



THE UNIVERSITY OF TEXAS
AT AUSTIN

EE381V LARGE SCALE OPTIMIZATION

Problem Set 2

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Part I

Matlab and Computational Assignment

1 Five flavors for Eq. (9.20) in B & V

1.1 Standard Gradient Descent with Backtracking

Command to be executed in matlab:

```
>> x_init = [1 1]'; alpha = 0.3; bta = 0.8;
>> [x, iter, all_costs] = gd_btls(x_init, @func, @func_grad, alpha, bta);
```

Dump

```
Iter: 1, Cost: 5.292007e+00, Conv_Rate: 0.106142, gamma: 0.800000
Iter: 2, Cost: 3.936296e+00, Conv_Rate: 0.743819, gamma: 0.409600
Iter: 3, Cost: 3.440868e+00, Conv_Rate: 0.874138, gamma: 0.327680
Iter: 4, Cost: 3.047567e+00, Conv_Rate: 0.885697, gamma: 0.262144
Iter: 5, Cost: 2.851476e+00, Conv_Rate: 0.935657, gamma: 0.262144
Iter: 6, Cost: 2.698084e+00, Conv_Rate: 0.946206, gamma: 0.209715
Iter: 7, Cost: 2.621951e+00, Conv_Rate: 0.971783, gamma: 0.134218
Iter: 8, Cost: 2.589490e+00, Conv_Rate: 0.987619, gamma: 0.107374
Iter: 9, Cost: 2.571558e+00, Conv_Rate: 0.993075, gamma: 0.068719
...
...
Iter: 40, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.000000
Iter: 41, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.000000
Iter: 42, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.000000
Iter: 43, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.000000
Iter: 44, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.000000
Convergence reached!
```

Minima

$x = [-0.3379, -0.0031]$, $obj = 2.559267$

Plot

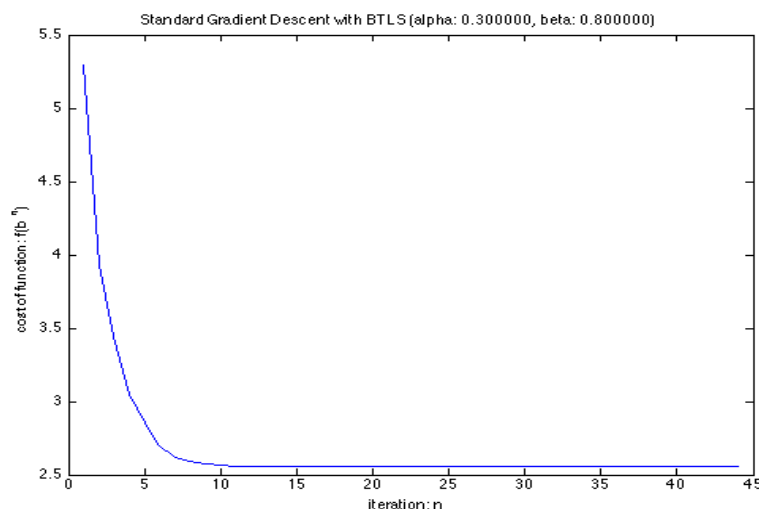


Figure 1: Standard gradient descent with BTLS on Eq. 9.20 with $\alpha = 0.3$ and $\beta = 0.8$

1.2 Steepest Descent with P_1

Command to be executed in matlab:

```
>> P1 = [8 0; 0 2];
>> x_init = [1 1]'; alpha = 0.3; bta = 0.8;
>> [x, iter, all_costs] = sd_btls(x_init, @func, @func_grad, P1, alpha, bta);
```

Dump

```
Iter: 1, Cost: 4.109800e+00, Conv_Rate: 0.082430, gamma: 0.035184
Iter: 2, Cost: 2.574134e+00, Conv_Rate: 0.626340, gamma: 0.409600
Iter: 3, Cost: 2.561394e+00, Conv_Rate: 0.995051, gamma: 1.000000
Iter: 4, Cost: 2.559319e+00, Conv_Rate: 0.999190, gamma: 0.800000
Iter: 5, Cost: 2.559268e+00, Conv_Rate: 0.999980, gamma: 0.800000
Iter: 6, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 7, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 8, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 9, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 10, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 11, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.800000
Iter: 12, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 1.000000
Convergence reached!
```

