State space description of the nonlinear dynamical system. Specific nonlinear properties and typical nonlinear phenomena. Nonlinear control techniques outlook

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1. State space description of the nonlinear dynamical system. Specific nonlinear properties and typical nonlinear phenomena. Nonlinear control techniques outlook.

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6. Control design using structural methods. Exact feedback linearization. Zero dynamics and minimum phase property.

7. Structure of single-input single-output systems. Exact feedback linearization, relative degree, partial and input-output linearization, zero dynamics computation and minimum phase property test. Examples.

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9. Multi-input multi-output systems. Examples, dynamical feedback, example of its application in the case study of the planar vertical take-off and landing plane.

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NONLINEAR DYNAMICAL SYSTEM

State space model

We consider dynamical systems (DS) modeled by a finite number of the first-order ordinary differential equations (ODE):

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_m)
\end{aligned}$$

$$\dot{x}_i := \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}, \ \ i = 1, \ldots, n,$$

$$n, m \in \mathbb{Z}$$

The **time variable** t, the **input variables** $u_1, \ldots, u_m$, the **state variables** ("memory" of DS past) $x_1, \ldots, x_n$.

Compact form of equations using vector notation:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}$$

$$\dot{x} = f(t, x, u), \quad x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m$$

NONLINEAR DYNAMICAL SYSTEM

State space model

Often, the **output** of the dynamical system is considered:

$$y = h(t, x, u), \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}, \quad q \in \mathbb{Z}, \quad y \in \mathbb{R}^q$$

So, the nonlinear dynamical system with the input $u \in \mathbb{R}^m$, the state $x \in \mathbb{R}^n$ and the output $y \in \mathbb{R}^q$ is given as

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned}$$

Also:
controlled dynamical system, **forced dynamical system**, ...

Unforced state equation:

$$\dot{x} = f(t, x)$$

Input u either not specified, or:

$$\dot{x} = f(t, x) \quad \Longleftarrow \quad \dot{x} = \tilde{f}(t, x, u) \leftarrow \ u = \gamma(t),\ u = \gamma(x),\ u = \gamma(t, x).$$

Open loop control $u = \gamma(t)$
closed loop control (feedback) $u = \gamma(x)$
or $u = \gamma(t, x)$ (the so-called time-varying feedback).

Linear systems

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

Consider domain (open connected set) $\mathcal{D} \subset \mathbb{R}^n$, interval $J \subset \mathbb{R}$ and

$$\dot{x} = f(t, x), \quad x \in \mathcal{D}, \; t \in J, \quad \text{where}$$

(1) $f(t, x)$ is piecewise continuous in t;
(2) $f(t, x)$ is locally Lipschitz in x.

More specifically:

(1) For a fixed $x \in \mathcal{D}$, mapping $f(\cdot, x) : \mathbb{R} \mapsto \mathbb{R}^n$ is continuous at any $t \in J$ except, possibly, finite number of finite-jump discontinuities.

Motivation:

$f(t, x) = \tilde{f}(t, x, u)$, where $u = \gamma(t)$ experiences step changes with time.

To specify **(2)**, introduce the following:

Definition. Given $r > 0$, ball-like neighborhood $\mathcal{N}(x_0, r)$ of $x_0 \in \mathbb{R}^n$ is:
$\mathcal{N}(x_0, r) := \{x \in \mathbb{R}^n | \, \|x - x_0\| < r\}$; $\|x\| := \sqrt{x^\top x} = \sqrt{x_1^2 + \cdots + x_n^2}$.

(2) $f(t,x)$ **is called locally Lipschitz at** x_0 **if:**

$\exists r>0, L>0$ such that $\forall t \in J$ and $\forall x, \overline{x} \in \mathcal{N}(x_0, r)$ it holds

$$\|f(t,x)-f(t,\overline{x})\| \leq L\|x-\overline{x}\|.$$

$f(t,x)$ **is called locally Lipschitz on a domain** $\mathcal{D}$**, if:** it is locally Lipschitz at every point $x_0 \in \mathcal{D}$.

$f(t,x)$ **is called Lipschitz on a set** $W \subset \mathbb{R}^n$ **if:** there exists $L>0$ such that $\forall t \in J$ and $\forall x, \overline{x} \in W$

$$\|f(t,x)-f(t,\overline{x})\| \leq L\|x-\overline{x}\|.$$

Note: the same Lipschitz constant everywhere on set W**!**

Globally Lipschitz = Lipschitz on $\mathbb{R}^n$.

"Lipschitz on a domain $\mathcal{D}$" $\neq$ "Locally Lipschitz on a domain $\mathcal{D}$"!!!

