The University of Texas at Austin

CS383C Numerical Analysis

Homework 04

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Exercises

Exercise 1.

Show that, for *a consistent matrix norm*, $\kappa(A) \geq 1$.

Proof.

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| \geq ||AA^{-1}|| = ||I|| = 1 \tag{1}$$

Note that the above $||\cdot||$ was for arbitrary induced matrix norm. □

Lemma 1. *For arbitrary matrix A and B, $||AB|| \leq ||A|| \cdot ||B||$.*

Proof.

$$\begin{aligned} ||AB|| = \sup_{x\neq 0} \frac{||ABx||}{||x||} &= \sup_{x\neq 0} \frac{||A(Bx)||}{||x||} && (2)\\ &\leq \sup_{x\neq 0} \frac{||A|| \cdot ||Bx||}{||x||} && (3)\\ &\leq \sup_{x\neq 0} \frac{||A|| \cdot ||B|| \cdot ||x||}{||x||} && (4)\\ &= ||A|| \cdot ||B|| && (5) \end{aligned}$$

Hence, it is concluded that $||AB|| \leq ||A|| \cdot ||B||$. □

Lemma 2. *For abitrary norm $||\cdot||$ and identity matrix I, $||I|| = 1$.*

Proof.

$$||I|| = \sup_{x\neq 0} \frac{||I \cdot x||}{||x||} = \sup_{x\neq 0} \frac{||x||}{||x||} = 1 \tag{6}$$

□

Exercise 2.

If A has lineraly independent columns, show that $||(A^H A)^{-1} A^H||_2 = \frac{1}{\sigma_{n-1}}$, where σ_{n-1} equals *the smallest sigular value* of A.

Proof. Let U, Σ and V be singular value decomposition of A, such that $A = U\Sigma V^H$.

$$\begin{aligned}
||(A^H A)^{-1} A^H||_2 &= ||\big((U\Sigma V^H)^H U\Sigma V^H\big)^{-1} (U\Sigma V^H)^H||_2 && (7)\\
&= ||\big(V\Sigma^H U^H U\Sigma V^H\big)^{-1} V\Sigma^H U^H||_2 && (8)\\
&= ||\big(V\Sigma^H \Sigma V^H\big)^{-1} V\Sigma^H U^H||_2 && (9)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}V^{-1}V\Sigma^H U^H||_2 && (10)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}\Sigma^H U^H||_2 && (11)\\
&= ||V^{-H}\Sigma^{-1} U^H||_2 && (12)\\
&= ||V\Sigma^{-1} U^H||_2 && (13)\\
&= ||\Sigma^{-1}||_2 && (14)\\
&= \frac{1}{\sigma_{n-1}} && (15)
\end{aligned}$$

□

Lemma 3. *(Unitary Invariance) For arbitrary unitary matrix U,*

$$||UA||_2 = ||AU||_2 = ||A||_2 \tag{16}$$

Lemma 4. *For arbitrary diagonal matrix Σ,*

$$||\Sigma^{-1}||_2 = \frac{1}{\sigma_{n-1}} \tag{17}$$

where, σ_{n-1} is the least entry of Σ.

Note that above two lemmas have been proven in exercises of previou notes.

Exercise 3.

Let A have linearly independent columns. Show that $\kappa_2(A^H A) = \kappa_2(A)^2$.

Proof. We achieve the proof by employing SVD over A. Let unitary matrix U, diagonal matrix Σ and unitary matrix V be singular value decomposition of A, such that $A = U\Sigma V^H$. We start from the definition of condition number $\kappa_2(\cdot)$.

$$\kappa_2(A^H A) = ||A^H A||_2 \cdot ||(A^H A)^{-1}||_2 \tag{18}$$

Then we discuss the term $||A^H A||_2$ and $||(A^H A)^{-1}||_2$ respectively.

$$\begin{aligned}
||A^H A||_2 &= ||(U\Sigma V^H)^H U\Sigma V^H||_2 &(19)\\
&= ||V\Sigma^H U^H U\Sigma V^H||_2 &(20)\\
&= ||V\Sigma^H \Sigma V^H||_2 &(21)\\
&= ||\Sigma^H \Sigma||_2 &(22)\\
&= \sigma_0^2 &(23)\\
&= ||A||_2^2 &(24)
\end{aligned}$$

Note that σ_0 is the largest singular value of matrix A and also the largest entry of Σ.

$$\begin{aligned}
||(A^H A)^{-1}||_2 &= ||\big((U\Sigma V^H)^H U\Sigma V^H\big)^{-1}||_2 &(25)\\
&= ||\big(V\Sigma^H U^H U\Sigma V^H\big)^{-1}||_2 &(26)\\
&= ||\big(V\Sigma^H \Sigma V^H\big)^{-1}||_2 &(27)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}V^{-1}||_2 &(28)\\
&= ||\Sigma^{-1}\Sigma^{-H}||_2 &(29)\\
&= ||\Sigma^{-1}\Sigma^{-1}||_2 &(30)\\
&= \frac{1}{\sigma_{n-1}^2} &(31)\\
&= ||A^{-1}||_2^2 &(32)
\end{aligned}$$

Now we have

$$\begin{aligned}
||A^H A||_2 &= ||A||_2^2 &(33)\\
||(A^H A)^{-1}||_2 &= ||A^{-1}||_2^2 &(34)
\end{aligned}$$

Then

$$\begin{aligned}
\kappa_2(A^H A) &= ||A^H A||_2 \cdot ||(A^H A)^{-1}||_2 &(35)\\
&= ||A||_2^2 \cdot ||A^{-1}||_2^2 &(36)\\
&= \big(||A||_2 \cdot ||A^{-1}||_2\big)^2 &(37)\\
&= \kappa_2(A)^2 &(38)
\end{aligned}$$

Hence, it can be concluded that

$$\kappa_2(A^H A) = \kappa_2(A)^2 \tag{39}$$

□

Exercise 4.

Exercise 5.

Let $U \in \mathbb{C}^{n \times n}$ be unitary. Show that $\kappa_2(U) = 1$.

Proof.

$$\begin{aligned}
\kappa_2(U) &= ||U||_2 ||U^{-1}||_2 && (40)\\
&= \sup_{x \neq 0} \frac{||Ux||_2}{||x||_2} \cdot \sup_{y \neq 0} \frac{||U^{-1}y||_2}{||y||_2} && (41)\\
&= \sup_{x \neq 0} \frac{||x||_2}{||x||_2} \cdot \sup_{y \neq 0} \frac{||y||_2}{||y||_2} && (42)\\
&= 1 \cdot 1 && (43)\\
&= 1 && (44)
\end{aligned}$$

□

Lemma 5. *For arbitrary unitary matrix U, its inverse U^{-1} is still unitary.*