THE UNIVERSITY OF TEXAS
AT AUSTIN

EE381V LARGE SCALE OPTIMIZATION

Problem Set 7

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Part I
Matlab and Computational Assignment

1 MaxCut

1.1 Formulation

The original maxcut problem is

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\sum_{i<j}(1-u_iu_j)w_{ij} \\ \text{subject to} \quad & u_i \in \{-1,1\}, \forall i \end{aligned} \tag{1}$$

The relaxed SDP problem can be written as

$$\begin{aligned} \text{minimize} \quad & Trace(X^TW) \\ \text{subject to} \quad & X_{ii}=1, \forall i \\ & X \succeq 0 \end{aligned} \tag{2}$$

where $X_{ij} = u_iu_j$.

1.2 Graphs

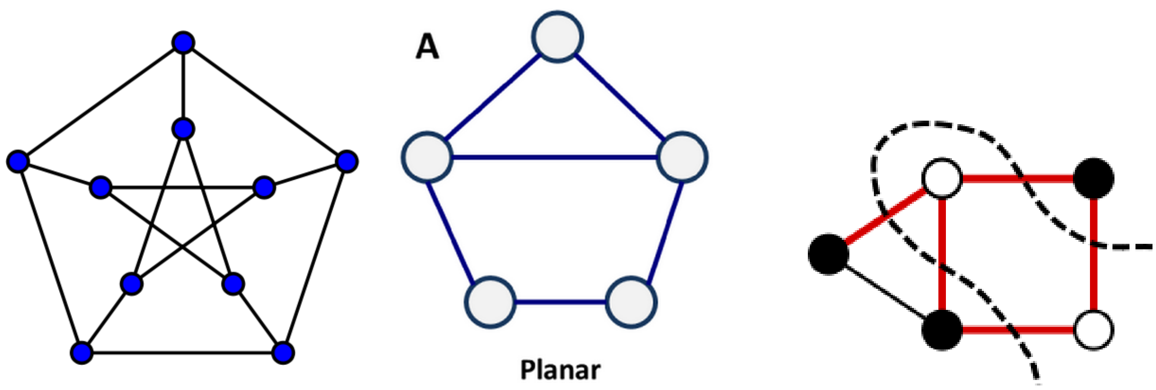

Figure 1: Three graphs for experimentation. (1) Peterson Graph (2) Simple Planar Graph I (3) Simple Planar Graph II

Though solved X, we can recover the cut that separate the whole set of vertex.

1.3 Implementation

```
function maxcut(W)

n = size(W,1);
cvx_begin sdp
    variable X(n,n) symmetric
    minimize trace( X' * W )
    subject to
        diag(X) = ones(n, 1)
        X >= 0
cvx_end

end
```

Part II
Written Assignment

1 Network Congestion Control

1.1 Problem Formulation

The overall system problem – to maximize utility minus cost – can be formulated as a convex optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{s\in S} U_s(x_s) - \sum_{j\in J} C_j(f_j) \\ \text{subject to} & Hy = x, Ay \le f \\ \text{over} & x, y \ge 0 \end{array} \tag{3}$$

1.2 Problem Decoupling

Lagrangian:

$$L(x, y; \lambda, \mu) = \sum_{s\in S} U_s(x_s) - \sum_{j\in J} C_j(f_j) - \lambda^T(x - Hy) + \mu^T(f - Ay - z) \tag{4}$$

$$= \sum_{s\in S} \big(U_s(x_s) - \lambda_s x_s\big) - \sum_{r\in R} y_r\big(\lambda_{s(r)} - \sum_{j\in r} \mu_j\big) + \sum_{j\in J} \mu_j(f_j - z_j) - \sum_{j\in J} C_j(f_j) \tag{5}$$

where λ and μ are lagrange multipliers.
According to optimality conditions

$$\frac{\partial L}{\partial x_s} = U_s'(x_s) - \lambda_s \tag{6}$$

$$\frac{\partial L}{\partial y_r} = \lambda_{s(r)} - \sum_{j\in r} \mu_j \tag{7}$$

$$\frac{\partial L}{\partial z_j} = -\mu_j \tag{8}$$

$$\lambda \ge U_s'(x_s), Hy = x, (\lambda - U'(x))^x = 0 \tag{9}$$

$$\mu \ge 0, Ax \le C, \mu^T(C - Ax) = 0 \tag{10}$$

$$\lambda^T H \le \mu^T A, y \ge 0, (\mu^T A - \lambda^T H)y = 0 \tag{11}$$

$USER_s(U_s; \lambda_s)$

$$\begin{array}{ll} \text{maximize} & \sum_{s\in S} U_s(x_s) - \lambda_s x_s \\ \text{subject to} & x_s \ge 0 \end{array} \tag{12}$$