Minima

$x = [-0.3466 \ -0.0000]$, $obj = 2.559267$

Plot

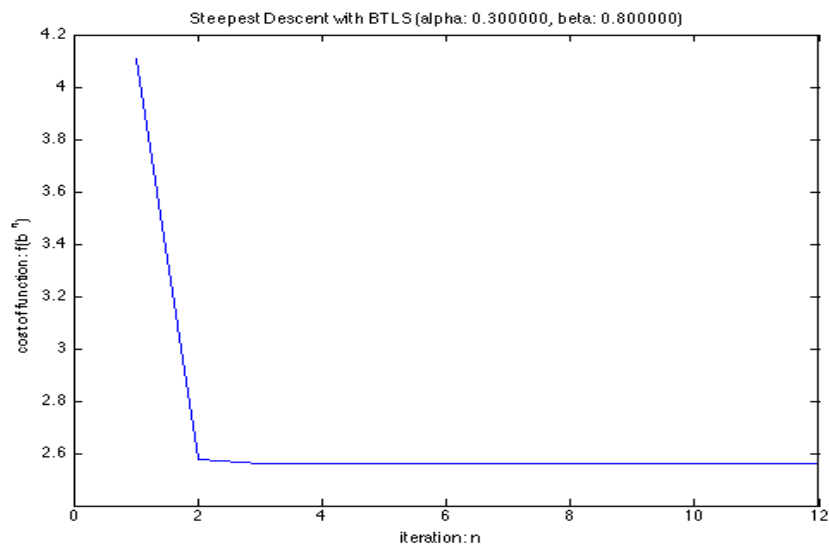


Figure 2: Steepest Descent with BTLS on Eq. 9.20 with P_1 , $\alpha = 0.3$ and $\beta = 0.8$

1.3 Steepest Descent with P_2

Command to be executed in matlab:

```
>> P2 = [2 0; 0 8];
>> x_init = [1 1]'; alpha = 0.3; bta = 0.8;
>> [x, iter, all_costs] = sd_btls(x_init, @func, @func_grad, P2, alpha, bta);
```

Dump

```
Iter: 1, Cost: 5.397521e+00, Conv_Rate: 0.108258, gamma: 0.011529
Iter: 2, Cost: 4.867283e+00, Conv_Rate: 0.901763, gamma: 0.068719
Iter: 3, Cost: 4.673428e+00, Conv_Rate: 0.960172, gamma: 0.107374
Iter: 4, Cost: 4.447617e+00, Conv_Rate: 0.951682, gamma: 0.085899
Iter: 5, Cost: 4.276423e+00, Conv_Rate: 0.961509, gamma: 0.134218
Iter: 6, Cost: 4.100684e+00, Conv_Rate: 0.958905, gamma: 0.107374
Iter: 7, Cost: 3.955941e+00, Conv_Rate: 0.964703, gamma: 0.134218
Iter: 8, Cost: 3.820349e+00, Conv_Rate: 0.965724, gamma: 0.134218
Iter: 9, Cost: 3.687908e+00, Conv_Rate: 0.965333, gamma: 0.134218
Iter: 10, Cost: 3.560463e+00, Conv_Rate: 0.965442, gamma: 0.134218
Iter: 11, Cost: 3.444372e+00, Conv_Rate: 0.967394, gamma: 0.134218
Iter: 12, Cost: 3.347890e+00, Conv_Rate: 0.971989, gamma: 0.167772
Iter: 13, Cost: 3.259404e+00, Conv_Rate: 0.973570, gamma: 0.167772
...
...
Iter: 164, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.262144
Iter: 165, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.409600
Iter: 166, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.327680
Iter: 167, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.262144
Iter: 168, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.167772
Convergence reached!
```

Minima

$x = [-0.3466 \ -0.0000]$, $obj = 2.559267$

Plot

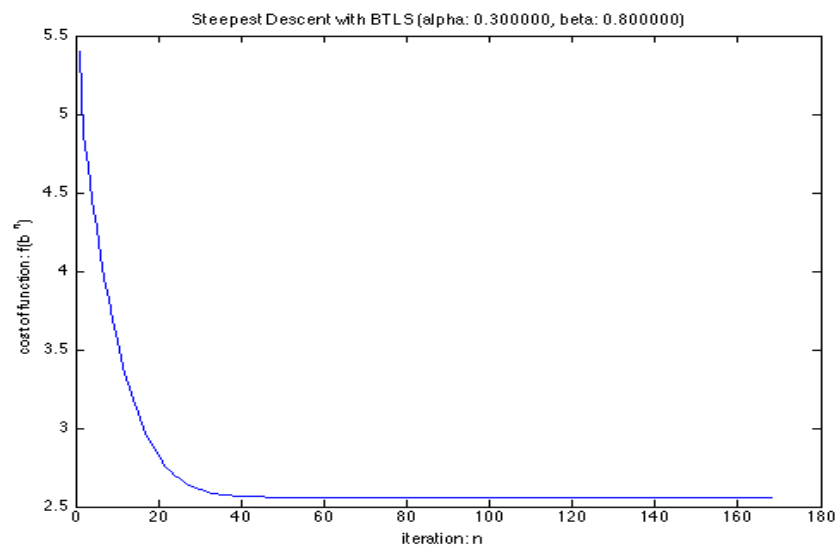


Figure 3: Steepest Descent with BTLS on Eq. 9.20 with P_2 , $\alpha = 0.3$ and $\beta = 0.8$

