Rent Division - The Couple Problem

SHIH-TING HUANG, Rochester Institute of Technology
KASHMIRA DOLAS, Rochester Institute of Technology

ABSTRACT Rent division is one of the most practical and trivial problems that are being researched as a part of the fair division problem in the crossroads of social theory, mathematics, economics and computer science. We have considered a special case of rent division problem where the participating agents are all couples or can be a combination of couples and individuals. We performed experiments and simulations on the two cases while assuming that if an agent is a couple then they want to live in the same room and they will provide common valuations for all the rooms. The couple-envy-freeness property on the basis of envy-freeness property is defined to suit the problem space. This paper, proves that, finding an allocation in which the individual envy-freeness and couple-envy-freeness can be satisfied at the same time can not be guaranteed.

Additional Key Words and Phrases: The couple problem, Rent Division, Room allocation, Rental harmony, Sperner's Lemma, Market-based algorithm, Fair Division, Divisible goods, Resource allocation.

1 INTRODUCTION

A society where issues such as high rent in busy cities and school or work life prone to relocation arise the need to share. Ordinarily, a person who is looking for a place to live finds an apartment that is to be shared with a few others. This person can be an individual that does not wish to share a room or may be open to sharing the room with another individual. This person can be in a relationship and would want to share his/her room with his/her partner. A question that arises here, is how to decide on, which room to be allocated to whom and how should the rent be divided between the couples and individuals. In order to avoid dispute, it is important to find an efficient, envy-free solution.

Broadly, rent division is a type of fair division problem and the target is to yield an efficient and envy-free allocation along with non-negative rent-partitioning while satisfying the agents with their assigned room. We also want to find a fair allocation. To satisfy fairness, the mechanism must be strategy-proof. An efficient algorithm is defined as an approach the yields optimal, desired results in a polynomial time. We say an allocation is efficient when no other allocation can bring more satisfaction to the agents. An allocation is said to be envy-free when all agents are satisfied with their allocation or feel that their allocation is at least as good as the allocation of any other agent. Strategy-proof mechanisms are resistant to manipulation by the participating agents. In this paper, we have defined a certain property and have provided proofs to support our claims. Our main contributions are that we show, (1) the available Market-based algorithm for rent division can be used for couples and that (2) there is no guarantee that the allocation will always satisfy the two properties of envy-freeness and couple-envy-freeness simultaneously. The couple-envy-freeness property is discussed in detail, ahead in the paper.

The construction of this paper is as follows. Section 2 shows the related work on rent division problem and comparisons of each mechanism. In this section we concisely cover the currently available solutions by discussing the algorithms, models and limitations, majorly in papers by Su[7], Abdulkadiroğlu et al.[1], Frick et al.[3], Procaccia et al.[6] and Lachlan et al.[5] on rent division, a fair division problem. Section 3 introduces the algorithms and models for rent division with the couple constraint. Section 4 examines envy-freeness among all couples. Section 5 investigates

Table 1. Rent Division papers: comparison

No.	Paper Title	Algorithm	Pro	Con
1.	Rental harmony: Sperner's lemma lemma in fair division[7]	Sperner's lemma	Envy-free	Negative pricing possibility, Manipulable, NP-hard
2.	Achieving rental harmony with a secretive roommate[3]	Sperner's lemma	Envy-free	Manipulable, NP-hard
3.	Room assignment-rent division: A market approach[1]	Market-based mechanism	Envy-free, non-negative pricing, efficient, individually-rational	Manipulable
4.	Spliddit: Unleashing Fair Division Algorithms[4]	Market-based mechanism	Envy-free, Pareto efficient	Manipulable
5.	Fair Rent Division on a Budget[6]	Polynomial Time algorithm	Includes budget constraints	Manipulable, envy-freeness not guaranteed
6.	Bidding for envy-freeness: A procedural approach to n-player fair-division problems[2]	Compensation procedure	Overcomes constraint: Number of items = Number of agents	Either envy-free or Strategy-proof
7.	A general strategy proof fair allocation mechanism[8]	Optimal fair allocation mechanism with compensation limits	Envy-free, Strategy-proof	Not incompatible for rent division model
8.	Randomised Room Assignment - Rent Division[5]	Introduces randomization after finding an envyfree solution and before allocation	Pareto efficient, Strategy-proof	Compromises on measure of envy-freeness in worst case

envy-freeness among couples and individuals. Section 6 summarizes the results of our research, conclusion, and future work.

