An R Companion to James Hamilton's Times Series Analysis

with R

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Preliminary

Foreword

`RcompHam94` is an R package that implements many of the worked examples in `Time Series Analysis` (Hamilton, 1994) as well as providing access to the code and datasets used. In many cases `RcompHam94` provides both simplified implmentations "from scratch" to allow the reader to explore the underlying logic and calculations, and more realistic implementations that make use of the large body of contributed packages in the Comprehensive R Archive Network (CRAN). Thus readers who have cut their teeth on the textbook can use this package as a stepping stone to doing their own analysis and/or research. Readers looking for additional introductory treatment of facilities available in CRAN can explore other excellent introductions such as `http://cran.r-project.org/doc/contrib/Farnsworth-EconometricsInR.pdf` and `http://cran.r-project.org/web/packages/AER/AER.pdf` (Kleiber and Zeileis, 2008).

We assume the reader has downloaded the R language, and package "RcompHam94" from `http://www.r-project.org/` and has read "An Introduction to R" available here `http://cran.r-project.org/doc/manuals/R-intro.html` and also available as a PDF from the "Help" menu of the R package.

To load the package, just use:

R code
```
library("RcompHam94")
```

Code shown in this document (and some not shown for brevity) can be executed using the R "demo" function. For a list of available demos, use:

R code
```
demo(package = "RcompHam94")
```

To invoke a specific demo, say the demo called "p112", use:

R code
```
demo(topic = "p112", package = "RcompHam94")
```

In general the demos are written so that the results of individual calculations can be examined after the fact by examining variables containing the results of those calculations.

Page references in the body of this document refer to `Time Series Analysis` .

Contents

1 Difference Equations

1.1 Dynamic Multipliers for First Order Difference Equations

Page 3 describes calculations for dynamic multipliers for first order difference equations. An example of these calculations in action is given on page 4. A simple method to calculate dynamic multipliers is to simulate the difference equation calculating forward based on an initial shock at time t=1, assuming the value of y at time 0 is 0. R indexes arrays starting at 1 instead of 0, so subscripts are one more than the convention used in the text, meaning that the shock will be said to occur at time 2.

R code

```
> T <- 20
> w <- 1 * (1:T == 2)
```

In the examples shown on page 4 there are actually four different equations being simulated, so we will use a matrix, rather than a vector, to store the results.

R code

```
> phis <- c(0.8, -0.8, 1.1, -1.1)
> y <- array(dim = c(T, length(phis)))
> y[1, ] <- rep(0, length(phis))
> for (j in 2:T) y[j, ] <- phis * y[j - 1, ] + w[j]
```

We can check this calculation against the closed form expression on page 3.

R code

```
> print(y[2:T, 1])
```

output

```
 [1] 1.00000000 0.80000000 0.64000000 0.51200000 0.40960000 0.32768000
 [7] 0.26214400 0.20971520 0.16777216 0.13421773 0.10737418 0.08589935
[13] 0.06871948 0.05497558 0.04398047 0.03518437 0.02814750 0.02251800
[19] 0.01801440
```

R code

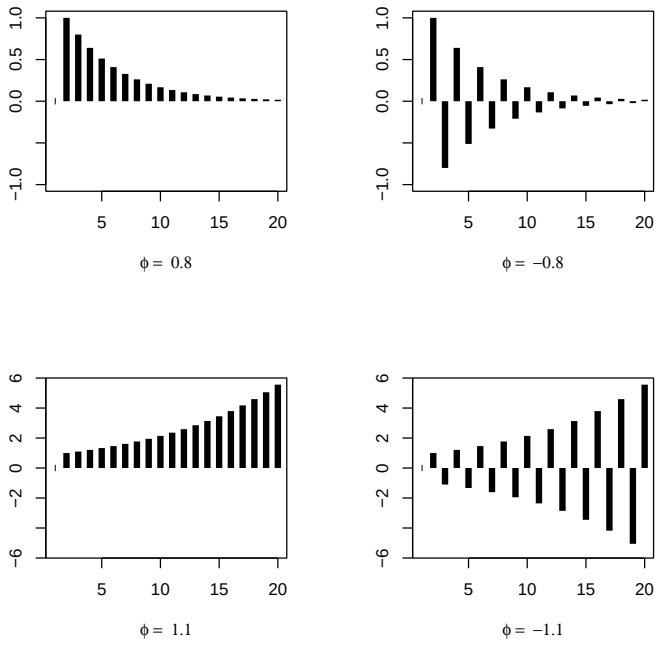
```
> print(phis[[1]]^seq(0, T - 2))
```

output

```
 [1] 1.00000000 0.80000000 0.64000000 0.51200000 0.40960000 0.32768000
 [7] 0.26214400 0.20971520 0.16777216 0.13421773 0.10737418 0.08589935
```

```
[13] 0.06871948 0.05497558 0.04398047 0.03518437 0.02814750 0.02251800
[19] 0.01801440
```

Finally we can plot the results using a histogram plot reproducing figure 1.1.



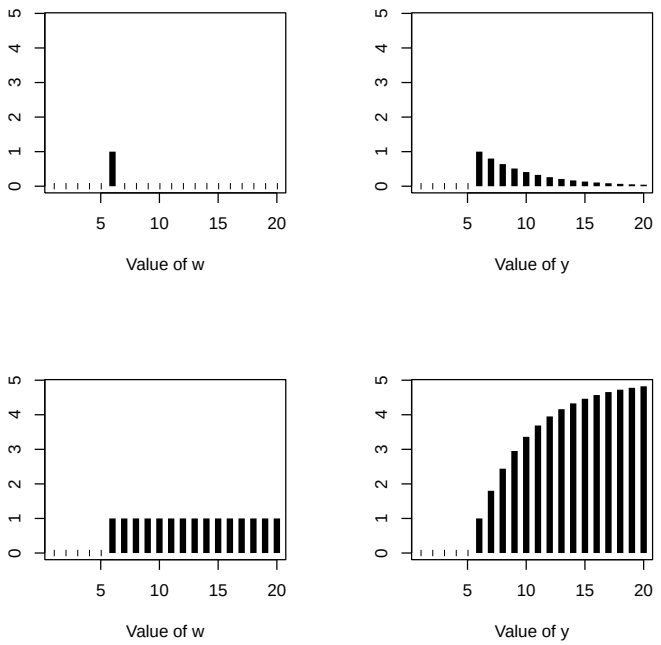
1.2 Comparing Transitory Versus Permanent Changes

The above example examined the effect changing ϕ on the dynamic multiplier. Pages 5 and 6 describe what happens when the permanence of the change is varied with a fixed multiplier, i.e. while leaving ϕ unchanged.

R code

```
> phi <- 0.8
> T <- 20
> w <- 1 * cbind(1:T == 6, 1:T >= 6)
> y <- array(dim = c(T, 2))
> y[1:5, ] <- 0
> for (j in 6:T) y[j, ] <- phi * y[j - 1, ] + w[j, ]
```

The results can be plotted reproducing figures 1.2 and 1.3.



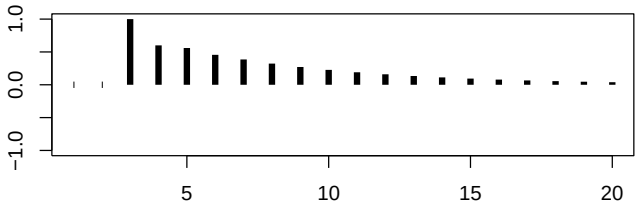
1.3 Dynamic Multipliers for Second Order Difference Equations

Finally we use similar techniques to calculate the effects of an impulse on a second order system. Here each column of phi represents the coefficients of a second order system.

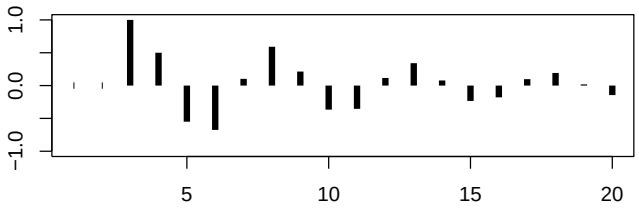
R code

```
> T <- 20
> w <- 1 * (1:20 == 3)
> y <- array(dim = c(T, 2))
> y[1:2, ] <- 0
> phi <- array(c(0.6, 0.2, 0.5, -0.8), c(2, 2))
> for (j in 3:T) y[j, ] <- apply(X = phi * y[(j - 1):(j - 2), ],
+     MARGIN = 2, FUN = sum) + w[j]
```

The results can be plotted reproducing figure 1.4.



$\phi 1 = 0.6, \phi 2 = 0.5$



$\phi 1 = 0.2, \phi 2 = -0.8$

2 Stationary ARMA Processes

2.1 Autocorrelations for AR and MA Processes

Pages 50 to 59 describe the calculation of autocorrelation functions of AR and MA processes. Following the expressions in the text we can calculate results using separate formulae for white noise, moving average, and autoregressive processes.

R code

```
> T <- 20
> specifications <- list(list(label = "White Noise", MA = vector(mode = "numeric"),
+     AR = vector(mode = "numeric")), list(label = "MA(1)", MA = c(0.8),
+     AR = vector(mode = "numeric")), list(label = "MA(4)", MA = c(-0.6,
+     0.5, -0.5, 0.3), AR = vector(mode = "numeric")), list(label = "AR(1) with 0.8",
+     MA = vector(mode = "numeric"), AR = c(0.8)), list(label = "AR(1) with -0.8",
+     MA = vector(mode = "numeric"), AR = c(-0.8)))
> sigmasq <- 1
```

White noise calculations are described on bottom of page 47 and the top of page 48.

R code

```
> specifications[[1]]$rho <- c(1, rep(0, T - 1))
```

Moving average calculations are described on page 51.

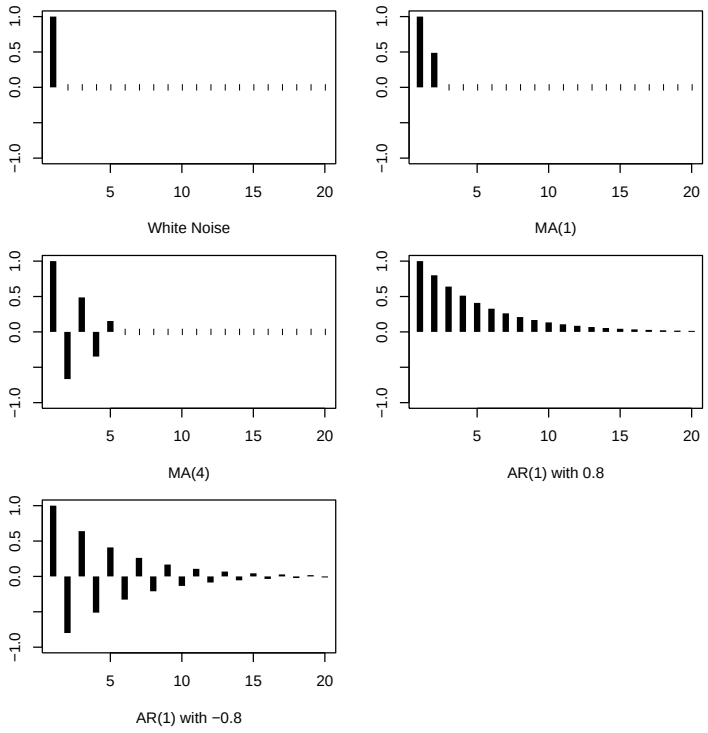
R code

```
> for (i in 2:3) {
+     MA <- specifications[[i]]$MA
+     q <- length(MA)
+     gamma <- vector(mode = "numeric", length = T)
+     gamma[1] <- sigmasq * t(c(1, MA)) %*% c(1, MA)
+     for (j in 1:q) gamma[j + 1] <- sigmasq * t(MA[j:q]) %*% c(1,
+         MA)[1:(q - j + 1)]
+     gamma[(q + 2):T] <- 0
+     specifications[[i]]$rho <- gamma/gamma[1]
+ }
```

Autocorrelation calculations are described on page 59

R code

```
> for (i in 4:5) {
+     AR <- specifications[[i]]$AR
+     p <- length(AR)
+     F <- rbind(AR, cbind(diag(p - 1), rep(0, p - 1)))
+     gamma <- vector(mode = "numeric", length = T)
+     gamma[1:p] <- sigmasq * solve(diag(p^2) - F %x% F)[1:p, 1]
+     for (j in (p + 1):T) gamma[[j]] <- t(gamma[(j - 1):(j - p)]) %*%
+         AR
+     specifications[[i]]$rho <- gamma/gamma[1]
+ }
```



2.2 R Facilities for ARMA Autocorrelations

Function ARMAacf can be used to calculate autocorrelations for an arbitrary ARMA process.

