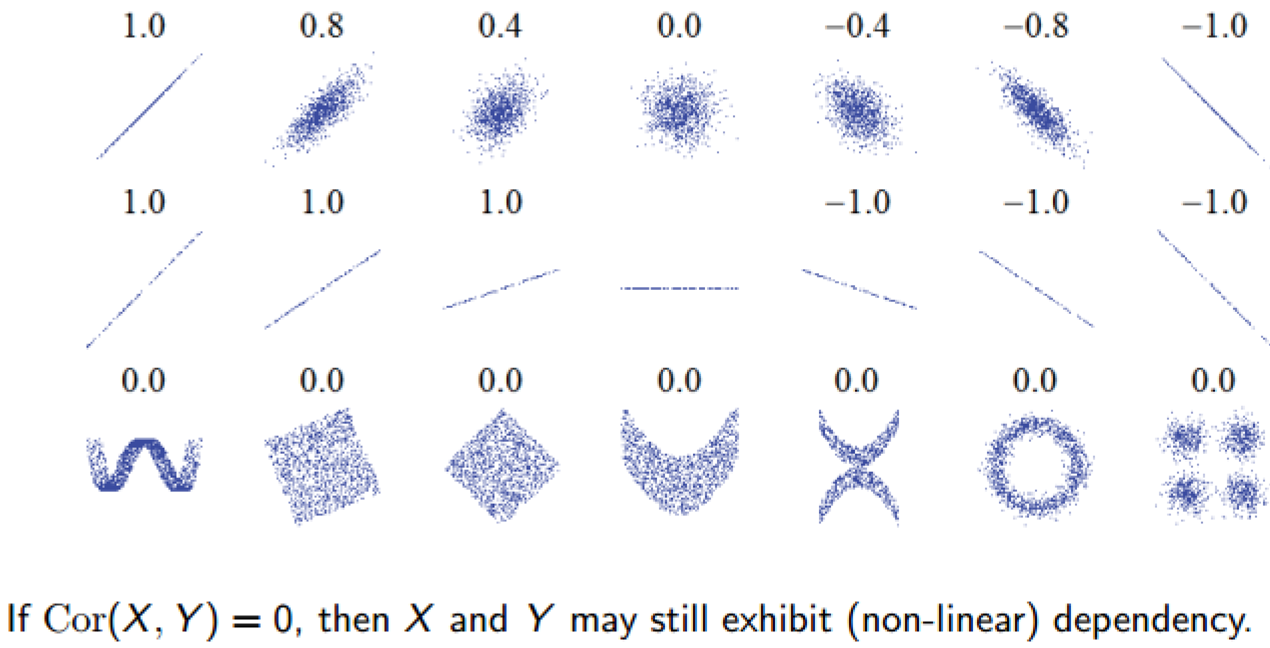


Figure 5: Correlations

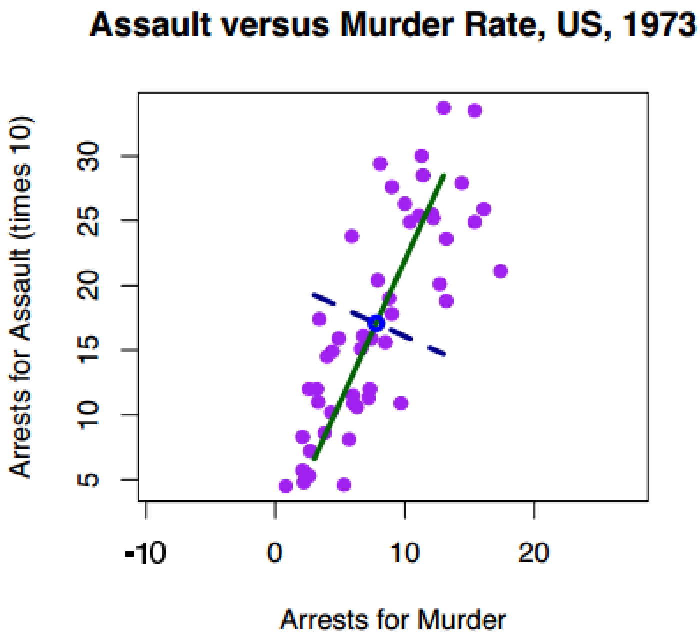
2.3.3 Bivariate Normal Distribution

Bivariate Normal Distribution	
Expected values and variances of the marginal distribution Covariance between X and Y	μ_X, σ_X^2 and μ_Y, σ_Y^2 $Cov(X,Y) = \rho_{XY}\sigma_X\sigma_Y$
Joint Density	$f_{X,Y}(x,y) =$ $\frac{1}{2\pi\sqrt{det(\sum)}} exp\Big(-\frac{1}{2}(x-\mu_X, y-\mu_Y) \sum^{-1} \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}\Big)$
Covariance Matrix	$\sum = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$

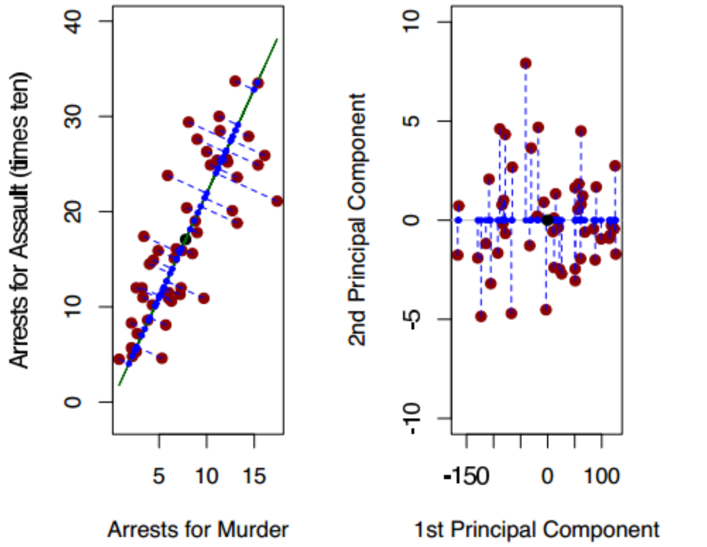
2.3.4 Principal Component Analysis (PCA)

PCA is a popular approach for deriving a low-dimensional set of features from a large set of variables. PCA is a technique for reducing the dimension of a n x p data matrix X where n corresponds to the number of observations and p to the number of variables.

2.3.4.1 Example: USArrests

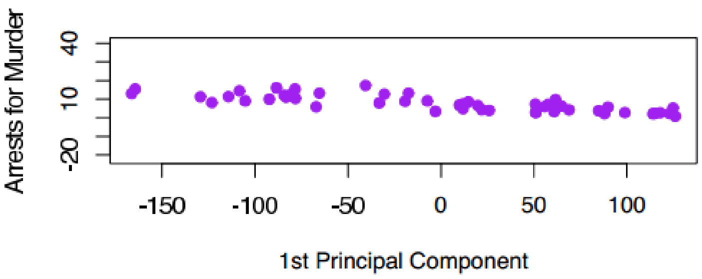


(a) The data vary the most along the first principal component

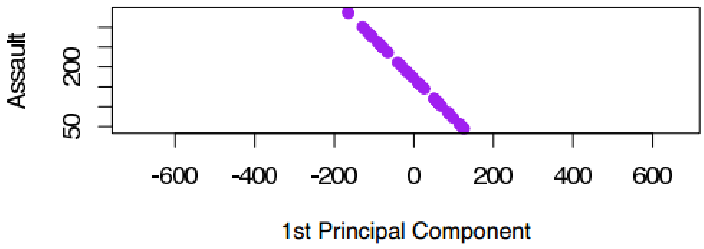


(b) Counter Clockwise Rotation that 1. PC coincides with x-axis

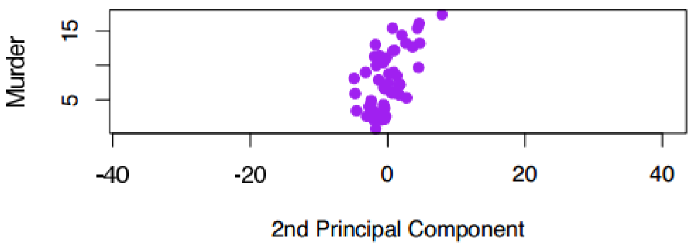
Figure 6: 1st and 2nd Principal Component



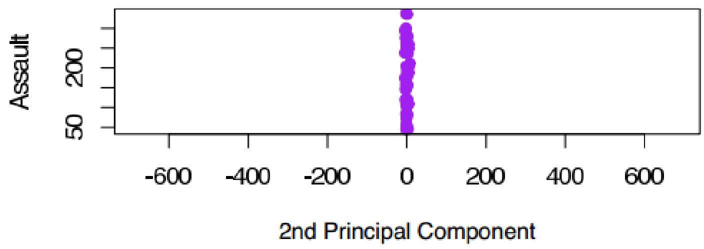
(a) z_{i1} versus Murder



(b) z_{i1} versus Assault



(a) z_{i2} versus Murder



(b) z_{i2} versus Assault

Figure 8: The fact that the 2nd principal component scores are much closer to zero indicates that this component captures far less information as the 1st principal component.

Theory

$Z_1 = -0.0419126(\text{Murder - }\overline{\text{Murder}}) - 0.9991213(\text{Assault-}\overline{\text{Assault}})$

$\phi_{11} = -0.00419126$ and $\phi_{21} = -0.9991213$ are the *principal component loadings*

The idea is that every that out of every linear combination of Murder and Assault sucht that

$$\phi_{11}^2 + \phi_{21}^2 = 1$$

and

$$Var(\phi_{11})(\text{Murder} - \overline{\text{Murder}}) + \phi_{21}(\text{Assault} - \overline{\text{Assault}})$$

is maximized.

prcomp() centers the variables to have mean zero. This corresponds to how the first principal component is defined.