$NETWORK(H, F; \lambda)$

$$\begin{array}{ll} \text{maximize} & \sum_{s\in S} \lambda_s x_s - \sum_{j\in J} C_j(f_j) \\ \text{subject to} & Hy = x, Ay \le f \\ \text{over} & x, y \ge 0 \end{array} \tag{13}$$

Theorem 1. *There exists a price vector $\lambda = (\lambda_s, s \in S)$ such that the vector $x = (x_s, s \in S)$, formed from the unique solution x_s to $USER_s(U_s; \lambda_s)$ for each $s \in S$, solves $NETWORK(H, A, C; \lambda)$. The vector x then also solves $SYSTEM(U, H, A, f)$.*

Proof. First note that $USER_s(U_s;\lambda_s)$ has unique solution for each s. Then we observe that the lagrangian form for $NETWORK(H,F;\lambda)$ is

$$L(x,y;\lambda,\mu) = \sum_{s\in S} U_s(x_s) - \sum_{j\in J} C_j(f_j) - p^T(x - Hy) + q^T(f - Ay - z) \tag{14}$$

$$= \sum_{s\in S}\big(U_s(x_s) - p_s x_s\big) - \sum_{r\in R} y_r\big(p_{s(r)} - \sum_{j\in r} q_j\big) + \sum_{j\in J} q_j(f_j - z_j) - \sum_{j\in J} C_j(f_j) \tag{15}$$

Hence, any quadruple (λ,μ,x,y), which satisfies optimality of 9-11 (solution of $SYSTEM$) identifies $p=\lambda$ and $q=\mu$, which establish that (x,y) solves $NETWORK(H,F;\lambda)$.
Conversely, for any solution x to $NETWORK(H,F;\lambda)$, then exists a p and q, where $x_s \geq 0$ then $p_s = \lambda_s$ and if $x_s = 0$, then $p_s \geq \lambda_s$. Thus if x_s solves $USER_s(U_s;\lambda_s)$, then it also solves $USER_s(U_s;p_s)$. Based on p and q, we can then construct a quadruple that satisfies optimality of 9-11. This quadruple gives x that solves $SYSTEM(U,H,A,f)$. □

2 Problem 7.12

It is obvious that the problem is separable by each pair of individual row of P and column of S.

$$\begin{aligned} &\text{maximize}_s && \min_i p_i^T s_i \\ &\text{subject to} && \mathbf{1}^T s_i = 1, s_i \geq 0, \forall i \end{aligned} \tag{16}$$

where p_i^T and s_i denote the ith row of P and column of S. Since it is known that

$$s_i = e_k \tag{17}$$

where k is the index of maxima among all probabilty values. Hence, we can do a maximum likelihood estimation.

$$S_{ji} = \begin{cases} 1, & \text{if } j = arg\max_k P_{ik} \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

which means that the overall problem can be separate as a set of problems. In each individual problem, we choose parameters s_i such that P_{ik} is optimized.

3 Problem 7.13

The dual problem can be written as follows

$$\begin{aligned} &\text{maximize} && \sum_{j:\alpha_j\in S} p_j = \mathbf{prob}(X\in S) \\ &\text{subject to} && \sum_{j=1}^{m} p_j = 1 \\ &&& \sum_{j=1}^{m} f_i(\alpha_i)p_j = b_i, \forall i \\ &&& p_j \geq 0, \forall j \end{aligned} \tag{19}$$

Obviously, this dual problem has its optima as right upper bound on $\mathbf{prob}(X\in S)$ and there exists a distribution that achieves the bound, in that the dual problem satisfies all conditions for strong duality.

4 Problem 8.8

4.1 Find a point in the intersection

LP Formulation.