R code

```
> g3 <- ARMAacf(ar = numeric(0), ma = specifications[[3]]$MA, lag.max = T,
+     pacf = FALSE)
> print(specifications[[3]]$rho)
```

output

```
 [1]  1.0000000 -0.6666667  0.4871795 -0.3487179  0.1538462  0.0000000
 [7]  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000
[13]  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000
[19]  0.0000000  0.0000000
```

R code

```
> print(g3)
```

output

```
         0          1          2          3          4          5          6
 1.0000000 -0.6666667  0.4871795 -0.3487179  0.1538462  0.0000000  0.0000000
         7          8          9         10         11         12         13
```

```
 0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000
        14         15         16         17         18         19         20
 0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000  0.0000000
```

R code

```
> g4 <- ARMAacf(ar = specifications[[4]]$AR, ma = numeric(0), lag.max = T -
+     1, pacf = FALSE)
> print(specifications[[4]]$rho)
```

output

```
 [1] 1.00000000 0.80000000 0.64000000 0.51200000 0.40960000 0.32768000
 [7] 0.26214400 0.20971520 0.16777216 0.13421773 0.10737418 0.08589935
[13] 0.06871948 0.05497558 0.04398047 0.03518437 0.02814750 0.02251800
[19] 0.01801440 0.01441152
```

R code

```
> print(g4)
```

output

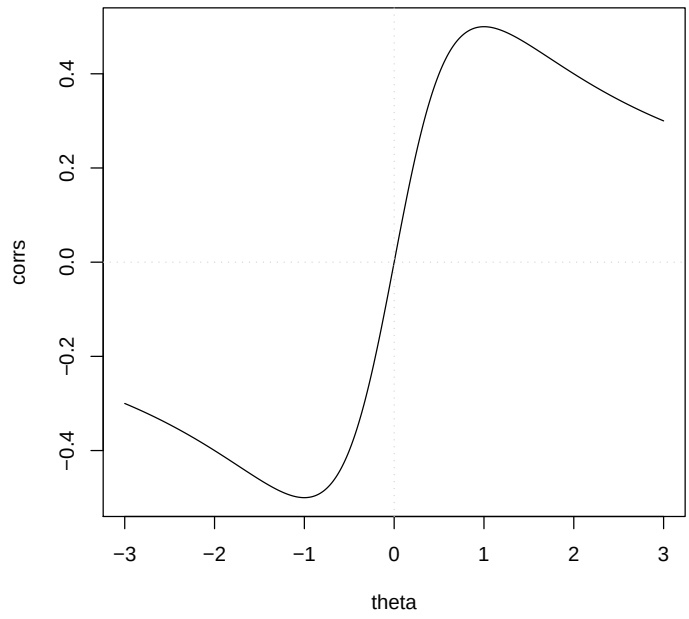
```
         0          1          2          3          4          5          6
1.00000000 0.80000000 0.64000000 0.51200000 0.40960000 0.32768000 0.26214400
         7          8          9         10         11         12         13
0.20971520 0.16777216 0.13421773 0.10737418 0.08589935 0.06871948 0.05497558
        14         15         16         17         18         19
0.04398047 0.03518437 0.02814750 0.02251800 0.01801440 0.01441152
```

2.3 Autocorrelations as a Function of the Moving Average Parameter

Figure 3.2 is easily generated from the formula for autocorrelations of an MA(1) process.

R code

```
> theta <- (-300:300) * 0.01
> corrs <- theta/(1 + theta^2)
> plot(theta, corrs, type = "l")
> grid(nx = 2, ny = 2)
```



2.4 Realizations of ARMA Processes

Pages 55 shows some realizations of AR processes. We will assume the innovations are drawn from a standard normal distribution.

R code

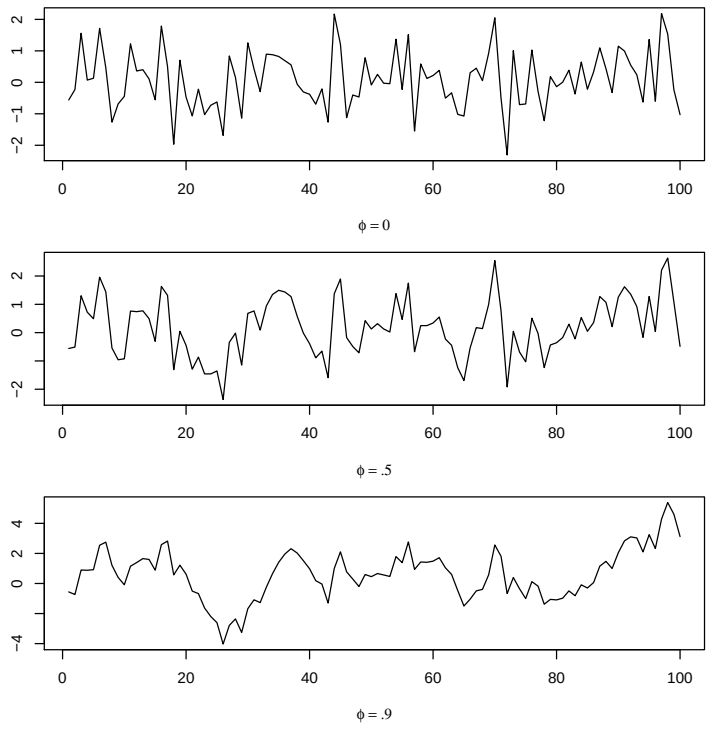
```
> specifications <- list(list(label = "f = 0", MA = vector(mode = "numeric"),
+     AR = vector(mode = "numeric")), list(label = "f = .5", MA = vector(mode = "numeric"),
+     AR = c(0.5)), list(label = "f = .9", MA = vector(mode = "numeric"),
+     AR = c(0.9)))
> T <- 100
> set.seed(123)
> epsilon <- rnorm(T, 0, 1)
```

These can be calculated by iterating forward on the defining equations.

R code

```
> simulate.forward <- function(specification, epsilon) {
+     T <- length(epsilon)
+     AR <- specification$AR
+     MA <- specification$MA
+     presample <- rep(0, max(length(AR), length(MA)))
+     epsilon <- c(presample, epsilon)
```

```
+     Y <- vector(mode = "numeric", length = T + length(presample))
+     Y[1:length(presample)] <- 0
+     for (i in (length(presample) + 1):(T + length(presample))) Y[i] <- epsilon[[i]] +
+         ifelse(length(AR) > 0, t(AR) %*% Y[(i - 1):(i - length(AR))],
+             0) + ifelse(length(MA) > 0, t(MA) %*% epsilon[(i -
+         1):(i - length(MA))], 0)
+     Y[(length(presample) + 1):(T + length(presample))]
+ }
> for (i in 1:length(specifications)) specifications[[i]]$Y <- simulate.forward(specifications[[i]],
+     epsilon)
```



2.5 R Facilities for simulating ARMA process

Function "simulate.forward" is a special case of capabilities provided by the function arima.sim in package stats, as the following code verifies.

R code

```
> for (specification in specifications) {
+     AR <- specification$AR
+     MA <- specification$MA
+     shift <- max(length(AR), length(MA))
+     Y <- arima.sim(model = list(order = c(length(AR), 0, length(MA)),
```

```
+         ar = AR, ma = MA), n = T, innov = epsilon[1:T], n.start = max(shift,
+         1), start.innov = rep(0, max(shift, 1)))
+     print(specification$Y[1:10])
+     print(Y[1:10])
+ }
```

output

```
 [1] -0.56047565 -0.23017749  1.55870831  0.07050839  0.12928774  1.71506499
 [7]  0.46091621 -1.26506123 -0.68685285 -0.44566197
 [1] -0.56047565 -0.23017749  1.55870831  0.07050839  0.12928774  1.71506499
 [7]  0.46091621 -1.26506123 -0.68685285 -0.44566197
 [1] -0.5604756 -0.5104153  1.3035007  0.7222587  0.4904171  1.9602735
 [7]  1.4410530 -0.5445347 -0.9591202 -0.9252221
 [1] -0.5604756 -0.5104153  1.3035007  0.7222587  0.4904171  1.9602735
 [7]  1.4410530 -0.5445347 -0.9591202 -0.9252221
 [1] -0.56047565 -0.73460557  0.89756330  0.87831536  0.91977156  2.54285939
 [7]  2.74948966  1.20947946  0.40167866 -0.08415118
 [1] -0.56047565 -0.73460557  0.89756330  0.87831536  0.91977156  2.54285939
 [7]  2.74948966  1.20947946  0.40167866 -0.08415118
```

4 Forecasting

4.1 A Box Jenkins Example

Example 4.1 from page 112 illustrates the Box-Jenkins approach based on autocorrelations. Here the data series is log changes of seasonally adjusted real US GNP from 1947 to 1988, available by simple transformations of the data in object "gnp1996". The data is prepared by selecting quarterly date from as shown, then computing the log of differences.

R code

```
> data(gnp1996, package = "RcompHam94")
> selection <- subset(gnp1996, Quarter >= "1947-01-01" & Quarter <=
+     "1988-10-01")
> y <- diff(log(selection$GNPH))
```

Page 110 shows how to compute sample autocorrelations - we will generate the first 20 to be used in plotting the results below.

R code

```
> max.lags <- 20
> T <- length(y)
> threshold <- 2/sqrt(T)
```

```
> gammas <- vector(mode = "numeric", length = max.lags + 1)
> gammas[[1]] <- 1/T * t(y - mean(y)) %*% (y - mean(y))
> for (j in 1:max.lags) gammas[j + 1] <- 1/T * t((y - mean(y))[(j +
+     1):T]) %*% (y - mean(y))[1:(T - j)]
> rhos <- gammas/gammas[[1]]
```

Page 111 shows how to compute sample partial autocorrelations.

R code

```
> subscripts <- outer(seq(1, max.lags), seq(1, max.lags), function(i,
+     j) {
+     abs(i - j)
+ })
> GAMMA <- array(gammas[as.vector(subscripts) + 1], c(max.lags,
+     max.lags))
> alphas <- vector(mode = "numeric", length = max.lags)
> for (m in 1:max.lags) alphas[m] <- solve(GAMMA[1:m, 1:m], gammas[2:(m +
+     1)])[[m]]
```

A plot of the outputs reproducing figure 4.2 is shown below. The source code is provided in the demo.

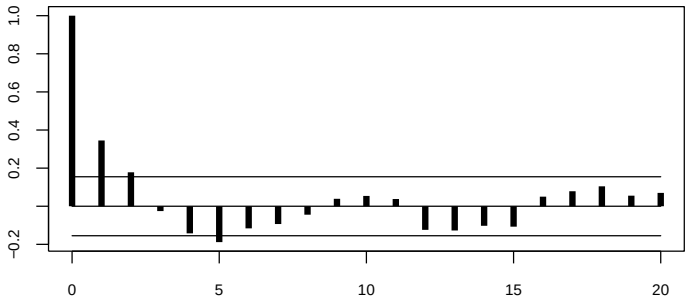


Figure 4.2(a) Sample autocorrelations

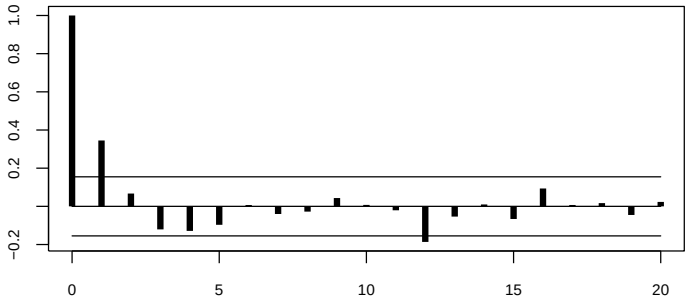


Figure 4.2(b) Sample partial autocorrelations

4.2 R Facilities for Sample Autocorrelations

Function acf from R package "stats" performs the same function as acf, as we can readily confirm.

R code

```
> acf.correlation <- acf(y, lag.max = max.lags, type = "correlation",
+     plot = FALSE, demean = TRUE)
> print(as.vector(acf.correlation$acf))
```

output

```
 [1]  1.00000000  0.34509475  0.17817758 -0.02537843 -0.14230681 -0.18827409
 [7] -0.11613672 -0.09335581 -0.04441490  0.03902657  0.05412612  0.03788102
[13] -0.12386994 -0.12725888 -0.10256196 -0.10719806  0.05022865  0.07874423
[19]  0.10451845  0.05540046  0.07001701
```

R code

```
> print(rhos)
```

output

```
 [1]  1.00000000  0.34509475  0.17817758 -0.02537843 -0.14230681 -0.18827409
 [7] -0.11613672 -0.09335581 -0.04441490  0.03902657  0.05412612  0.03788102
[13] -0.12386994 -0.12725888 -0.10256196 -0.10719806  0.05022865  0.07874423
[19]  0.10451845  0.05540046  0.07001701
```

R code

```
> acf.partial <- acf(y, lag.max = max.lags, type = "partial", plot = FALSE,
+     demean = TRUE)
> print(as.vector(acf.partial$acf))
```

output

```
 [1]  0.345094750  0.067075208 -0.120748043 -0.128609341 -0.096659383
 [6]  0.006935269 -0.040052970 -0.027544630  0.043507786  0.007543470
[11] -0.020592065 -0.186352407 -0.053599417  0.009939122 -0.066137883
[16]  0.093638650  0.007111983  0.016895000 -0.045185857  0.023227306
```

R code

```
> print(alphas)
```

output

```
 [1]  0.345094750  0.067075208 -0.120748043 -0.128609341 -0.096659383
 [6]  0.006935269 -0.040052970 -0.027544630  0.043507786  0.007543470
[11] -0.020592065 -0.186352407 -0.053599417  0.009939122 -0.066137883
[16]  0.093638650  0.007111983  0.016895000 -0.045185857  0.023227306
```

6 Spectral Analysis

Pages 167 to 170 give an example of the uses of spectral analysis, as applied to US Industrial Production from January 1947 to November 1989, available in data source "indprod". We will analyze the actual raw data, as well as one month and one year log changes.