$z_{i1} = -0.0419126(\text{Murder - }\overline{\text{Murder}}) - 0.9991213(\text{Assault-}\overline{\text{Assault}})$

The values of $z_{i1}, ..., z_{n1}$ are known as *principal component scores*, seen in the right-hand panel in Figure 6. $z_{i1} > 0$ indicates a state with below-average arrests for murder and below average for assault. A negative score suggests the opposite.

$Z_2 = 0.9991213(\text{Murder - }\overline{\text{Murder}}) - 0.0419126(\text{Assault-}\overline{\text{Assault}})$

Code Example

```r
#First principal component
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
pr.out$rotation[,1]

## Murder   Assault
## -0.0419126  -0.9991213

#principal component scores (z_i1 to z_in)
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
head(pr.out$x)

##      PC1(z_i1)  PC2(z_i2)
## Alabama    -65.40950  2.6728663
## Alaska    -92.25166 -1.6559620
## Arizona    -123.14478  -4.8535831
## Arkansas  -19.26551  0.2047123
## California  -105.19832  -3.1999471
## Colorado  -33.21549  -1.2812733

#Second principal component
pr.out <- prcomp(USArrests[,c("Murder","Assault"
    )])
pr.out$rotation[,2]

## Murder   Assault
## 0.9991213  -0.0419126
```

With two-dimensional data, such as in our USArrests example, we can construct at most two principal components. However, if we had other variables, such as Rape, then additional components could be constructed.

2.3.4.2 PCA and Covariance Matrix

The covariance matrix of two random variables X and Y is defined as

$$\sum = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(Y,X) & \sigma_Y^2 \end{pmatrix}$$

Since Z_1 and Z_2 are required to be uncorrelated, this implies for their covariance matrix $\sum$ to have vanishing off diagonal elements. Therefore the covariance has to be diagonalized. This can be done with by a rotation matrix Φ so that

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = (X - \mu_X, Y - \mu_Y) \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} \phi_{11}(X - \mu_X) & \phi_{12}(Y - \mu_Y) \\ \phi_{21}(X - \mu_X) & \phi_{22}(Y - \mu_Y) \end{pmatrix}$$

and

$$\begin{pmatrix} Cov(Z_1,Z_1) & Cov(Z_1,Z_2) \\ Cov(Z_2,Z_1) & Cov(Z_2,Z_2) \end{pmatrix} = \begin{pmatrix} \sigma_{Z_1}^2 & 0 \\ 0 & \sigma_{Z_2}^2 \end{pmatrix}$$

The rotation matrix Φ needs to satisfy the condition $\phi_{11}^2 + \phi_{21}^2 = 1$ and $\phi_{12}^2 + \phi_{22}^2 = 1$ It is straightforward to generalize the case of $p = 2$ to an arbitrary p.

2.3.4.3 Proportion of Variance Explained by Principal Components

Theory	Code Example

There is an information loss of the given data by projecting the observations onto the first few principal components. Therefore we want to know the *proportion of variance explained (PVE)*. The *total variance* is defined as $$\sum_{j=1}^{p} Var(X_j) = \sum_{j=1}^{p} \frac{1}{n} \sum_{i=1}^{n} x_{ij}^2$$ and the variance of the mth principal component is $$\frac{1}{n} \sum_{i=1}^{n} z_{im}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{p} \phi_{jm} x_{ij} \right)^2$$ Therefore the PVE by the mth principal component is given by $$\frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{p} \phi_{jm} x_{ij} \right)^2}{\sum_{j=1}^{p} \sum_{i=1}^{n} x_{ij}^2}$$	`pr.out <- prcomp(USArrests[,c("Murder","Assault")],scale=FALSE)`
	`pr.var <- pr.out$sdev^2`
	`pve <- pr.var/sum(pr.var)`
	`pve`
	
	`## [1] 0.9990292327 0.0009707673`
	
	`## Most of the information of the data about the arrests for Murder and Assault is contained in the first principal component.`

3 Regression Analysis

3.1 Simple Linear Regression

TO DO: Chapter 5

...

3.1.1 Estimating the Coefficients

Theory	Code Example
Estimation of response variable Y based on a predictor variable X. $$Y \simeq \beta_0 + \beta_1 X$$	`lm(Y ~ X, data=someData)`

Source code:	Output:

<pre>advertising <- read.csv("../Data/Advertising.csv")
model <- lm(sales ~ TV, data=advertising)
summary(model)</pre>
	<pre>##
	## Call:
	## lm(formula = sales ~ TV, data = Advertising)
	##
	## Residuals:
	## Min 1Q Median 3Q Max
	## -8.3860 -1.9545 -0.1913 2.0671 7.2124
	##
	## Coefficients:
	## Estimate Std. Error t value Pr(>|t|)
	## (Intercept) 7.032594 0.457843 15.36 <2e-16 ***
	## TV 0.047537 0.002691 17.67 <2e-16 ***
	## ---
	## Signif. codes:
	## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
	##
	## Residual standard error: 3.259 on 198 degrees of freedom
	## Multiple R-squared: 0.6119,Adjusted R-squared: 0.6099
	## F-statistic: 312.1 on 1 and 198 DF, p-value: < 2.2e-16</pre>

Interpretation of output:

TO DO: interpretation here

3.2 Residual Analysis

TO DO: Chapter 6

3.3 Multiple Linear Regression

TO DO: Chapter 7

3.4 Linear Model Selection

TO DO: Chapter 8

4 Classification

4.1 Logistic Regression

TO DO: Chapter 10

4.2 Decision Trees

Decision trees are applied to both, classification and regression. **TO DO: Chapter 11**

4.2.1 Classification Trees

4.2.1.1 Binary Splitting

In binary splitting, a training set is used to split up the predictor domain into regions which contain data for which the response variable belongs to the same class. By **binary** it is meant that a region is split into **two** subregions (i.e. "is a predictor less or greater than a threshold value?" → yes/no).

Theory	Code Example

Theory

Algorithm:

1. Initialise the set of regions $\mathcal{R} = R$ by the predictor domain R
2. Choose the optimal region R in $\mathcal{R}$ and the optimal predictor X_i such that a binary split of R with respect to X

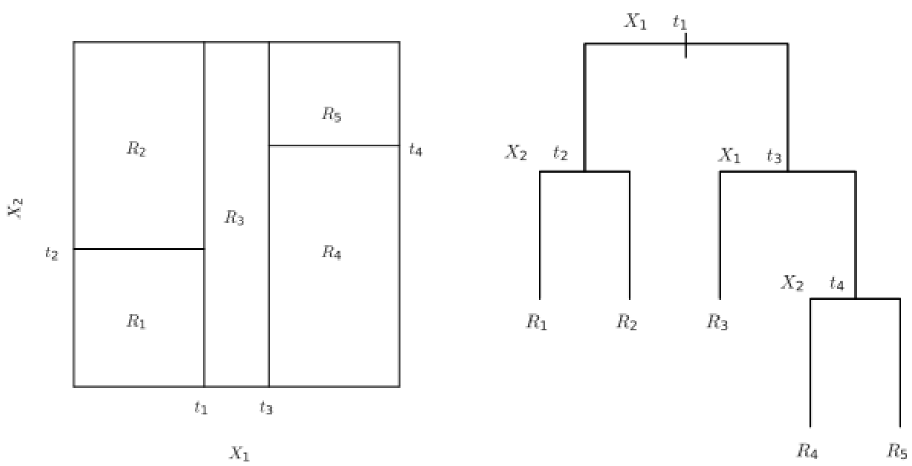
   $$R_1 = \{\vec{x} \in R | x_i > t\} \quad \text{and} \quad R_2 = \{\vec{x} \in R | x_i \leqslant t\}$$

   gives the highest gain in purity (for some threshold t).
3. Replace R in $\mathcal{R}$ with R_1 and R_2 and return to 2.

The iteration is stopped if the current splitting fulfils a predefined stopping criterion.

Code Example

```r
require(tree)
#default controls
tc = tree.control(nobs = 303, mincut = 5,
    minsize = 10, mindev = 0.01)

#grow tree
tree.model = tree(AHD~MaxHR+Age, data = heart,
    control = tc)

#plot tree and label splits
plot(tree.model)
text(tree.model, cex=0.8)

#plot partition (only for two predictor case)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label
    ], pch=20)
```



(a) Example regions resulting from binary splitting

(b) Example decision tree resulting from binary splitting

4.2.1.2 Node Purity

Notation:

Variable	Description
Y	Response variable
K	Levels (categories) of the response variable
T	The decision tree
M	Amount of terminal nodes
$\hat{p}_{mk}$	proportion of the training data in region m from level k

Purity Measures:

Classification error rate	$E_m(T) = 1 - \max_k(\hat{p}_{mk})$

Gini index	$G_m(T) = \sum_{k=1}^{K} \hat{p}_{mk} \cdot (1 - \hat{p}_{mk})$
Cross-entropy	$D_m(T) = -\sum_{k=1}^{K} \hat{p}_{mk} \cdot \log(\hat{p}_{mk})$

Code example: Cross Entropy and Gini measures in R

```
require(tree)
# deviance or cross entropy
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "deviance")
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)

# Gini index
tc = tree.control(303, mincut = 5, minsize = 60, mindev = 0.01)
tree.model = tree(AHD~MaxHR+Age, data = heart, split = "gini", control = tc)
plot(tree.model)
text(tree.model, cex=0.8)
partition.tree(tree.model)
points(Age~MaxHR, data = heart, col = cols[label], pch=20)
```

4.3 Random Forests

TO DO: Chapter 12

5 Time Series Analysis

5.1 Introduction to Time Series

Models are not always *independent of the order* of the training data. Many real life measuring and data recording processes result in data sets that are *serially correlated.* For example *machine monitoring, stock, environmental observations or federal statistics.* These kind of data is called *time series data.* Usually there are several goals that one wants to achieve in time series data.