Notations: $\frac{\partial f}{\partial x}$ is the so-called **Jacobian matrix** representing derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \ \overline{L} = \begin{bmatrix} \overline{L}_{11} & \cdots & \overline{L}_{1n} \\ \vdots & \vdots & \vdots \\ \overline{L}_{n1} & \cdots & \overline{L}_{nn} \end{bmatrix}, \ \|\overline{L}\| = \sqrt{\sum_{i,j=1}^{n} \overline{L}_{ij}^2},$$

Lemma LIP. Let for every fixed $t \in J \subset \mathbb{R}$ the mapping $f(t,x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable with respect to x on the **convex** domain $\mathcal{D} \subset \mathbb{R}^n$ and let for every fixed $x \in \mathcal{D}$ the function $f(t,x) : \mathbb{R} \mapsto \mathbb{R}^n$ is piecewise continuous on $J \subset \mathbb{R}$, where J is interval. Then:

- $f(t,x)$ is locally Lipschitz at every $x_0 \in \mathcal{D}$,
- $f(t,x)$ is Lipschitz on $\mathcal{D}$ if and only if there exists $\overline{L}_{ij} > 0$ such that $|\partial f_i/\partial x_j(t,x)| \leq \overline{L}_{ij}, \forall i,j = 1,\ldots,n, t \in J, x \in \mathcal{D}$. Moreover, the respective global Lipschitz constant is less or equal to $\sqrt{\sum_{i,j=1}^{n} \overline{L}_{ij}^2}$.

Proof. As $\mathcal{D}$ is convex, the segment between x and $\overline{x}$ is entirely contained inside $\mathcal{D}$ and the first order Taylor expansion can be used:

$$f(t,x) - f(t,\overline{x}) = \frac{\partial f}{\partial x}(t, \overline{x} + \Theta(x - \overline{x}))(x - \overline{x}),\ \Theta \in [0,1].$$

$$\|f(t,x) - f(t,\overline{x})\| = \|\frac{\partial f}{\partial x}(t, \overline{x} + \Theta(x - \overline{x}))(x - \overline{x})\|,\ \Theta \in [0,1].$$

$$\|f(t,x) - f(t,\overline{x})\|^2 = \left[\|\frac{\partial f}{\partial x}(t, \overline{x} + \Theta(x - \overline{x}))(x - \overline{x})\|\right]^2,\ \Theta \in [0,1].$$

$$\sum_{i=1}^{n}(f_i(t,x) - f_i(t,\overline{x}))^2 =$$

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}(t, \overline{x} + \Theta(x - \overline{x}))(x_j - \overline{x}_j)\right)^2 \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\overline{L}_{ij}||x_j - \overline{x}_j|\right)^2.$$

$$\|f(t,x) - f(t,\overline{x})\|^2 \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\overline{L}_{ij}||x_j - \overline{x}_j| \right)^2$$

Cauchy–Bunyakovsky–Schwarz inequality: $\forall a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$

$$\sum_{j=1}^{n} a_j b_j \leq \sqrt{\sum_{j=1}^{n} a_j^2} \sqrt{\sum_{j=1}^{n} b_j^2} \Leftrightarrow a^\top b \leq \|a\|\|b\|, \;\; a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\overline{L}_{ij}||x_j - \overline{x}_j| \right)^2 \leq \sum_{i=1}^{n} \left(\sqrt{\sum_{j=1}^{n} \overline{L}_{ij}^2} \sqrt{\sum_{j=1}^{n} |x_j - \overline{x}_j|^2} \right)^2 = \|x - \overline{x}\|^2 \|\overline{L}\|^2,$$

$$\Longrightarrow \quad \|f(t,x) - f(t,\overline{x})\| \leq \|\overline{L}\|\|x - \overline{x}\| \qquad \square.$$

NONLINEAR DYNAMICAL SYSTEM

Unforced equation properties - how to check all these Lipschitz properties?

Clearly, if $f(t,x)$ is differentiable with respect to $x \in \overline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ is bounded convex set, then $f(t,x)$ is Lipschitz on $\overline{\mathcal{D}}$.

By the Lemma LIP, linear and constant mappings are globally Lipschitz and linear combination preserves global Lipschitz property.

The Lemma LIP can be used only for **continuously differentiable** wrt x mapping $f(t,x)$. **Non-differentiable mappings may be Lipschitz, even globally!**

Typical examples of non-differentiable, but Lipschitz mappings, are those containing saturations, absolute value, *etc.* other piecewise linear mappings.

Sinuses and cosines, as well as their inverses are Lipschitz even on unbounded domains.

The estimate of the Lipschitz constant obtained during the proof of Lemma LIP is too conservative, in exercises it will be shown how to obtain better ones.