R code

```
> data(indprod, package = "RcompHam94")
> selection <- subset(indprod, Month >= "1947-01-01" & Month <=
+     "1989-11-01")
> raw.data <- selection$IPMFG6
> logdiff.data <- 100 * diff(log(raw.data), lag = 1)
> yeardiff.data <- 100 * diff(log(raw.data), lag = 12)
```

For plotting purposes, generate frequencies at regular intervals as show on page 159. The first spectrum uses unsmoothed estimates, the last two use a Bartlett kernel.

We show this in two ways:

- Step by step function (page 16)
- Built-in function (page 17)

Step by step function

R code

```
> s.Y.omega <- function(omega, gammas, params) {
+     1/(2 * pi) * (gammas[[1]] + 2 * as.numeric(t(gammas[-1]) %*%
+         cos(1:(length(gammas) - 1) * omega)))
+ }
> s.Y.omega.Bartlett <- function(omega, gammas, params) {
+     1/(2 * pi) * (gammas[[1]] + 2 * as.numeric(t((1 - 1:params/(params +
+         1)) * gammas[2:(params + 1)]) %*% cos(1:params * omega)))
+ }
> generate.plot.data <- function(values, estimator, params) {
+     T <- length(values)
+     acf.covariance <- acf(values, lag.max = T - 1, type = "covariance",
+         plot = FALSE, demean = TRUE)
+     sapply(2 * pi/T * 1:((T - 1)/2), estimator, as.vector(acf.covariance$acf),
+         params)
+ }
> raw.s.Y.omega <- generate.plot.data(raw.data, s.Y.omega, NULL)
> logdiff.s.Y.omega <- generate.plot.data(logdiff.data, s.Y.omega.Bartlett,
```

```
+     12)
> yeardiff.s.Y.omega <- generate.plot.data(yeardiff.data, s.Y.omega.Bartlett,
+     12)
```

The resulting output is shown below.

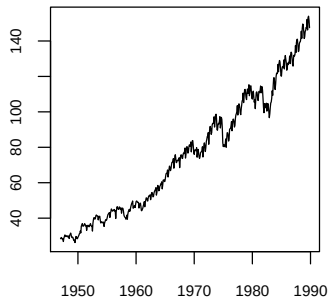


Figure 6.3 − FRB IP Index, NSA

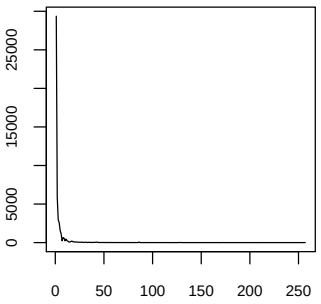


Figure 6.4 − Value of j

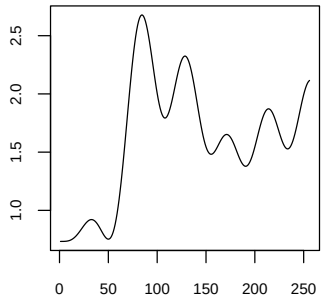


Figure 6.5 − Value of j

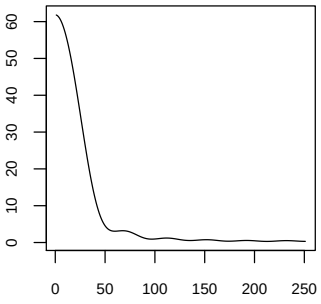


Figure 6.6 − Value of j

Built-in function We use here the function spectrum:

R code

```
> args(spectrum)
```

output

```
function (x, ..., method = c("pgram", "ar"))
NULL
```

R code

```
> sp <- spectrum(raw.data, plot = FALSE, span = 10)
> x <- 100 * diff(log(raw.data))
> sp2 <- spectrum(x, span = 6, plot = FALSE)
> x12 <- 100 * diff(log(raw.data), lag = 12)
> sp3 <- spectrum(x12, span = 20, plot = FALSE)
```

The resulting output is shown below.

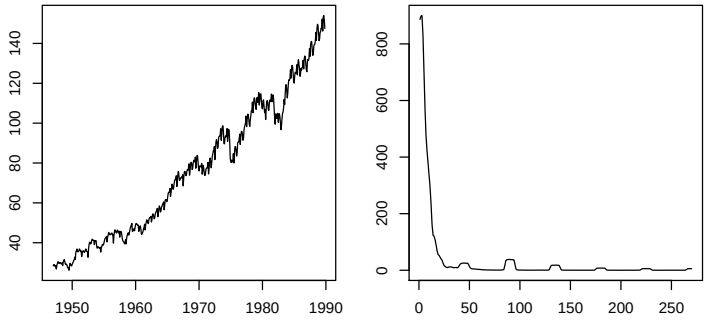


Figure 6.3 − FRB IP Index, NSA

Value of j

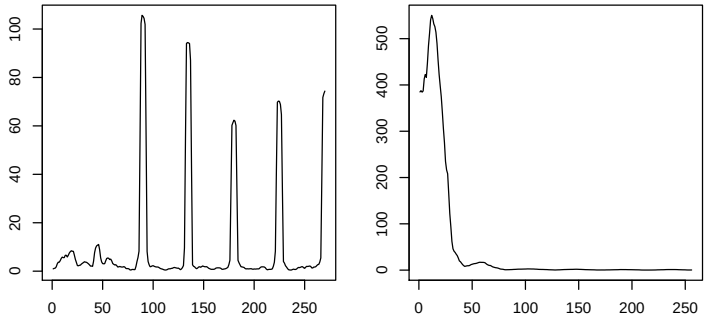


Figure 6.5 − Value of j

Figure 6.6 − Value of j

7 Asymptotic distribution theory

Further features in R

There exist a number of packages that illustrate interactively the central limit theorem or the law of large numbers, that can be useful to have an intuition on those theorems:

- Package `animation` has *clt.ani* and *lln.ani*
- Package `TeachingDemos` has *clt.examp*
- Package `ResearchMethods` has *cltDemo*

In addition, package `ConvergenceConcepts`, described in the R journal 1/2 2009 *enables one to investigate graphically the four classical modes of convergence of a sequence of random variables: convergence almost surely, convergence in probability, convergence in law and convergence in r-th mean.* (de Micheaux and Liquet (2009))

13 The Kalman Filter

13.1 Kalman Filtering Example Applied to Detecting Business Cycles

Page 376 describes an application of Kalman filtering to business cycles by James Stock and Mark Watson.

This can be implemented in two steps. The first is to implement the Kalman algorithm as described in the text. The following function follows the notation in Chapter 13.

R code

```
> kalman <- function(H, R, F, x, A, y, Q, xi.1.0, P.1.0) {
+     T <- dim(x)[[2]]
+     P.t.t_1 <- array(dim = c(dim(P.1.0), T + 1))
+     P.t.t_1[, , 1] <- P.1.0
+     P.t.t <- array(dim = c(dim(P.1.0), T))
+     K.t <- array(dim = c(dim(H), T))
+     xi.t.t_1 <- array(dim = c(length(xi.1.0), T + 1))
+     xi.t.t_1[, 1] <- xi.1.0
+     xi.t.t <- array(dim = c(length(xi.1.0), T))
+     L <- 0
+     for (tt in 1:T) {
+         V <- solve(t(H) %*% P.t.t_1[, , tt] %*% H + R)
+         K.t[, , tt] <- P.t.t_1[, , tt] %*% H %*% V
+         P.t.t[, , tt] <- P.t.t_1[, , tt] - K.t[, , tt] %*% t(H) %*%
+             P.t.t_1[, , tt]
+         P.t.t_1[, , tt + 1] <- F %*% P.t.t[, , tt] %*% t(F) +
+             Q
+         w <- y[, tt] - t(A) %*% x[, tt] - t(H) %*% xi.t.t_1[,
+             tt]
+         xi.t.t[, tt] <- xi.t.t_1[, tt] + K.t[, , tt] %*% w
+         xi.t.t_1[, tt + 1] <- F %*% xi.t.t[, tt]
+         L <- L - 1/2 * dim(y)[[1]] * log(2 * pi) + 1/2 * log(det(V)) -
+             1/2 * t(w) %*% V %*% w
+     }
+     xi.t.T <- array(dim = c(length(xi.1.0), T))
+     xi.t.T[, T] <- xi.t.t[, T]
+     P.t.T <- array(dim = c(dim(P.1.0), T))
+     P.t.T[, , T] <- P.t.t[, , T]
+     for (tt in (T - 1):1) {
+         Jt <- P.t.t[, , tt] %*% t(F) %*% solve(P.t.t_1[, , tt +
```

```
+             1])
+         xi.t.T[, tt] <- xi.t.t[, tt] + Jt %*% (xi.t.T[, tt +
+             1] - xi.t.t_1[, tt + 1])
+         P.t.T[, , tt] <- P.t.t[, , tt] + Jt %*% (P.t.T[, , tt +
+             1] - P.t.t_1[, , tt + 1]) %*% t(Jt)
+     }
+     list(xi.t.t = xi.t.t, xi.t.t_1 = xi.t.t_1, P.t.t = P.t.t,
+         P.t.t_1 = P.t.t_1, K.t = K.t, log.likelihood = L, xi.t.T = xi.t.T,
+         P.t.T = P.t.T)
+ }
```

The second is to specify the state space model as described on pp376-377 and estimate the parameters via maximum likelihood. Data for this analysis is consumption and income data form dataset "coninc" in log differences.

R code

```
> data(coninc, package = "RcompHam94")
> YGR <- diff(log(coninc$GYD82))
> CGR <- diff(log(coninc$GC82))
> y <- t(cbind(YGR - mean(YGR), CGR - mean(CGR)))
```

The following helper function converts the parameters from a vector of labeled components into the correct inputs for the filter as shown in equations [13.1.28], [13.1.29], and [13.1.30].

R code

```
> THETA <- c(phic = 0.9, phi1 = 0.9, phi2 = 0.9, g1 = 0.5, g2 = 0.5,
+     sigc = 0.05^0.5, sig11 = 0.05^0.5, sig22 = 0.05^0.5, r11 = sd(YGR),
+     r22 = sd(CGR))
> theta.y.to.params <- function(THETA, y) {
+     params <- list(F = diag(THETA[c("phic", "phi1", "phi2")]),
+         Q = diag(THETA[c("sigc", "sig11", "sig22")]^2), H = rbind(THETA[c("g1",
+             "g2")], diag(2)), R = diag(THETA[c("r11", "r22")]^2),
+         A = diag(c(0, 0)), x = c(1, 1) %o% rep(1, dim(y)[[2]]),
+         xi.1.0 = c(0, 0, 0))
+     c(params, list(P.1.0 = array(solve(diag(length(params$xi.1.0)^2) -
+         params$F %x% params$F, as.vector(params$Q)), c(length(params$xi.1.0),
+         length(params$xi.1.0)))))
+ }
```

The objective function is the log.likelihood obtained from the Kalman iteration.

R code

```
> objective <- function(THETA, y) {
+     params <- theta.y.to.params(THETA, y)
```

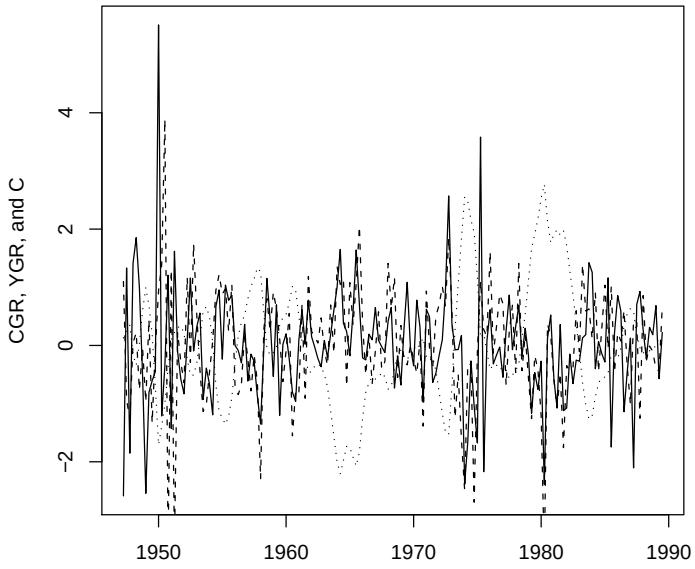
```
+     kalman(params$H, params$R, params$F, params$x, params$A,
+         y, params$Q, params$xi.1.0, params$P.1.0)$log.likelihood
+ }
> optimizer.results <- optim(par = THETA, fn = objective, gr = NULL,
+     y = y, control = list(trace = 0))
```

Finally calculate the smoothed results based on the ML estimated parameters.

R code

```
> params <- theta.y.to.params(optimizer.results$par, y)
> smoothed.results <- kalman(params$H, params$R, params$F, params$x,
+     params$A, y, params$Q, params$xi.1.0, params$P.1.0)
> smoothed.data <- smoothed.results$xi.t.T[1, ]
```

The results of the smoothed inference are shown below.



13.2 R facilities for Kalman Filtering

There are several different packages in R for Kalman filtering, some that provide univariate support, others multivariate support. For example, package `FKF` is a fast implementation, but there are others. One key aspect of using such packages is specifying an interface to allow for time varying inputs, and providing results

under those conditions. Some packages use caller supplied functions, others check for dimensions of (up to three dimensional) arrays, etc.