- **Descriptive Analysis**
- **Modelling and Interpretation**
- **Decomposition**
- **Predection**
- **Regression**

5.1.1 Time Series with R

Theory	Code Example

All data in R are stored in objects, which provide a range of methods. The class of an object can be found using the class function. For example, we have already encountered the data.frame class. It has a series of methods, such as names or nrow:
(The data set iris contains 50 samples of three types of Iris flowers, measured along four variables.)
	`class(iris); names(iris); nrow(iris)`
	
	`## [1] "data.frame"`
	`## [1] "Sepal.Length" "Sepal.Width" "Petal.Length" "Petal.Width"`
	`## [5] "Species"`
	`## [1] 150`

5.1.1.1 The ts Class

Theory	Code Example

Basic properties:

The AirPassengers-data is a built in set of class ts. Most important methods for ts class are:

1. start() returns the start time of the series.
2. end() returns the end time of the series.
3. frequency() returns the number of samples per unit time.
4. plot() displays the time series as a function over the time axis. plot function calls plot.ts which is tailored for time series. plot.ts joins discrete time points automatically with lines. See Figure AirPassengers.

```
class(AirPassengers)
## [1] "ts"

start(AirPassengers); end(AirPassengers);
    frequency(AirPassengers)
## [1] 1949 1
## [1] 1960 12
## [1] 12

#1/frequency = 1/12 = 0.0833
deltat(AirPassengers)
## [1] 0.0833

#output in figure AirPassengers.
plot(AirPassengers, main = "Passengers", ylab="
    Number (in 1000s)")
grid()
```

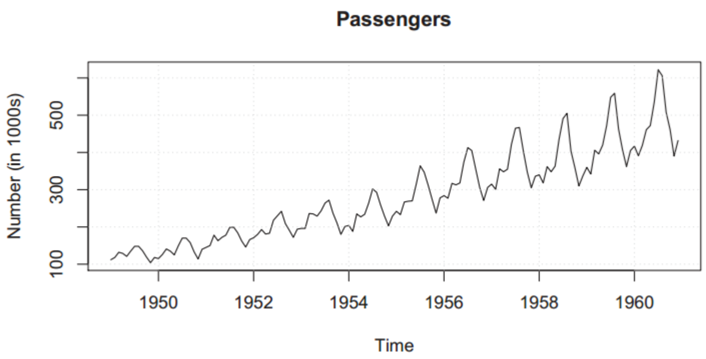
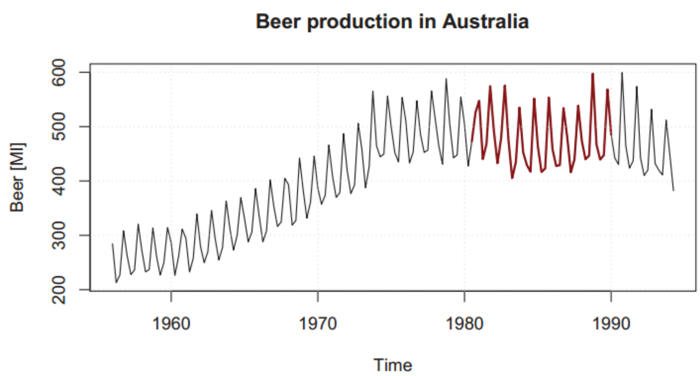


Figure 10: AirPassengers



(a) Subset of a Time Series (seasonal behaviour)

Theory	Code Example

Defining a ts class

If data is not in a time series form we can make a ts object by using the ts function. This is not necessary for AirPassengers, therefore the example AustralianBeer is used.

1. summary() gives the five-number summary as well as the mean of the time series.This function shows the minimum, the first quartile, the median, the second quartile and the maximum of the time series. This is called the *five-number-summary* of a data set. Additionally the mean is also computed.
2. window() returns a subset of the time series defined by a start and an end time.

```
X.beer = read.table("../Daten/AustralianBeer.csv
    ", sep=";", header = T)

X.beer.ts = ts(X.beer[,2], start = c(1956,1),
    end = c(1994, 2),frequency = 4)
summary(X.beer.ts)
## Min. 1st Qu. Median Mean 3rd Qu. Max.
## 213  325  427  408  467  600

#Figure Subset of Time Series
plot(X.beer.ts, ylab="Beer [Ml]", main="Beer
    production in Australia")
X.ts.w = window(X.beer.ts, start = c(1980,3),
    end = c(1990, 1))
summary(X.ts.w)
## Min. 1st Qu. Median  Mean   3rd Qu. Max.
## 405  437  467  478    530     598
lines(X.ts.w, col = "darkred", lwd=2)
grid()
```

Theory

Multivariate Time Series
A few important ideas and concepts related to multivariate time series data illustrated with the following example:

The quaterly supply of electricity in Australia compared to the quaterly beer production see Figure 12a.
The plots show increasing trends in production for both goods, partly due to the rising population in Australia from about 10 million to about 18 million over the same period. But notice that electricity production has risen by a factor of 7 during which the population has not quite doubled.

There are many functions in R for handling more than one series, including ts.intersect to obtain the intersection of two series that overlap in time. There are some **pitfalls** shown with AirPassenger and electricity in Figure 12b.

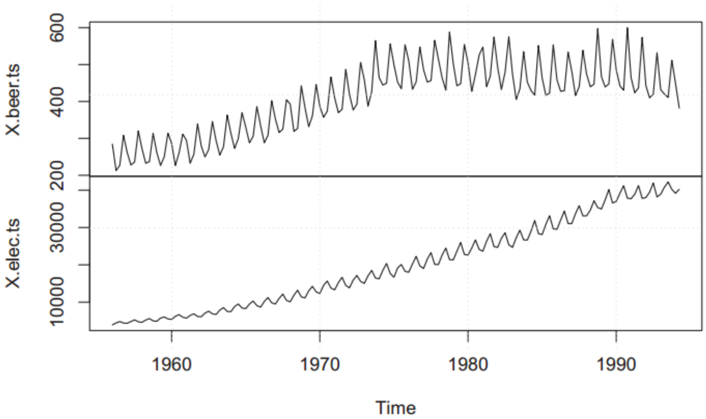
The two series are correlated but there is of course **no causal dependence** of the two series. They are confounded by seasonal effects.

Non-equdistant time series are **not covered** by the ts class. There are further packages:

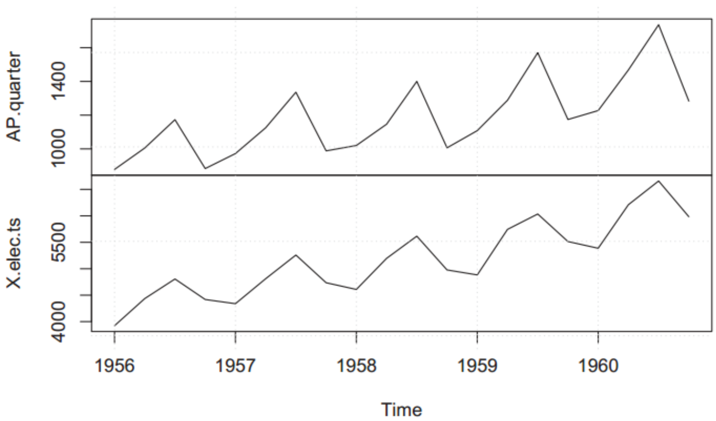
- **zoo-package**:It provides methods for regular and irregular spaced times series as well as arbitrary date formats.
- **xts-package**: It is an extension of the zoo-package which allows for further customzation.

Code Example

```r
##Example beer vs electricity (Australia)

## First load the electricity data from file and
     create a time series out of it.
X.elec = read.table("../Daten/
    AustralianElectricity.csv", sep=";",
header = T)
X.elec.ts = ts(X.elec[,2], start = c(1956,1),
    end = c(1994, 2),
frequency

##Bind the two separate series together by means
     of the cbind command and plot the series
X.ts = cbind(X.beer.ts, X.elec.ts)
plot(X.ts, main="Beer and electricity production
     in Australia")
grid()

##aggregate the monthly data of the
    AirPassengers data to quarterly data
##aggregate sums the data set to the desired
    frequency up
AP.quarter = aggregate(AirPassengers, nfrequency
     = 4)

#Extract common time points and combine the
    corresponding data values to a new,
    bivariate time series
AP.elec = ts.intersect(AP.quarter, X.elec.ts)
plot(AP.elec, main="Air Passenger bookings and
    electricity production")
grid()
```



(a) Beer and electricity production in Australia



(b) Air Passenger bookings and electricity production

5.1.2 Basic transformation, visualization and decomposition of time series

5.1.2.1 Data transformation

In many situtions it is desirable or necessary to transform a time series before the application of models and predictions. Many methods require

- **Gaussian** or **symmetric** distribution of the data.
- A **linear** trend relationship between time and data.

- A **constant variance** across time.

Theory

Box-Cox-transformation

For highly skewed or heteroskedastic data - data whose variance is not constant across time - it is often better to use not the original series $\{x_1, x_2, ...\}$ but a transformed series $\{g(x_1), g(x_2), ...\}$. The Box-Cox-transformation is well suited for correcting skewness and variance.

For a times series $\{x_1, x_2, ...\}$ with positive values the Box-Cox transformations are defined as

$$g(x) = \begin{cases} \frac{x^\lambda - 1}{\lambda} & \lambda \neq 0 \\ log(x) & \lambda = 0 \end{cases}$$

As in Figure 13 to see the original data exhibits clear seasonal effects and an upward trend.The intensity of the seasonal influence, i.e. the variance over time, is also increasing. The parameter $\lambda = 0$, i.e. the log-transform of the data, yields a stabilized image: a seemingly linear trend with homogeneus seasonal effects.