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & Ax \leq b \\ & Fx \leq g \end{array} \tag{20}$$

Strong Alternative.

$$\begin{array}{ll} \text{find} & \lambda, \mu \\ \text{subject to} & \lambda, \mu \geq 0 \\ & b^T\lambda + g^T\mu < 0, \\ & A^T\lambda + F^T\mu = 0 \end{array} \tag{21}$$

Geometric intepretation.
If the intersection is a empty set, then we can find a hyperplane separating P_1 and P_2 from the strong alternative problem. Otherwise, what we find from the alternative problem is a hyperplane that minimizes the "maximum margin" to two polyhedra.

4.2 Determine whether $P_1 \subset P_2$

LP Formulation.
Strong Alternative.
Geometric intepretation.

5 Problem 8.8 Repeat

5.1 Find a point in the intersection

LP Formulation.

$$\begin{array}{ll} \text{find} & x, \lambda, \mu \\ \text{subject to} & x = \sum_{i=1}^{K} \lambda_i v_i, \sum_{i=1}^{K} \lambda_i = 1, \lambda_i \geq 0 \\ & x = \sum_{j=1}^{L} \mu_j w_j, \sum_{j=1}^{L} \mu_j = 1, \mu_j \geq 0 \end{array} \tag{22}$$

Strong Alternative.

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & a^T v_i \leq b, \forall i \\ & a^T w_j > b, \forall j \end{array} \tag{23}$$

Geometric intepretation.
The strong dual alternative problem finds a hyperplan that separate two sets of points, $\{v_1, ..., v_K\}$ and $\{w_1, ..., w_L\}$. That is, the hyperplane separates P_1 and P_2.

5.2 Determine whether $P_1 \subset P_2$

6 Problem 8.9

The Closest Euclidean distance matrix problem can be formulated as

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} (\sqrt{D_{ij}} - \hat{d}_{ij})^2 \\ \text{subject to} \quad & D_{ii} = 0, \forall i = 1, ..., n \\ & D_{ij} \geq 0, \forall i, j = 1, ..., n \\ & 0 \succeq (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T) D (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T) \end{aligned} \tag{24}$$

Since the D is symmetric, then the above optimization is a SDP problem.
The vector $x_1, ..., x_n$ can be recovered from eigenvalue decomposition of the corresponding Gram matrix (dependent on solved D^*).

$$G = -\frac{1}{2}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T) D^* (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T) \tag{25}$$
$$= X^T \Lambda X \tag{26}$$

Once an eigenvalue is zero, the corresponding dimension can be removed and thus k is reduced by one and smaller than n.

7 Problem 8.25

7.1 t^* and separability

Lemma 1. *Show that if $t^* > 0$, then two sets of points are separable.*

Proof. If $t^* > 0$, then

$$a^{*T} x_i \geq b^* + t^* > b^*, i = 1, ..., N \tag{27}$$
$$a^{*T} y_i \geq b^* - t^* < b^*, i = 1, ..., M \tag{28}$$
$$\tag{29}$$

where the (a^*, b^*) pair is the optimal solution of (8.23) that corresponds to t^*. Hence, it is obvious that $f(x) = a^{*T} x - b^*$ can separate two given set of points. □

Lemma 2. *Show that if two sets of points are separable, then $t^* > 0$.*

Proof. Let the linear function $f(x) = a^T x - b$ be the linear separator, such that

$$a^T x_i > b, i = 1, ..., N \tag{30}$$
$$a^T y_i < b, i = 1, ..., M \tag{31}$$

Then we have arbitrary margin t satisfying

$$t = \min\{\inf_i (a^T x_i - b), \inf_i (b - a^T y_i)\} > 0 \tag{32}$$

Hence, the optimal margin t^* is guaranteed to be positive. That is, $t^* > 0$. □

Lemma 3. *The constraint $||a^*||_2 \leq 1$ is tight, that is, $||a^*||_2 = 1$.*

Proof. We prove the tighness by contradiction. Assume that $||a^*||_2 < 1$. Let (a^*, b^*, t^*) be a optimal solution of (8.23). Then

$$a^{*T} x_i - b^* \geq t^*, i = 1, ..., N \tag{33}$$
$$a^{*T} y_i - b \leq t^*, i = 1, ..., M \tag{34}$$