For example, a simple implementation of the example on page 382 using function *kalman* above might look like:

R code

```
> sigmasq <- 2
> params <- list(F = array(c(0, 1, 0, 0), c(2, 2)), Q = diag(c(sigmasq,
+     0)), H = array(c(1, 0.8), c(2, 1)), R = array(0, c(1, 1)),
+     A = array(0.5, c(1, 1)), x = 1 %o% rep(1, 5), y = 1 %o% c(1,
+         seq(0.5, 4)), xi.1.0 = c(0, 0))
> params <- c(params, list(P.1.0 = array(solve(diag(length(params$xi.1.0)^2) -
+     params$F %x% params$F, as.vector(params$Q)), c(length(params$xi.1.0),
+     length(params$xi.1.0)))))
> myResults <- kalman(params$H, params$R, params$F, params$x, params$A,
+     params$y, params$Q, params$xi.1.0, params$P.1.0)
```

We can perform the some operations using package FKF with a slight alteration of the function arguments. In particular, many of the arguments using an outer product as a quick way to convert them into a structure of one additional dimension, with the length of the additional dimension being 1. This is a convenient calling convention to specifying a *non* time varying parameter. If the parameter *were* time varying then the full extra dimension would be used. For example, the F matrix can be time varying in FKF (called Tt). A call exploiting this would then have a vector of two dimensional F matrices, one for each time index, i.e. a three dimensional array. If F is not time varying, (as in the case of the simple example above) then a three dimensional array with the third dimension being of length 1 is used.

R code

```
> fkfResults <- FKF::fkf(a0 = params$xi.1.0, P0 = params$P.1.0,
+     dt = rep(0, length(params$xi.1.0)) %o% 1, Tt = params$F %o%
+         1, HHt = params$Q %o% 1, ct = t(params$A) %*% params$x,
+     Zt = t(params$H) %o% 1, GGt = params$R %o% 1, yt = params$y,
+     check.input = TRUE)
```

The results can be confirmed by examing the output:

R code

```
> print(myResults$xi.t.t)
```

output

```
          [,1]       [,2]        [,3]     [,4]     [,5]
[1,] 0.3048780 -0.1951600  1.02502699 1.100137 2.031900
[2,] 0.2439024  0.2439500 -0.03128374 1.124828 1.210125
```

R code

```
> print(fkfResults$att)
```

output

```
          [,1]       [,2]        [,3]     [,4]     [,5]
[1,] 0.3048780 -0.1951600  1.02502699 1.100137 2.031900
[2,] 0.2439024  0.2439500 -0.03128374 1.124828 1.210125
```

14 Generalized Method of Moments

14.1 Classical Method of Moments

Pages 409-410 gives a simple example of estimating the degrees of freedom of a standard t distribution. To illustrate, first generate a sample of 500 observations from a t distribution with 10 degrees of freedom.

R code

```
> Y <- rt(500, 10)
```

Then maximize the sum of logs of a t density evaluated on the sample points.

R code

```
> objective <- function(nu, Y) {
+     -sum(log(dt(Y, df = nu)))
+ }
> classical.results <- optimize(interval = c(1, 30), f = objective,
+     Y = Y)
> mu2 <- mean(Y^2)
> nu <- 2 * mu2/(mu2 - 1)
> print(classical.results)
```

output

```
$minimum
[1] 6.453701

$objective
[1] 786.0864
```

R code

```
> print(nu)
```

output

```
[1] 6.825244
```

14.2 Generalized Method of Moments

Using the sample sample, we can estimate the degrees of freedom using GMM. To this end define a function following the GMM recipe in the text.

R code

```
> compute.estimates <- function(Y, h, interval) {
+     g <- function(Y, THETA) {
+         apply(X = apply(X = Y, MARGIN = 1, FUN = h, THETA = THETA),
+             MARGIN = 1, FUN = mean)
+     }
+     objective <- function(THETA, Y, W) {
+         g.value <- g(Y, THETA)
+         t(g.value) %*% W %*% g.value
+     }
+     r <- length(h(Y[1, ], interval[[1]]))
+     a <- length(interval[[1]])
+     T <- dim(Y)[[1]]
+     stage.1.results <- optimize(interval = interval, f = objective,
+         Y = Y, W = diag(r))
+     temp <- apply(X = Y, MARGIN = 1, FUN = h, THETA = stage.1.results$objective)
+     S <- 1/T * temp %*% t(temp)
+     stage.2.results <- optimize(interval = interval, f = objective,
+         Y = Y, W = solve(S))
+     J.test <- 1 - pchisq(T * stage.2.results$objective, r - a)
+     list(stage.1.results = stage.1.results, stage.2.results = stage.2.results,
+         overidentifying = J.test)
+ }
```

Using this function is then a matter of specifying an appropriate function h to define an observation of the set of moments being targeted.

R code

```
> h <- function(Yt, THETA) {
+     nu <- THETA
+     c(Yt^2 - nu/(nu - 2), Yt^4 - 3 * nu^2/((nu - 2) * (nu - 4)))
+ }
> estimates <- compute.estimates(Y %o% 1, h, interval = c(5, 30))
> print(estimates)
```

output

```
$stage.1.results
$stage.1.results$minimum
[1] 7.660023

$stage.1.results$objective
            [,1]
[1,] 0.003742687


$stage.2.results
$stage.2.results$minimum
[1] 8.148267

$stage.2.results$objective
            [,1]
[1,] 0.001194489


$overidentifying
          [,1]
[1,] 0.4396313
```

A second example estimates the shape parameter of a two-sided gamma distribution.

R code

```
> Yg <- rgamma(500, 10) * sign(runif(500, -1, 1))
> hg <- function(Yt, THETA) {
+     k <- THETA
+     nu <- k
+     mu <- k
+     sigma <- k
+     skew <- 2/sqrt(k)
+     kurt <- 6/k
+     c(Yt^2 - sigma - mu^2, Yt^4 - (kurt * (sigma^2) + 3) - 4 *
+         (skew * sigma^1.5) * mu - 6 * sigma * mu^2 - mu^4)
+ }
> gestimates <- compute.estimates(Yg %o% 1, hg, interval = c(5,
+     30))
> print(gestimates)
```

output

```
$stage.1.results
$stage.1.results$minimum
[1] 10.58884

$stage.1.results$objective
         [,1]
[1,] 17.59123


$stage.2.results
$stage.2.results$minimum
[1] 10.14325

$stage.2.results$objective
             [,1]
[1,] 0.0006710071


$overidentifying
          [,1]
[1,] 0.5624362
```

14.3 R Facilities for Generalized Method of Moments

TBD

15 Models of Nonstationary Time Series

15.1 Fractional Integration

This example uses package package *fracdiff* to compute the exponent of fractional integration as described on pp 448-449. We use the function *fdGPH*:

R code

```
> library(fracdiff)
> args(fdGPH)
```

output

```
function (x, bandw.exp = 0.5)
NULL
```

Applied on US GDP and Treasury Yields data:

R code

```
> data(gnptbill, package = "RcompHam94")
> print(fdGPH(gnptbill$GNP))
```

output

```
$d
[1] 0.9588756

$sd.as
[1] 0.2427173

$sd.reg
[1] 0.04061276
```

R code

```
> print(fdGPH(gnptbill$TBILL))
```

output

```
$d
[1] 0.9511594

$sd.as
[1] 0.2427173

$sd.reg
[1] 0.227921
```

17 Univariate Processes with Unit Roots

19 Cointegration

19.1 Testing Cointegration when the Cointegrating Vector is Known

Section 19.2, beginning on page 582 describes cointegration testing of purchasing power parity between Italian lire and US dollars. The data used is 100 times log monthly price levels and spot nominal and real exchange rates, normalized to a value of zero at the start of the series.

R code

```
> data(ppp, package = "RcompHam94")
> selection <- subset(ppp, Month >= "1973-01-01" & Month <= "1989-10-01")
```

```
> ppp.data <- data.frame(Month = selection$Month, pstar = 100 *
+     log(selection$PC6IT/selection$PC6IT[[1]]), p = 100 * log(selection$PZUNEW/selection$PZUNEW[[1]
+     ner = -100 * log(selection$EXRITL/selection$EXRITL[[1]]))
> ppp.data[["rer"]] <- ppp.data$p - ppp.data$ner - ppp.data$pstar
```

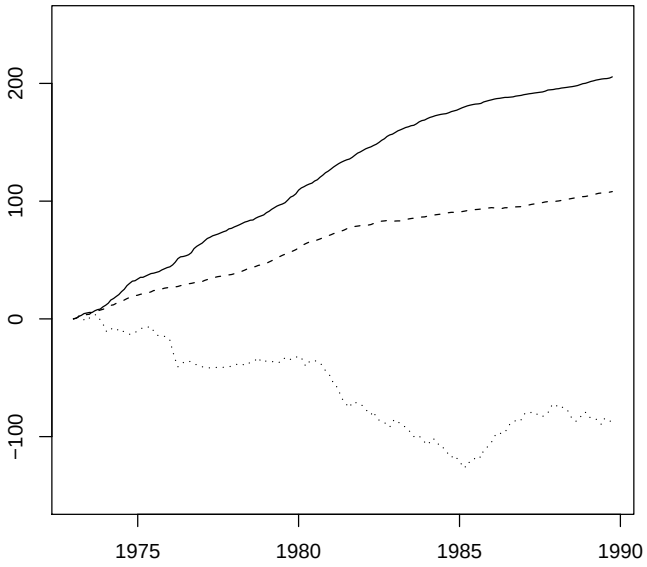


Figure 19.2

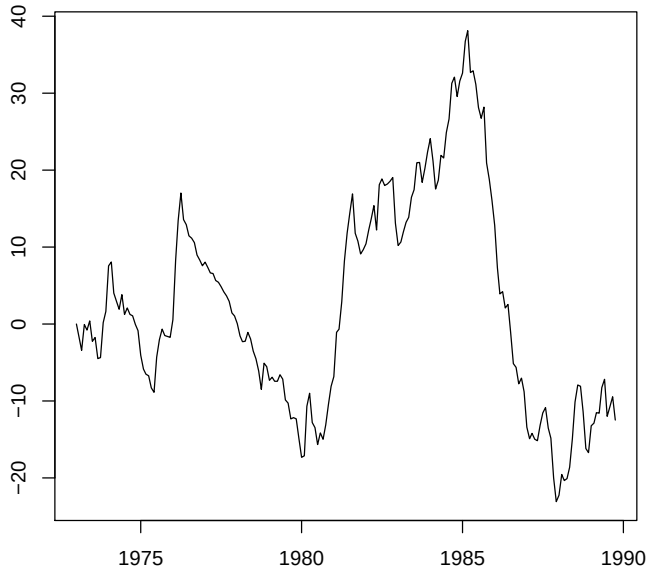


Figure 19.3

To save time define a simple utility function to perform augmented Dickey Fuller analysis according to the conventions in the text.

R code

```
> do.DF <- function(series, lag) {
+     T <- length(series)
+     df.lms <- summary(lm(yt ~ yt_1 + tt + delta.yt_ + 1, list(yt = series[-1:-(lag +
+         1)], delta.yt_ = embed(diff(series[-T]), lag), yt_1 = series[c(-1:-lag,
+         -(T:T))], tt = (lag + 2):T)))
+     df.results <- Dickey.Fuller(T = length(series) - lag - 1,
+         rho = df.lms$coefficients[["yt_1", "Estimate"]], sigma.rho = df.lms$coefficients[["yt_1",
+             "Std. Error"]], zeta = df.lms$coefficients[paste("delta.yt_",
+             1:lag, sep = ""), "Estimate"])
+     F <- Wald.F.Test(R = cbind(rep(0, 2), diag(2), rep(0, 2) %o%
+         rep(0, lag)), b = df.lms$coefficients[, "Estimate"],
+         r = c(1, 0), s2 = df.lms$sigma^2, XtX_1 = df.lms$cov.unscaled)
+     print(df.lms$coefficients)
+     print(df.results)
+     print(F)
+ }
```

Following the text, check each series with a Dickey Fuller test with a regression estimated with twelve lags.