2 RELATED WORK

In this section, we briefly compare and contrast the algorithms and theorems proved by Su[7], Abdulkadiroğlu et al.[1], Frick et al.[3], Procaccia et al.[6] and Lachlan et al.[5] for rent division, by listing the methods, algorithms, pros and cons in a tabular format. See Table 1.

Room assignment - Rent Division problem is a subset of fair division problem which was first addressed using combinatorial results known as Spener's Lemma. Using this and assuming full knowledge of agents preferences Su showed that there always exits an envy-free solution.

Referencing the proof by Su and the lemma, an auction method was developed by Abdulkadiroğlu et al. for resolving the room allocation and deciding the prices for each room. This mechanism is now used as the basic algorithm for Spliddit website[4], this website also computes shared room allocation and rent problem by considering preferences of one room twice for that couple. Later, Su with Haake et al. devised a generic mechanism while lifting the restriction on number agent being equal to the number of items. All these mechanisms are deterministic in nature.

For a mechanism to be envy-free, the participating agents should give their true preferences. All deterministic mechanisms are said to be envy-free assuming that agents are truthful, but are prone to manipulation. Considering the compensation limit as a restriction, Sun et al. designed an optimal fair allocation mechanism that is strategy-proof and envy-free. In a paper by Dufton et al., randomization is used to achieve strategy-proofness at an expense of envy-freeness in worst case scenario. Up to our knowledge, there is no significant work presented for rent division problem that involves couples.

3 ALGORITHMS AND MODELS

3.1 Notation

We assume that the total number of agents and the capacity of the apartment is equal. Consider that we have m number of agents and n number of rooms in an apartment. Since, the problem at hand involves couples, $m \geq n$ must hold. We represent an agent with an i, and a set $I = \{1, 2, ..., m\}$ to represent an agent set. If agent i is a couple, we use i_1 and i_2 to represent the two individuals in the couple. We use j to represent a room, and a set $J = \{1, 2, ..., n\}$ to represent a room set. If room j is a double-bed room, we use j_1 and j_2 to represent the two capacities of the room. The total rent is represented as R.

v_{ij} denotes agent i's valuation for room j. For every agent $i \in I$, the total valuation for the entire apartment must be great than or equal to the total rent i.e. $\sum_{j \in J} v_{ij} \geq R$. We do this to make sure every agent thinks the house is worth renting.

Vector P represents the price of every room and comprises of n elements. p_j is an element of P and represents the payment price for room j such that the sum of payment price should equal to the rent, $\sum_{j \in J} p_j = R$. We initialize P, with uniform distribution of the total rent into n elements.

The u_{ij} denotes agent i's utility for room j. The utility is defined as the difference between the valuation and payment price, $u_{ij} = v_{ij} - p_j$. μ is a vector that represents the allocation for every agent. μ_i is an element of μ and indicates the room allocated to agent i.

3.2 Market-Based Algorithm

A market-based algorithm for rent division problem was proposed by Abdulkadiroğlu et al[1]. If an allocation exists, this mechanism is efficient, envy-free and individually-rational in generating a non-negative price for each room. A limitation to this mechanism is that it is not strategy-proof. One can easily provide a strategic preference since the valuation function is open and hence is highly manipulable.

The algorithm mimics the market-based pricing process. Every agent provides his/her valuation for each room as per his/her demand. The mechanism basically adjusts the prices of the rooms according to the valuations of the agents. We say a room is over-demanded if number of agents to choose this room is greater than the demand for other rooms. The rooms that are not in over-demanded room set are in non-over-demanded room set. The over-demanded room set is represented as OD. The algorithm iteratively increases the prices of the rooms in over-demanded set and decreases the prices of the rooms in non-over-demanded set by the same rate over the process till we find an envy-free allocation.

