THE UNIVERSITY OF TEXAS
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CS383C NUMERICAL ANALYSIS

Midterm Exam

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Exercises

Exercise 1.

(a) Prove that $||\cdot||$ is a vector norm.

To prove $||\cdot||$ is a vector norm, we have to show all the following three property holds:

- If $x \neq 0$, then $||x|| > 0$.
- For arbitrary scalar α and vector x, $||\alpha x|| = |\alpha| \cdot ||x||$.
- For arbitrary vector x and y, $||x + y|| \leq ||x|| + ||y||$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i,\ \delta_i > 0$ holds. Besides, since $x \neq 0$, then $\exists i,\ x_i^2 > 0$.

$$||x|| = \sqrt{x^T D x} = \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} > 0 \qquad x \neq 0 \tag{1}$$

Therefore, we proved that if $x \neq 0$, then $||x|| > 0$. □

Proof.

$$||\alpha x|| = \sqrt{(\alpha x)^T D(\alpha x)} = \sqrt{\sum_{i=0}^{m-1} \alpha^2 \delta_i x_i^2} = \sqrt{\alpha^2 \sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \cdot ||x|| \tag{2}$$

□

Proof.

$$||x + y||^2 = (\sqrt{(x + y)^T D(x + y)})^2 = x^T Dx + x^T Dy + 2x^T Dy \tag{3}$$

$$(||x|| + ||y||)^2 = (\sqrt{x^T Dx})^2 + (\sqrt{y^T Dy})^2 = x^T Dx + y^T Dy + 2\sqrt{x^T Dx}\sqrt{y^T Dy} \tag{4}$$

Now we show that $x^T Dy \leq \sqrt{x^T Dx}\sqrt{y^T Dy}$, which is equivalent to

$$\Big(\sum_{i=0}^{m-1} \delta_i x_i y_i\Big)^2 \leq \Big(\sum_{i=0}^{m-1} \delta_i x_i^2\Big)\Big(\sum_{i=0}^{m-1} \delta_i y_i^2\Big) \tag{5}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \delta_i\delta_j x_i y_i x_j y_j \leq \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \delta_i\delta_j x_i^2 y_j^2 \tag{6}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i\delta_j (2x_i y_i x_j y_j) + \sum_{i=0}^{m-1} \delta_i^2 x_i y_i x_i y_i \leq \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i\delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) + \sum_{i=0}^{m-1} \delta_i^2 x_i^2 y_i^2 \tag{7}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i\delta_j (2x_i y_i x_j y_j) \leq \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i\delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) \tag{8}$$

which obivously holds if we apply basic inequality on each (i, j) pair,

$$2x_i y_i x_j y_j = 2(x_i y_j)(x_j y_i) \leq x_i^2 y_j^2 + x_j^2 y_i^2 \tag{9}$$

Since we have proved $x^T Dy \leq \sqrt{x^T Dx}\sqrt{y^T Dy}$, then

$$||x + y||^2 \leq (||x|| + ||y||)^2 \tag{10}$$

which leads to desired triangle inequality since $||\cdot|| \geq 0$,

$$||x + y|| \leq ||x|| + ||y|| \tag{11}$$

□

Since all three property have been proved, it can be concluded that

$$||\cdot|| \text{ defined by } ||x|| = \sqrt{x^T Dx} \text{ is a norm.} \tag{12}$$

(b) Show that $||\cdot||$ is a matrix norm.

To prove $||\cdot||$ is a matrix norm, we have to show all the following three property holds:

- If $A \neq 0$, then $||A|| > 0$.
- For arbitrary scalar α and matrix A, $||\alpha A|| = |\alpha| \cdot ||A||$.
- For arbitrary matrix A and B, $||A+B|| \leq ||A|| + ||B||$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i,\ \delta_i > 0$ holds. And Since $A \neq 0$, then there is at least one column vector that is not zero vector. Let us assume it is the j-th column, a_j, that is non-zero. Then

$$||A|| = ||DA||_2 = \max_{||x||_2 \neq 0} \frac{||DAx||_2}{||x||_2} \geq \frac{||DAe_j||_2}{||e_j||_2} > 0 \tag{13}$$

Therefore, we proved that if $A \neq 0$, then $||A|| > 0$. $\square$

Proof.

$$\begin{aligned}
||\alpha A|| &= ||D(\alpha A)||_2 && (14)\\
&= ||\alpha DA||_2 && (15)\\
&= |\alpha| \cdot ||DA||_2 && ||\cdot||_2 \text{ is a norm.} \quad (16)\\
&= |\alpha| \cdot ||A|| && (17)
\end{aligned}$$