Code Example

```
# define the Box-Cox transformation
box.cox <- function(x, lambda) {
if (lambda==0) log(x) else (x^lambda - 1)/lambda
}

# plot the original and the transformed data
# --> see figure Box-Cox-transformation for
    different values of lambda
layout(matrix(c(1,2,3,4), 2,2))
plot(AirPassengers, main = "Original", ylab="",
    xlab="")
plot(box.cox(AirPassengers, 2), main = "lambda =
     2",
ylab="", xlab="")
plot(box.cox(AirPassengers, -0.5), main = "
    lambda = -0.5",
ylab="", xlab="")
plot(box.cox(AirPassengers, 0), main = "lambda =
     0",
ylab="", xlab="")
```

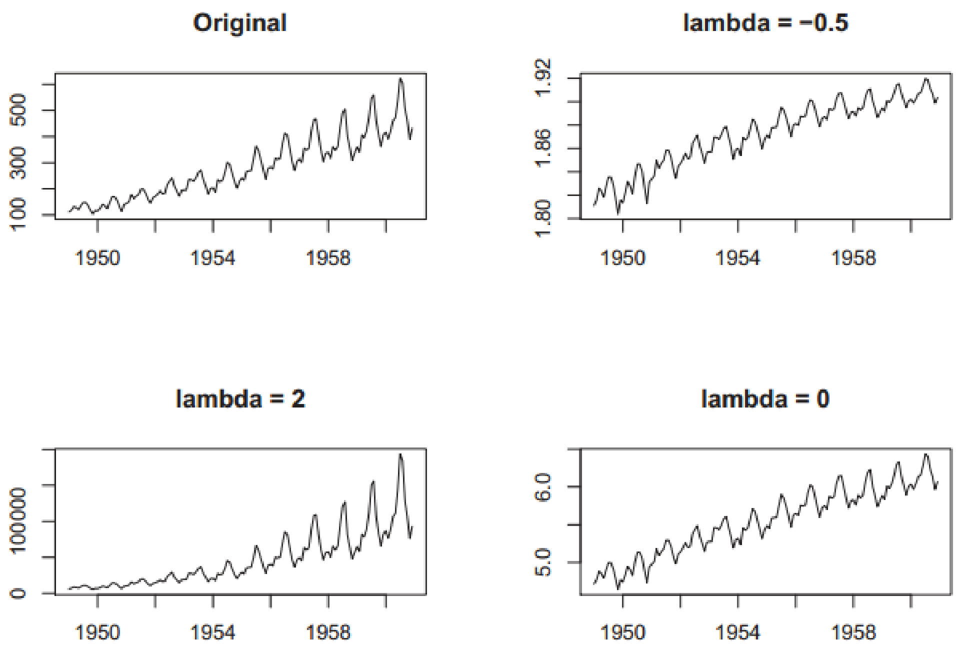


Figure 13: Box-Cox-transformation for different values of λ

Theory	Code Example

Theory

Time-shift transformation
Sometimes it is necessary to transform the time-axis as well. The most simple form version of time transforms is shifting.

Let $\{x_1, x_2, ...\}$ be a time series.

1. The time-shift by a *lag* of $k \in \mathbb{Z}$ is defined by

$$g(x_i) = x_{i-k}$$

2. For the particular case where $k = 1$ the time-shift is called *backshift*

$$B(x_i) = x_{i-1}$$

In other words, applying a time-shift to a time series amounts to go back k steps (if $k > 0$) or go ahead $-k$ steps (if $k < 0$) in the series.

In R the function lag is used to apply a time shift for various values of k.

The back-shift operator is applied if *differences* of times series are computed, since $x_i - x_{i-1} = x_i - B(x_i)$. In particular, differencing is often combined with Box-Cox transformations. For example in the *log-returns* of a (financial) time series are defined as

$$y_i = log(x_i) - log(x_{i-1}) = log\left(\frac{x_i}{x_{i-1}}\right)$$

Code Example

```r
#lag function --> see figure time-shift transformations
AP = AirPassengers
AP.back = lag(AP, k = 4)
AP.ahead = lag(AP, k = -5)
plot(AP, lwd=2)
lines(AP.back, col="darkcyan")
lines(AP.ahead, col = "darkred")
grid()
legend("topleft", legend=c("original", "shifted back", "shifted ahead"),
lty=1, col = c("black", "darkcyan", "darkred"))
```

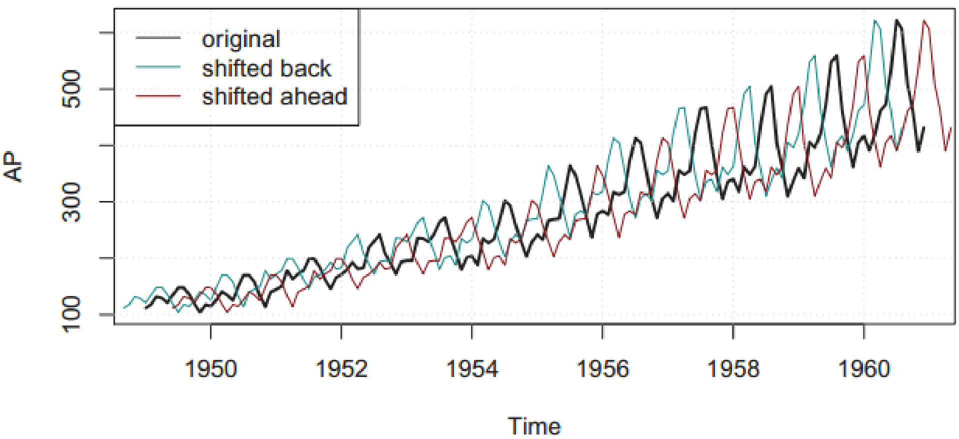


Figure 14: Time-shift transformation

5.1.2.2 Visualizations

Theory

Visualization Example

We have hourly measurements of several sensors, where we now focus on the air temperature. The time series is defined with a basic units of days starting at day 1 and frequency 24. The end()-command shows that the time series lasts 396 days and 8 hours, or in other words, has 9512 data points. The plot in Figure 15a shows the complete time series. The ylim option limits the temperature axis to non-negative values.

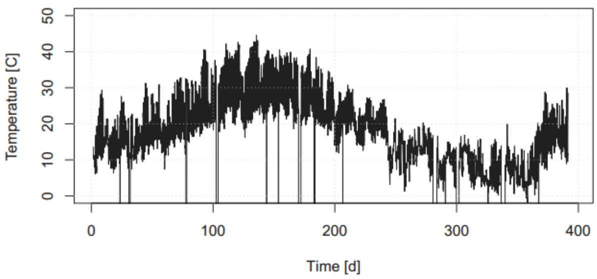
We focus on a period of 20 days to analyse the temperature behaviour in more detail. Figure 15b shows the data.

Figure 15c shows how data aggregation can be visualized with the boxplot-command. We are going to generate a boxplot for each full hour in one figure. To this end the cycle() function is very convenient: it returns for a given time series the positions in the cycle of each observation. In our example, a cycle is one day consisting of 24 hours. This means, that the first entry in the time series is at cycle position 18, i.e. measuredat 6 p.m. and that the 875-th measurement is at cycle position 4, which corresponds to 4 a.m. The subset of observations that share a common cycle are called cycle-subseries and will be used later for time series decomposition.

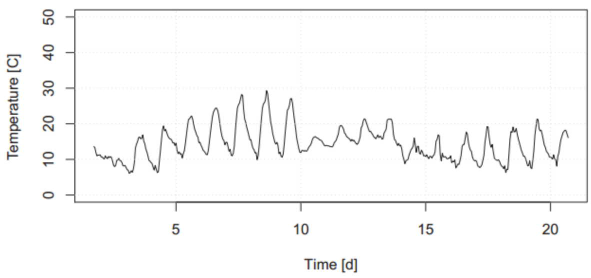
A useful graphical approach for visually inspecting correlations of consecutive observations are *lagged scatterplots*. They amount to produce scatterplots of the original time series values against a time-shiftet version, i.e. plotting the data pairs (x_i, x_{i-k}). This can be done in R by the lag.plot command. Figure 15d shows that the scatterplot with lag 1 shows a linear pattern which indicates a correlation. A lag of 10 hours results in a unspecific scatter plot.

Code Example

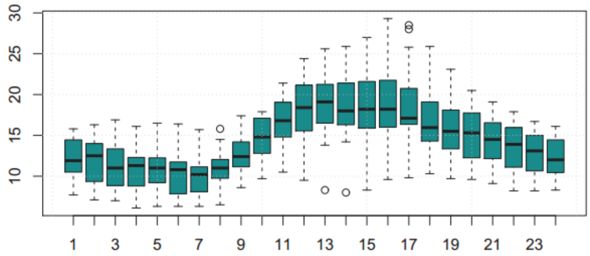
```r
:##Figure Air Temperature measurement: 9512 points
AirData = read.table("../Daten/AirQualityUCI/AirQualityUCI.csv",
sep=";", header=T,dec = ",")
AirTmp.ts = ts(AirData[,c(13)], start = c(1,18), frequency = 24)
end(AirTmp.ts)
## [1] 396 8

plot(AirTmp.ts, main = "Air Temperature measurement: full data set",
ylab="Temperature [C]", xlab="Time [d]", ylim = c(0,50))
grid()

##Figure Air Temperature measurement: 480 points
AirTmpWin.ts = window(AirTmp.ts, start = c(1, 18), end=c(20, 18))
plot(AirTmpWin.ts, main = "Air Temperature measurement: detail",
ylab="Temperature [C]", xlab="Time [d]", ylim = c(0,50))
grid()

##Figure Air temperature: Boxplot
 cycle(AirTmp.ts)[1]; cycle(AirTmp.ts)[875]
## [1] 18
## [1] 4

boxplot(AirTmpWin.ts ~ cycle(AirTmpWin.ts),
col = "darkcyan", main = "Air temperature")
grid()

##Figure lag.plot
lag.plot(AirTmpWin.ts, pch=20, main = "")
lag.plot(AirTmpWin.ts, pch=20, main = "", set.lags = 10)
```



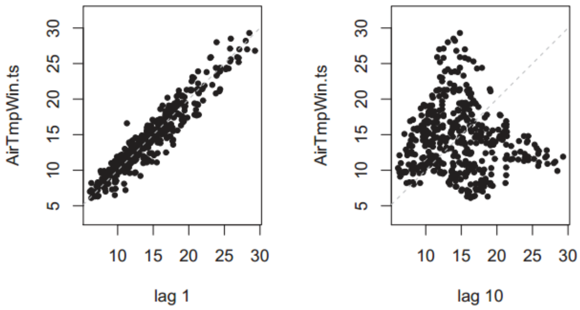
(a) Air temperature measurement: 9512 points