Initially, the price of each room is set to R/n, P^0 is a initial price vector such that,

$$P^0 = [R/n, ..., R/n]$$

If the set of over-demanded rooms is empty, $OD(P^0) = \emptyset$, this indicates we have an efficient match μ for all agents and sets $P = P^0$ and terminates the procedure.
If $OD(P^0) \neq \emptyset$, then the algorithm proceeds to the next step. The next step i.e. t^{th} step, is finding the minimum utility gap, $x(P^t)$, between over-demanded room and non-over-demanded room among all agents. $x(P^t)$ is divided using the following rules:
For j^{th} over-demanded room, the price is increased by,

$$P'_j = \frac{n - |OD(P^{t-1})|}{n} * x(P^t)$$

For j^{th} non-over-demanded room, the price is decreased by,

$$P'_j = \frac{|OD(P^{t-1})|}{n} * x(P^t)$$

Now, we have new price vector P^t of P'_j for every room. If $OD(P^t) = \emptyset$, this indicated the mechanism found an allocation μ, sets $P = P^t$ and terminates the procedure. If not, the mechanism repeats this step until an allocation is found. This mechanism guarantees to generate a non-negative envy-free rent division, if there exits one. That is , the resultant price for each room will always be positive, which reflects the real-life situation.

4 ENVY-FREENESS BETWEEN COUPLES

First, we consider the situation where all participants are couples.

Definition 4.1. We say, an allocation is envy-free if every agent's utility for the allocated room is greater than or equal to his/her utility for every other room.

The property of envy-freeness can be represent as $u_{ij} \geq u_{ij'}$, where j' represents all other room besides room j.

For every agent, the room he/she gets has a utility for that room at least equal to every other room. We say, couples have same valuation while considering a specific case where the two agents in a couple provide one common valuation for each room i.e. $V_i = V_{i_1} = V_{i_2}$. We assume a double-bed room has two equal valued single-beds. The valuation of a double-bed room, v_{ij}, considering couples having equal valuation can be divided into two equal valuations and can be expressed as, $v_{ij} = v_{ij_1} + v_{ij_2}$.

Definition 4.2. The allocation is couple-envy-free if all couples among the participating agent are allocated the same double-bed room as his/her partner, if there exists an allocation.

For a double-bed room j, if $\mu_i = j$, either agent i is a couple, $\{\mu_{i_1}, \mu_{i_2}\} = \{j_1, j_2\}$, or there exist an another μ_k, where $\{\mu_i, \mu_k\} = \{j_1, j_2\}$ and both agent i and agent k are single. That is, only a couple or two single agents can get a double-bed room to satisfy the couple-envy-freeness property.

LEMMA 4.3. *To satisfy definition 4.1, for each agent i, the difference between the valuation for his/her allocation μ_i and the other agent, i''s valuation for μ_i, is greater or equal to the difference between the i's valuations for a room other than μ_i, i.e. $\mu_{i'}$ and the agent i''s valuations who gets $\mu_{i'}$,*

$$\sum_{i,i'=1}^{m} (v_{i\mu_i} - v_{i'\mu_i}) \geq \sum_{i,i'=1}^{m} (v_{i\mu_{i'}} - v_{i'\mu_{i'}})$$

Table 2. Valuation table of 2 agents (couples)

room	1	2	3
agent 1	v_{11}	v_{12}	v_{13}
agent 2	v_{21}	v_{22}	v_{23}

Table 3. Derived valuation table of 2 agents (couples)

room	1_1	1_2	2	3
agent 1_1	$v_{1_1 1_1}$	$v_{1_1 1_2}$	v_{12}	v_{13}
agent 1_2	$v_{1_2 1_1}$	$v_{1_2 1_2}$	v_{12}	v_{12}
agent 2_1	$v_{2_1 1_1}$	$v_{2_1 1_2}$	v_{22}	v_{23}
agent 2_2	$v_{2_2 1_1}$	$v_{2_2 1_2}$	v_{22}	v_{23}

.

