The University of Texas
at Austin

CS383C Numerical Analysis

Homework 04

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Exercises

Part I

Exercises on Solving LLS Problems

Exercise 2.

We can prove the existence of LQ factorization for $A_{m\times n}(m<n)$ by induction on m.

- Base Case: $m=1$

$$A = (a_0^H) = l_{00}q_0^H = \left(\begin{array}{c|c} l_{00} & 0 \end{array}\right)\left(\begin{array}{c} q_0^H \\ \hline 0 \end{array}\right) \tag{1}$$

  Let $L_L = l_{00}$ and $Q_T = q_0^H$, we have

$$A = \left(\begin{array}{c|c} L_L & 0 \end{array}\right)\left(\begin{array}{c} Q_T \\ \hline 0 \end{array}\right) = L_LQ_T \tag{2}$$

  holds for $m=1$.

- Inductive Case: assume that existence of LQ factorization holds for $m-1$, then we show that it also holds for m.
  That is, we partition

$$A = \left(\begin{array}{c} A_0 \\ \hline a_1^H \end{array}\right) \tag{3}$$

  Assume that $A_0 = L_{00}Q_0$ holds, then we show that $\exists L_L, Q_T,\ s.t.\ A = L_LQ_T$ holds.
  Let

$$L_L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^H & \rho_{11} \end{array}\right),\ Q_T = \left(\begin{array}{c} Q_0 \\ \hline q_1^H \end{array}\right) \tag{4}$$

  where $l_{10}^H = a_1^HQ_0^H$ and $\rho_{11} = ||a_1^H - l_{10}^HQ_0||_2$. Then we have

$$L_LQ_T = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^H & \rho_{11} \end{array}\right)\left(\begin{array}{c} Q_0 \\ \hline q_1^H \end{array}\right) = \left(\begin{array}{c} L_{00}Q_0 \\ \hline l_{10}^HQ_0 + \rho_{11}q_1^H \end{array}\right) = \left(\begin{array}{c} L_{00}Q_0 \\ \hline a_1^H \end{array}\right) = \left(\begin{array}{c} A_0 \\ \hline a_1^H \end{array}\right) = A \tag{5}$$

- By principle of induction, $A = L_LQ_T$ holds for arbitrary m.

Exercise 3.

$$\begin{aligned}
\min_z ||Az-y||_2 &= \min_z ||LQz-y||_2 && (6)\\
&= \min_{\{z=Q^Hw\}} ||LQQ^Hw-y||_2 && (7)\\
&= \min_{\{z=Q^Hw\}} ||Lw-y||_2 && (8)\\
&= \min_{\{z=Q^Hw\}} ||\left(\begin{array}{c|c} L_L & 0 \end{array}\right)\left(\begin{array}{c} w_T \\ \hline w_B \end{array}\right) - y||_2 && (9)\\
&= \min_{\{z=Q^Hw\}} ||L_Lw_T-y||_2 && (10)
\end{aligned}$$

which derives solution for $w_T = L_L^{-1}y$. And then the general solution is

$$z = Q^Hw = \left(\begin{array}{c|c} Q_T^H & Q_B^H \end{array}\right)\left(\begin{array}{c} w_T \\ \hline w_B \end{array}\right) = Q_T^Hw_T + Q_B^Hw_B = Q_T^HL_L^{-1}y + Q_B^Hw_B \tag{11}$$

where w_B can be arbitrary vector in $\mathbb{C}^{n-r}$.