(b) Air temperature measurement: 480 points



(c) Air temperature: Boxplot



(d) lag.plot

Figure 15: Visualization

5.1.2.3 Decomposition of time series

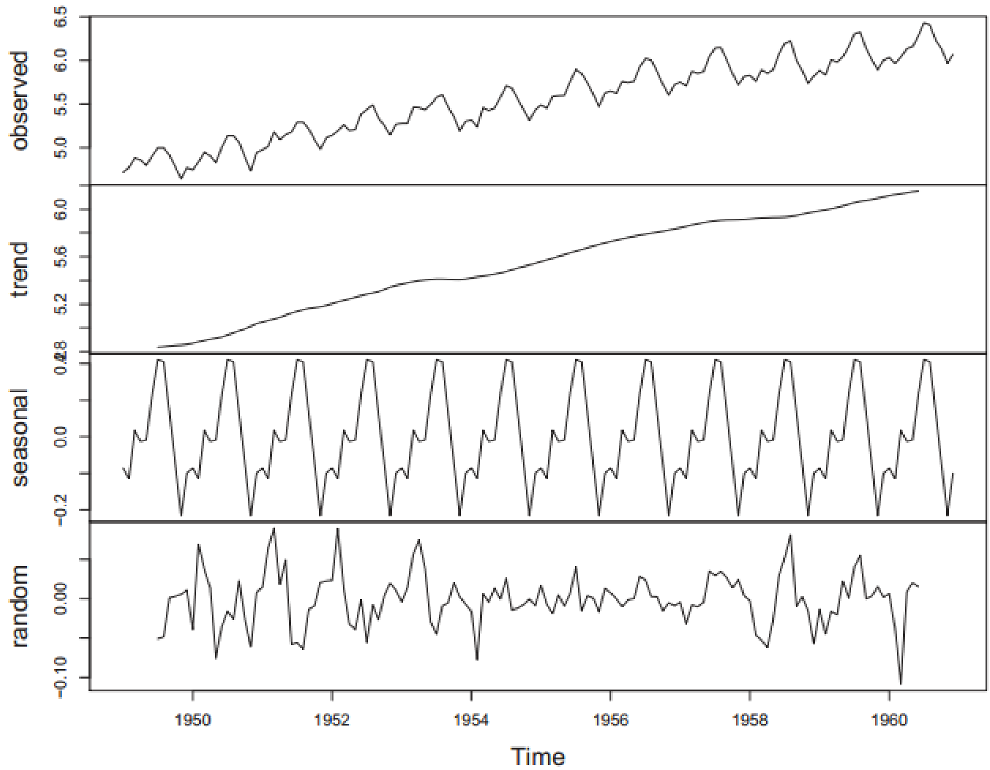


Figure 16: Decomposition of additive time series

Theory

Moving Average

A simple additive decomposition model is given by

$$x_k = m_k + s_k + z_k$$

where at time index k, x_k is the observed series, m_k is the trend, s_k is the seasonal effect, and z_k is an error term that is, in general, a sequence of *correlated* random variables with mean zero.

In the AirPassenger data the seasonal effects may increase as the trend increase. Thus a multiplicative model is more convenient

$$x_k = m_k \cdot s_k + y_k$$

If the noise is multiplicative as well, the logarithm of x_k is a linear model again

$$log(x_k) = log(m_k) + log(s_k) + log(y_k)$$

A simple method for estimating m_k and s_k is by means of the moving average filter. Assume that $\{x_1, x_2, ..., x_k\}$ is a time series and that $p \in \mathbf{N}$. The *moving averaage filter* of length p is defined as follows

- If p is odd, then $p = 2l + 1$ and the filtered sequence is defined by

$$g(x_i) = \frac{1}{p}(x_{i-l} + ... + x_i + ... + x_{i+l})$$

- If p is even, then $p = 2l$ and the filtered sequence is defined by

$$g(x_i) = \frac{1}{p}(\frac{1}{2}x_{i-l}+x_{i-l+1}+...+x_i+...+i+l-1+\frac{1}{2}x_{i+l})$$

The value p is referred to as *window width*.

Estimate seasonal additive effect: $\hat{s_k} = x_k - \hat{m_k}$

Remainder: $\hat{r_i} = x_i - \hat{m_i} - \hat{s_i}$

To diminsh the non-random parts the steps are repeated with the logarithm model. AirPassenger amounts to a multiplicative model.

Code Example

```r
##moving average can be done by filter function
##weights = c(0.5, rep(1,11), 0.5)/12 shows that for an even p=12 a window with length p+1 (odd) is constructed but counts the end points only by one half
##figure decomposition of additive time series (trend)
weights = c(0.5, rep(1,11), 0.5)/12
est.trend <- filter(AirPassengers, filter = weights, sides = 2)
plot(est.trend, lwd=2, ylim=c(100, 700))
lines(AirPassengers, col = "darkcyan")
legend("topleft", legend = c("data", "trend"), lty=1,
col = c("darkcyan", "black"))
grid()

##estimate seasonal effects
##figure decomposition of additive time series (seasonal)
est.season = AirPassengers - est.trend
cyc = factor(cycle(AirPassengers))
est.season.month = tapply(est.season, cyc, mean, na.rm=T)
est.season = est.season.month[cyc]
plot(est.season, type="l")
abline(h=0)

##remainder
est.rem = AirPassengers - est.trend - est.season
plot(as.vector(est.rem), type="l", ylab = "rem") #needs fix

##figure decomposition of additive time series (random)
##logarithm (amounts to multiplicative model)
log.data = log(AirPassengers)
#trend estimation of log data
est.trend.log <- filter(log.data, filter = weights, sides = 2)
# seasonality estimation for log data
est.season.log = log.data - est.trend.log
est.season.month = tapply(est.season.log, cyc, mean, na.rm=T)
est.season.log = est.season.month[cyc]
# remainder term estimation for log data
est.rem.log = log.data - est.trend.log - est.season.log
plot(as.vector(est.rem.log), type="l", ylab = "rem") #needs fix

## all in one with decompose function
##figure decomposition of additive time series
decomposed.data = decompose(log(AirPassengers))
plot(decomposed.data)
```

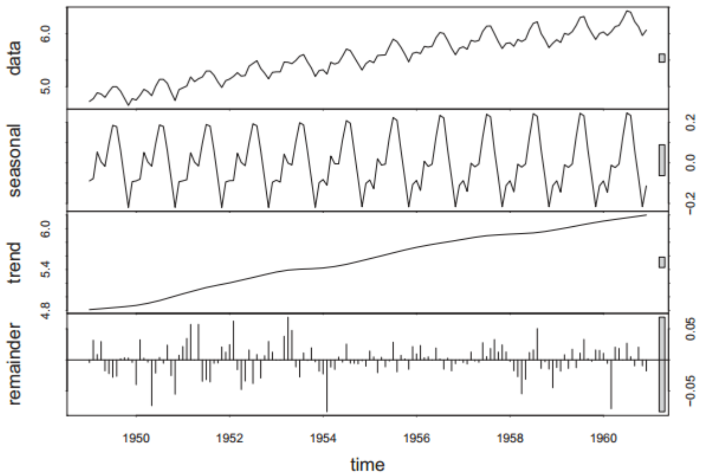

Theory	Code Example

Seasonal Decomposition of Time Series by Loss (STL)

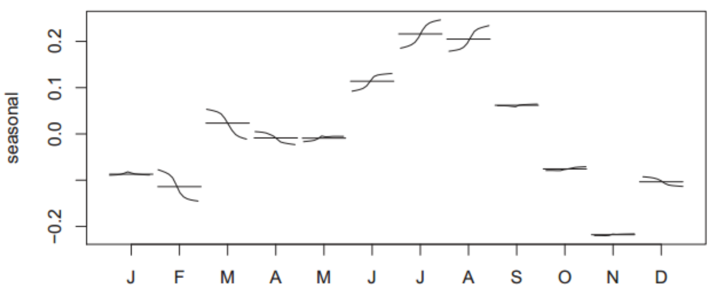
The decomposition method above is seldomly used because of several reasons. Hence the stl() function is used. Two mandatory parameters have to be passed to it:

1. x he time series to be decomposed
2. s.window he loess window size for the seasonality component. The larger the value the slower the change of seasonality in the data set over time.

```
#State of the art method for decomposing time
    series.
stl.fit = stl(log(AirPassengers), s.window = 10)
plot(stl.fit)

##Plots the cyle-subseries in a common plot
monthplot(stl.fit)
```



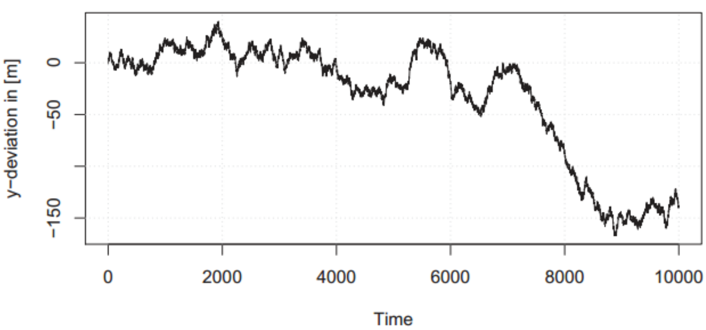
(a) STL decomposition of the log of AirPassengers data



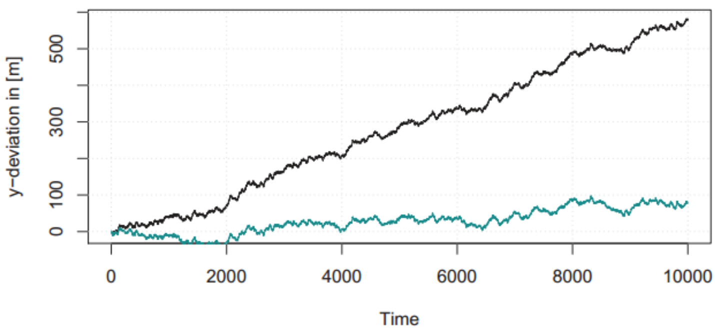
(b) Cycle-subseries of the STL decomposition

5.2 Mathematical Models for Time Series

5.2.1 Mathematical Concepts of Time Series



(a) Random walk process



(b) Random walk with drift (black)

Figure 18: Time series observation of a random walk process

Theory

Time series and discrete stochastic process Let T be a set of equidistant time points $T = \{t_1, t_2, ...\}$.