R code

```
> for (series.name in c("p", "pstar", "ner", "rer")) do.DF(series = ppp.data[[series.name]],
+     lag = 12)
```

output

```
                Estimate  Std. Error     t value      Pr(>|t|)
(Intercept)  0.136160926 0.085779070   1.5873444  1.142502e-01
yt_1         0.994004087 0.003067474 324.0464885 6.323397e-244
tt           0.002927051 0.001766655   1.6568325  9.935541e-02
delta.yt_1   0.553397837 0.075217880   7.3572644  7.109482e-12
delta.yt_2  -0.056908322 0.085440124  -0.6660609  5.062543e-01
delta.yt_3   0.070125117 0.084906900   0.8259060  4.099884e-01
delta.yt_4   0.060389596 0.081969953   0.7367284  4.622797e-01
delta.yt_5  -0.078232496 0.078488461  -0.9967388  3.202754e-01
delta.yt_6  -0.048376861 0.070721885  -0.6840437  4.948576e-01
delta.yt_7   0.165843348 0.068915448   2.4064757  1.715410e-02
delta.yt_8  -0.070207448 0.070014467  -1.0027563  3.173709e-01
delta.yt_9   0.244644550 0.070161410   3.4868819  6.187074e-04
delta.yt_10 -0.110047172 0.072579707  -1.5162251  1.312771e-01
delta.yt_11  0.117580628 0.072937432   1.6120753  1.087579e-01
delta.yt_12  0.046702346 0.068650314   0.6802933  4.972230e-01
$T
[1] 189

$rho
[1] 0.994004

$sigma.rho
[1] 0.003067474

$zeta
 delta.yt_1  delta.yt_2  delta.yt_3  delta.yt_4  delta.yt_5  delta.yt_6
 0.55339784 -0.05690832  0.07012512  0.06038960 -0.07823250 -0.04837686
 delta.yt_7  delta.yt_8  delta.yt_9 delta.yt_10 delta.yt_11 delta.yt_12
 0.16584335 -0.07020745  0.24464455 -0.11004717  0.11758063  0.04670235

$rho.stat
[1] -10.78352

$t.stat
[1] -1.954675
```

```
[1] 2.412933
               Estimate  Std. Error     t value      Pr(>|t|)
(Intercept)  0.768007976 0.253071035   3.0347526  2.776788e-03
yt_1         0.999456707 0.004116999 242.7633949 3.768702e-222
tt          -0.002406065 0.004989081  -0.4822662  6.302229e-01
delta.yt_1   0.420701728 0.076110499   5.5275124  1.170691e-07
delta.yt_2  -0.011592127 0.081521266  -0.1421976  8.870885e-01
delta.yt_3   0.013439685 0.080162382   0.1676558  8.670488e-01
delta.yt_4   0.077206365 0.080125530   0.9635676  3.366000e-01
delta.yt_5  -0.036494296 0.080087139  -0.4556824  6.491866e-01
delta.yt_6   0.145282237 0.078670504   1.8467180  6.648647e-02
delta.yt_7  -0.099118088 0.078839877  -1.2572075  2.103634e-01
delta.yt_8   0.046717520 0.078598766   0.5943798  5.530301e-01
delta.yt_9  -0.049982364 0.078111841  -0.6398820  5.230909e-01
delta.yt_10 -0.034638353 0.078168372  -0.4431249  6.582258e-01
delta.yt_11  0.075555037 0.077993666   0.9687330  3.340230e-01
delta.yt_12  0.021863739 0.073346671   0.2980877  7.659919e-01
$T
[1] 189

$rho
[1] 0.9994567

$sigma.rho
[1] 0.004116999

$zeta
 delta.yt_1  delta.yt_2  delta.yt_3  delta.yt_4  delta.yt_5  delta.yt_6
 0.42070173 -0.01159213  0.01343968  0.07720637 -0.03649430  0.14528224
 delta.yt_7  delta.yt_8  delta.yt_9 delta.yt_10 delta.yt_11 delta.yt_12
-0.09911809  0.04671752 -0.04998236 -0.03463835  0.07555504  0.02186374

$rho.stat
[1] -0.2382095

$t.stat
[1] -0.1319633

[1] 4.249956
               Estimate  Std. Error     t value      Pr(>|t|)
(Intercept) -0.389337356 0.413800921 -0.94088084  3.480703e-01
yt_1         0.982941298 0.010766440 91.29678192 6.506909e-149
```

```
tt          -0.007384125 0.006883901 -1.07266573  2.849066e-01
delta.yt_1   0.348829755 0.074439036  4.68611329  5.595654e-06
delta.yt_2  -0.025567401 0.079110764 -0.32318485  7.469433e-01
delta.yt_3   0.002617322 0.078947706  0.03315261  9.735909e-01
delta.yt_4   0.011689457 0.080007934  0.14610372  8.840086e-01
delta.yt_5   0.099314112 0.079948258  1.24222983  2.158234e-01
delta.yt_6   0.001387289 0.080819939  0.01716518  9.863245e-01
delta.yt_7   0.063205400 0.080614348  0.78404653  4.340788e-01
delta.yt_8   0.117223384 0.080560981  1.45508883  1.474464e-01
delta.yt_9  -0.061127657 0.080788556 -0.75663757  4.502903e-01
delta.yt_10  0.081739596 0.080696462  1.01292665  3.125017e-01
delta.yt_11  0.037261364 0.080646524  0.46203311  6.446347e-01
delta.yt_12 -0.030363466 0.076740775 -0.39566275  6.928385e-01
$T
[1] 189

$rho
[1] 0.9829413

$sigma.rho
[1] 0.01076644

$zeta
  delta.yt_1   delta.yt_2   delta.yt_3   delta.yt_4   delta.yt_5   delta.yt_6
 0.348829755 -0.025567401  0.002617322  0.011689457  0.099314112  0.001387289
  delta.yt_7   delta.yt_8   delta.yt_9  delta.yt_10  delta.yt_11  delta.yt_12
 0.063205400  0.117223384 -0.061127657  0.081739596  0.037261364 -0.030363466

$rho.stat
[1] -9.112996

$t.stat
[1] -1.584433

[1] 1.489674
                Estimate  Std. Error     t value      Pr(>|t|)
(Intercept)  0.0532014210 0.390557357  0.13621923  8.918054e-01
yt_1         0.9712932573 0.014145189 68.66597772 5.679805e-128
tt          -0.0004612496 0.003237185 -0.14248477  8.868620e-01
delta.yt_1   0.3178370194 0.074163266  4.28563944  3.010943e-05
delta.yt_2  -0.0149166870 0.078078854 -0.19104644  8.487119e-01
delta.yt_3   0.0127973250 0.077727723  0.16464299  8.694161e-01
```

```
delta.yt_4   0.0224258044 0.078676900  0.28503671  7.759550e-01
delta.yt_5   0.0845155831 0.078339518  1.07883716  2.821536e-01
delta.yt_6  -0.0030653274 0.079071534 -0.03876651  9.691210e-01
delta.yt_7   0.0299137752 0.078750797  0.37985362  7.045173e-01
delta.yt_8   0.0824197050 0.078641636  1.04804158  2.960730e-01
delta.yt_9  -0.0478615036 0.078647910 -0.60855405  5.436137e-01
delta.yt_10  0.0755667133 0.078405880  0.96378886  3.364893e-01
delta.yt_11  0.0504082264 0.078279945  0.64394816  5.204570e-01
delta.yt_12 -0.0124704308 0.075997755 -0.16408946  8.698512e-01
$T
[1] 189

$rho
[1] 0.9712933

$sigma.rho
[1] 0.01414519

$zeta
  delta.yt_1   delta.yt_2   delta.yt_3   delta.yt_4   delta.yt_5   delta.yt_6
 0.317837019 -0.014916687  0.012797325  0.022425804  0.084515583 -0.003065327
  delta.yt_7   delta.yt_8   delta.yt_9  delta.yt_10  delta.yt_11  delta.yt_12
 0.029913775  0.082419705 -0.047861504  0.075566713  0.050408226 -0.012470431

$rho.stat
[1] -13.48204

$t.stat
[1] -2.029435

[1] 2.078078
```

Now check the real exchange rate with a Phillips Perron test

R code

```
> pp.lms <- summary(lm(zt ~ zt_1 + 1, data.frame(zt = ppp.data$rer[-1],
+     zt_1 = ppp.data$rer[-length(ppp.data$rer)])))
> PP.results <- Phillips.Perron(T = length(pp.lms$residuals), rho = pp.lms$coefficients[["zt_1",
+     "Estimate"]], sigma.rho = pp.lms$coefficients[["zt_1", "Std. Error"]],
+     s = pp.lms$sigma, lambda.hat.sq = as.numeric(Newey.West(pp.lms$residuals %o%
+         1, 12)), gamma0 = mean(pp.lms$residuals^2))
> print(pp.lms$coefficients)
```

output

```
              Estimate Std. Error    t value      Pr(>|t|)
(Intercept) -0.0297931 0.17835718 -0.1670418  8.675068e-01
zt_1         0.9865420 0.01275287 77.3584248 1.854719e-150
```

R code

```
> print(PP.results)
```

output

```
$T
[1] 201

$rho
[1] 0.986542

$sigma.rho
[1] 0.01275287

$s.sq
[1] 6.205887

$lambda.hat.sq
[1] 13.03064

$gamma0
[1] 6.144137

$rho.stat
[1] -6.35068

$t.stat
[1] -1.706128
```

Estimating the impulse response function gives a sense of the persistence of deviations from PPP.

R code

```
> ar.results <- ar(ppp.data$rer, aic = FALSE, order.max = 13, method = "ols",
+     demean = TRUE)
> tt <- seq(1, 72)
> start.innov <- rep(0, 13)
> et <- c(start.innov, 1, rep(0, length(tt) - 14))
> arima.sim.output <- arima.sim(list(order = c(13, 0, 0), ar = ar.results$ar),
```

```
+     n = length(tt), innov = et, n.start = length(start.innov),
+     start.innov = start.innov)
> irf <- as.vector(arima.sim.output)
```

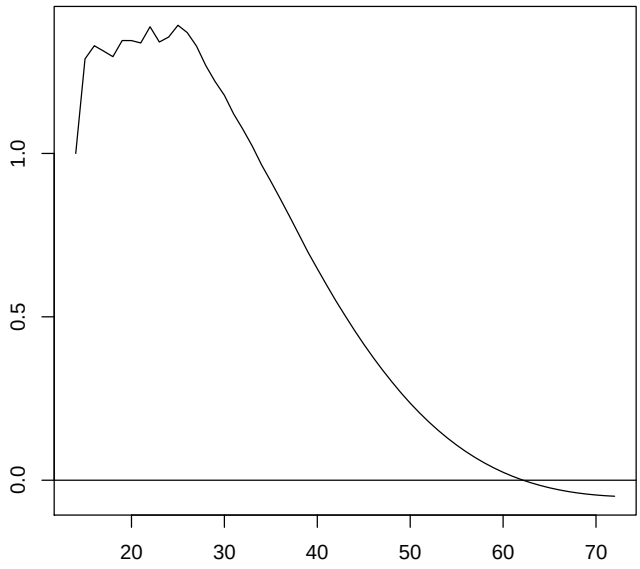


Figure 19.4

19.2 Estimating the Cointegrating Vector

Page 598 shows an example of the Phillips Ouliaris Hansen procedure for estimating a cointegrating vector.

R code

```
> poh.cointegration.lm <- lm(p ~ 1 + ner + pstar, ppp.data)
> poh.residual.lms <- summary(lm(u ~ 0 + u_1, data.frame(u = poh.cointegration.lm$residuals[-1],
+     u_1 = poh.cointegration.lm$residuals[-length(poh.cointegration.lm$residuals)])))
> POH.results <- Phillips.Perron(T = length(poh.residual.lms$residuals),
+     rho = poh.residual.lms$coefficients[["u_1", "Estimate"]],
+     sigma.rho = poh.residual.lms$coefficients[["u_1", "Std. Error"]],
+     s = poh.residual.lms$sigma, lambda.hat.sq = as.numeric(Newey.West(poh.residual.lms$residuals %
+         1, 12)), gamma0 = mean(poh.residual.lms$residuals^2))
> print(summary(poh.cointegration.lm)$coefficients)
```

output

```
              Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 2.71231296 0.367695493  7.376519  4.298888e-12
```

```
ner          0.05134848 0.012045369  4.262923  3.114337e-05
pstar        0.53004097 0.006708385 79.011705 3.148050e-152
```

R code

```
> print(poh.residual.lms$coefficients)
```

output

```
     Estimate Std. Error  t value     Pr(>|t|)
u_1 0.9833108 0.01171956 83.90338 7.71577e-158
```

R code

```
> print(POH.results)
```

output

```
$T
[1] 201

$rho
[1] 0.9833108

$sigma.rho
[1] 0.01171956

$s.sq
[1] 0.1630028

$lambda.hat.sq
[1] 0.4082242

$gamma0
[1] 0.1621919

$rho.stat
[1] -7.542281

$t.stat
[1] -2.020981
```

A second example performs a similar analysis on quarterly US consumption and income data from 1947Q1 to 1989Q4.

R code

```
> data(coninc, package = "RcompHam94")
> selection <- subset(coninc, Quarter >= "1947-01-01" & Quarter <=
```

```
+     "1989-07-01")
> coninc.data <- data.frame(Quarter = selection$Quarter, cons = 100 *
+     log(selection$GC82), inc = 100 * log(selection$GYD82))
```

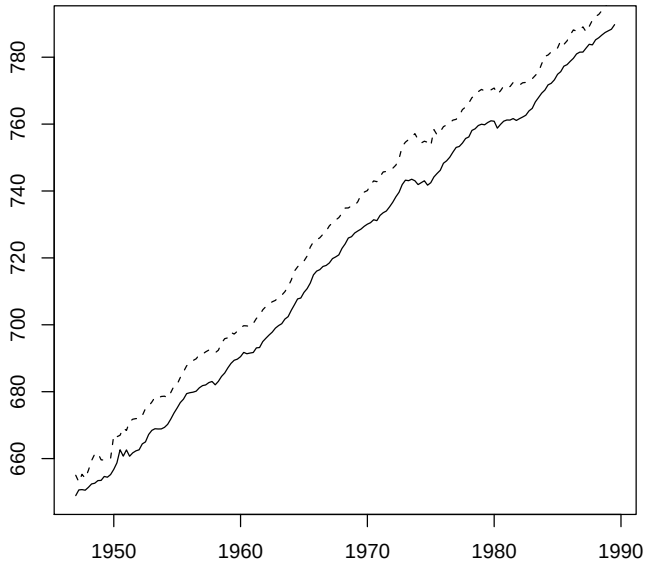


Figure 19.5

Test individual series for unit root status using Dickey Fuller.