Proof. Let's consider a case with two couples. Here, one agent represents two individuals in a couple. Agent 1 and agent 2, try to split a one double-bed room, room 1, and two single-bed rooms, room 2 and room 3. The valuation can be represented as Table 2. Since the valuation for a double-bed room can be divided into two equal valuations, and a couple can be divided into the two agents and have same valuations for every room, the valuation table can be adjusted into Table 3. We assume there is a matching allocation $\mu(1_1, 1_2, 2_1, 2_2) = [1_1, 1_2, 2, 3]$ and the payment price $P(1_1, 1_2, 2, 3) = [p_{1_1}, p_{1_2}, p_2, p_3]$. In order to satisfy envy-freeness:

For agent 1_1: $(v_{1_1 1_1} - p_{1_1}) \geq (v_{1_1 1_2} - p_{1_2})$, $(v_{1_1 1_1} - p_{1_1}) \geq (v_{12} - p_2)$, $(v_{1_1 1_1} - p_{1_1}) \geq (v_{13} - p_3)$

For agent 1_2: $(v_{1_2 1_2} - p_{1_2}) \geq (v_{1_2 1_1} - p_{1_1})$, $(v_{1_2 1_2} - p_{1_2}) \geq (v_{12} - p_2)$, $(v_{1_2 1_2} - p_{1_2}) \geq (v_{13} - p_3)$

For agent 2_1: $(v_{22} - p_2) \geq (v_{2_1 1_1} - p_{1_1})$, $(v_{22} - p_2) \geq (v_{2_1 1_2} - p_{1_2})$, $(v_{22} - p_2) \geq (v_{23} - p_3)$

For agent 2_2: $(v_{23} - p_3) \geq (v_{2_2 1_1} - p_{1_1})$, $(v_{23} - p_3) \geq (v_{2_2 1_2} - p_{1_2})$, $(v_{23} - p_3) \geq (v_{22} - p_2)$

Since the couple share same valuation of each room, we know that,

$$v_{1_1 1_1} = v_{1_1 1_2} = v_{1_2 1_1} = v_{1_2 1_2} \text{ and } v_{2_1 1_1} = v_{2_1 1_2} = v_{2_2 1_1} = v_{2_2 1_2}$$

.

Summing up agent 1_1's inequalities, we can get:

$$3 * (v_{1_1 1_1} - p_{1_1}) \geq (v_{1_1 1_2} + v_{12} + v_{13}) - (p_{1_2} + p_2 + p_3) \tag{1}$$

Summing up other agent's first inequality, we get:

$$(v_{1_2 1_2} + v_{22} + v_{23}) - (p_{1_2} + p_2 + p_3) \geq (v_{1_2 1_1} + v_{2_1 1_1} + v_{2_2 1_1}) - 3 * (p_{1_1}) \tag{2}$$

Summing up the two inequalities 1 and 2, we get:

$$2 * (v_{1_1 1_1} - v_{2_1 1_1}) \geq (v_{12} - v_{22}) + (v_{13} - v_{23}) \tag{3}$$

Using same steps for agent 2_1, we can get:

$$(v_{22} - v_{12}) \geq (v_{2_1 1_1} - v_{1_1 1_1}) \tag{4}$$

$$(v_{23} - v_{13}) \geq (v_{2_2 1_1} - v_{1_1 1_1}) \tag{5}$$

From equations 3, 4 and 5, we proved that Lemma 4.3 holds.

□

THEOREM 4.4. *In a case where all the agents are couples, it is not guaranteed to generate an allocation that satisfies both definition 4.1 and definition 4.2 at the same time.*

PROOF. Consider an example of two couple agents that try to split one double-bed room and two single-bed rooms. The agents' valuations are shown in Table 4, and we can generate the derived valuations as shown in Table 5. By lemma 4.3, only the following allocations are able to satisfied envy-freeness:

$$\mu(1_1, 1_2, 2_1, 2_2) = \{[1_1, 3, 1_2, 2], [1_1, 3, 2, 1_2], [1_2, 3, 1_1, 2], [1_2, 3, 2, 1_1], [3, 1_1, 1_2, 2], [3, 1_1, 2, 1_2], [3, 1_2, 1_1, 2], [3, 1_2, 2, 1_1]\}$$

However, none of these allocate room 1_1 and room 1_2 to agent 1 or agent 2 at the same time.

If we use market-based algorithm, the initial price vector is,

$$P^0 = [750/4, 750/4, 750/4, 750/4]$$

Initial utility matrix is shown in Table 6. The $OD(P^0) = \{2, 3\}$.
The minimal utility gap $x(P^0)$ is, $x(P^0) = (12.5 - (-37.5)) = 50$.
Hence, the prices of room 2 and room 3 increase by 50, and the prices of room 1_1 and room 1_2 decrease by 50.
Now, we get new price vector P^1 and the new utility matrix is shown in 7.

$$P^1 = [137.5, 137.5, 237.5, 237.5]$$

In this iteration, the $OD(P^1) = \emptyset$, hence the final payment price vector $P = P^1$. In the set of all possible allocations shown in Table 7, there is no allocation that can assign both room 1_1 and room 1_2 to agents 1_1 and 1_2 or agent 2_1 and 2_2 at the same time.

