1 Introduction

Visualize an infinitely long line of dominoes. If you knock the first one down yourself, you know the next one will fall, and so will the one after that. Let's call any domino k. We can say if the $k = 1$ domino falls, the $k = 2$ domino falls. More generally, if the k'th domino falls, the $k + 1$'th domino will as well.

This idea is very powerful. We can use it to prove a rule true. Speaking arbitrarily, a rule can have k cases. If we can prove the rule holds true for the $k + 1$'th case, then its true for all cases.

2 Mathematical Induction

A domino cannot fall by itself; it needs an initial push. This initial push is the **base case**. We must prove the base case true because we must have a k'th case to have a $k + 1$'th case.

Once we prove the base case true, we can try and prove the $k + 1$'th case true, the **inductive step**. The $k + 1$'th case is the infinitely long line of dominoes falling down one by one.

We can apply this logic to something like a sequence. Let's write down the domino effect in mathematical terms.

For a sequence $P(n)$ up to the n'th term, let the equation $S(n)$ represent the summation of $P(n)$'s elements:

$$a_1 + a_2 + \cdots + a_n = S(n)$$

Set $n = k$, where k is any real integer.

Base case:

$$k = \text{some base value}$$

Assuming the equation $S(n)$ holds true to the sequence for $n = 1$, prove $S(n)$ holds true for $n = k + 1$.

Inductive step:

$$n = k + 1$$

If the inductive step is successful, $S(n)$ is a valid rule.

3 Example

Prove:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \text{ for } n \geq 0$$

Base case:

$$k = 0 \tag{1}$$

$$0^3 = \frac{0^2(n+1)^2}{4} \tag{2}$$

$$0 = 0 \tag{3}$$

Inductive step:

Adding the $k+1$'th term to $S(k)$ yields

$$\frac{k^2(k+1)^2}{4} + (k+1)^3$$

We must prove that

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

To explain the above step in plain English, we want to prove that $S(k)$ in terms of k plus the $k+1$'th term is equal to $S(k+1)$ in terms of the $k+1$'th term.

In other words, we know $S(k)$ is true in terms of k, but we need to prove its true for all cases, $k+1$.

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4} \tag{4}$$

$$k^2(k+1)^2 + 4(k+1)^3 = (k+1)^2(k+2)^2 \tag{5}$$

$$(k+1)\frac{k^2(k+1)^2 + 4(k+1)^3}{k+1} = (k+1)^2(k+2)^2 \tag{6}$$

$$(k+1)(k+1)(k+2)^2 = (k+1)^2(k+2)^2 \tag{7}$$

$$\frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4} \tag{8}$$

Note: In (6), we use the rational roots theorem to see that $k = -1$ is a root of the LHS, so we know we can divide by $k+1$ to create a simpler polynomial.

S(n) is a valid summation formula of the sequence P(n) for all $n \geq 0$.

4 Conclusion

The hardest part of mathematical induction is the algebra in the inductive step. You're trying to prove a formula $S(k)$ plus $k + 1$ is equivalent to what we're trying to prove is true, $S(k + 1)$. Once you've got that down, induction is just some algebraic gymnastics.

Finally, mathematical induction isn't limited to proving domino-sequence like relationships. We can use it to prove many things, from summation formulas to the maximum regions a square plane can be broken up into by n lines.

5 References

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