1.4 Cyclic Coordinate Descent

Command to be executed in matlab:

```
>> x_init = [1 1]'; alpha = 0.3; bta = 0.8;
>> [x, iter, all_costs] = ccd_btls(x_init, @func, @func_grad, alpha, bta);
```

Dump

```
Iter: 1, Cost: 1.217687e+01, Conv_Rate: 0.244232, gamma: 0.043980
Iter: 2, Cost: 3.551842e+00, Conv_Rate: 0.071239, gamma: 0.035184
Iter: 3, Cost: 3.064142e+00, Conv_Rate: 0.862691, gamma: 0.209715
Iter: 4, Cost: 2.732279e+00, Conv_Rate: 0.769257, gamma: 0.134218
Iter: 5, Cost: 2.664867e+00, Conv_Rate: 0.975328, gamma: 0.209715
Iter: 6, Cost: 2.600923e+00, Conv_Rate: 0.951925, gamma: 0.134218
Iter: 7, Cost: 2.585633e+00, Conv_Rate: 0.994121, gamma: 0.262144
Iter: 8, Cost: 2.563802e+00, Conv_Rate: 0.985727, gamma: 0.107374
Iter: 9, Cost: 2.561867e+00, Conv_Rate: 0.999245, gamma: 0.209715
Iter: 10, Cost: 2.560171e+00, Conv_Rate: 0.998584, gamma: 0.107374
Iter: 11, Cost: 2.559836e+00, Conv_Rate: 0.999869, gamma: 0.167772
Iter: 12, Cost: 2.559551e+00, Conv_Rate: 0.999758, gamma: 0.107374
Iter: 13, Cost: 2.559478e+00, Conv_Rate: 0.999971, gamma: 0.167772
...
...
Iter: 60, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.134218
Iter: 61, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.262144
Iter: 62, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.085899
Iter: 63, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.327680
Iter: 64, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.134218
Convergence reached!
```

Minima

$x = [-0.3466 \ 0.0000]$, $obj = 2.559267$

Plot

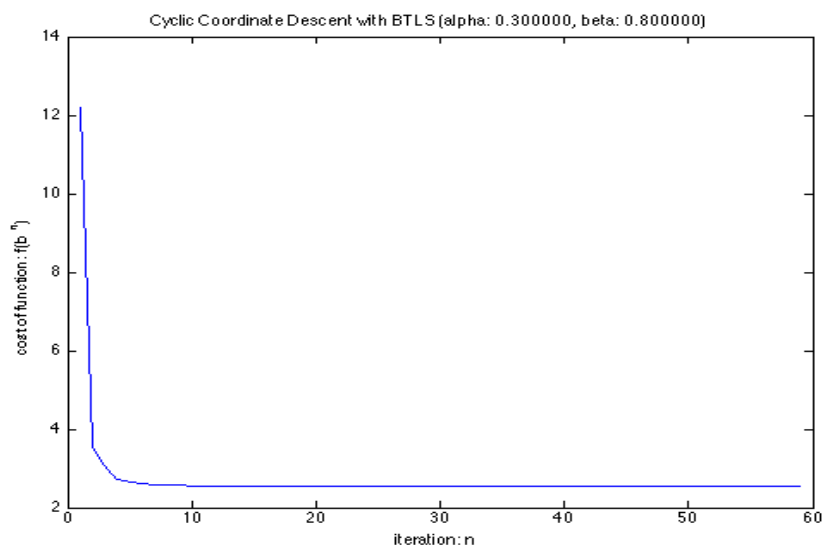


Figure 4: Cyclical Coordinate Descent with BTLS on Eq. 9.20 with $\alpha = 0.3$ and $\beta = 0.8$

1.5 Greedy Coordinate Descent

Command to be executed in matlab:

```
>> x_init = [1 1]'; alpha = 0.3; bta = 0.8;
>> [x, iter, all_costs] = gcd_btls(x_init, @func, @func_grad, alpha, bta);
```