Exercise 4.

```
%% Homework 03: LQ Factorization
% Copyright 2014 The University of Texas at Austin
%
% For licensing information see
%                http://www.cs.utexas.edu/users/flame/license.html
%
% Programmed by: Jimmy Lin
%                jimmylin@utexas.edu
function [ A_out, L_out, Q_out ] = LG_CGS_unb( A, L, Q )
  [ AT, ...
    AB ] = FLA_Part_2x1( A, ...
                         0, 'FLA_TOP' );
  [ LTL, LTR, ...
    LBL, LBR ] = FLA_Part_2x2( L, ...
                               0, 0, 'FLA_TL' );
  [ QT, ...
    QB ] = FLA_Part_2x1( Q, ...
                         0, 'FLA_TOP' );
  while ( size( AT, 1 ) < size( A, 1 ) )
    [ A0, ...
      a1t, ...
      A2 ] = FLA_Repart_2x1_to_3x1( AT, ...
                                    AB, ...
                                    1, 'FLA_BOTTOM' );
    [ L00,  l01,      L02,  ...
      l10t, lambda11, l12t, ...
      L20,  l21,      L22 ] = FLA_Repart_2x2_to_3x3( LTL, LTR, ...
                                                     LBL, LBR, ...
                                                     1, 1, 'FLA_BR' );
    [ Q0, ...
      q1t, ...
      Q2 ] = FLA_Repart_2x1_to_3x1( QT, ...
                                    QB, ...
                                    1, 'FLA_BOTTOM' );
    %------------------------------------------------------------%
    l10t = a1t * Q0';
    temp = a1t - l10t * Q0;
    lambda11 = norm(temp, 2);
    q1t = temp / lambda11;
    %------------------------------------------------------------%
    [ AT, ...
      AB ] = FLA_Cont_with_3x1_to_2x1( A0, ...
                                       a1t, ...
                                       A2, ...
                                       'FLA_TOP' );
    [ LTL, LTR, ...
      LBL, LBR ] = FLA_Cont_with_3x3_to_2x2( L00,  l01,      L02,  ...
                                             l10t, lambda11, l12t, ...
                                             L20,  l21,      L22,  ...
                                             'FLA_TL' );
    [ QT, ...
      QB ] = FLA_Cont_with_3x1_to_2x1( Q0, ...
                                       q1t, ...
                                       Q2, ...
                                       'FLA_TOP' );
  end
  A_out = [ AT
            AB ];
  L_out = [ LTL, LTR
            LBL, LBR ];
  Q_out = [ QT
            QB ];
return
```

Exercise 5.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Homework 03: Householder LQ Transformation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

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%
% Programmed by: Jimmy Lin
%                jimmylin@utexas.edu

function [ A_out, T_out ] = HLQ_unb( A, T )

  [ ATL, ATR, ...
    ABL, ABR ] = FLA_Part_2x2( A, ...
                               0, 0, 'FLA_TL' );
  [ TT, ...
    TB ] = FLA_Part_2x1( T, ...
                         0, 'FLA_TOP' );

  while ( size( ATL, 1 ) < size( A, 1 ) )

    [ A00,  a01,     A02,  ...
      a10t, alpha11, a12t, ...
      A20,  a21,     A22 ] = FLA_Repart_2x2_to_3x3( ATL, ATR, ...
                                                    ABL, ABR, ...
                                                    1, 1, 'FLA_BR' );
    [ T0, ...
      t1t, ...
      T2 ] = FLA_Repart_2x1_to_3x1( TT, ...
                                    TB, ...
                                    1, 'FLA_BOTTOM' );

    %------------------------------------------------------------%

    [ alpha11, temp, t1t] = Housev( alpha11, a12t');
    a12t = temp';
    w21 = (a21 +  A22 * a12t') / t1t;
    a21 = a21 - w21;
    A22 = A22 - w21 * a12t;

    %------------------------------------------------------------%

    [ ATL, ATR, ...
      ABL, ABR ] = FLA_Cont_with_3x3_to_2x2( A00,  a01,     A02,  ...
                                             a10t, alpha11, a12t, ...
                                             A20,  a21,     A22,  ...
                                             'FLA_TL' );
    [ TT, ...
      TB ] = FLA_Cont_with_3x1_to_2x1( T0, ...
                                       t1t, ...
                                       T2, ...
                                       'FLA_TOP' );

  end

  A_out = [ ATL, ATR
            ABL, ABR ];
  T_out = [ TT
            TB ];