1. A *discrete stoachstic process* is a set of random variables $\{X_1, X_2, ...\}$. Each single random variable X_i has a univariate distribution function F_i and can be observed at time t_i.

2. A *time series* $\{x_1, x_2, ...\}$ is a realization of a discrete stochastic process $\{X_1, X_2, ...\}$. In other words, the value x_i is a realization of the random variable X_i measured at time t_i.

In other words, a time series is a concrete observation of values, and a stochastic process is a theoretical construct to model the underlyingn mechanism that generates the values. To illusrate this, the *random walk* is a simple example.

1. Choose n independent Bernoulli random variables $\{D_1, ..., D_n\}$ that take on the values 1 and -1 with equal probability, i.e. $p = 0.5$.

2. Define the random variables$\{X_i = D_1, ..., D_i\}$ for each $1 \leqslant i \leqslant n$. Then $\{X_1, X_2, ...\}$ is a discrete stochastic process modelling the random walk

   Figure 18a $\rightarrow X_i = X_{i-1} + D_i$, $X_0 = 0$

   Random walk with drift

   Figure 18b $\rightarrow Y_i = \delta + Y_{i-1} + D_i$, $Y_0 = 0$

Code Example

```r
##Random Walk (Figure Random walk process)
##The function cumsum() computes the cumulated sum of a given vector.
d = sample(c(-1,1), replace = T, size = 10000)
x = ts(cumsum(d))
plot(x, main="Random Walk", ylab="y-deviation in [m]")
grid()

##Random Walk with drift (Figure Random walk with drift (black)
set.seed(12); d = sample(c(-1,1), replace = T, size = 10000)
delta = 5e-2
y = rep(0,1,10000)
for (i in 2:10000){
y[i] = delta + y[i-1] + d[i]
}
plot(ts(y), main="Random Walk with drift", ylab="y-deviation in [m]")
lines(cumsum(d), col="darkcyan")
grid()
```

Theory

White noise processes

A white noise process consists of i.i.r. variables $\{W_1, W_2, ...\}$ where W_i has mean 0 and variance σ^2.

If we apply a sliding window filter to the white noise process $\{W_1, W_2, ...\}$ we obtain a *moving average process*. With a window length of 3 we obtain

$$V_i = \frac{1}{3}(W_{i-1} + W_i + W_{i+1})$$

.

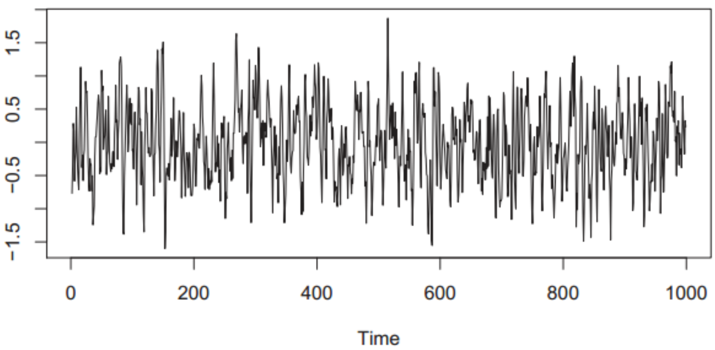
Considering again the white noise process, computing the following sequence

$$X_i = 1.5X_{i-1} - 0.9X_{i-2} + W_i$$

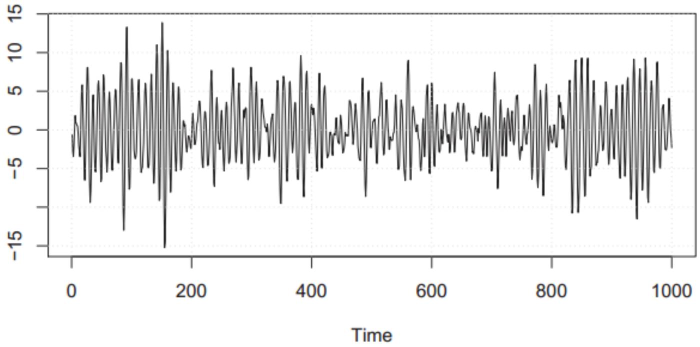
The value of the process at time instance i is modelled as linear combination of the past two values plus some random component. This process is called *autoregressive*.

Code Example

```r
##White noise process
w = ts(rnorm(1000))
plot(w, main="White noise", ylab="value")
grid()

##Moving average process of white noise
window = c(1,1,1)/3;
v = filter(w, sides=2, window)
plot(v,main="MA process", ylab="")

##Autoregressive process of white noise
##filter function can the parameter method set to recursive to compute the autoregressive model
ar = filter(w, filter = c(1.5,-0.9), method="recursive")
plot(ar, main="AR(2) process", ylab="")
grid()
```



(a) Moving average process of white noise



(b) Autoregressive process of white noise

5.2.2 Measures of Dependence

Autocovariance and autocorrelation

Let $\{X_1, X_2, ...\}$ be a discrete stochastic process

1. The *autocovariance* γ_X is defined as

$$\gamma_X(i,j) = Cov(X_i, X_j) = E[(X_i - \mu(i))(X_j - \mu(j))]$$

2. The *autocorrelation* ρ_X is defined as

$$\rho_X(i,j) = \frac{\gamma_X(i,j)}{\sqrt{\gamma_X(i,i)\gamma_X(j,j)}}$$

Note that the autocovariance and the autocorrelatiion are symmetric, i.e. $\gamma(i,j) = \gamma(j,i)$. The autocovariance measures the *linear dependence* of two points on the same process observed at different times. For $i = j$ the autocovariance reduces to the variance of X_i.

The autocorrelation hence gives a rough measure how well the series at time i can be forecast by the value at time j.

It is important to consider the properties of the process model. For example a white noise process has the autocovariance function

$$\gamma(i,j) = \begin{cases} 0 & i \neq j \\ \sigma^2 & i = j \end{cases}$$

Accordingly the autocorrelation is 1 if $i = j$ and 0 else.

Example

The autocovariance of the three point moving average process is computed as follows.

$$\gamma(i,j) = Cov(X_i, X_j) =$$
$$Cov\left(\tfrac{1}{3}(W_{i-1} + W_i + W_{i+1}), \tfrac{1}{3}(W_{j-1} + W_j + W_{j+1})\right)$$

When $i = j$

$$Cov(X_i, X_i) = \tfrac{1}{9}Cov(W_{i-1}+W_i+W_{i+1}, W_{i-1}+W_i+W_{i+1})$$
$$= \tfrac{1}{9}(Cov(W_{i-1}, W_{i-1}) + Cov(W_i, W_i) + Cov(W_{i+1}, W_{i+1}))$$
$$= \tfrac{3\sigma^2}{9}$$

This follows from the fact, that W_i, W_{i-1} and W_{i+1} are *mutually uncorrelated*. For $i + 1 = j$ we find

$$Cov(X_i, X_{i+1}) =$$
$$\tfrac{1}{9}Cov(W_{i-1} + W_i + W_{i+1}, W_i + W_{i+1} + W_{i+2})$$
$$= \tfrac{1}{9}(Cov(W_i, W_i) + Cov(W_{i+1}, W_{i+1})) = \tfrac{2\sigma^2}{9}$$

Summarized:

$$\gamma(i,j) = \begin{cases} \frac{3\sigma^2}{9} & i = j \\ \frac{2\sigma^2}{9} & |i-j| = 1 \\ \frac{\sigma^2}{9} & |i-j| = 2 \\ 0 & else \end{cases}$$

In this example the autocovariance only depends on the distance of the observations, but not on their value. Hence the autocorrelation $\rho(i,j) = \gamma(i,j)/\sqrt{\gamma(i,i)\gamma(j,j)} = \gamma(i,j)/\gamma(i,i)$ This gives

$$\rho(i,j) = \begin{cases} 1 & i = j \\ \frac{2}{3} & |i-j| = 1 \\ \frac{1}{3} & |i-j| = 2 \\ 0 & else \end{cases}$$

Mean sequence (or mean function)

Example

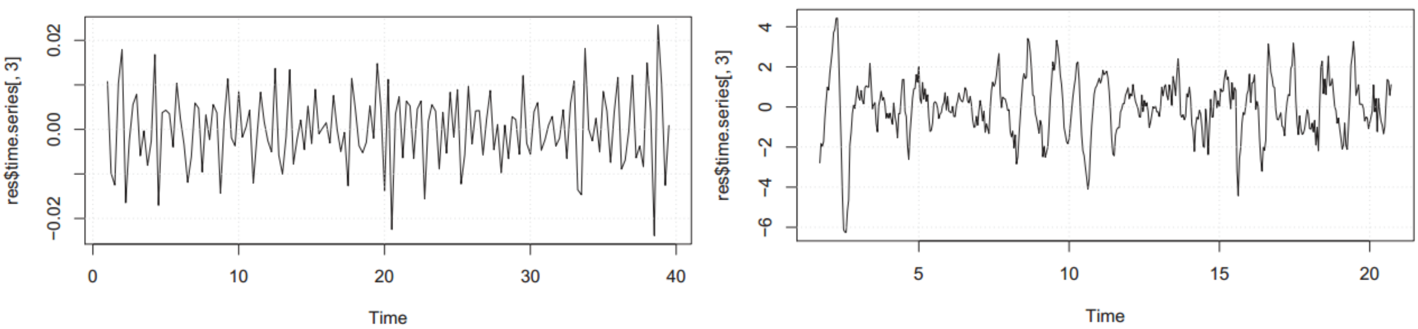
The mean sequence $\{\mu(1), \mu(2), ...\}$ of a discrete stochastic process $\{X_1, X_2, ...\}$ is defined as the sequence of the means: $$\mu(i) = E[X_i].$$	If X_i is a random walk with drift, i.e. X_0 and $X_i = \delta + X_{i-1} + W_i$ then we find that $$E[X_1] = \delta + E[X_0] + E[W_1] = \delta$$ $$E[X_2] = \delta + E[X_1] + E[W_2] = 2\delta$$ $$E[X_3] = \delta + E[X_2] + E[W_3] = 3\delta$$ *etc.* This means that $\mu(i) = i\delta$.

