THE UNIVERSITY OF TEXAS
AT AUSTIN

CS383C NUMERICAL ANALYSIS

HW07 Cholesky Factorization

Edited by LaTeX

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RELEASE DATE

Oct. 24 2014

DUE DATE

Oct. 30 2014

TIME SPENT

2 hours

October 26, 2014

Exercises

Exercise 3. Prove that $A = B^H B$ is HPD.

To prove that $A = B^H B$ is HPD, we need to show both of the following:

- $A^H = A$.

  Proof.
  $$A^H = (B^H B)^H = B^H (B^H)^H = B^H B = A \tag{1}$$
  □

- $\forall x \neq 0,\ x^H A x > 0$.

  Proof. Let x be arbitrary non-zero vector in $\mathbb{C}^n$
  $$x^H A x = x^H B^H B x = (Bx)^H Bx = ||Bx||_2^2 > 0 \tag{2}$$
  Note that for $x \neq 0$, $Bx = \sum_i B_i x_i \neq 0$, otherwise B is not linearly independent columns. □

Since two properties above are proven, then we can conclude that

If $B \in \mathbb{C}^{m \times n}$ has linearly independent columns, then $A = B^H B$ is HPD.

Exercise 4. Show that diagonal elements are real and positive.

- diagonal elements of $A \in \mathbb{C}^{m \times m}$ are real.

  Proof. Since A is HPD, then $A^H = A$. Hence, for diagonal elements $\theta_0, \theta_1, ..., \theta_{m-1}$, then
  $$\forall i = 0, ..., m-1,\ \theta_i^H = \theta_i \tag{3}$$
  Let $\theta_i = x_i + y_i j$, where j denotes imaginary unit, then
  $$\forall i = 0, ..., m-1,\ -y_i = y_i \tag{4}$$
  That tells us
  $$\forall i = 0, ..., m-1,\ y_i = 0 \tag{5}$$
  Then it can be concluded that

  all diagonal elements $\theta_0, \theta_1, ..., \theta_{m-1}$ are real. (all imaginary part is zero.)

  □

- diagonal elements of $A \in \mathbb{C}^{m \times m}$ are positive.

  Proof. Since $A \in \mathbb{C}^{m \times m}$ is HPD, then
  $$\forall x \neq 0,\ x^H A x > 0 \tag{6}$$
  Let e_0, ... e_{m-1} denotes unit vector (whose imaginary part is zero) that spans through the whole $\mathbb{C}^m$. And let $\theta_0, \theta_1, ..., \theta_{m-1}$ denotes diagonal elements of HPD matrix A.
  $$\forall i = 0, ..., m-1,\ \theta_i = e_i^H A e_i > 0 \tag{7}$$
  Note that the $e_i^H A e_i \neq 0$ since $e_i \neq 0$. Hence, it can be conclude that

  all diagonal elements $\theta_0, \theta_1, ..., \theta_{m-1}$ are positive.

  □

Exercise 14. Implement Cholesky Factorization

```
% Copyright 2014 The University of Texas at Austin
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% For licensing information see
%                http://www.cs.utexas.edu/users/flame/license.html
%
% Programmed by: Jimmy Lin
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function [ A_out ] = CHOL_unb( A )

  [ ATL, ATR, ...
    ABL, ABR ] = FLA_Part_2x2( A, ...
                               0, 0, 'FLA_TL' );

  while ( size( ATL, 1 ) < size( A, 1 ) )

    [ A00,  a01,     A02,  ...
      a10t, alpha11, a12t, ...
      A20,  a21,     A22 ] = FLA_Repart_2x2_to_3x3( ATL, ATR, ...
                                                    ABL, ABR, ...
                                                    1, 1, 'FLA_BR' );

    %------------------------------------------------------------%

    a01 = zeros(size(a01));
    A02 = zeros(size(A02));
    a12t = zeros(size(a12t));
    alpha11 = sqrt(alpha11);
    a21 = a21 / alpha11;
    A22 = A22 - tril (a21 * a21');

    %------------------------------------------------------------%

    [ ATL, ATR, ...
      ABL, ABR ] = FLA_Cont_with_3x3_to_2x2( A00,  a01,     A02,  ...
                                             a10t, alpha11, a12t, ...
                                             A20,  a21,     A22, ...
                                             'FLA_TL' );

  end

  A_out = [ ATL, ATR
            ABL, ABR ];

return
```

Exercise 15. Relationship of Cholesky Factorization and QR

Proof. For matrix $B \in \mathbb{C}^{m\times n}$ with linearly independent columns, it has an unique QR factorization such that $B = QR$, where $Q \in \mathbb{C}^{m\times n}$ and $R \in \mathbb{C}^{n\times n}$. And then for HPD matrix A, we have

$$A = B^H B = (QR)^H QR = R^H Q^H QR = \underbrace{R^H}_{L}\underbrace{R}_{L^H} \tag{8}$$

□