Dump

```
Iter: 1, Cost: 5.615225e+00, Conv_Rate: 0.112625, gamma: 0.005903
Iter: 2, Cost: 4.227075e+00, Conv_Rate: 0.752788, gamma: 0.028147
Iter: 3, Cost: 3.306101e+00, Conv_Rate: 0.782125, gamma: 0.035184
Iter: 4, Cost: 2.823271e+00, Conv_Rate: 0.853958, gamma: 0.054976
Iter: 5, Cost: 2.666299e+00, Conv_Rate: 0.944401, gamma: 0.068719
Iter: 6, Cost: 2.577867e+00, Conv_Rate: 0.966834, gamma: 0.107374
Iter: 7, Cost: 2.566546e+00, Conv_Rate: 0.995608, gamma: 0.107374
Iter: 8, Cost: 2.561822e+00, Conv_Rate: 0.998160, gamma: 0.107374
Iter: 9, Cost: 2.560570e+00, Conv_Rate: 0.999511, gamma: 0.107374
Iter: 10, Cost: 2.559369e+00, Conv_Rate: 0.999531, gamma: 0.107374
Iter: 11, Cost: 2.559298e+00, Conv_Rate: 0.999972, gamma: 0.107374
Iter: 12, Cost: 2.559277e+00, Conv_Rate: 0.999992, gamma: 0.107374
    ...
    ...
Iter: 32, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.107374
Iter: 33, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.107374
Iter: 34, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.107374
Iter: 35, Cost: 2.559267e+00, Conv_Rate: 1.000000, gamma: 0.107374
Convergence reached!
```

Minima

```
x = [-0.3466 -0.0000], obj = 2.559267
```

Plot

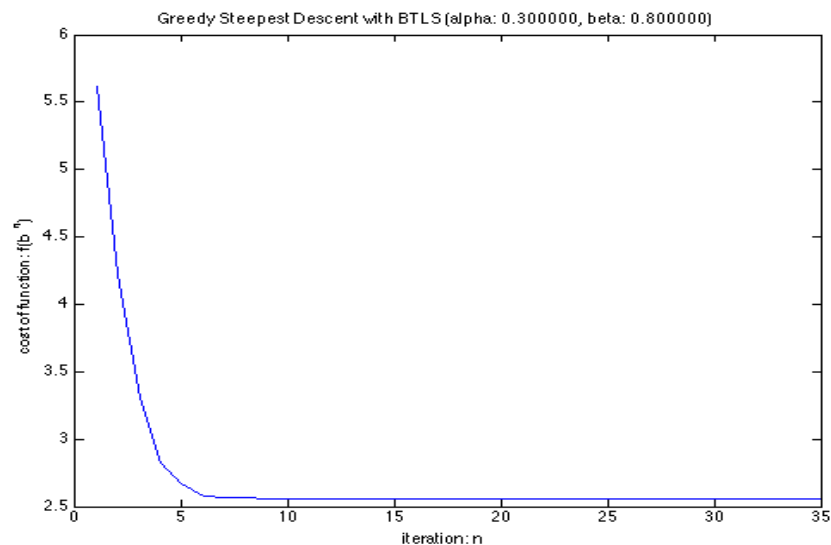


Figure 5: Greedy Coordinate Descent with BTLS on Eq. 9.20 with $\alpha = 0.3$ and $\beta = 0.8$

1.6 Conclusions

- $(1, 1)$ is a decent initial point for all five flavors of optimization methods.
- Steepest descent method could both enhance and impair the convergence speed, comparing to standard gradient descent (uniform heuristic or unheuristic). The specific effect depends on what heuristic matrix is provided.
- Greedy coordinate descent does converge to optima in less number of iterations than the cyclic coordinate descent but in larger computational cost in each iteration.

Part II

Written Problems

1 Coordinate Descent

(a) Give an example

The example to illustrate failure of coordinate descent in converging to global minimum of function f at point x is

$$f(x, y) = \|(x, y)\|_\infty = \max(x, y) \quad (1)$$

at point $(x, y) = (-2, -2)$.