Lemma LEU. Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x at x_0 for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state unforced equation $\dot{x} = f(t,x)$ with the **initial condition** $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Remark: Lemma LEU is sufficient condition only. Examples of systems with $f(t,x)$ that are not Lipschitz even locally, but solutions exists and are unique, will be studied in exercises. But systems violating local Lipschitz property are candidates for non-uniqueness, nevertheless, further analysis (*e.g.* solving the equation) is needed.

Example: $\dot{x} = x^{1/3}$ with the initial condition $x(0) = 0$ has at least two solutions: (I) $x(t) \equiv 0$ and (II) $x(t) = (2t/3)^{3/2}$.

Peano theorem: For continuous $f(t,x)$ at least one solution exists, detailed and precise formulation is skipped.

Lemma LEU = "Lemma on Local Existence and Uniqueness".

Why "local"?

Example: For the system $\dot{x} = -x^2$, the function $f(x) = -x^2$ is locally Lipschitz for all $x \in \mathbb{R}$. But it is not globally Lipschitz. The solution with the initial condition $x(0) = -1$ is
$x(t) = 1/(t-1) \to -\infty$ as $t \to 1$.

"Finite escape time": Solution escapes to infinity in finite time (*e.g.* $t = 1$ in the previous example).

Lemma GEU. Let $f(t,x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, there the state unforced equation $\dot{x} = f(t,x)$ with the **initial condition** $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

Lemma GEU = "Lemma on Global Existence and Uniqueness".

Example: Linear system $\dot{x} = A(t)x + g(t)$, where $\|A(t)\| \leq L$, $L > 0$, for all $t \geq t_0$, satisfies conditions of Lemma GEU, but that case is too restrictive as we aim to study **nonlinear systems**.

Lemma GEULL. Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on domain $\mathcal{D}$ for all $t \in [t_0, t_1]$. Let W be a compact (closed and bounded) subset of $\mathcal{D}$ and $x_0 \in W$. Further, suppose it is known that every solution of $\dot{x} = f(t,x)$ with the **initial condition** $x(t_0) = x_0 \in W$ lies entirely in W. Then there is a unique solution that is defined for all $t \geq t_0$.

Example: One-dimensional system $\dot{x} = -x^3$. Any solution can not leave the set $[-a, a]$, $a > 0$, if $x(0) \in [-a, a]$, so by Lemma GEULL solution exists for all $t \geq t_0$ and is unique. Note, that similar system $\dot{x} = x^3$ suffers finite escape time phenomenon. In these simple cases, everything can be double checked by explicit solving the equations - more details during the exercises.

Problem to solve $\dot{x} = f(t, x)$ with the **initial condition** $x(t_0) = x_0$ is called as the **initial value problem**, or also **Cauchy problem**.

Any solution of Cauchy problem with some initial condition is called **system trajectory**.

To prove Lemma LEU, one can use *e.g.* fixed point theorem for contracting mappings, or prove convergence of functional sequence of Euler-like approximations.

To prove Lemma GEU, one uses the fact that δ in LEU is proportional to $1/L$, so if Lipschitz constant is globally bounded, one can repeatedly prolong the solution in time by step having guaranteed minimal length.

Lemma GEULL is then obvious consequence of Lemma GEU.

Time-invariant (autonomous) unforced system:

$$\dot{x} = f(x),\ x \in \mathbb{R}^n.$$

Why terminology "autonomous"? Let $n_1, n_2 \in N$, $n_1 + n_2 = n$.

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} f^1(x^1, x^2) \\ f^2(x^2) \end{bmatrix},\ x^1 \in \mathbb{R}^{n_1},\ x^2 \in \mathbb{R}^{n_2},\ f^1 \in \mathbb{R}^{n_1},\ f^2 \in \mathbb{R}^{n_2}.$$

Substituting the solution $x^2(t)$ of $\dot{x}^2 = f^2(x^2)$ into $f^1(x^1, x^2)$ gives

$$\dot{x}^1 = f^1(x^1, x^2(t)) := \overline{f}(x^1, t).$$

So, time dependence is usually consequence of some external influence - *i.e.* system is **non-autonomous**.

Basic property of autonomous systems: Let $x(t)$ be a solution, then for every $a \in \mathbb{R}$ $\overline{x}(t) = x(t-a)$ is also the solution.

Indeed: $\dot{\overline{x}} = \dot{x}(t-a) = f(x(t-a)) = f(\overline{x}(t))$.

Consequence of basic property of autonomous system: No trajectory of the system $\dot{x} = f(x),\ x \in \mathcal{D}$, can intersect itself, if $f(x)$ is locally Lipschitz on $\mathcal{D}$.