Table 4. Valuation table of the 2 agents (couples)

room	1	2	3
agent 1	300	200	250
agent 2	300	250	200

Table 5. Derived Valuation table of the 2 agents (couples)

room	1_1	1_2	2	3
agent 1_1	150	150	200	250
agent 1_2	150	150	200	250
agent 2_1	150	150	250	200
agent 2_2	150	150	250	200

Table 6. Initial utility of the 2 agents (couples)

room	1_1	1_2	**2**	**3**
agent 1_1	-37.5	-37.5	12.5	**62.5**
agent 1_2	-37.5	-37.5	12.5	**62.5**
agent 2_1	-37.5	-37.5	**62.5**	12.5
agent 2_2	-37.5	-37.5	**62.5**	12.5

□

Table 7. 1st utility of the 2 agents (couples)

room	1_1	1_2	2	3
agent 1_1	**12.5**	**12.5**	-37.5	**12.5**
agent 1_2	**12.5**	**12.5**	-37.5	**12.5**
agent 2_1	**12.5**	**12.5**	**12.5**	-37.5
agent 2_2	**12.5**	**12.5**	**12.5**	-37.5

Table 8. Valuation table of 1 couple and 2 individual agents

room	1	2	3
agent 1	v_{11}	v_{12}	v_{13}
agent 2	v_{21}	v_{22}	v_{23}
agent 3	v_{31}	v_{32}	v_{33}

Table 9. Derived valuation table of 1 couple and 2 individual agents

room	1_1	1_2	2	3
agent 1_1	$v_{1_1 1_1}$	$v_{1_1 1_2}$	v_{12}	v_{13}
agent 1_2	$v_{1_2 1_1}$	$v_{1_2 1_2}$	v_{12}	v_{13}
agent 2	v_{21_1}	v_{21_2}	v_{22}	v_{23}
agent 3	v_{31_1}	v_{31_2}	v_{32}	v_{33}

5 ENVY-FREENESS BETWEEN COUPLES AND INDIVIDUALS

In this section, we examine the situation where the participating agents include couples and individuals.

LEMMA 5.1. *Even when the participating agents include couples and individuals, the lemma 4.3 holds.*

PROOF. Let's consider a case with one couples, agent 1, and two individuals, agent 2 and agent 3, are trying to split a double-bed room, room 1, and two single-bed rooms, room 2 and room 3. The valuation can be represented as in Table 8. Since the valuation for a double-bed room can be divided into two equal valuations, and a couple can be divided into the two agents and have same valuations for every room, the valuation table can be adjusted into Table 9. We assume there is a matching allocation $\mu(1_1, 1_2, 2, 3) = [1_1, 1_2, 2, 3]$ and the payment price $P(1_1, 1_2, 2, 3) = [p_{1_1}, p_{1_2}, p_2, p_3]$. In order to satisfy envy-freeness:
For agent 1_1: $(v_{1_1 1_1} - p_{1_1}) \geq (v_{1_1 1_2} - p_{1_2})$, $(v_{1_1 1_1} - p_{1_1}) \geq (v_{12} - p_2)$, $(v_{1_1 1_1} - p_{1_1}) \geq (v_{13} - p_3)$,
For agent 1_2: $(v_{1_2 1_2} - p_{1_2}) \geq (v_{1_2 1_1} - p_{1_1})$, $(v_{1_2 1_2} - p_{1_2}) \geq (v_{12} - p_2)$, $(v_{1_2 1_2} - p_{1_2}) \geq (v_{13} - p_3)$,
For agent 2: $(v_{22} - p_2) \geq (v_{21_1} - p_{1_1})$, $(v_{22} - p_2) \geq (v_{21_2} - p_{1_2})$, $(v_{22} - p_2) \geq (v_{23} - p_3)$,
For agent 3: $(v_{33} - p_3) \geq (v_{31_1} - p_{1_1})$, $(v_{33} - p_3) \geq (v_{31_2} - p_{1_2})$, $(v_{33} - p_3) \geq (v_{32} - p_2)$.