And the contour is drawn as follows:

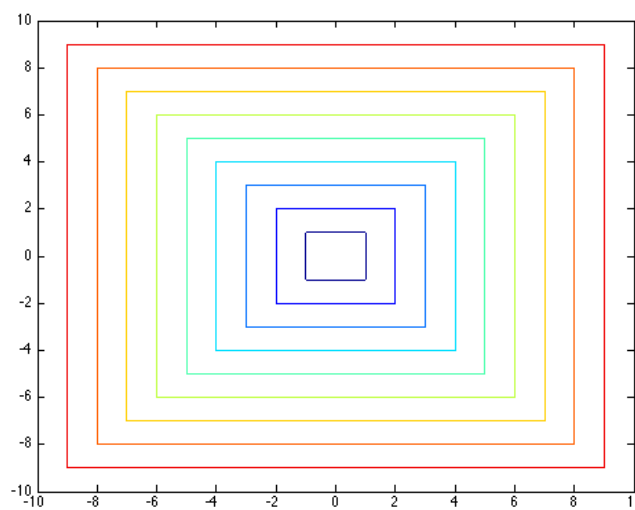


Figure 6: Contour of function $f(x, y) = \|(x, y)\|_\infty$

Note that since $f(x, y) = \|(x, y)\|_\infty$ is not differentiable, any coordinate descent that employs gradient of $f(x, y)$ is not available. But we can still apply non-gradient-based coordinate descent algorithm on this problem.

Remark 1. *There is a bug for this question. Function can be chosen as $f(x) = x$, which is convex but not strongly convex and the point x can be arbitrary point. Since $f(x) = x$ has its global minimum $-\infty$, then the coordinate descent starting from arbitrary point $x \in \mathbb{R}$ can never get to the global minimum of f forever.*

(b) $f(x, y) = x^2 + y^2 + 3xy$

The coordinate descent with exact line search **will not** always converge to a stationary point. The stationary point is $(x, y) = (0, 0)$. Consider the coordinate descent at $(10, -10)$:

$$\frac{\partial f(x, y)}{\partial x} = 2x + 3y = -10 \quad f(x, y) \text{ will descend if } x \text{ increase} \quad (2)$$

$$\frac{\partial f(x, y)}{\partial y} = 2y + 3x = 10 \quad f(x, y) \text{ will descend if } y \text{ decrease} \quad (3)$$

Obviously, the coordinate descent at $(10, -10)$ will go away from $(0, 0)$. Therefore, $f(x, y)$ has no guarantee to converge to stationary point by coordinate descent with exact line search.

2 Condition Number

For arbitrary step size t , the strongly convex function $f(x) = \frac{1}{t}x^2$ starting from $x_0 = 1$ does not converge to optimal solution.

Proof. We begin with showing $f(x) = \frac{1}{t}x^2$ is strongly convex. Obviously, $0 \leq f(x) \leq \frac{2}{t}$. Hence, function $f(x) = \frac{1}{t}x^2$ is strongly convex with $m = 0$ and $M = \frac{1}{t}$. (The hessian of $f(x)$ is bounded.)

Then we show that $f(x) = \frac{1}{t}x^2$ starting from $x_0 = 1$ does not converge to optimal solution. The gradient of $f(x)$ at x_0 is given by

$$\nabla f(x_0) = \nabla f(x)|_{x=1} = \frac{2}{t}x|_{x=1} = \frac{2}{t} \quad (4)$$

By gradient algorithm, the update direction at x_0 is given by

$$\delta x_0 = -\nabla f(x_0) = -\frac{2}{t} \quad (5)$$

Apply the update step

$$x^+ = x_0 - \Delta x_0 = 1 - t \cdot \frac{2}{t} = 1 - 2 = -1 \quad (6)$$

It is obvious that

$$f(x^+) = f(-1) = \frac{1}{t} = f(1) = f(x_0) \quad (7)$$

Continue the gradient algorithm repeatedly. And then we find that the function at k -step always equals its initial value $f(x_0)$ and never reach its optimum.

$$f(x_k) = f(x_0) \neq f(0) = 0 \quad (8)$$

Hence, it is proved that *for any fixed step size t , there exists a smooth strongly convex function with bounded Hessian, such that a fixed stepsize gradient algorithm starting from some point x_0 , does not converge to the optimal solution.* \square

3 Decreasing Stepsize

Proof. Let constant m and M be the lower and upper bound on hessian of $f(x)$. That is

$$mI \leq \nabla^2 f(x) \leq MI \quad (9)$$

The given conditions are

$$\lim_{k \rightarrow \infty} t_k = 0 \quad (10)$$

$$\sum_{k=0}^{\infty} t_k = \infty \quad (11)$$

which tells us that t_k will start from very large value and finally decrease to 0 (sufficiently small step size). Then we have

$$\exists k, 0 < t_k < \frac{1}{M} \quad (12)$$

By second-order approximation and lower bound of Hessian on step k , we have

$$f(x_{k+1}) \leq f(x) - t_k \| \nabla f(x_k) \|_2^2 + \frac{Mt_k^2}{2} \| \nabla f(x_k) \|_2^2 \quad (13)$$

$$f(x_{k+1}) \leq f(x) \left(\frac{Mt_k}{2} - 1 \right) t_k \| \nabla f(x_k) \|_2^2 \quad (14)$$