Indeed, let $x(t)$ is trajectory intersecting itself, *i.e.* $\exists t_1, t_2 \in \mathbb{R}, \varepsilon > 0$ such that $x(t_1) = x(t_2), t_1 \neq t_2$ and $x(t_1 + \tau) \neq x(t_2 + \tau),\ \forall \tau \in (0, \varepsilon)$. Consider the Cauchy problem $\dot{x} = f(x),\ x(0) = x(t_1)$, then it has two distinct solutions on $[0, \varepsilon)$: (I) $\bar{x}(t) = x(t + t_1)$ and (II) $\tilde{x}(t) = x(t + t_2)$. This contradicts to Lemma LEU.

Observation: Without any loss of generality, the Cauchy problem for autonomous systems can be considered with initial condition at zero time only:

$$\dot{x} = f(x),\ x(0) = x_0.$$

Indeed, the solution for $x(t_0) = x_0$ is obtained via forward time shift by t_0.

Example: $\dot{x} = x, x(t_0) = x_0 \in \mathbb{R}$, has the solution $x_0 \exp(t - t_0)$.

Time-invariant (autonomous) controlled dynamical system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \\ x &\in \mathbb{R}^n \\ u &\in \mathbb{R}^m \\ y &\in \mathbb{R}^q \end{aligned}$$

Unlike in case of unforced autonomous system, $f(x,u)$ may depend on t through $u(t)$. Even despite that, the initial conditions can be again taken as $x(0) = x_0$ as one can shift adequately the input waveform in time. Indeed, the input is considered to be totally in our disposal.

Practical viewpoint: Time-invariant system does not change its properties in time. Time-variant system, changes properties with time, *e.g.* some parts of appliances may change their performance by getting older, *etc.*

For the unforced autonomous system $\dot{x} = f(x)$, the **equilibrium point** x^E is defined by $f(x^E) = 0$, and its Cauchy problem with initial condition $x(0) = x^E$ has obviously the solution $x(t) \equiv x^E$ for all $t \geq 0$.

For the unforced time-variant system $\dot{x} = f(t, x)$, the **equilibrium point** x^E is defined by $f(t, x^E) = 0 \; \forall t \geq t_0$, and its Cauchy problem with initial condition $x(t_0) = x^E$ has obviously the solution $x(t) \equiv x^E$ for all $t \geq t_0$.

For the time-invariant controlled dynamical system $\dot{x} = f(x, u)$, **equilibrium pair** x^E, u^E is defined by $f(x^E, u^E) = 0$, and its Cauchy problem with initial condition $x(t_0) = x^E$ and applying constant control $u(t) \equiv u^E$ has obviously the solution $x(t) \equiv x^E$ for all $t \geq t_0$. The value u^E is actually the well-known trim control value, so usually u is already pre-adjusted in such a way, that there is some available $u^E = 0$.

For the time-invariant controlled dynamical system with outputs $\dot{x} = f(x, u), \; y = h(x, u)$, one usually adds requirement $y(x^E, u^E) = 0$.

For the time-variant controlled system (with outputs) - analogously.

NONLINEAR DYNAMICAL SYSTEM

More general models

We will study only the model $\dot{x} = f(x, u)$, $y = h(x, u)$. Usually, it is possible to obtain such a model.

Example: Equation with higher order time derivatives, solvable with respect to the highest derivative (here $x, y, u \in \mathbb{R}$):

$$x^{(n)}(t) = \Phi(x^{(n-1)}(t), \cdots, \ddot{x}(t), \dot{x}(t), x(t), u(t)),$$
$$y = \Psi(x^{(n-1)}(t), \cdots, \ddot{x}(t), \dot{x}(t), x(t), u(t)).$$

Introduce $x_1 = x, x_2 = \dot{x}, x_2 = \ddot{x}, \ldots, x_n = x^{(n-1)}$, then

$$f(x, u) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \Phi(x_n, \ldots, x_1, u) \end{bmatrix}, \quad h(x, u) = \Psi(x_n, \ldots, x_1, u).$$

Example: The following input-output relation can be also handled:

$$\Phi(y^{(k)}(t), \cdots, \dot{y}(t), y(t), u^{(l)}(t), \cdots, \dot{u}(t), u(t)) = 0, \quad y, u \in \mathbb{R},$$

provided it is solvable uniquely with respect to $y^{(k)}(t)$.

“Essentially nonlinear phenomena”:

- Finite time escape
- Multiple isolated equilibria
- Limit cycles
- Subharmonic, harmonic, or almost periodic oscillations
- Chaos
- Multiple modes of behaviour
- Bifurcations

Some examples are to be studied during exercises.

- Approximate nonlinearity: use the first order approximation around equilibria or along some given system trajectory
- Compensate nonlinearity: use the exact feedback linearization to cancel nonlinearities
- Dominate nonlinearity: use the robust control techniques, the high-gain feedback
- Use intrinsic properties: example is passivity based control
- Divide and conquer: use the backstepping

Only some of these techniques will be addressed during this course in detail.

Illustrative examples are to be given during exercises.