return
```

Part II
Exercises on Conditioning

Exercise 1.

Show that, for *a consistent matrix norm*, $\kappa(A) \geq 1$.

Proof.

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| \geq ||AA^{-1}|| = ||I|| = 1 \tag{12}$$

Note that the above $||\cdot||$ was for arbitrary induced matrix norm. □

Lemma 1. *For arbitrary matrix A and B, $||AB|| \leq ||A|| \cdot ||B||$.*

Proof.

$$||AB|| = \sup_{x\neq 0} \frac{||ABx||}{||x||} = \sup_{x\neq 0} \frac{||A(Bx)||}{||x||} \tag{13}$$
$$\leq \sup_{x\neq 0} \frac{||A|| \cdot ||Bx||}{||x||} \tag{14}$$
$$\leq \sup_{x\neq 0} \frac{||A|| \cdot ||B|| \cdot ||x||}{||x||} \tag{15}$$
$$= ||A|| \cdot ||B|| \tag{16}$$

Hence, it is concluded that $||AB|| \leq ||A|| \cdot ||B||$. □

Lemma 2. *For abitrary norm $||\cdot||$ and identity matrix I, $||I|| = 1$.*

Proof.

$$||I|| = \sup_{x\neq 0} \frac{||I \cdot x||}{||x||} = \sup_{x\neq 0} \frac{||x||}{||x||} = 1 \tag{17}$$

□

Exercise 2.

If A has lineraly independent columns, show that $||(A^H A)^{-1} A^H||_2 = \frac{1}{\sigma_{n-1}}$, where σ_{n-1} equals *the smallest sigular value* of A.

Proof. Let U, Σ and V be singular value decomposition of A, such that $A = U\Sigma V^H$.

$$
\begin{aligned}
||(A^H A)^{-1} A^H||_2 &= ||\big((U\Sigma V^H)^H U\Sigma V^H\big)^{-1} (U\Sigma V^H)^H||_2 && (18)\\
&= ||\big(V\Sigma^H U^H U\Sigma V^H\big)^{-1} V\Sigma^H U^H||_2 && (19)\\
&= ||\big(V\Sigma^H \Sigma V^H\big)^{-1} V\Sigma^H U^H||_2 && (20)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}V^{-1}V\Sigma^H U^H||_2 && (21)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}\Sigma^H U^H||_2 && (22)\\
&= ||V^{-H}\Sigma^{-1} U^H||_2 && (23)\\
&= ||V\Sigma^{-1} U^H||_2 && (24)\\
&= ||\Sigma^{-1}||_2 && (25)\\
&= \frac{1}{\sigma_{n-1}} && (26)
\end{aligned}
$$

□

Lemma 3. *(Unitary Invariance) For arbitrary unitary matrix U,*

$$||UA||_2 = ||AU||_2 = ||A||_2 \tag{27}$$

Lemma 4. *For arbitrary diagonal matrix Σ,*

$$||\Sigma^{-1}||_2 = \frac{1}{\sigma_{n-1}} \tag{28}$$

where, σ_{n-1} is the least entry of Σ.

Note that above two lemmas have been proven in exercises of previou notes.

Exercise 3.

Let A have linearly independent columns. Show that $\kappa_2(A^HA) = \kappa_2(A)^2$.

Proof. We achieve the proof by employing SVD over A. Let unitary matrix U, diagonal matrix Σ and unitary matrix V be singular value decomposition of A, such that $A = U\Sigma V^H$. We start from the definition of condition number $\kappa_2(\cdot)$.

$$\kappa_2(A^HA) = ||A^HA||_2 \cdot ||(A^HA)^{-1}||_2 \tag{29}$$

Then we discuss the term $||A^HA||_2$ and $||(A^HA)^{-1}||_2$ respectively.

$$\begin{aligned}
||A^HA||_2 &= ||(U\Sigma V^H)^H U\Sigma V^H||_2 && (30)\\
&= ||V\Sigma^H U^H U\Sigma V^H||_2 && (31)\\