Thus, we proved that $||\alpha A|| = |\alpha| \cdot ||A||$. $\square$

Proof.

$$\begin{aligned}
||A+B|| &= ||D(A+B)||_2 && (18)\\
&= \max_{x\neq 0} \frac{||DAx + DBx||_2}{||x||_2} && (19)\\
&\leq \max_{x\neq 0} \frac{||DAx||_2 + ||DBx||_2}{||x||_2} && (20)\\
&= \max_{x\neq 0} \left(\frac{||DAx||_2}{||x||_2} + \frac{||DBx||_2}{||x||_2} \right) && (21)\\
&\leq \max_{x\neq 0} \frac{||DAx||_2}{||x||_2} + \max_{y\neq 0} \frac{||DBy||_2}{||y||_2} && (22)\\
&= ||DA||_2 + ||DB||_2 = ||A|| + ||B|| && (23)
\end{aligned}$$

Hence, we proved that $||A+B|| \leq ||A|| + ||B||$. $\square$

Since all three property have been proved, it can be concluded that

$$||\cdot|| \text{ defined by } ||A|| = ||DA||_2 \text{ is a norm.} \tag{24}$$

Exercise 2.

(a) Show that $(..)^{-1} = (..)$.

Proof.

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array}\right) = \left(\begin{array}{c|c} L_{00}L_{00}^{-1} & 0 \\ \hline l_{10}^T L_{00}^{-1} - \frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} \cdot \lambda 11 & \frac{1}{\lambda_{11}} \cdot \lambda_{11} \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array}\right) = I \tag{25}$$

Hence, it is concluded that (assume all inversion exists)

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^{-1} = \left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array}\right) \tag{26}$$

□

(b) Show that $H_1 H_0 = ..$

According to given formula for H_i, we have

$$H_0 = I - \frac{1}{\tau_0} u_0 u_0^T \tag{27}$$

$$H_1 = I - \frac{1}{\tau_1} u_1 u_1^T \tag{28}$$

Now we manipulate $H_1 H_0$,

$$\begin{aligned} H_1 H_0 &= (I - \frac{1}{\tau_1} u_1 u_1^T)(I - \frac{1}{\tau_0} u_0 u_0^T) & (29) \\ &= I - \frac{1}{\tau_1} u_1 u_1^T - \frac{1}{\tau_0} u_0 u_0^T + \frac{1}{\tau_0 \tau_1} u_1 u_1^T u_0 u_0^T & (30) \\ &= I - (u_0|u_1)\left(\begin{array}{c|c} \frac{1}{\tau_0} & 0 \\ \hline -\frac{u_0^T u_1}{\tau_0 \tau_1} & \frac{1}{\tau_1} \end{array}\right)(u_0|u_1)^T & (31) \\ &= I - (u_0|u_1)\left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array}\right)^{-1}(u_0|u_1)^T & (32) \end{aligned}$$

Hence, it is concluded that (assume all inversion exists)

$$H_1 H_0 = I - (u_0|u_1)\left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array}\right)^{-1}(u_0|u_1)^T \tag{33}$$

(c) Show that $(1 - \frac{1}{\tau_1 u_1 u_1^T}).. = I - ...$

$$\begin{aligned} (I - \frac{1}{\tau_1} u_1 u_1^T)(I - U_0 L_{00}^T U_0^T) &= I - \frac{1}{\tau_1} u_1 u_1^T - U_0 L_{00}^T U_0^T + \frac{1}{\tau_1} u_1 u_1^T U_0 L_{00}^T U_0^T & (34) \\ &= I - (U_0|u_1)\left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -u_1^T U_0 L_{00}^{-1}/\tau_1 & \frac{1}{\tau_1} \end{array}\right)(U_0|u_1)^T & (35) \\ &= I - (U_0|u_1)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array}\right)^{-1}(U_0|u_1)^T & (36) \end{aligned}$$

Thus, it is concluded that (assume all inversion exists)

$$(I - \frac{1}{\tau_1} u_1 u_1^T)(I - U_0 L_{00}^T U_0^T) = I - (U_0|u_1)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array}\right)^{-1}(U_0|u_1)^T \tag{37}$$

(d) Show that $H_{k-1}\cdots H_1 H_0 = I - UL^{-1}U^H$

We can do mathematical induction on m. We separate the discussion of $k=1$ and $k>1$.