In terms of (12), then we have

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2} t_k \| \nabla f(x_k) \|_2^2 \quad (15)$$

Subtracted both sides with p^*

$$f(x_{k+1}) - p^* \leq f(x_k) - p^* - \frac{1}{2} t_k \| \nabla f(x_k) \|_2^2 \quad (16)$$

Combined with previously derived inequality

$$\| \nabla f(x_k) \|_2^2 \geq 2m(f(x_k) - p^*) \quad (17)$$

Then we have

$$f(x_{k+1}) - p^* \leq (1 - mt_k)(f(x_k) - p^*) \quad (18)$$

Since $m > 0$ and $t_k > 0$, then

$$f(x_{k+1}) - p^* \leq c(f(x_k) - p^*) \quad (19)$$

where $c = (1 - mt_k) < 1$. And let us apply this formula recursively and get

$$f(x_{k+1}) - p^* \leq c^k(f(x_0) - p^*) \quad (20)$$

which guarantee the convergence of $f(x)$ since $c < 1$. Hence, it can be concluded that with the infinity decreasing step size sequence, the gradient descent must converge to the global optimal solution. \square

4 Convex Functions

(a) $f(x) = \sup_i f_i(x)$

Let a and b be two arbitrary distinct points such that $a \in \text{dom} f$ and $b \in \text{dom} f$. And let $\lambda \in [0, 1]$.

$$f(\lambda a + (1 - \lambda)b) = \sup_i f_i(\lambda a + (1 - \lambda)b) \quad (21)$$

$$\leq \sup_i \lambda f_i(a) + (1 - \lambda) f_i(b) \quad (22)$$

$$\leq \sup_i \lambda f_i(a) + \sup_i (1 - \lambda) f_i(b) \quad (23)$$

$$= \lambda \sup_i f_i(a) + (1 - \lambda) \sup_i f_i(b) \quad (24)$$

$$= \lambda f(a) + (1 - \lambda) f(b) \quad (25)$$

Since $f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$ hold for arbitrary distinct points a and b , we can conclude that

$$f_i(x) \text{ is convex } \forall i \Rightarrow f(x) = \sup_i f_i(x) \text{ is convex.} \quad (26)$$

(b) $\lambda_{\max}(M)$

(c) **Weighted shortest path from a to b**

5 Convex Functions: Jensen's Inequality

- (a) $\text{epi}(f)$ is also convex
- (b) Finite version of Jensen's inequality

6 Projection

7 Computing Projections

- (a) $\chi = \{x : L_i \leq x_i \leq U_i, i = 1, \dots, n\}$
- (b) $\chi = \mathbb{R}_+^n$
- (c) **Euclidean ball:** $\{x : \|x\|_2 \leq 1\}$
- (d) **1-norm ball:** $\{x : \sum_i |x_i| \leq 1\}$
- (e) **Positive semidefinite cone:** $S_+^n = \{M \in S^n : x^T M x \geq 0, \forall x \in \mathbb{R}^n\}$
- (f) **Probability Simplex:** $\chi = \{\sum_i x_i = 1, x_i \geq 0, i = 1, \dots, n\}$

A Codes Printout

(a) Eq. 20 and its gradient

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% objective function for optimization
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
%   y = func (x)
% Parameter:
%   x: input vector, must be column vector

function y = func (x)
assert (size(x, 1) == 2);
assert (size(x, 2) == 1);
x_1 = x(1);
x_2 = x(2);
term1 = x_1 + 3*x_2 - 0.1;
term2 = x_1 - 3*x_2 - 0.1;
term3 = -1*x_1 - 0.1;
y = exp(term1) + exp(term2) + exp(term3);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% gradient of function EQ. 9.20 in B & V
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
%   gradient = func_grad (x)
% Parameter:
%   b: variable vector

