THE UNIVERSITY OF TEXAS
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CS383C NUMERICAL ANALYSIS

Homework 03

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Exercise 2. Show that if H is a reflector, then

2.1 $HH = I$

Since H is a reflector, we have

$$H = I - 2uu^H \tag{1}$$

where u is unit vector ($||u||_2^2 = 1$).

$$\begin{aligned} HH &= (I - 2uu^H)(I - 2uu^H) \\ &= I \cdot I - 2uu^H - 2uu^H + 4(uu^H)(uu^H) \\ &= I - 4uu^H + 4||u||_2^2 uu^H \\ &= I - 4uu^H + 4uu^H \\ &= I \end{aligned} \tag{2}$$

Lemma 1. *For arbitrary vector $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $ab^Tc = (b^Tc)a$.*

Proof.

$$ab^Tc = \begin{pmatrix} a_0 b^T \\ a_1 b^T \\ \vdots \\ a_{n-1} b^T \end{pmatrix} c = \begin{pmatrix} a_0 b^T c \\ a_1 b^T c \\ \vdots \\ a_{n-1} b^T c \end{pmatrix} = \begin{pmatrix} (b^Tc) a_0 \\ (b^Tc) a_1 \\ \vdots \\ (b^Tc) a_{n-1} \end{pmatrix} = (b^Tc) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = (b^Tc)a \tag{3}$$

□

Lemma 2. $(uu^H)(uu^H) = ||u||_2^2 uu^H$

Proof.

$$\begin{aligned} (uu^H)(uu^H) &= (uu^H u)u^H \\ &= ((u^H u)u)u^H \\ &= (||u||_2^2 u)u^H \\ &= ||u||_2^2 uu^H \end{aligned} \tag{4}$$

□

2.2 $H = H^H$

$$\begin{aligned} H^H &= (I - 2uu^H)^H \\ &= I^H - (2uu^H)^H \\ &= I - 2(u^H)^H u^H \\ &= I - 2uu^H \\ &= H \end{aligned} \tag{5}$$

2.3 $HH^H = I$

In terms of (5), multiply both sides with H and then reuse conclusion in (2)

$$H^H H = HH = I \tag{6}$$

Exercise 4. Show that if $x \in \mathbb{R}^n$, $v = x \mp ||x||_2 e_0$ and $\tau = v^T v/2$, then $(I - \frac{1}{\tau} vv^T)x = \pm||x||_2 e_0$

We start from the reflector $I - \frac{1}{\tau} vv^T$ on x,

$$\begin{aligned}
& (I - \frac{1}{\tau} vv^T)x \\
= & \left(I - \frac{2vv^T}{v^T v}\right)x \\
= & \left(I - \frac{2(x \mp ||x||_2 e_0)(x \mp ||x||_2 e_0)^T}{(x \mp ||x||_2 e_0)^T (x \mp ||x||_2 e_0)}\right)x \\
= & \left(I - 2\frac{xx^T \mp ||x||_2 (xe_0^T + e_0 x^T) + ||x||_2^2 e_0 e_0^T}{2||x||_2^2 \mp 2||x||_2 e_0^T x}\right)x \\
= & \left(I - \frac{xx^T \mp ||x||_2 (xe_0^T + e_0 x^T) + ||x||_2^2 e_0 e_0^T}{||x||_2^2 \mp ||x||_2 e_0^T x}\right)x \\
= & \frac{||x||_2^2 \mp ||x||_2 e_0^T x - xx^T \pm ||x||_2 (xe_0^T + e_0 x^T) - ||x||_2^2 e_0 e_0^T}{||x||_2^2 \mp ||x||_2 e_0^T x} x \\
= & \frac{||x||_2^2 x \mp ||x||_2 e_0^T x x - xx^T x \pm ||x||_2 (xe_0^T + e_0 x^T)x - ||x||_2^2 e_0 e_0^T x}{||x||_2^2 \mp ||x||_2 e_0^T x} \\
= & \frac{(||x||_2^2 x - xx^T x) + (\mp||x||_2 e_0^T x x \pm ||x||_2 x e_0^T x) \pm ||x||_2 e_0 x^T x - ||x||_2^2 e_0 e_0^T x}{||x||_2^2 \mp ||x||_2 e_0^T x} \\
= & \frac{\pm||x||_2 e_0 x^T x - ||x||_2^2 e_0 e_0^T x}{||x||_2^2 \mp ||x||_2 e_0^T x} \\
= & \frac{\pm e_0 x^T x - ||x||_2 e_0 e_0^T x}{||x||_2 \mp e_0^T x} \\
= & \frac{\pm||x||_2^2 e_0 - ||x||_2 (e_0^T x) e_0}{||x||_2 \mp e_0^T x} \\
= & \frac{\pm(||x||_2 - (e_0^T x))||x||_2 e_0}{||x||_2 \mp e_0^T x} \\
= & \pm||x||_2 e_0
\end{aligned} \tag{7}$$

Note that above derivation frequently makes use of the lemma 1.

