The University of Texas
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CS331 Algorithm

Assignment 05

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1 Exercise 1

Let $G = (U, V, E)$ be a configuration and let p be the price vector for G such that p_v is equal to the start price of v for all items v in V . Assume that the MCM M of G associated with this execution of Algorithm A is an MWMCM of G. Prove that $P(G, M, p)$ holds. Hint: Make use of Exercise 1 from Assignment 4.

To prove the $P(G, M, p)$ holds, we need to correctly provide the proof for all of three predicates holds: $P_0(G, p)$, $P_1(G, M, p)$ and $P_2(G, M, p)$.

Proof for $P_0(G, p)$. From the exercise 1 of assignment 4, we know that for any stable price vector q, all its components q_v must be at least the start price of item v, that is p_v in this case. Hence, $P_0(G, p)$ holds. □

Proof for $P_1(G, M, p)$. Proof for the . □

Proof for $P_2(G, M, p)$ by Contradiction. Let us assume that $P_2(G, M, p)$does not holds. That is to say, there exists a item v whose start price p_v is larger than weight of one edge (u, v) in M. Formally,

$$\exists (u, v) \in M,\ p_v > w(u, v) \tag{1}$$

However, we have that the given M is an MWMCM. Hence, weight of any edge must be larger than or equal to the reserve price of the associated item. Otherwise, item v will remained unmatched in M. (Here we do not presume the existence of dummy bid, which is just for programming part.) And since we already know that the start price is at most reserve price. Formally,

$$\forall (u, v) \in M,\ p_v \leq \text{reserve price of item } v \leq w(u, v) \tag{2}$$

That is,

$$\forall (u, v) \in M,\ p_v \leq w(u, v) \tag{3}$$

Note that (1) contradicts the (3). Therefore, we should negate the assumption at the very beginning and conclude that $P_2(G, M, p)$ does holds. □

2 Exercise 2

Consider an iteration of Algorithm A on a configuration $G = (U, V, E)$ that increments the price of an item due to a violation of Stability Condition 2. Assume that the MCM M of G associated with this execution of Algorithm A is an MWMCM of G. Let $p(resp., p_0)$ denote the price vector of G maintained by Algorithm A just before (resp., after) this iteration. Prove that if $P_0(G, p)$ holds, then so does $P_0(G, p_0)$. Hint: Make use of Lemma 1 of Assignment 4.

In order to prove that if $P_0(G, p)$ holds, then $P_0 G, p'$ holds after one iteration of algorithm A, we should prove it for all three possible violation cases of algorithm A.

Stability Condition 1 is violated. In this case, the algorithm A will halts "unsuccessfully". Hence, there is no change for price vector after the iteration. That is, we have $p = p'$. And obviously, it is true that if $P_0(G, p)$ holds, then $P_0(G, p')$ holds. □

Stability Condition 2 is violated. Since the violation of stability condition 2 causes the iteration of algorithm A, it is true that

$$\exists u,\ (u, v) \in M,\ (u, v^*) \in E,\ s.t.\ w(u, v) - p_v < w(u, v^*) - p_{v^*} \tag{4}$$

Let us instantiate the v to v_0 and v^* to v_1, and have

$$w(u, v_0) - p_{v_0} < w(u, v_1) - p_{v_1} \tag{5}$$

In this case, the p'_{v_1} will increments by 1, it is true that

$$w(u, v_0) - p_{v_0} \leq w(u, v_1) - p'_{v_1} \tag{6}$$

that is,

$$p'_{v_1} \leq w(u, v_1) - w(u, v_0) + p_{v_0} \tag{7}$$

For the stable price vector q, according the stability condition 2, we have

$$\forall v^*,\ w(u, v^*) - q_{v^*} \leq w(u, v_0) - q_{v_0} \tag{8}$$

Let us instantiate the v^* to v_1 and have

$$w(u, v_1) - q_{v_1} \leq w(u, v_0) - q_{v_0} \tag{9}$$

That is,

$$w(u, v_1) - w(u, v_0) + q_{v_0} \leq q_{v_1} \tag{10}$$

Since $p_{v_0} \leq q_{v_0}$ holds,

$$w(u, v_1) - w(u, v_0) + p_{v_0} \leq q_{v_1} \tag{11}$$

Combined with (7), we have

$$p'_{v_1} \leq q_{v_1} \tag{12}$$

Since p'_{v_1} is the only item whose price is changed in this iteration, $\forall v \neq v_1,\ p'_v \leq q_{v_1}$ holds. Hence, we can conclude that after the iteration of algorithm A with the cause of stability violation 2, Predicate 0 still holds. That is, $P_0(G, p) \Rightarrow P_0(G, p')$ holds if one iteration is caused by stability violation 2. □

Stability Condition 3 is violated. In the violation of stability condition 3, we have

$$\exists v^*,\ (u^*, v^*) \in E,\ u^* isunmatchedinM,\ p_{v^*} < w(u^*, v^*) \tag{13}$$

Let u_0 to be a unmatched bid in M, and $(u_0, v_0) \in E$. Hence, we can instantiate v^* to v_0, and u^* to u_0

$$(u_0, v_0) \in E,\ p_{v_0} < w(u_0, v_0) \tag{14}$$

Since the algorithm A in this case will increment p_{v_0} by one and other price component remains unchanged, we have

$$p_{v_0} \leq w(u, v_0) \tag{15}$$

Since M is already MWMCM, and q is stable price vector, then (M, q) is stable solution. According to the third property of stable solution, we have

$$\forall u^* \in U,\ u^* isunmatchedinM,\ w(u^*, v_0) \leq q_{v_0} \tag{16}$$

We instantiate u^* to u_0, which is reasonable because u_0 Then we have

$$w(u_0, v_0) \leq q_{v_0} \tag{17}$$

Combined with (15), it is true that

$$p_{v_0} \leq q_{v_0} \tag{18}$$

Since other price component does not vary at that iteration, we can conclude that $p \leq q$ holds in this case. That is, $P_0(G, p) \Rightarrow P_0(G, p')$ holds if one iteration is caused by stability violation 3.

$\square$

Since one iteration in algorithm A will only be caused by one of three cases, and in all of these three cases, $P_0(G, p) \Rightarrow P_0(G, p')$ holds for that iteration, it can be concluded that if $P_1(G, M, p)$ holds, then so does $P_1(G, M, p')$.