- Particular Case: $k=1$,

  Proof.

$$H_0 = I - \frac{1}{\tau_0}u_0 u_0^T = I - \underbrace{(u_0)}_{U_{(0)}}\underbrace{(\frac{1}{\tau_0})}_{L_{(0)}^{-1}}\underbrace{(u_0)^T}_{U_{(0)}^T} \tag{38}$$

  Hence, $H_{k-1}\cdots H_1 H_0 = I - UL^{-1}U^H$ holds for $k=1$. □

  Note that this case is particular because it derives matrix which is indeed scalar.

- Base Case: $k=2$, this can be proved by using result of (b).
- Inductive Case: assume that (I.H.)

$$H_{k-1}\cdots H_1 H_0 = I - U_{(k-1)}L_{(k-1)}^{-1}U_{(k-1)}^H \tag{39}$$

  show that

$$H_k H_{k-1}\cdots H_1 H_0 = I - U_{(k)}L_{(k)}^{-1}U_{(k)}^H \tag{40}$$

  holds for lower triangular matrix $L_{(k)}$.

  Proof. With application of the result of (c), we have

$$\begin{aligned} H_k(H_{k-1}H_{k-2}\cdots H_1 H_0) &= (1 - \frac{1}{\tau_k}u_k u_k^T)(I - U_{(k-1)}L_{(k-1)}^{-1}U_{(k-1)}^H) & (41)\\ &= I - \underbrace{(U_{(k-1)}|u_k)}_{U_{(k)}}\underbrace{\left(\begin{array}{c|c} L_{(k-1)} & 0 \\ \hline u_k^T U_{(k-1)} & \tau_k \end{array}\right)^{-1}}_{L_{(k)}^{-1}}\underbrace{(U_{(k-1)}|u_k)^T}_{U_{(k)}^T} & (42)\\ &= I - U_{(k)}L_{(k)}^{-1}U_{(k)}^T & (43)\end{aligned}$$

  □

- By the principle of mathematical induction, it can be concluded that

$$H_{k-1}\cdots H_1 H_0 = I - UL^{-1}U^H \tag{44}$$

  holds for all $k>0$.

Exercise 3.

(a) Relationship between m, n and r.

$$rank(A) = r \leq \min(m, n) \tag{45}$$

(b) Reduced SVD in terms of U, V and Σ.

- $A = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}$ and $V_r = V_L$.

$$A = U\Sigma V^T = (U_L|U_R)\left(\begin{array}{c|c}\Sigma_{TL} & 0 \\ \hline 0 & 0\end{array}\right)(V_L|V_R)^T = U_L \Sigma_{TL} V_L^T \tag{46}$$

- $(A^TA)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1}\Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$\begin{aligned}
(A^TA)^{-1} &= \left((U\Sigma V^T)^T U\Sigma V^T\right)^{-1} && (47)\\
&= \left(V\Sigma^T U^T U \Sigma V^T\right)^{-1} && (48)\\
&= \left(V\Sigma\Sigma V^T\right)^{-1} && (49)\\
&= \left((V_L|V_R)\left(\begin{array}{c|c}\Sigma_{TL} & 0 \\ \hline 0 & 0\end{array}\right)\left(\begin{array}{c|c}\Sigma_{TL} & 0 \\ \hline 0 & 0\end{array}\right)(V_L|V_R)^T\right)^{-1} && (50)\\
&= \left(V_L\Sigma_{TL}\Sigma_{TL}V_L^T\right)^{-1} && (51)\\
&= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (52)
\end{aligned}$$

- $(A^TA)^{-1}A^T = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1}$ and $V_r = U_L$.

$$\begin{aligned}
(A^TA)^{-1}A^T &= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1}V_L\Sigma_{TL}^T U_L^T && (53)\\
&= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}\Sigma_{TL}^T U_L^T && (54)\\
&= V_L^{-T}\Sigma_{TL}^{-1}U_L^T && (55)
\end{aligned}$$

- $A(A^TA)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}^{-1}\Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$\begin{aligned}
A(A^TA)^{-1} &= U_L\Sigma_{TL}V_L^T V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (56)\\
&= U_L\Sigma_{TL}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (57)\\
&= U_L\Sigma_{TL}^{-1}V_L^{-1} && (58)
\end{aligned}$$

- $A(A^TA)^{-1}A^T = U_r \Sigma_r V_r^T$, where $U_r = I^{m\times r}$, $\Sigma_r = I^{r\times r}$ and $V_r = I^{n\times r}$.

$$\begin{aligned}
A(A^TA)^{-1}A^T &= (U_L\Sigma_{TL}^{-1}V_L^{-1})V_L\Sigma_{TL}^T U_L^T && (59)\\
&= U_L\Sigma_{TL}^{-1}\Sigma_{TL}^T U_L^T && (60)\\
&= U_L U_L^T && (61)\\
&= I && (62)
\end{aligned}$$

Exercise 4.