Exercise 5. Complex

It is easy to show that the conclusion in (7) can extend to complex space. That is,

$$(I - \frac{1}{\tau}vv^H)x = \oplus||x||_2 e_0 \tag{8}$$

where $x \in \mathbb{C}^n,\ v = x \oplus ||x||_2 e_0$ and $\tau = v^H v/2$.
Let

$$v = \left(\frac{1}{u_2}\right),\ x = \left(\frac{\chi}{x_2}\right),\ e = \left(\frac{1}{0}\right) \tag{9}$$

Then have

$$\left(I - \frac{1}{\tau}\left(\frac{1}{u_2}\right)\left(\frac{1}{u_2}\right)^H\right) \cdot \left(\frac{\chi}{x_2}\right) = \left(\frac{\rho}{0}\right) \tag{10}$$

where $\tau = v^H v/2 = (1 + u_2^H u_2)/2$ and $\rho = \oplus||x||_2$.

Lemma 3. *If $x \in \mathbb{C}^n,\ v = x \oplus ||x||_2 e_0$ and $\tau = v^H v/2$, then $(I - \frac{1}{\tau}vv^H)x = \oplus||x||_2 e_0$.*

Proof. We start from the reflector $I - \frac{1}{\tau}vv^H$ on x,

$$\begin{aligned}
& (I - \frac{1}{\tau}vv^H)x \\
= & \left(I - \frac{2vv^H}{v^H v}\right)x \\
= & \left(I - \frac{2(x \oplus ||x||_2 e_0)(x \oplus ||x||_2 e_0)^H}{(x \oplus ||x||_2 e_0)^H (x \oplus ||x||_2 e_0)}\right)x \\
= & \left(I - 2\frac{xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) + ||x||_2^2 e_0 e_0^H}{2||x||_2^2 \oplus 2||x||_2 e_0^H x}\right)x \\
= & \left(I - \frac{xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) + ||x||_2^2 e_0 e_0^H}{||x||_2^2 \oplus ||x||_2 e_0^H x}\right)x \\
= & \frac{||x||_2^2 \oplus ||x||_2 e_0^H x - xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) - ||x||_2^2 e_0 e_0^H}{||x||_2^2 \oplus ||x||_2 e_0^H x}x \\
= & \frac{||x||_2^2 x \oplus ||x||_2 e_0^H xx - xx^H x \oplus ||x||_2(xe_0^H + e_0 x^H)x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{(||x||_2^2 x - xx^H x) + (\oplus||x||_2 e_0^H xx \oplus ||x||_2 xe_0^H x) \oplus ||x||_2 e_0 x^H x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{\oplus||x||_2 e_0 x^H x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{\oplus e_0 x^H x - ||x||_2 e_0 e_0^H x}{||x||_2 \oplus e_0^H x} \\
= & \frac{\oplus||x||_2^2 e_0 - ||x||_2 (e_0^H x) e_0}{||x||_2 \oplus e_0^H x} \\
= & \frac{\oplus(||x||_2 - (e_0^H x))||x||_2 e_0}{||x||_2 \oplus e_0^H x} \\
= & \oplus||x||_2 e_0
\end{aligned} \tag{11}$$

□

Exercise 6. Matrix Equivalence

We start from right hand side

$$
\begin{aligned}
RHS &= I - \frac{1}{\tau_1}\left(\begin{array}{c} 0 \\ \hline 1 \\ \hline u_2 \end{array}\right)\left(\begin{array}{c} 0 \\ \hline 1 \\ \hline u_2 \end{array}\right)^H \\
&= \left(I - \frac{1}{\tau_1} \begin{array}{c|c} 0 & 0 \\ \hline 0 & \left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)\left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)^H \end{array} \right) \\
&= \left(I - \begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{\tau_1}\left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)\left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)^H \end{array} \right) \\
&= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I - \frac{1}{\tau_1}\left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)\left(\begin{array}{c} 1 \\ \hline u_2 \end{array}\right)^H \end{array} \right) \\
&= LHS
\end{aligned} \tag{12}
$$

Exercise 11. Expensive Algorithm

As indicatd by **Theorem 10** in the note, we have cost of the algorithm in Figure 6 for $A \in \mathbb{C}^{m\times n}$

$$C_{FormQ}(m,n) = 2mn^2 - \frac{2}{3}n^3 \tag{13}$$

For $m = n$, the cost can be simplified as

$$C_{FormQ}(A) = \frac{4}{3}n^3 = \mathcal{O}(n^3) \tag{14}$$

However, if we accumulate Q by using n householder transformation with

$$Q = (\ldots((IH_0)H_1)\ldots H_{n-1}) \tag{15}$$

Then the cost we have is at least

$$C_{accumulation}(A) = n^3 \cdot (n-1) = \mathcal{O}(n^4) \tag{16}$$

where n^3 comes from each one matrix multiplication, and $n-1$ comes from the total number of householder matrix H_i ($i \in \mathbb{Z}, i \in [0, n-1]$).
Comparing formula (14) and (16), it is obvious that the accumulation method is much more expensive than algorithm in Figure 6.