&= ||V\Sigma^H \Sigma V^H||_2 && (32)\\
&= ||\Sigma^H \Sigma||_2 && (33)\\
&= \sigma_0^2 && (34)\\
&= ||A||_2^2 && (35)
\end{aligned}$$

Note that σ_0 is the largest singular value of matrix A and also the largest entry of Σ.

$$\begin{aligned}
||(A^HA)^{-1}||_2 &= ||\big((U\Sigma V^H)^H U\Sigma V^H\big)^{-1}||_2 && (36)\\
&= ||\big(V\Sigma^H U^H U\Sigma V^H\big)^{-1}||_2 && (37)\\
&= ||\big(V\Sigma^H \Sigma V^H\big)^{-1}||_2 && (38)\\
&= ||V^{-H}\Sigma^{-1}\Sigma^{-H}V^{-1}||_2 && (39)\\
&= ||\Sigma^{-1}\Sigma^{-H}||_2 && (40)\\
&= ||\Sigma^{-1}\Sigma^{-1}||_2 && (41)\\
&= \frac{1}{\sigma_{n-1}^2} && (42)\\
&= ||A^{-1}||_2^2 && (43)
\end{aligned}$$

Now we have

$$\begin{aligned}
||A^HA||_2 &= ||A||_2^2 && (44)\\
||(A^HA)^{-1}||_2 &= ||A^{-1}||_2^2 && (45)
\end{aligned}$$

Then

$$\begin{aligned}
\kappa_2(A^HA) &= ||A^HA||_2 \cdot ||(A^HA)^{-1}||_2 && (46)\\
&= ||A||_2^2 \cdot ||A^{-1}||_2^2 && (47)\\
&= \big(||A||_2 \cdot ||A^{-1}||_2\big)^2 && (48)\\
&= \kappa_2(A)^2 && (49)
\end{aligned}$$

Hence, it can be concluded that

$$\kappa_2(A^HA) = \kappa_2(A)^2 \tag{50}$$

□

Exercise 4.

4.1

(Only-If) If $Ax = y$, multiply both side with A^H and then we have $A^H Ax = A^H y$.
(If) If $A^H Ax = A^H y$, then multiply both side with $(A^H)^{-1}$ (inverse exists because A is full-rank) and we have $(A^H)^{-1}A^H Ax = (A^H)^{-1}A^H y$. Since $(A^H)^{-1}A^H = I$, it comes out $Ax = y$.

4.2

For normal equation method, the solution of LLS can be derived by

$$x = (A^H A)^{-1}A^H y \tag{51}$$

Let $B = (A^H A)^{-1}A^H$, then the condition number of normal equation program is

$$\begin{aligned}
\kappa(B^{-1}) &= ||((A^H A)^{-1}A^H)^{-1}|| \cdot ||(A^H A)^{-1}A^H|| &(52)\\
&= \sigma_0 \cdot \frac{1}{\sigma_{n-1}} &(53)\\
&= \frac{\sigma_0}{\sigma_{n-1}} &(54)\\
&= \kappa(A) &(55)
\end{aligned}$$

By this, we can conclude that the condition number of normal equation program is not necessarily square of $\kappa(A)$.

Exercise 5.

Let $U \in \mathbb{C}^{n\times n}$ be unitary. Show that $\kappa_2(U) = 1$.

Proof.

$$\begin{aligned}
\kappa_2(U) &= ||U||_2||U^{-1}||_2 &(56)\\
&= \sup_{x\neq 0}\frac{||Ux||_2}{||x||_2} \cdot \sup_{y\neq 0}\frac{||U^{-1}y||_2}{||y||_2} &(57)\\
&= \sup_{x\neq 0}\frac{||x||_2}{||x||_2} \cdot \sup_{y\neq 0}\frac{||y||_2}{||y||_2} &(58)\\
&= 1 \cdot 1 &(59)\\
&= 1 &(60)
\end{aligned}$$

□

Lemma 5. *For arbitrary unitary matrix U, its inverse U^{-1} is still unitary.*

Proof.

$$UU^H = I \tag{61}$$

We multiply both side on the left with U^{-1} and U^{-H}, it comes

$$U^{-H}U^{-1}UU^H = U^{-H}U^{-1} \tag{62}$$

That is

$$(U^{-1})^H U^{-1} = I \tag{63}$$

Since U is unitary, then U is square matrix and so as U^{-1}. Then it is concluded that U^{-1} is unitary. □