R code

```
> for (series.name in c("inc", "cons")) do.DF(series = coninc.data[[series.name]],
+     lag = 6)
```

output

```
               Estimate  Std. Error     t value     Pr(>|t|)
(Intercept) 20.336729221 15.04162460  1.35203010 1.783352e-01
yt_1         0.970584904  0.02306293 42.08419931 1.029200e-86
tt           0.023796844  0.01985318  1.19864142 2.324968e-01
delta.yt_1  -0.006528755  0.08092856 -0.08067307 9.358060e-01
delta.yt_2  -0.035846316  0.08025935 -0.44663103 6.557649e-01
delta.yt_3   0.102128545  0.07758036  1.31642276 1.899755e-01
delta.yt_4  -0.187536343  0.07699406 -2.43572477 1.599577e-02
delta.yt_5  -0.037187883  0.07813842 -0.47592314 6.347992e-01
delta.yt_6   0.027855951  0.07662877  0.36351818 7.167132e-01
$T
[1] 164
```

```
$rho
[1] 0.970585

$sigma.rho
[1] 0.02306293

$zeta
  delta.yt_1   delta.yt_2   delta.yt_3   delta.yt_4   delta.yt_5   delta.yt_6
-0.006528755 -0.035846316  0.102128545 -0.187536343 -0.037187883  0.027855951

$rho.stat
[1] -4.242382

$t.stat
[1] -1.275428

[1] 1.132134
               Estimate  Std. Error    t value     Pr(>|t|)
(Intercept) 29.46860131 15.19248322  1.9396830 5.423391e-02
yt_1         0.95552168  0.02360001 40.4881863 2.508405e-84
tt           0.03721088  0.02006161  1.8548306 6.552012e-02
delta.yt_1   0.03624864  0.07979877  0.4542506 6.502840e-01
delta.yt_2   0.25964745  0.07935028  3.2721680 1.315743e-03
delta.yt_3   0.06273192  0.08172798  0.7675697 4.439106e-01
delta.yt_4  -0.05234112  0.08122252 -0.6444163 5.202580e-01
delta.yt_5  -0.04791625  0.07956524 -0.6022260 5.479037e-01
delta.yt_6  -0.06782142  0.07919698 -0.8563637 3.931186e-01
$T
[1] 164

$rho
[1] 0.9555217

$sigma.rho
[1] 0.02360001

$zeta
 delta.yt_1  delta.yt_2  delta.yt_3  delta.yt_4  delta.yt_5  delta.yt_6
 0.03624864  0.25964745  0.06273192 -0.05234112 -0.04791625 -0.06782142

$rho.stat
[1] -9.011597
```

```
$t.stat
[1] -1.884673

[1] 1.858290
```

Estimate cointegration vector, then check for unit root status of the residual using Phillips Perron.

R code

```
> poh.cointegration.lm <- lm(cons ~ 1 + inc, coninc.data)
> poh.residual.lms <- summary(lm(u ~ 0 + u_1, data.frame(u = poh.cointegration.lm$residuals[-1],
+     u_1 = poh.cointegration.lm$residuals[-length(poh.cointegration.lm$residuals)])))
> POH.results <- Phillips.Perron(T = length(poh.residual.lms$residuals),
+     rho = poh.residual.lms$coefficients[["u_1", "Estimate"]],
+     sigma.rho = poh.residual.lms$coefficients[["u_1", "Std. Error"]],
+     s = poh.residual.lms$sigma, lambda.hat.sq = as.numeric(Newey.West(poh.residual.lms$residuals %
+         1, 6)), gamma0 = mean(poh.residual.lms$residuals^2))
> print(summary(poh.cointegration.lm)$coefficients)
```

output

```
             Estimate  Std. Error     t value      Pr(>|t|)
(Intercept) 0.6675807 2.350348907   0.2840347  7.767315e-01
inc         0.9864943 0.003217444 306.6080542 5.567137e-234
```

R code

```
> print(poh.residual.lms$coefficients)
```

output

```
     Estimate Std. Error  t value     Pr(>|t|)
u_1 0.7818542 0.04788553 16.32757 1.402076e-36
```

R code

```
> print(POH.results)
```

output

```
$T
[1] 170

$rho
[1] 0.7818542

$sigma.rho
[1] 0.04788553
```

```
$s.sq
[1] 1.22395

$lambda.hat.sq
[1] 1.030594

$gamma0
[1] 1.216750

$rho.stat
[1] -32.04525

$t.stat
[1] -4.27529
```

19.3 Testing Hypotheses About the Cointegrating Vector

Page 608-612 illustrate a technique that uses leads and lags to produce a stationary vector for hypothesis testing.

R code

```
> T <- length(coninc.data$Quarter)
> lead.lag.data <- list(ct = coninc.data$cons[c(-1:-5, -((T - 3):T))],
+     yt = coninc.data$inc[c(-1:-5, -((T - 3):T))], delta.yt = diff(coninc.data$inc[c(-1:-4,
+         -((T - 3):T))]), delta.yt_ = embed(diff(coninc.data$inc[-((T -
+         4):T)]), 4), delta.yt. = embed(diff(coninc.data$inc[-1:-5])[(T -
+         6):1], 4)[(T - 9):1, ], tt = 6:(T - 4))
```

The regression is estimated with both no trend and trend, and the corrected t-stat is calculated.

R code

```
> no.trend.lm <- lm(ct ~ 1 + yt + delta.yt. + delta.yt + delta.yt_,
+     lead.lag.data)
> trend.lm <- lm(ct ~ 1 + yt + tt + delta.yt. + delta.yt + delta.yt_,
+     lead.lag.data)
> for (model in list(no.trend.lm, trend.lm)) {
+     lags <- 2
+     cms <- summary(model)
+     T <- length(cms$residuals)
+     cfs <- cms$coefficients
+     t.rho <- (cfs[["yt", "Estimate"]] - 1)/cfs[["yt", "Std. Error"]]
```

```
+     rms <- summary(lm(u ~ 0 + u_, list(u = cms$residuals[-c(1:lags)],
+         u_ = embed(cms$residuals[-T], lags))))
+     sigma1.hat.sq <- mean(rms$residuals^2)
+     lambda.11 <- sigma1.hat.sq^0.5/(1 - sum(rms$coefficients[paste("u_",
+         1:lags, sep = ""), "Estimate"]))
+     t.a <- t.rho * cms$sigma/lambda.11
+     print(cfs)
+     print(rms$coefficients)
+     print(T)
+     print(cms$sigma)
+     print(t.rho)
+     print(sigma1.hat.sq)
+     print(lambda.11)
+     print(t.a)
+ }
```

output

```
               Estimate  Std. Error     t value      Pr(>|t|)
(Intercept) -4.51922906 2.340224673  -1.9311091  5.534290e-02
yt           0.99215853 0.003063317 323.8837231 1.617626e-216
delta.yt.1   0.48592391 0.115704789   4.1996871  4.551158e-05
delta.yt.2   0.26411856 0.114892015   2.2988418  2.288546e-02
delta.yt.3   0.28614193 0.115594505   2.4753939  1.441397e-02
delta.yt.4   0.14530952 0.118799555   1.2231487  2.231790e-01
delta.yt    -0.24036007 0.117415901  -2.0470828  4.238356e-02
delta.yt_1  -0.01101143 0.113899420  -0.0966768  9.231113e-01
delta.yt_2   0.06969114 0.111505773   0.6250003  5.329142e-01
delta.yt_3   0.04055551 0.111155199   0.3648548  7.157303e-01
delta.yt_4   0.02150153 0.110083985   0.1953193  8.454056e-01
     Estimate Std. Error  t value     Pr(>|t|)
u_1 0.7179687 0.07722647 9.296924 1.127578e-16
u_2 0.2057401 0.07684783 2.677241 8.207043e-03
[1] 162
[1] 1.516006
[1] -2.559799
[1] 0.3809180
[1] 8.089864
[1] -0.4796954
                Estimate  Std. Error    t value     Pr(>|t|)
(Intercept) 198.87166510 15.01478288 13.2450577 5.215628e-27
yt            0.68117915  0.02292367 29.7150967 9.919458e-65
tt            0.26895671  0.01974617 13.6207037 5.213974e-28
```

```
delta.yt.1    0.40061828  0.07787309  5.1445023 8.270282e-07
delta.yt.2    0.15407283  0.07749787  1.9880910 4.862147e-02
delta.yt.3    0.16559666  0.07805023  2.1216678 3.550904e-02
delta.yt.4    0.02782397  0.08016237  0.3470952 7.290063e-01
delta.yt     -0.05124600  0.07998305 -0.6407108 5.226882e-01
delta.yt_1    0.12737594  0.07708222  1.6524685 1.005308e-01
delta.yt_2    0.23116996  0.07573754  3.0522506 2.687346e-03
delta.yt_3    0.20472613  0.07553655  2.7102923 7.505953e-03
delta.yt_4    0.18997478  0.07487875  2.5370986 1.219919e-02
     Estimate Std. Error  t value     Pr(>|t|)
u_1 0.6871713 0.07786238 8.825460 1.937474e-15
u_2 0.1291820 0.07666487 1.685022 9.395837e-02
[1] 162
[1] 1.017016
[1] -13.90793
[1] 0.3439489
[1] 3.193478
[1] -4.429212
```

20 Full-Information Maximum Likelihood Analysis of Cointegrated Systems

20.1 An Application of the Johansen Approach to the PPP data

Section 20.3 reanalyzes the data used in Chapter 19 using the FIML approach.

R code

```
> data(ppp, package = "RcompHam94")
> selection <- subset(ppp, Month >= "1973-01-01" & Month <= "1989-10-01")
> ppp.data <- data.frame(pstar = 100 * log(selection$PC6IT/selection$PC6IT[[1]]),
+     p = 100 * log(selection$PZUNEW/selection$PZUNEW[[1]]), ner = -100 *
+         log(selection$EXRITL/selection$EXRITL[[1]]))
> y <- as.matrix(ppp.data)
```

First conduct the auxiliary regressions. Given that the right hand sides consists of lagged values of the changes in y for both [20.2.4] and [20.2.5], construct a regression with both lagged y and lagged changes of y as left hand side.

R code

```
> delta.y <- diff(y)
> lags <- 12
```

```
> X <- embed(delta.y[-dim(delta.y)[[1]], ], lags)
> T <- dim(X)[[1]]
> n <- dim(y)[[2]]
> lhs <- cbind(delta.y[-1:(-lags), ], y[c(-1:-lags, -(T + lags +
+     1)), ])
> aux.lm <- lm(lhs ~ 1 + X, list(lhs = lhs, X = X))
> uv <- sapply(summary(aux.lm), FUN = function(x) {
+     x$residuals
+ })
> u <- uv[, 1:n]
> v <- uv[, (n + 1):(2 * n)]
```

Now calculate the canonical correlations according to [20.2.6], [20.2.7], [20.2.8], and calculate eigenvalues according to [20.2.9], and log likelihood as in [20.2.10]. Note that u is T rows by n columns so that ut is the t-th row of matrix u, so only a single inner product, rather than sum of outer products, is needed.

R code

```
> SigmaUU <- 1/T * t(u) %*% u
> SigmaVV <- 1/T * t(v) %*% v
> SigmaUV <- 1/T * t(u) %*% v
> eigen.results <- eigen(solve(SigmaVV) %*% t(SigmaUV) %*% solve(SigmaUU) %*%
+     SigmaUV)
> lambda <- eigen.results$values
> LRT <- -T * sum(log(1 - lambda))
> print(SigmaUU)
```

output

```
               Response pstar  Response p Response ner
Response pstar     0.17931504  0.01531134   0.02715177
Response p         0.01531134  0.04341512  -0.03267373
Response ner       0.02715177 -0.03267373   4.60842626
```

R code

```
> print(SigmaVV)
```

output

```
               Response pstar Response p Response ner
Response pstar      1503.5545   794.7041    -697.4981
Response p           794.7041   421.5535    -365.1883
Response ner        -697.4981  -365.1883     414.1322
```

R code

```
> print(SigmaUV)
```

output

```
               Response pstar Response p Response ner
Response pstar     -3.5787320 -1.7958934    1.5095381
Response p         -0.8602478 -0.4969721    0.5243431
Response ner       -3.1461173 -2.0636489   -2.2685853
```

R code

```
> print(lambda)
```

output

```
[1] 0.12002316 0.05077020 0.03174158
```

R code

```
> print(T * log(1 - lambda))
```

output

```
[1] -24.165480  -9.847724  -6.096434
```

R code

```
> print(LRT)
```

output

```
[1] 40.10964
```

Finally following page 648, calculate the first cointegrating vector normalized as in [20.3.9], and also normalized to have unity for the first coefficient.

R code

```
> ahat1 <- eigen.results$vectors[, 1]
> ahat1.tilde <- ahat1/sqrt(t(ahat1) %*% SigmaVV %*% ahat1)
> ahat1.normal <- ahat1/ahat1[[1]]
> print(ahat1)
```

output

```
[1] -0.48885151  0.87144476 -0.04010268
```

R code

```
> print(ahat1.tilde)
```

output

```
[1] -0.44788450  0.79841545 -0.03674197
```

R code

```
> print(ahat1.normal)
```

output

```
[1]  1.00000000 -1.78263694  0.08203448
```

20.2 Likelihood Ratio Tests on the Cointegration Vector

Page 649 shows how to conduct hypothesis tests on the cointegration vector. The follow code implements [20.3.10] - [20.3.14] and subsequent calculations.