function gradient = func_grad (x)
assert(all(size(x)==[2 1]))
x_1 = x(1);
x_2 = x(2);
grad_1 = exp(x_1+3*x_2-0.1) + exp(x_1-3*x_2-0.1) -1*exp(-1*x_1-0.1);
grad_2 = 3*exp(x_1+3*x_2-0.1) -3*exp(x_1-3*x_2-0.1);
gradient = [grad_1 grad_2]';
end

```

(b) Standard Gradient Descent with BackTracking Line Search

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% HW2: Gradient Descent with Backtrack Line Search
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
% [b, iter, all_costs] = gd_btls (b_init, f, fgrad, alpha, discount)
% Parameters:
%   b_init: initial value of variable
%   f: objective function
%   fgrad: gradient of objective function
%   alpha: parameter for evaluating "OK" step size
%   discount(beta): discounting factor for not-OK step size
%% Note that the distinguishable period eps = 10e-16

function [b, iter, all_costs] = gdbtls (b_init, f, fgrad, alpha, discount)
assert (discount > 0 && discount < 1);
assert (alpha > 0 && alpha < 0.5);
iter = 1; % iteration count
b = b_init; % variable vector
last_cost = func(b); % value of objective function in most recent iteration
all_costs = []; % all values of objective function, for plotting
while true,
    %% compute essential numerics and do gradient descent
    gradient = fgrad(b);
    delta_b = -1.0 * gradient / norm(gradient);
    %% do backtrack line search
    gamma = 1.0; % step size
    while f(b+gamma*delta_b) > f(b) + alpha*gamma*gradient'*delta_b,
        gamma = discount * gamma;
        % disp(sprintf('BTLS: new gamma is %f', gamma));
    end
    b = b + gamma * delta_b;
    cost = func(b);
    rate = (cost / last_cost);
    all_costs = [all_costs cost];
    %% output numeric information of this iteration
    disp (sprintf('Iter: %d, Cost: %e, Conv.Rate: %f, gamma: %f', iter, cost, rate, gamma));
    %% quadratic optimization converges to zero
    if abs((cost - last_cost) / last_cost) < eps,
        disp('Convergence reached!')
        break
    end
    %% prepare for next iteration
    last_cost = cost;
    iter = iter + 1;
end
%% uncomment following code for plotting individual gradient descent run
% plot f(b^(n)) with regard to n
plot (1:iter, all_costs)
title (sprintf ('Standard Gradient Descent with BTLS (alpha: %f, beta: %f)', alpha, discount));
xlabel ('iteration: n');
ylabel ('cost of function: f(b^n)');
end

```

(c) Steepest Descent with BackTracking Line Search

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Steepest Descent Algorithm with Backtrack Line Search
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
% [b, iter, all_costs] = sd_btls (b_init, f, fgrad, P, alpha, discount)
% Parameters:
%   b_init: starting search point of variable
%   f: objective function
%   fgrad: gradient of objective function
%   P: matrix that defines norm of steepest descent
%   alpha: parameter for evaluating "OK" step size
%   discount(beta): discounting factor for not-OK step size
%% Note that the distinguishable period eps = 10e-16

function [b, iter, all_costs] = sd_btls (b_init, f, fgrad, P, alpha, discount)
assert (discount > 0 && discount < 1);
assert (alpha > 0 && alpha < 0.5);
iter = 1; % iteration count
b = b_init; % variable vector
last_cost = func(b); % value of objective function in most recent iteration
all_costs = []; % all values of objective function, for plotting
while true,
    %% compute essential numerics and do gradient descent
    gradient = fgrad(b);
    delta_b = -1.0 * inv(P) * gradient;
    %% do backtrack line search
    gamma = 1.0;
    while f(b+gamma*delta_b) > f(b) + alpha*gamma*gradient'*delta_b,
        gamma = discount * gamma;
        % disp(sprintf('BTLS: new gamma is %f', gamma));
    end
    b = b + gamma * delta_b;
    cost = func(b);
    rate = (cost / last_cost);
    all_costs = [all_costs cost];
    %% output numeric information of this iteration
    disp (sprintf('Iter: %d, Cost: %e, Conv_Rate: %f, gamma: %f', iter, cost, rate, gamma));
    %% quadratic optimization converges to zero
    if abs((cost - last_cost) / last_cost) < eps,
        disp('Convergence reached!')
        break
    end
    %% prepare for next iteration
    last_cost = cost;
    iter = iter + 1;
end
%% uncomment following code for plotting individual gradient descent run
% plot f(b^(n)) with regard to n
plot (1:iter, all_costs)
title (sprintf ('Steepest Descent with BTLS (alpha: %f, beta: %f)', alpha, discount));
xlabel ('iteration: n');
ylabel ('cost of function: f(b^n)');
end

```

(d) Cyclic Coordinate Descent

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Cyclic Coordinate Descent Algorithm with Backtrack Line Search
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
%   [b, iter, all_costs] = ccd.btls (b_init, f, fgrad, alpha, discount)
% Parameters:
%   b_init: starting search point of variable
%   f: objective function
%   fgrad: gradient of objective function
%   P: matrix that defines norm of steepest descent
%   alpha: parameter for evaluating "OK" step size
%   discount(beta): discounting factor for not-OK step size
% Note:
%   a) the minimal distinguishable value eps = 10e-16
%   b) coordinate descent in cyclical epoch is one iteration

function [b, iter, all_costs] = ccd.btls (b_init, f, fgrad, alpha, discount)
assert (discount > 0 && discount < 1);
assert (alpha > 0 && alpha < 0.5);
iter = 1; % iteration count
b = b_init; % variable vector
ndim = size(b, 1);
last_cost = func(b); % value of objective function in most recent iteration
all_costs = []; % all values of objective function, for plotting
while true,
    gradient = fgrad(b);
    for d = 1:ndim,
        %% compute essential numerics and do gradient descent
        delta_b = zeros (d, 1);
        delta_b(d) = -1.0 * gradient(d);
        %% do backtrack line search
        gamma = 1.0;
        while f(b+gamma*delta_b) > f(b) + alpha*gamma*gradient'*delta_b,
            gamma = discount * gamma;
        end
        b = b + gamma * delta_b;
    end
    cost = func(b);
    rate = (cost / last_cost);
    all_costs = [all_costs cost];
    %% output numeric information of this iteration
    disp (sprintf('Iter: %d, Cost: %e, Conv-Rate: %f, gamma: %f', iter, cost, rate, gamma));
    %% quadratic optimization converges to zero
    if abs((cost - last_cost) / last_cost) < eps,
        disp('Convergence reached!')
        break
    end
    %% prepare for next iteration
    iter = iter + 1;
    last_cost = cost;
end
%% uncomment following code for plotting individual gradient descent run
% plot f(b^(n)) with regard to n
plot (1:iter, all_costs)
title (sprintf ('Cyclic Coordinate Descent with BTLS (alpha: %f, beta: %f)', alpha, discount));
xlabel ('iteration: n');
ylabel ('cost of function: f(b^n)');
end