5.2.3 Stationarity

Stationarity is a concept of regularity that allows us to infer information from a single time series.

Strict stationarity	**Weak stationarity**
A discrete stochastic process is called strictly stationary if for each finite collection $\{X_{i_1}, ..., X_{i_n}\}$ and each lag $h \in \mathbb{Z}$ the shifted collection $$\{X_{i_1+h}, ..., X_{i_n+h}\}$$ has the **same distribution**. Put differently $$P(X_{i_1} \leqslant c_1, ..., X_{i_n} \leqslant c_n) = P(X_{i_1+h} \leqslant c_1, ..., X_{i_n+h} \leqslant c_n)$$ This means that the probabilistic character of the process does not change over time. For many applications is strict stationarity a too strong assumption and hard to assess from a single data set.	A discrete stochastic process X_i is called *weakly stationary* if 1. the mean sequence $\mu_X(i)$ is constant and does not depend on the time index i and 2. the autocovariance sequence $\gamma_X(i, j)$ depends on i and j only through their difference $\|i - j\|$. Each strictly stationary time series is also weakly stationary. But the opposite is in general not true. Since the autocovariance/-correlation for a (weakly) stationary process only depends on the **time *lag*** $h = i - j$ one considers these sequences as a function of h alone: $$\gamma(h) = \gamma(i, i + h)$$ $$\rho(h) = \rho(i, i + h)$$ See also the example of the moving average process above.

5.2.3.1 Testing Stationarity



(a) Remainder: Australian Electricity (b) Remainder: Air Temperature

Figure 20: Testing stationarity

Theory	Code Example

Theory

In practice we have to *test* or at least *guess* whether or not the underlying process of a given time series is stationary. There are several statistical hypothesis test for stationarity.

The first and simplest test one can apply to check for stationarity is to plot the time series and look for evidence of trend in mean, variance, autocorrelation and seasonality. If such patterns are present then these are signs of non-stationarity and different mechanism exit to turn the series into a stationary one, as data transformation or time series decomposition.

A further possibility is to compute the mean and autocovariance sequences seperately for several windows and compare their behaviour. When there is a dramatic change, then the hypotheses of stationarity can be rejected.

The series in Figure 20a does not exhibit any seasonal patterns, has a constant mean and roughly constant variance. From visual inspection one would conclude that the underlying process is stationary.

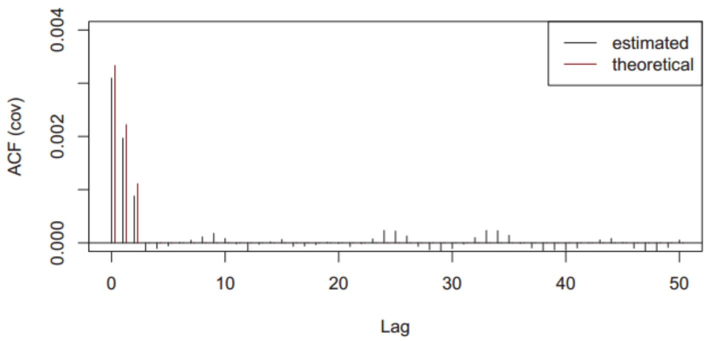
The series in Figure 20b still exhibits some seasonality, such that stationarity of the underlying process seems unlikely.

Code Example

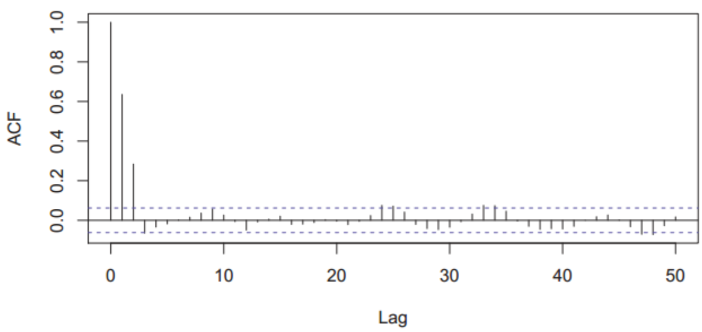
```r
##Figuer: Remainder AustralianElectricity
AusEl = read.table("../Daten/AustralianElectricity.csv",
sep=";", header=T,dec = ",")
AusEl.ts = ts(AusEl[,2], frequency = 4)
res = stl(log(AusEl.ts), s.window = 16)
# the time series component contains 3 series where the
# third constitutes the remainder sequence
plot(res$time.series[,3], main="Remainder: Australian Electricity")
grid()

##Figure: Remainder AirTemperature
AirData = read.table("../Daten/AirQualityUCI/AirQualityUCI.csv",
sep=";", header=T,dec = ",")
AirTmp.ts = ts(AirData[,c(13)], start = c(1,18), frequency = 24)
AirTmpWin.ts = window(AirTmp.ts, start = c(1, 18), end=c(20, 18))
res = stl(AirTmpWin.ts, s.window = 10)
plot(res$time.series[,3], main="Remainder: Air Temperature")
grid()
```

5.2.4 Estimation of Correlation

In this section it is assumed that the time series $\{x_1, x_2, ..., x_n\}$ is a realization of a weakly stationary process $\{X_1, X_2, ..., X_n\}$.



(a) Sample autocovariance function of a MA(5) process (black) and the theoretical values (red)



(b) Sample autocorrelation function of MA(5) process

Figure 21: Estimation of Correlation

Theory

Estimation of the mean sequence

Due to stationarity we know that the mean sequence $\mu(k) = \mu$ is constant. A canonical estimator hence is

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

In the present situation the observations are dependent. Hence the standard error $\sigma/\sqrt{n}$ is not applicable. We have to recompute the standard error.

$$Var(\hat{\mu}) = \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma(h)$$

Estimation of the autocovariance

The theoretical autocovariance function is estimated by the sample autocovariance sequence which we define as

1. The *sample autocovariance* is defined by

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n-h}(x_{i+h} - \hat{x})(x_i - \bar{x})$$

with $\hat{\gamma}(h) = \hat{\gamma}(-h)$ for $h = 0, 1, ..., n-1$

2. The *sample autocorrelation* is defined by

$$\hat{\rho} = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

Note that the sum runs over a restricted range of $n-h$ points but we nevertheless normalize by n rather than $n-h$. Neither choice results in an unbiased estimator.

Code Example

```
##Computing the estimators for simulated data. First we simulate a realization of a moving average
# create white noise
set.seed(123)
w = rnorm(1000, mean = 2, sd = 0.1)
# filter and define time series
# rep(1,3) = [1,1,1]
MA = ts(filter(w, filter = rep(1,3)/3, sides = 2))
##In the filter command, missing values NA at the boundaries are generated. The function na.omit omits these cases where NAs are involved
MA= na.omit(MA)

##The autocovariance and -correlation can be computed with the acf function. The tyoe parameter chooses either correlation (default) or covariance. lag.max gives the maxiamal lag upto which the autocovariance is computed.
##Figure: sample autocorrelation
ac = acf(MA, type="covariance", lag.max = 50, ylim=c(0,0.004))
# theoretical autocovariance
sigma = 0.1
cv.true = rep(0, 1, length(ac$acf))
cv.true[1] = 3*sigma^2/9;
cv.true[2] = 2*sigma^2/9;
cv.true[3] = sigma^2/9;
points(ac$lag+0.3, cv.true, pch=18, col="darkred", type="h")
legend("topright", legend=c("estimated", "theoretical"),
lty=1, col = c("black", "darkred"))

##computing the correlogram Figure: sample autocorrelation
acf(MA, lag.max = 50)
## The two-sigma confidence bands are drawn automatically
```

5.3 Forecasting ime Series

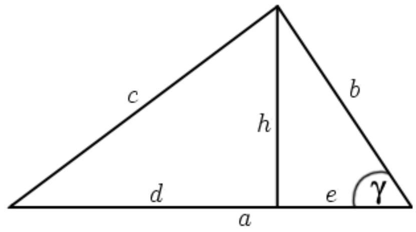
TO DO: Chapter 15

6 Idiotenseite

6.1 Dreiecksformeln

Cosinussatz

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos\gamma$$



Sinussatz

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r = \frac{u}{\pi}$$

Pythagoras beim Sinus

$$\sin^2(b) + \cos^2(b) = 1 \qquad \tan(b) = \frac{\sin(b)}{\cos(b)}$$

$\sin\beta = \frac{b}{a} = \frac{\text{Gegenkathete}}{\text{Hypotenuse}}$
$\cos\beta = \frac{c}{a} = \frac{\text{Ankathete}}{\text{Hypotenuse}}$

$\tan\beta = \frac{c}{b} = \frac{\text{Gegenkathete}}{\text{Ankathete}}$
$\cot\beta = \frac{c}{b} = \frac{\text{Ankathete}}{\text{Gegenkathete}}$

6.2 Funktionswerte für Winkelargumente

deg	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

deg	rad	sin	cos
90°	$\frac{\pi}{2}$	1	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

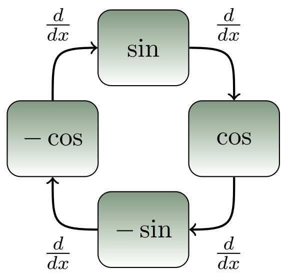
deg	rad	sin	cos
180°	π	0	-1
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

deg	rad	sin	cos
270°	$\frac{3\pi}{2}$	-1	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