(a) $\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1$ where δv_{11} is small.

Proof.

$$\psi_1 = [v_{11}\check{\chi}_1]_* = \frac{v_{11}\check{\chi}_1}{1+\epsilon_*} = (1 - \frac{\epsilon_*}{1+\epsilon_*})v_{11}\check{\chi}_1 = (v_{11} - \frac{\epsilon_* v_{11}}{1+\epsilon_*})\check{\chi}_1 \equiv (v_{11} + \delta v_{11})\check{\chi}_1 \tag{63}$$

where

$$\delta v_{11} = -\frac{\epsilon_* v_{11}}{1+\epsilon_*}, \text{ where } |\epsilon_*| \leq \mathbf{u} \tag{64}$$

Then

$$|\delta v_{11}| = |-\frac{\epsilon_*}{1+\epsilon_*}v_{11}| \tag{65}$$
$$= \frac{|\epsilon_*|}{|1+\epsilon_*|}|v_{11}| \tag{66}$$
$$\leq \frac{\mathbf{u}}{|1+\epsilon_*|}|v_{11}| \tag{67}$$
$$\leq \frac{\mathbf{u}}{1+|\epsilon_*|}|v_{11}| \tag{68}$$
$$\leq \frac{\mathbf{u}}{1+|\epsilon_*|-|\epsilon_*|-\mathbf{u}}|v_{11}| \tag{69}$$
$$= \frac{\mathbf{u}}{1-\mathbf{u}}|v_{11}| = \gamma_1|v_{11}| \tag{70}$$

Hence, we proved that

$$\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1 \text{ where } |\delta v_{11}| = \gamma_1 |v_{11}| \tag{71}$$

□

(b)

Proof. Start from dot product. We directly use the corollary in notes.

$$d \stackrel{\Delta}{=} [u_{12}^T \check{x_2}] = (u_{12}^T + \delta u_{12}^T)\check{x_2} \text{ , where } \delta u_{12}^T \leq \gamma_k |u_{12}^T| \leq \gamma_{k+1}|u_{12}^T| \tag{72}$$

Then we consider the subtraction step with Standard Computational Model (SCM).

$$s \stackrel{\Delta}{=} [\psi_1 - d] = (\psi_1 - d)(1+\epsilon_-) = (\psi_1 - d) + (\psi_1 - d)\epsilon_- \equiv (\psi_1 - d) + \delta\psi_1 = (\psi_1 + \delta\psi_1) - d \tag{73}$$

where $|\delta\psi_1| = |(\psi_1 - d)\epsilon_-| \leq |\psi_1 \epsilon_-| = |\epsilon_-||\psi_1| \leq \mathbf{u}|\psi_1|$. Then we consider the division step with Alternative Computational Model (ACM).

$$\check{\chi}_1 = [\frac{s}{v_{11}}] = \frac{s}{v_{11}}(1+\epsilon_/)_{-1} = \frac{s}{v_{11}(1+\epsilon_/)} \tag{74}$$

which can be rewritten as

$$(v_{11} + \delta v_{11})\check{\chi}_1 = s \tag{75}$$

where $|\delta v_{11}| = |\epsilon_/ v_{11}| \leq |\epsilon_/| \cdot |v_{11}| \leq \mathbf{u}|v_{11}|$. Then we have

$$(v_{11} + \delta v_{11})\check{\chi}_1 = (\psi_1 + \delta\psi_1) - d = (\psi_1 + \delta\psi_1) - (u_{12}^T + \delta u_{12}^T)\check{x}_2 \tag{76}$$

where

$$|\delta v_{11}| \leq \mathbf{u}|v_{11}|,\ |\delta\psi_1| \leq \mathbf{u}|\psi_1|,\ |\delta u_{12}^T| \leq \gamma_{k+1}|u_{12}^T| \tag{77}$$

□

(c) Induction for the desired result

To prove the desired result, let us do mathematical induction on k (the iteration of algorithm).