Since the couple share same valuation of each room, we know that,

$$v_{1_1 1_1} = v_{1_1 1_2} = v_{1_2 1_1} = v_{1_2 1_2}$$

Summing up agent 1_1's inequalities, we can get:

$$3 * (v_{1_1 1_1} - p_{1_1}) \geq [(v_{1_1 1_2} + v_{12} + v_{13}) - (p_{1_2} + p_2 + p_3)] \tag{6}$$

Table 10. Valuation table of 1 couple and 2 individual agents

room	1	2	3
agent 1	300	200	250
agent 2	200	250	300
agent 3	400	175	175

Table 11. Derived valuation table of 1 couple and 2 individual agents

room	1_1	1_2	2	3
agent 1_1	150	150	200	250
agent 1_2	150	150	200	250
agent 2	100	100	250	300
agent 3	200	200	175	175

Summing up other agent's first inequality, we can get:

$$[(v_{1_2 1_2} + v_{22} + v_{33}) - (p_{1_2} + p_2 + p_3)] \geq [(v_{1_2 1_1} + v_{21_1} + v_{31_1}) - 3 * (p_{1_1})] \tag{7}$$

Summing up the two inequalities 6 and 7, we get:

$$[(v_{1_1 1_1} - v_{21_1}) + (v_{1_1 1_1} - v_{31_1})] \geq [(v_{1_1 2} - v_{22}) + (v_{1_1 3} - v_{33})] \tag{8}$$

Using same steps for agent 2, we can get:

$$[(v_{22} - v_{12}) + (v_{22} - v_{12}) + (v_{22} - v_{32})] \geq [(v_{21_1} - v_{1_1 1_1}) + (v_{21_2} - v_{1_2 1_2}) + (v_{23} - v_{33})] \tag{9}$$

Using same steps for agent 3, we can get:

$$[(v_{33} - v_{13}) + (v_{33} - v_{13}) + (v_{33} - v_{23})] \geq [(v_{31_1} - v_{1_1 1_1}) + (v_{31_2} - v_{1_2 1_2}) + (v_{32} - v_{22})] \tag{10}$$

Hence, from equations 8, 9 and 10, we prove Lemma 5.1. □

Theorem 5.2. *In a case where participating agents include couples and individuals, it is not guaranteed to generate an allocation that satisfies both definition 4.1 and definition 4.2 at the same time.*

Proof. Consider an example of one couple and two individuals that try to split a one double-bed room and two single-bed rooms. The agents' valuations are shown in Table 10, and we can generate the derived valuations as shown in Table 11. By lemma 5.1, only the following allocations are able to satisfied envy-freeness:

$$\mu(1_1, 1_2, 2, 3) = \{[1_1, 2, 3, 1_2], [1_1, 3, 2, 1_2], [1_2, 2, 3, 1_1], [1_2, 3, 2, 1_1]$$
$$[2, 1_1, 3, 1_2], [2, 1_2, 3, 1_1], [3, 1_1, 2, 1_2], [3, 1_2, 2, 1_1]\}$$

However, none of these allocate room 1_1 and 1_2 to agent 1 (the couple) at the same time. If we use market-based algorithm, the price vector P^0 is initialized as,

$$P^0 = [750/4, 750/4, 750/4, 750/4]$$

Initial utility matrix is as shown in Table 12. By analyzing Table 12 we get, $OD(P^0) = \{3\}$.
The minimal utility gap $x(P^0)$, is $x(P^0) = (62.5 - 12.5) = 50$.
Hence, the prices of room 3 increase by 37.5 and the prices of other rooms decrease by 12.5. Now, we get new price vector P^1 and the new utility matrix is shown in Table 13.

$$P^1 = [175, 175, 175, 225]$$

Table 12. Initial utility of 1 couple and 2 individuals

room	1_1	1_2	2	**3**
agent 1_1	-37.5	-37.5	12.5	**62.5**
agent 1_2	-37.5	-37.5	12.5	**62.5**
agent 2	-87.5	-37.5	62.5	**113**
agent 3	**12.5**	**12.5**	-12.5	-12.5