Then multiplies both side with $\frac{1}{||a^*||_2}$,

$$\frac{a^{*T}}{||a^*||_2} x_i - \frac{b^*}{||a^*||_2} \geq \frac{t^*}{||a^*||_2}, i = 1, ..., N \tag{35}$$
$$\frac{a^{*T}}{||a^*||_2} y_i - \frac{b^*}{||a^*||_2} \leq \frac{t^*}{||a^*||_2}, i = 1, ..., M \tag{36}$$

which indicates that $(\frac{a^{*T}}{||a^*||_2}, \frac{b^*}{||a^*||_2}, \frac{t^*}{||a^*||_2})$ are also a optimal solution. However, it yields contradiction to the pre-defined optimality (a^*, b^*, t^*) since $0 < ||a^*||_2 < 1$. Thus, we conclude that $||a^*||_2 = 1$. □

7.2 QP equivalence

Due to the tightness of $||a||_2 \leq 1$, then we can rewrite the problem (8.23) as follows

$$\begin{aligned} \text{maximize} \quad & \frac{||a||_2}{||\tilde{a}||_2} \\ \text{subject to} \quad & \tilde{a}^T x_i - \tilde{b} \geq 1, i = 1, ..., N \\ & \tilde{a}^T y_i - \tilde{b} \leq -1, i = 1, ..., M \\ & ||a||_2 = 1 \end{aligned} \tag{37}$$

Since $||a||_2$ is fixed, (the constraint $||a||_2 = 1$), then the above problem can be further re-written as

$$\begin{aligned} \text{minimize} \quad & ||\tilde{a}||_2 \\ \text{subject to} \quad & \tilde{a}^T x_i - \tilde{b} \geq 1, i = 1, ..., N \\ & \tilde{a}^T y_i - \tilde{b} \leq -1, i = 1, ..., M \end{aligned} \tag{38}$$

which is a QP problem.

8 Problem 8.24

The original problem is as follows

$$\begin{array}{ll} \text{minimize} & \rho \\ \text{subject to} & (a+u)^T x_i \geq b, \forall u \in \{u : ||u||_2 \leq \rho\}, \forall i = 1, ..., N \\ & (a+u)^T y_j \leq b, \forall u \in \{u : ||u||_2 \leq \rho\}, \forall i = 1, ..., M \\ & ||a||_2 \leq 1 \end{array} \tag{39}$$

The constraints can be written as optimization problem for u

$$a^T x_i + \left(\begin{array}{c} \min_u u^T x_i \\ s.t. ||u||_2 \leq \rho \end{array} \right) \geq b, \forall i = 1, ..., N \tag{40}$$

$$a^T y_i + \left(\begin{array}{c} \min_u u^T y_i \\ s.t. ||u||_2 \leq \rho \end{array} \right) \leq b, \forall i = 1, ..., M \tag{41}$$

Solving the optimization problems gives

$$u_1^* = -\frac{\rho x_i}{||x_i||_2} \tag{42}$$

$$u_2^* = \frac{\rho y_i}{||y_i||_2} \tag{43}$$

Hence, the resulted linear program that find a and b to maximize the weight error margin is as follows

$$\begin{array}{ll} \text{minimize} & \rho \\ \text{subject to} & a^T x_i + \rho ||x_i||_2 \geq b, \forall i = 1, ..., N \\ & a^T y_i + \rho ||y_i||_2 \leq b, \forall i = 1, ..., M \\ & ||a||_2 \leq 1 \end{array} \tag{44}$$

9 Problem 8.25

Let ϕ be a separating ellipsoid, described by $A \succ 0$, b and c, such that

$$x_i^T A x_i + b^T x_i + c < 0, \forall i = 1, ..., N \tag{45}$$

$$y_i^T A y_i + b^T y_i + c > 0, \forall i = 1, ..., M \tag{46}$$

The condition number minimization problem can be solved by minimizing the maximum maximum eigenvalue with regard to a lower-bound on the minimum eigenvalues. Then the optimization problem can be written as a SDP problem

$$\begin{array}{ll} \text{minimize} & \beta \\ \text{subject to} & x_i^T A x_i + b^T x_i + c < 0, \forall i = 1, ..., N \\ & y_i^T A y_i + b^T y_i + c > 0, \forall i = 1, ..., M \\ & I \succeq A \succeq \beta I \end{array} \tag{47}$$