6.3 Periodizität

$\cos(a + k \cdot 2\pi) = \cos(a) \qquad \sin(a + k \cdot 2\pi) = \sin(a) \qquad (k \in \mathbb{Z})$

6.5 Ableitungen



6.4 Quadrantenbeziehungen

$\sin(-a) = -\sin(a)$
$\sin(\pi - a) = \sin(a)$
$\sin(\pi + a) = -\sin(a)$
$\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right) = \cos(a)$

$\cos(-a) = \cos(a)$
$\cos(\pi - a) = -\cos(a)$
$\cos(\pi + a) = -\cos(a)$
$\cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\frac{\pi}{2} + a\right) = \sin(a)$

6.6 Additionstheoreme

$\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b)$
$\cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$
$\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)}$

6.7 Doppel- und Halbwinkel

$\sin(2a) = 2\sin(a)\cos(a)$
$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\cos^2\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{2} \qquad \sin^2\left(\dfrac{a}{2}\right) = \frac{1-\cos(a)}{2}$

6.8 Produkte

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
$\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

6.9 Euler-Formeln

$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \qquad \cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$
$e^{x+jy} = e^x \cdot e^{jy} = e^x \cdot (\cos(y) + j\sin(y))$
$e^{j\pi} = e^{-j\pi} = -1$

6.10 Summe und Differenz

$\sin(a)+\sin(b)=2\cdot\sin\left(\frac{a+b}{2}\right)\cdot\cos\left(\frac{a-b}{2}\right)$
$\sin(a)-\sin(b)=2\cdot\sin\left(\frac{a-b}{2}\right)\cdot\cos\left(\frac{a+b}{2}\right)$

$\cos(a)+\cos(b)=2\cdot\cos\left(\frac{a+b}{2}\right)\cdot\cos\left(\frac{a-b}{2}\right)$
$\cos(a)-\cos(b)=-2\cdot\sin\left(\frac{a+b}{2}\right)\cdot\sin\left(\frac{a-b}{2}\right)$
$\tan(a)\pm\tan(b)=\dfrac{\sin(a\pm b)}{\cos(a)\cos(b)}$

6.12 Ableitungen elementarer Funktionen S436

Funktion	Ableitung	Funktion	Ableitung
C (Konstante)	0	$\sec x$	$\frac{\sin x}{\cos^2 x}$
x	1	$\sec^{-1} x$	$\frac{-\cos x}{\sin^2 x}$
$x^n \ (n\in\mathbb{R})$	nx^{n-1}	$\arcsin x \quad (\lvert x\rvert<1)$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\arccos x \quad (\lvert x\rvert<1)$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$	$\arctan x$	$\frac{1}{1+x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\mathrm{arccot}\ x$	$-\frac{1}{1+x^2}$
$\sqrt[n]{x} \quad (n\in\mathbb{R}, n\neq 0, x>0)$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$	$\mathrm{arcsec}\ x$	$\frac{1}{x\sqrt{x^2-1}}$
e^x	e^x	$\mathrm{arcossec}\ x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\mathrm{e}^{bx} \quad (b\in\mathbb{R})$	$b\mathrm{e}^{bx}$	$\sinh x$	$\cosh x$
$a^x \quad (a>0)$	$a^x \ln a$	$\cosh x$	$\sinh x$
$a^{bx} \quad (b\in\mathbb{R}, a>0)$	$ba^{bx}\ln a$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\ln x$	$\frac{1}{x}$	$\coth x \quad (x\neq 0)$	$-\frac{1}{\sinh^2 x}$
$\log_a x \quad (a>0, a\neq 1, x>0)$	$\frac{1}{x}\log_a \mathrm{e} = \frac{1}{x\ln a}$	$\mathrm{Arsinh}\ x$	$\frac{1}{\sqrt{1+x^2}}$
$\lg x \quad (x>0)$	$\frac{1}{x}\lg \mathrm{e} \approx \frac{0.4343}{x}$	$\mathrm{Arcosh}\ x \quad (x>1)$	$\frac{1}{\sqrt{x^2-1}}$
$\sin x$	$\cos x$	$\mathrm{Artanh}\ x \quad (\lvert x\rvert<1)$	$\frac{1}{1-x^2}$
$\cos x$	$-\sin x$	$\mathrm{Arcoth}\ x \quad (\lvert x\rvert>1)$	$-\frac{1}{x^2-1}$
$\tan x \quad (x\neq(2k+1)\frac{\pi}{2}, k\in\mathbb{Z})$	$\frac{1}{\cos^2 x}=\sec^2 x$	$[f(x)]^n \quad (n\in\mathbb{R})$	$n[f(x)]^{n-1}f'(x)$
$\cot x \quad (x\neq k\pi, k\in\mathbb{Z})$	$\frac{-1}{\sin^2 x}=-cosec^2 x$	$\ln f(x) \quad (f(x)>0)$	$\frac{f'(x)}{f(x)}$

6.11 Einige unbestimmte Integrale S1074

$\int dx = x + C$	$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$
$\int \frac{1}{x}dx = \ln\lvert x\rvert + C,\ x \neq 0$	$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C,\ a\epsilon\mathbb{R}^+\backslash\{1\}$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C,\ x \neq k\pi$ mit $k\epsilon\mathbb{Z}$
$\int \frac{dx}{\cos^2 x} = \tan x + C,\ x \neq \frac{\pi}{2} + k\pi$ mit$k\epsilon\mathbb{Z}$	$\int \sinh x dx = \cosh x + C$
$\int \cosh x dx = \sinh x + C$	$\int \frac{dx}{\sinh^2 x} = -\coth x + C,\ x \neq 0$
$\int \frac{dx}{\cosh^2 x} = \tanh x + C$	$\int \frac{dx}{ax+b} = \frac{1}{a}\ln\lvert ax+b\rvert + C,\ a \neq 0, x \neq -\frac{b}{a}$
$\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\arctan\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0$	$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left\lvert\frac{ax-b}{ax+b}\right\rvert + C,\ a \neq 0,\ b \neq 0,\ x \neq \frac{b}{a},\ x \neq -\frac{b}{a}$
$\int \sqrt{a^2x^2+b^2}dx = \frac{x}{2}\sqrt{a^2x^2+b^2} + \frac{b^2}{2a}\ln\left(ax + \sqrt{a^2x^2+b^2}\right) + C,\ a \neq 0,\ b \neq 0$	$\int \sqrt{a^2x^2-b^2}dx = \frac{x}{2}\sqrt{a^2x^2-b^2} - \frac{b^2}{2a}\ln\left\lvert ax + \sqrt{a^2x^2-b^2}\right\rvert + C,\ a \neq 0,\ b \neq 0, a^2x^2 \geqq b^2$
$\int \sqrt{b^2-a^2x^2}dx = \frac{x}{2}\sqrt{b^2-a^2x^2} + \frac{b^2}{2a}\arcsin\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 \leqq b^2$	$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \frac{1}{a}\ln(ax + \sqrt{a^2x^2+b^2}) + C,\ a \neq 0,\ b \neq 0$
$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \frac{1}{a}\ln\left\lvert ax + \sqrt{a^2x^2-b^2}\right\rvert + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 > b^2$	$\int \frac{dx}{\sqrt{b^2-a^2x^2}} = \frac{1}{a}\arcsin\frac{a}{b}x + C,\ a \neq 0,\ b \neq 0,\ a^2x^2 < b^2$
Die Integrale $\int\frac{dx}{X}, \int\sqrt{X}dx, \int\frac{dx}{\sqrt{X}}$ mit $X = ax^2+2bx+c,\ a \neq 0$ werden durch die Umformung $X = a(x+\frac{b}{a})^2 + (c-\frac{b^2}{a})$ und die Substitution $t = x + \frac{b}{a}$ in die oberen 4 Zeilen transformiert.	$\int \frac{xdx}{X} = \frac{1}{2a}\ln\lvert X\rvert - \frac{b}{a}\int\frac{dx}{X},\ a \neq 0,\ X = ax^2+2bx+c$
$\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a}\cdot\sin 2ax + C,\ a \neq 0$	$\int \cos^2 ax dx = \frac{x}{2} + \frac{1}{4a}\cdot\sin 2ax + C,\ a \neq 0$
$\int \sin^n ax dx = -\frac{sin^{n-1}ax\cdot\cos ax}{na} + \frac{n-1}{n}\int\sin^{n-2}ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int \cos^n ax dx = \frac{\cos^{n-1}ax\cdot\sin ax}{na} + \frac{n-1}{n}\int\cos^{n-2}ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln\left\lvert\tan\frac{ax}{2}\right\rvert + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$	$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left\lvert\tan(\frac{ax}{2} + \frac{\pi}{4})\right\rvert + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$ mit $k\epsilon\mathbb{Z}$
$\int \tan ax dx = -\frac{1}{a}\ln\lvert\cos ax\rvert + C,\ a \neq 0,\ x \neq \frac{\pi}{2a} + k\frac{\pi}{a}$mit $k\epsilon\mathbb{Z}$	$\int \cot ax dx = \frac{1}{a}\ln\lvert\sin ax\rvert + C,\ a \neq 0,\ x \neq k\frac{\pi}{a}mitk\epsilon\mathbb{Z}$
$\int x^n \sin ax dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int x^n \cos ax dx = \frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax dx,\ n\epsilon\mathbb{N},\ a \neq 0$
$\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx,\ n\epsilon\mathbb{N},\ a \neq 0$	$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C,\ a \neq 0,\ b \neq 0$
$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C,\ a \neq 0,\ b \neq 0$	$\int \ln x dx = x(\ln x - 1) + C,\ x\epsilon\mathbb{R}^+$
$\int x^\alpha\cdot\ln x dx = \frac{x^{\alpha+1}}{(\alpha+1)^2}[(\alpha+1)\ln x - 1] + C,\ x\epsilon\mathbb{R}^+,\ \alpha\epsilon\mathbb{R}\backslash\{-1\}$	