Table 13. 1st utility of 1 couple and 2 individuals

room	1_1	1_2	**2**	**3**
agent 1_1	-25	-25	**25**	**25**
agent 1_2	-25	-25	**25**	**25**
agent 2	-75	-75	**75**	**75**
agent 3	**25**	**25**	0	-50

Table 14. 2nd utility of 1 couple and 2 individuals

room	1_1	1_2	2	3
agent 1_1	**0**	**0**	**0**	**0**
agent 1_2	**0**	**0**	**0**	**0**
agent 2	-50	-50	**75**	**75**
agent 3	**50**	**50**	-25	-25

In the 2^{nd} iteration, the $OD(P^1) = 2, 3$.
The minimal utility gap $x(P^0) = (25 - (-25)) = 50$.
Hence, the prices of room 2 and room 3 increase by 25, and the prices of other rooms decrease by 25. We get a new price vector P^2 and the 2^{nd} iteration utility matrix is shown in 14.

$$P^2 = [150, 150, 200, 250]$$

In this iteration, the $OD(P^2) = \emptyset$. Hence, the final payment price vector $P = P^2$. In the set of all possible allocations, agent 3 is always assigned to room 1 and it is impossible to assign both room 1_1 and room 1_2 to agent 1 at the same time.

□

6 CONCLUSIONS

In this paper, we discussed rent division with the special case where either all agents are couples or some agents are couples. The traditional rent division problem only focuses on envy-freeness by individuals. We defined a new property that captures the sociology of couples in this special case called couple-envy-freeness. We used the inequalities of envy-freeness and market-based algorithm to prove that (1) in a case where all the agents are couples, it is not guaranteed to generate an allocation that satisfies both individual envy-freeness and couple-envy-freeness at the same time, and (2) in a case where participating agents include couples and individuals, it is not guaranteed to generate an allocation that satisfies both individual envy-freeness and couple-envy-freeness at the same time.

Even though our results are negative, the outcome still provided some information on the rent division problem with couples and can guide us for future research. For the future work, one direction we consider working on is finding a relationship between couples' and individuals' valuations that can guarantee the allocation is satisfied both individual envy-freeness and couple-envy-freeness. The other direction is proposing a rent division algorithm that accepts both couples' and individuals' valuations in an attempt to perform an allocation. This can also we extended to the couples' problem in hotel allocation where the task is focused on optimizing the room allocation, as the hotel charges are constant as per the room type.

REFERENCES

[1] Atila Abdulkadiroğlu, Tayfun Sönmez, and M. Utku Ünver. 2004. Room assignment-rent division: A market approach. *Social Choice and Welfare* 22, 3 (01 Jun 2004), 515–538. https://doi.org/10.1007/s00355-003-0231-0

[2] Francis Edward Su Claus-Jochen Haake, Matthias G. Raith. [n. d.]. Bidding for envy-freeness: A procedural approach to n-player fair-division problems. Social Choice and Welfare. ([n. d.]), 723fi?!749. https://www.math.hmc.edu/~su/papers.dir/bfe.pdf

[3] Florian Frick, Kelsey Houston-Edwards, and Frédéric Meunier. 2017. Achieving rental harmony with a secretive roommate. (05 2017). arXiv:1702.07325

[4] Jonathan Goldman and Ariel D. Procaccia. 2015. Spliddit: Unleashing Fair Division Algorithms. *SIGecom Exch.* 13, 2 (Jan. 2015), 41–46. https://doi.org/10.1145/2728732.2728738

[5] Kate Larson Lachlan Dufton. 2011. Randomised Room Assignment-Rent Division. (2011). http://research.illc.uva.nl/COMSOC/IJCAI-2011/papers/dufton-larson.pdf

[6] Ariel D. Procaccia, Rodrigo A. Velez, and Dingli Yu. 2017. Fair Rent Division on a Budget. (2017).

[7] Francis Edward Su. 1999. Rental Harmony: Sperner's Lemma in Fair Division. *The American Mathematical Monthly* 106, 10 (1999), 930–942. http://www.jstor.org/stable/2589747

[8] Ning Sun and Zaifu Yang. 2003. A general strategy proof fair allocation mechanism. *Economics Letters* 81, 1 (2003), 73 – 79. https://doi.org/10.1016/S0165-1765(03)00151-4