```

(e) Greedy Coordinate Descent

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Greedy Coordinate Descent Algorithm with Backtrack Line Search
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
% [b, iter, all_costs] = gcd.btls (b_init, f, fgrad, alpha, discount)
% Parameters:
%   b_init: starting search point of variable
%   f: objective function
%   fgrad: gradient of objective function
%   P: matrix that defines norm of steepest descent
%   alpha: parameter for evaluating "OK" step size
%   discount(beta): discounting factor for not-OK step size
% Note:
%   a) the minimal distinguishable value eps = 10e-16

function [b, iter, all_costs] = gcd.btls (b_init, f, fgrad, alpha, discount)
assert (discount > 0 && discount < 1);
assert (alpha > 0 && alpha < 0.5);
iter = 1; % iteration count
b = b_init; % variable vector
ndim = size(b, 1);
last_cost = func(b); % value of objective function in most recent iteration
all_costs = []; % all values of objective function, for plotting
while true,
    gradient = fgrad(b);
    waitlist = zeros(ndim, 1);
    gammalist = zeros(ndim, 1);
    for d = 1:ndim,
        %% compute essential numerics and do gradient descent
        delta_b = zeros (d, 1);
        delta_b(d) = -1.0 * gradient(d);
        %% do backtrack line search
        gamma = 1.0;
        while f(b+gamma*delta_b) > f(b) + alpha*gamma*gradient'*delta_b,
            gamma = discount * gamma;
        end
        waitlist(d) = f(b+gamma*delta_b);
        gammalist(d) = gamma;
    end
    [min_value, min_index] = min(waitlist);
    delta_b = zeros (d, 1);
    delta_b(min_index) = -1.0 * gradient(min_index);
    b = b + gammalist(min_index) * delta_b;
    cost = func(b);
    rate = (cost / last_cost);
    all_costs = [all_costs cost];
    %% output numeric information of this iteration
    disp (sprintf('Iter: %d, Cost: %e, ConvRate: %f, gamma: %f', iter, cost, rate, gamma));
    %% quadratic optimization converges to zero
    if abs((cost - last_cost) / last_cost) < eps,
        disp('Convergence reached!')
        break
    end
    %% prepare for next iteration
    iter = iter + 1;
    last_cost = cost;
end
%% uncomment following code for plotting individual gradient descent run
% plot f(b^(n)) with regard to n
plot (1:iter, all_costs)
title (sprintf ('Greedy Steepest Descent with BTLS (alpha: %f, beta: %f)', alpha, discount));
xlabel ('iteration: n');
ylabel ('cost of function: f(b^n)');
end

```