R code

```
> D = cbind(c(1, 0, 0), c(0, 0, 1))
> SigmaVV.tilde <- t(D) %*% SigmaVV %*% D
> SigmaUV.tilde <- SigmaUV %*% D
> eigen.results <- eigen(solve(SigmaVV.tilde) %*% t(SigmaUV.tilde) %*%
+     solve(SigmaUU) %*% SigmaUV.tilde)
> lambda.tilde <- eigen.results$values
> h <- 1
> LRT <- -T * sum(log(1 - lambda[1:h])) + T * sum(log(1 - lambda.tilde[1:h]))
> ahat1.normal.tilde <- eigen.results$vectors[, 1]/eigen.results$vectors[,
+     1][[1]]
> print(SigmaVV.tilde)
```

output

```
          [,1]      [,2]
[1,] 1503.5545 -697.4981
[2,] -697.4981  414.1322
```

R code

```
> print(SigmaUV.tilde)
```

output

```
                      [,1]       [,2]
Response pstar -3.5787320  1.5095381
Response p     -0.8602478  0.5243431
Response ner   -3.1461173 -2.2685853
```

R code

```
> print(lambda.tilde)
```

output

```
[1] 0.05828948 0.03295258
```

R code

```
> print(T * log(1 - lambda.tilde))
```

output

```
[1] -11.350839  -6.332964
```

R code

```
> print(LRT)
```

output

```
[1] 12.81464
```

R code

```
> print(ahat1.normal.tilde)
```

output

```
[1] 1.000000 1.012463
```

Page 650 shows a second example.

R code

```
> h <- 1
> D = c(1, -1, -1) %o% 1
> SigmaVV.tilde <- t(D) %*% SigmaVV %*% D
> SigmaUV.tilde <- SigmaUV %*% D
> eigen.results <- eigen(solve(SigmaVV.tilde) %*% t(SigmaUV.tilde) %*%
+     solve(SigmaUU) %*% SigmaUV.tilde)
> lambda.tilde <- eigen.results$values
> LRT <- -T * sum(log(1 - lambda[1:h])) + T * sum(log(1 - lambda.tilde[1:h]))
> print(SigmaVV.tilde)
```

output

```
         [,1]
[1,] 1414.452
```

R code

```
> print(SigmaUV.tilde)
```

output

```
                     [,1]
Response pstar -3.2923768
Response p     -0.8876187
Response ner    1.1861170
```

R code

```
> print(lambda.tilde)
```

output

```
[1] 0.04912925
```

R code
```
> print(T * log(1 - lambda.tilde))
```

output
```
[1] -9.521278
```

R code
```
> print(LRT)
```

output
```
[1] 14.64420
```

21 Time Series Models of Heteroskedasticity

21.1 Preamble

Page 658 and forward provide examples of ARCH models. Several utility functions are needed for these examples. The function "arch.fitted.values" calculates the value of ht given the conditional information set YT and a parameter vector THETA as described on page 660, [21.1.17] to [21.1.20].

R code
```
> arch.fitted.values <- function(THETA, YT) {
+     alpha <- THETA[grep("alpha.*", names(THETA))]
+     beta <- THETA[grep("beta.*", names(THETA))]
+     zeta <- THETA["zeta"]
+     m <- length(alpha)
+     h <- rep(as.vector(zeta), m)
+     u <- YT$y - YT$x %*% beta
+     indices <- (m + 1):length(YT$y)
+     z <- array(0, c(length(indices), 1 + m))
+     for (tt in indices) h[tt] <- t(c(zeta, alpha)) %*% c(1, u[(tt -
+         1):(tt - m)]^2)
+     list(u = u, h = h)
+ }
```

Function "arch.standard.errors" calculates values for standard errors according to the description on page 663, particularly equations [21.1.25], and also using [21.1.21] for the estimate of the outer product estimate of the information matrix.

R code
```
> arch.standard.errors <- function(THETA, YT) {
+     x <- YT$x
```

```
+     y <- YT$y
+     k <- dim(x)[[2]]
+     alpha <- THETA[grep("alpha.*", names(THETA))]
+     zeta <- THETA["zeta"]
+     m <- length(alpha)
+     T <- length(y) - m
+     a <- k + 1 + m
+     fv <- arch.fitted.values(THETA, YT)
+     h <- fv$h
+     u2 <- fv$u^2
+     S <- array(0, c(a, a))
+     D <- array(0, c(a, a))
+     for (tt in (m + 1):length(y)) {
+         temp <- c(t(alpha) %*% ((u2[(tt - 1):(tt - m)] %o% rep(1,
+             k)) * x[(tt - 1):(tt - m), ]), c(1, u[(tt - 1):(tt -
+             m)]^2))
+         st <- (u2[tt] - h[tt])/(2 * h[tt]^2) * temp + c(u2[tt]/h[tt] *
+             x[tt, ], rep(0, a - k))
+         S <- S + 1/T * st %*% t(st)
+         D <- D + 1/T * (1/(2 * h[tt]^2) * temp %*% t(temp) +
+             rbind(cbind(1/h[tt] * x[tt, ] %*% t(x[tt, ]), array(0,
+                 c(k, a - k))), array(0, c(a - k, a))))
+     }
+     diag(1/T * solve(D) %*% S %*% solve(D))^0.5
+ }
```

The following two helper functions calculate the likelihood values under different distributional assumptions. The normal likelihood is calculated according to [21.1.20], the scaled t according to [21.1.24].

R code

```
> arch.normal <- function(THETA, YT) {
+     fv <- arch.fitted.values(THETA, YT)
+     m <- length(THETA[grep("alpha.*", names(THETA))])
+     h <- fv$h[-1:-m]
+     u <- fv$u[-1:-m]
+     -1/2 * (length(h) * log(2 * pi) - sum(log(h)) - sum(u^2/h))
+ }
> arch.scaled.t <- function(THETA, YT) {
+     fv <- arch.fitted.values(THETA, YT)
+     m <- length(THETA[grep("alpha.*", names(THETA))])
+     h <- fv$h[-1:-m]
+     u <- fv$u[-1:-m]
```

```
+     nu <- THETA[grep("nu", names(THETA))]
+     result <- length(h) * log(gamma((nu + 1)/2)/(sqrt(pi) * gamma(nu/2)) *
+         (nu - 2)^-0.5) - 1/2 * sum(log(h)) - (nu + 1)/2 * sum(log(1 +
+         u^2/(h * (nu - 2))))
+ }
```

GMM estimates are calculated according to the recipe in Chapter 14, notably equations [14.1.7] and [14.1.10]. Functions h and S are specified by the caller.

R code

```
> GMM.estimates <- function(YT, h, THETA, S) {
+     g <- function(YT, THETA) {
+         apply(X = apply(X = YT, MARGIN = 1, FUN = h, THETA = THETA),
+             MARGIN = 1, FUN = mean)
+     }
+     objective <- function(THETA, YT, W) {
+         g.value <- g(YT, THETA)
+         as.numeric(t(g.value) %*% W %*% g.value)
+     }
+     r <- length(h(YT[1, ], THETA))
+     a <- length(THETA)
+     stage.1.results <- optim(par = THETA, fn = objective, gr = NULL,
+         YT = YT, W = diag(r))
+     temp <- t(apply(X = YT, MARGIN = 1, FUN = h, THETA = stage.1.results$par))
+     ST <- S(temp)
+     stage.2.results <- optim(par = stage.1.results$par, fn = objective,
+         gr = NULL, YT = YT, W = solve(ST))
+     list(stage.1.results = stage.1.results, stage.2.results = stage.2.results)
+ }
```

21.2 Application of ARCH Models to US Fed Funds Data

The dataset for these examples is the US Fed Funds Rate, monthly between Jan 1955 and December 2000, shown below.

R code

```
> data(fedfunds, package = "RcompHam94")
> selection <- subset(fedfunds, Month >= "1955-01-01" & Month <=
+     "2000-12-01")
> y <- selection$FFED
```

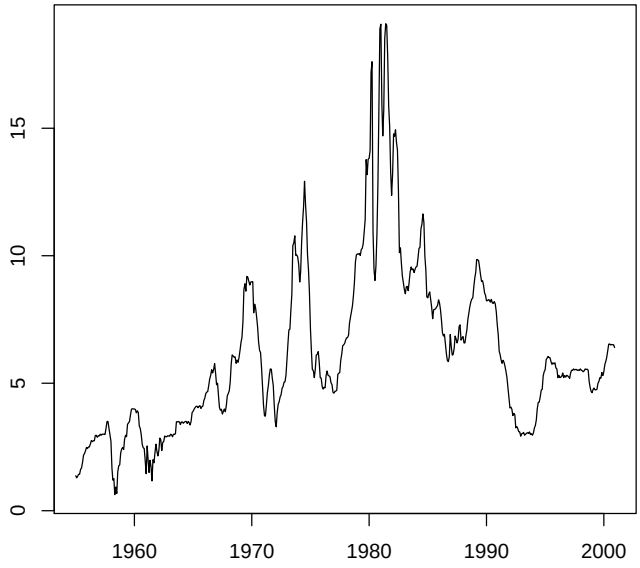


Figure 21.1 − US Fed Funds Rate

A first step is to characterize the autocorrelation structure of the squared residuals. These two regressions show that a second order AR process seems to fit the data pretty well.

R code

```
> y.lm <- lm(y ~ 1 + y_1, list(y = y[-1], y_1 = y[-length(y)]))
> u <- y.lm$residuals
> u2.lm <- lm(u2 ~ 1 + u2_lag, list(u2 = u[-1:-4]^2, u2_lag = embed(u[-length(u)]^2,
+     4)))
> F34 <- Wald.F.Test(R = cbind(rep(0, 2) %o% rep(0, 3), diag(2)),
+     b = u2.lm$coefficients, r = c(0, 0), s2 = summary(u2.lm)$sigma^2,
+     XtX_1 = summary(u2.lm)$cov.unscaled)
> F34.sig <- 1 - pf(F34, 2, length(u2.lm$residuals) - u2.lm$rank)
> F234 <- Wald.F.Test(R = cbind(rep(0, 3) %o% rep(0, 2), diag(3)),
+     b = u2.lm$coefficients, r = c(0, 0, 0), s2 = summary(u2.lm)$sigma^2,
+     XtX_1 = summary(u2.lm)$cov.unscaled)
> F234.sig <- 1 - pf(F234, 3, length(u2.lm$residuals) - u2.lm$rank)
> accept.arch <- pchisq(length(u2.lm$residuals) * summary(u2.lm)$r.squared,
+     4)
> print(F34)
```

output

```
[1] 0.8225742
```

R code

```
> print(F34.sig)
```

output

```
[1] 0.439847
```

R code

```
> print(F234)
```

output

```
[1] 11.88167
```

R code

```
> print(F234.sig)
```

output

```
[1] 1.513714e-07
```

R code

```
> print(accept.arch)
```

output

```
[1] 1
```

Next we use a maximum likelihood estimation to estimate the parameters for the second order equation assuming normal errors.

R code

```
> YT <- list(y = y[-1], x = cbind(rep(1, length(y) - 1), y[-length(y)]))
> THETA <- c(beta = y.lm$coefficients, zeta = var(y.lm$residuals),
+     alpha = c(0.1, 0.1))
> optimizer.results <- optim(par = THETA, fn = arch.normal, gr = NULL,
+     YT = YT)
> print(optimizer.results$par)
```

output

```
beta.(Intercept)         beta.y_1             zeta           alpha1
      0.25226382       0.94858488       0.02734929       0.95530391
          alpha2
      0.29858866
```

R code

```
> se <- arch.standard.errors(optimizer.results$par, YT)
> print(se)
```

output

```
[1] 0.048395622 0.010418563 0.004969357 0.108784800 0.082012310
```

Now use GMM to estimate the same parameters following page 664. The initial values for the regression coefficients are derived from the (homoskedastic) regression above, as is the presample variance. The estimator for S assumes no correlation at leads and lags.