- Base Case: $k = 0$

  Proof. In this case, $U^{0\times 0}$ does not participate in computation, v_{11} is the right-bottom-most element, and no pre-computed $\check{x}_2$. Hence, it is natural that

$$(v_{11} + \delta v_{11})\check{\chi}_1 = (\psi_1 + \delta\psi_1) \tag{78}$$

  which indicates that $(U + \Delta U)\check{x} = (y + \delta y)$ holds for $k = 0$. (first iteration) □

- Inductive case: Assume that at p-th iteration ($p > 0$), pre-computed $\check{x}_{2(p)}$ satisfy

$$(U_{(p)} + \Delta U_{(p)})\check{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \tag{79}$$

  where $|\Delta U_{(p)}| \leq \gamma_n |U_{(p)}|$ and $|\delta y_{2(p)}| \leq \mathbf{u}|y_{2(p)}|$, (I.H.) holds.
  Show that for the pre-computed $\check{x}_2$ at next iteration $\check{x}_{2(p+1)} = \begin{pmatrix} \check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)} \end{pmatrix}$ and $y_{2(p+1)} = \begin{pmatrix} \psi_{1(p)} \\ \hline y_{2(p)} \end{pmatrix}$

$$(U_{(p+1)} + \Delta U_{(p+1)})\check{x}_{2(p+1)} = (y_{(p+1)} + \Delta y_{(p+1)}) \tag{80}$$

  where $|\Delta U_{(p+1)}| \leq \gamma_n |U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u}|y_{2(p+1)}|$, holds for some upper triangle $U_{(p+1)}$.

  Proof. In terms of result of (b), we have

$$(v_{11(p)} + \delta v_{11(p)})\check{\chi}_{1(p)} + (u^T_{12(p)} + \delta u^T_{12(p)})\check{x}_{2(p)} = (\psi_{1(p)} + \delta\psi_{1(p)}) \tag{81}$$

  where $\delta v_{11(p)} \leq \mathbf{u}|v_{11(p)}|$, $\delta\psi_{1(p)} \leq \mathbf{u}|\psi_{1(p)}|$, and $\delta u^T_{12(p)} \leq \gamma_{p+1}|u^T_{12(p)}|$.
  Combined with (I.H.)

$$(U_{(p)} + \Delta U_{(p)})\check{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \tag{82}$$

  we have

$$\left(\begin{array}{c|c} v_{11(p)} + \delta v_{11(p)} & u^T_{12(p)} + \delta u^T_{12(p)} \\ \hline 0 & U_{(p)} + \Delta U_{(p)} \end{array}\right) \begin{pmatrix} \check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)} \end{pmatrix} = \begin{pmatrix} \psi_{1(p)} + \delta\psi_{1(p)} \\ \hline y_{2(p)} + \Delta y_{2(p)} \end{pmatrix} \tag{83}$$

  where $|\Delta U_{(p)}| \leq \gamma_n|U_{(p)}|$, $\delta v_{11(p)} \leq \mathbf{u}|v_{11(p)}| \leq \gamma_n|v_{11(p)}|$, $\delta u^T_{12(p)} \leq \gamma_{p+1}|u^T_{12(p)}| \leq \gamma_n|u^T_{12(p)}|$ and $\delta\psi_{1(p)} \leq \mathbf{u}|\psi_{1(p)}|$, $|\delta y_{2(p)}| \leq \mathbf{u}|y_{2(p)}|$. (Note that $0 \leq p \leq n-1$)
  which is equivalent to

$$\left(\underbrace{\left(\begin{array}{c|c} v_{11(p)} & u^T_{12(p)} \\ \hline 0 & U_{(p)} \end{array}\right)}_{U_{(p+1)}} + \underbrace{\left(\begin{array}{c|c} \delta v_{11(p)} & \delta u^T_{12(p)} \\ \hline 0 & \Delta U_{(p)} \end{array}\right)}_{\Delta U_{(p+1)}}\right) \underbrace{\begin{pmatrix} \check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)} \end{pmatrix}}_{\check{x}_{2(p+1)}} = \left(\underbrace{\begin{pmatrix} \psi_{1(p)} \\ \hline y_{2(p)} \end{pmatrix}}_{y_{2(p+1)}} + \underbrace{\begin{pmatrix} \delta\psi_{1(p)} \\ \hline \Delta y_{2(p)} \end{pmatrix}}_{\delta y_{2(p+1)}}\right) \tag{84}$$

$$(U_{(p+1)} + \Delta U_{(p+1)})\check{x}_{2(p+1)} = (y_{2(p+1)} + \delta y_{2(p+1)}) \tag{85}$$

  where $|\Delta U_{(p+1)}| \leq \gamma_n|U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u}|y_{2(p+1)}|$.
  Hence, we sucessfully proved (80). □

- By principle of mathematical induction, it is proved w.l.o.g. that

$$(U + \Delta U)\check{x} = (y + \delta y) \text{ , where } |\Delta U| \leq \gamma_n|U| \text{ and } |\delta y| \leq \mathbf{u}|y| \tag{86}$$