R code

```
> h <- function(wt, THETA) {
+     beta <- THETA[grep("beta.*", names(THETA))]
+     zeta <- THETA["zeta"]
+     alpha <- THETA[grep("alpha.*", names(THETA))]
+     m <- length(alpha)
+     k <- length(beta)
+     yt <- wt[grep("yt.*", names(wt))]
+     xt <- wt[grep("xt.*", names(wt))]
+     ylagt <- wt[grep("ylagt.*", names(wt))]
+     xlagt <- t(array(wt[grep("xlagt.*", names(wt))], c(k, m)))
+     ut <- yt - t(xt) %*% beta
+     zt <- c(1, (ylagt - t(xlagt) %*% beta)^2)
+     c(ut * xt, (ut^2 - t(zt) %*% c(zeta, alpha)) * zt)
+ }
> S.estimator <- function(ht) {
+     1/dim(ht)[[1]] * t(ht) %*% ht
+ }
> THETA <- c(beta = y.lm$coefficients, zeta = var(y.lm$residuals),
+     alpha = c(0.1, 0.1))
> m <- length(THETA[grep("alpha.*", names(THETA))])
> T <- length(YT$y) - m
> w <- as.matrix(data.frame(yt = YT$y[-1:-m], xt = YT$x[-1:-m,
+     ], ylagt = embed(YT$y[-(T + m)], m), xlagt = embed(YT$x[-(T +
+     m), ], m)))
> estimates <- GMM.estimates(YT = w, h = h, THETA = THETA, S.estimator)
> print(estimates$stage.1.results$par)
```

output

```
beta.(Intercept)         beta.y_1             zeta           alpha1
      0.05788674       0.98955937       0.32491651       0.01073606
          alpha2
      0.02105476
```

R code

```
> print(estimates$stage.2.results$par)
```

```
output
beta.(Intercept)          beta.y_1              zeta            alpha1
      0.02579794        0.99791508       -0.17911928        0.01239927
          alpha2
      0.07770754
```

21.3 R Facilities For GARCH models

TBD

22 Modeling Time Series with Changes in Regime

22.1 Statistical Analysis of i.i.d. Mixture Distributions

Figure 22.2:

```
R code
> curve(0.8 * dnorm(x, 0, 1), from = -2, to = 8, n = 100, col = 1,
+     ylab = "f(x)", main = "Density of mixture of 2 gaussians")
> curve(0.2 * dnorm(x, 4, 1), from = -2, to = 8, n = 100, add = TRUE,
+     col = 3)
> mixture <- function(x) 0.8 * dnorm(x, 0, 1) + 0.2 * dnorm(x,
+     4, 1)
> curve(mixture, from = -2, to = 8, n = 100, col = 2, add = TRUE)
```

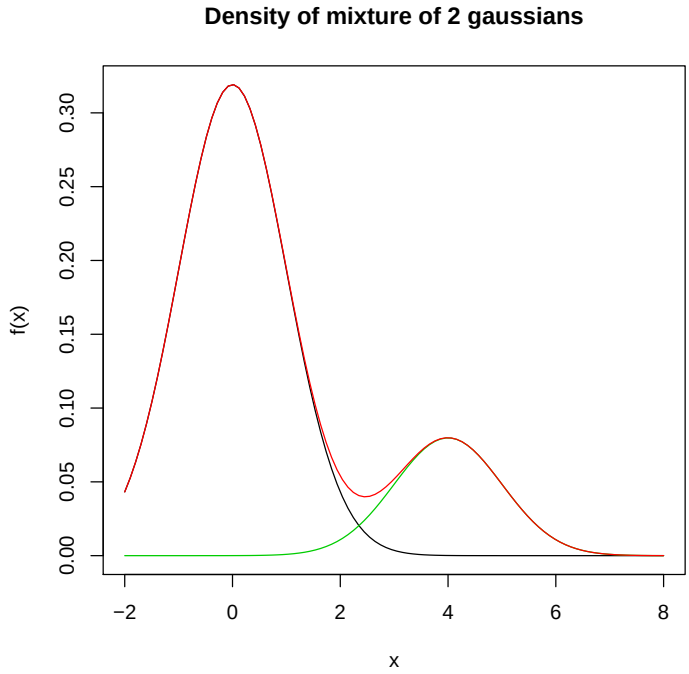
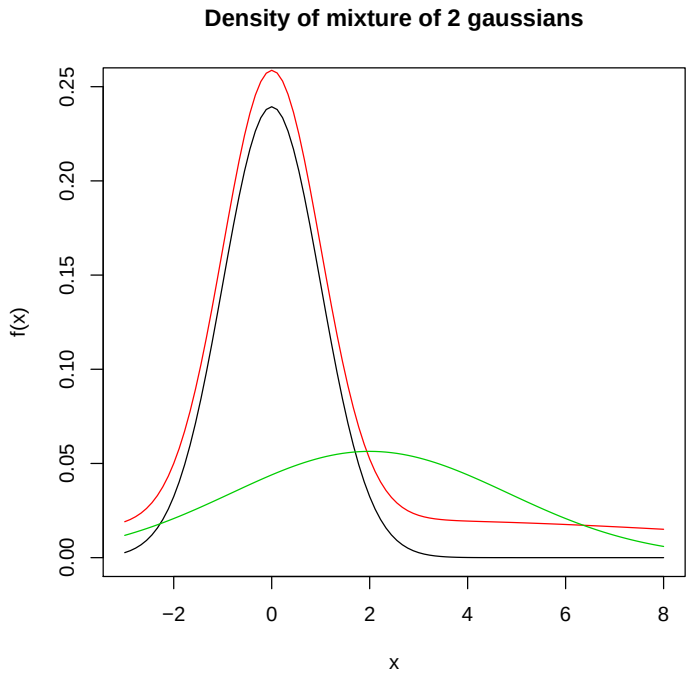


Figure 22.3:

R code

```
> mixture2 <- function(x) 0.6 * dnorm(x, 0, 1) + 0.4 * dnorm(x,
+     2, 8)
> curve(mixture2, from = -3, to = 8, n = 100, col = 2, ylab = "f(x)",
+     main = "Density of mixture of 2 gaussians", ylim = c(0, 0.25))
> curve(0.6 * dnorm(x, 0, 1), from = -3, to = 8, n = 100, col = 1,
+     add = TRUE)
> curve(0.4 * dnorm(x, 2, sqrt(8)), from = -3, to = 8, n = 100,
+     add = TRUE, col = 3)
```



22.2 Modeling Changes in Regime

Page 697 describes an example of the application of Markov switching models to US GNP from 1951Q1 to 1984Q4.

R code

```
> data(gnpdata, package = "RcompHam94")
> selection <- subset(gnpdata, Quarter >= "1951-01-01" & Quarter <=
+     "1984-04-01")
> d <- selection$Quarter[-1]
> g <- diff(100 * log(selection$GNP))
```

The actual implementation uses the technique of collapsing multi-period states into a single state, p691, p698. During the maximum likelihood estimation process the state probabilities will change, but the layout of the matrix is still the same. The following code fragment precalculates the transition matrix structure with the five possible values, then uses a separate 5 element lookup vector to populate it.

R code

```
> nlags <- 4
> nstates <- 2^(nlags + 1)
```

```
> lagstate <- 1 + outer(1:nstates, 1:(nlags + 1), FUN = function(i,
+     j) {
+     trunc((i - 1)/2^(nlags + 1 - j))%%2
+ })
> head(lagstate)
```

output

```
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    1    1    1    1
[2,]    1    1    1    1    2
[3,]    1    1    1    2    1
[4,]    1    1    1    2    2
[5,]    1    1    2    1    1
[6,]    1    1    2    1    2
```

R code

```
> transit <- outer(X = 1:nstates, Y = 1:nstates, FUN = function(i,
+     j) {
+     ((2 * lagstate[i, 1] + lagstate[j, 1] - 1) - 1) * (((i -
+         1)%%(2^nlags)) == trunc((j - 1)/2)) + 1
+ })
> head(transit)
```

output

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13] [,14]
[1,]    2    2    1    1    1    1    1    1    1     1     1     1     1     1
[2,]    1    1    2    2    1    1    1    1    1     1     1     1     1     1
[3,]    1    1    1    1    2    2    1    1    1     1     1     1     1     1
[4,]    1    1    1    1    1    1    2    2    1     1     1     1     1     1
[5,]    1    1    1    1    1    1    1    1    2     2     1     1     1     1
[6,]    1    1    1    1    1    1    1    1    1     1     2     2     1     1
     [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25] [,26]
[1,]     1     1     1     1     1     1     1     1     1     1     1     1
[2,]     1     1     1     1     1     1     1     1     1     1     1     1
[3,]     1     1     1     1     1     1     1     1     1     1     1     1
[4,]     1     1     1     1     1     1     1     1     1     1     1     1
[5,]     1     1     1     1     1     1     1     1     1     1     1     1
[6,]     1     1     1     1     1     1     1     1     1     1     1     1
     [,27] [,28] [,29] [,30] [,31] [,32]
[1,]     1     1     1     1     1     1
[2,]     1     1     1     1     1     1
[3,]     1     1     1     1     1     1
[4,]     1     1     1     1     1     1
```

```
[5,]     1     1     1     1     1     1
[6,]     1     1     1     1     1     1
```

The bulk of the work is done by the following function, based on the algorithm in section 22.4. Ergodic probabilities are defined as on page 684, including equation [22.2.26]. The loop uses equations [22.4.24], [22.4.2], [22.4.5], [22.4.8], [22.4.7], [22.4.6] and [22.4.14].

R code

```
> infer.regimes <- function(THETA, YT) {
+     phi <- THETA[grep("phi.*", names(THETA))]
+     mu <- THETA[grep("mu.*", names(THETA))]
+     sigma <- THETA["sigma"]
+     p11star <- THETA["p11star"]
+     p22star <- THETA["p22star"]
+     T <- length(YT)
+     tp <- c(0, p11star, 1 - p22star, 1 - p11star, p22star)
+     P <- array(tp[transit], c(nstates, nstates))
+     A <- rbind(diag(nstates) - P, rep(1, nstates))
+     ergodic.pi <- (solve(t(A) %*% A) %*% t(A))[, nstates + 1]
+     xi.t.t <- ergodic.pi %o% rep(1, nlags)
+     xi.t.t_1 <- xi.t.t
+     log.likelihood <- 0
+     for (tt in (nlags + 1):T) {
+         residuals <- as.vector(((rep(1, nstates) %o% YT[tt:(tt -
+             nlags)]) - array(mu[lagstate], c(nstates, nlags +
+             1))) %*% c(1, -phi))
+         eta.t <- dnorm(residuals, mean = 0, sd = sigma)
+         fp <- eta.t * xi.t.t_1[, tt - 1]
+         fpt <- sum(fp)
+         xi.t.t <- cbind(xi.t.t, fp/fpt)
+         log.likelihood <- log.likelihood + log(fpt)
+         xi.t.t_1 <- cbind(xi.t.t_1, P %*% xi.t.t[, tt])
+     }
+     xi.t.T <- xi.t.t[, T] %o% 1
+     for (tt in (T - 1):1) xi.t.T <- cbind(xi.t.t[, tt] * (t(P) %*%
+         (xi.t.T[, 1]/xi.t.t_1[, tt])), xi.t.T)
+     list(log.likelihood = log.likelihood, xi.t.t = xi.t.t, xi.t.T = xi.t.T)
+ }
```

Initial values of the parameters for transition probabilities are set from historical averages. The phi and sigma values are obtained from a (non-state) regression of change in GDP on 4 of its own lags.

R code

```
> g.lm <- lm(g ~ 1 + g_lag, list(g = g[-1:-nlags], g_lag = embed(g[-length(g)],
+     nlags)))
> THETA <- c(p11star = 0.85, p22star = 0.7, mu = c(1, 0), phi = as.vector(g.lm$coefficients[1 +
+     (1:nlags)]), sigma = summary(g.lm)$sigma)
```

Now we are in a position to optimize, then calculate the smoothed probabilities from the optimal parameters.

R code

```
> objective <- function(THETA, YT) {
+     -infer.regimes(THETA, YT)$log.likelihood
+ }
> optimizer.results <- optim(par = THETA, hessian = TRUE, fn = objective,
+     gr = NULL, YT = g, method = "BFGS")
> se <- diag(solve(optimizer.results$hessian))^0.5
> print(optimizer.results$par)
```

output

```
    p11star     p22star         mu1         mu2        phi1        phi2
 0.90030933  0.76062170  1.17515197 -0.31750266  0.02262260 -0.02950457
       phi3        phi4       sigma
-0.22818176 -0.20243029  0.77954523
```

R code

```
> print(se)
```

output

```
   p11star    p22star        mu1        mu2       phi1       phi2       phi3
0.04022558 0.09745502 0.08379353 0.27312797 0.12911244 0.14402459 0.11136972
      phi4      sigma
0.11306913 0.06950831
```

R code

```
> regimes <- infer.regimes(optimizer.results$par, g)
> recession.probability <- as.vector((1:nstates > nstates/2) %*%
+     regimes$xi.t.t)
> smoothed.recession.probability <- as.vector((1:nstates > nstates/2) %*%
+     regimes$xi.t.T)
```

The results are shown below.

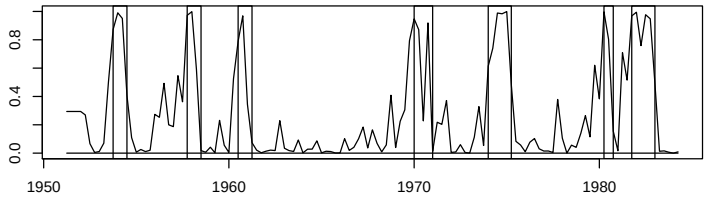
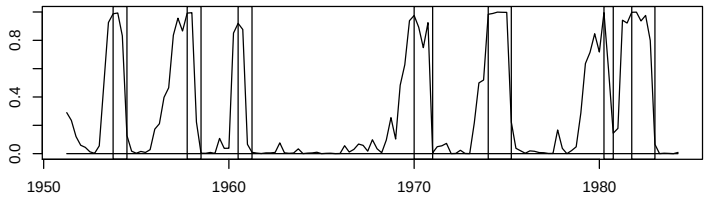


Figure 22.4a



Smoothed recession probabilities

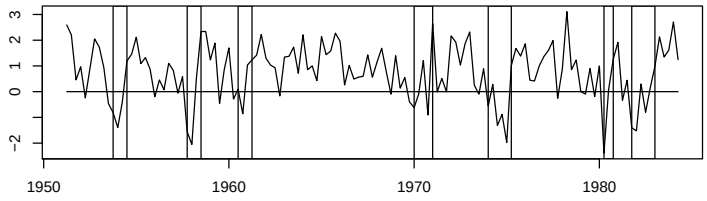


Figure 22.4b

References

de Micheaux, P. L., and B. Liquet (2009): "ConvergenceConcepts : an R Package to Investigate Various Modes of Convergence," *The R Journal*, 1(2), 18–25.

Hamilton, J. (1994): *Time series analysis*. Princeton University Press.

Kleiber, C., and A. Zeileis (2008): *Applied Econometrics with R*. Springer-Verlag, New York.