



THE UNIVERSITY OF TEXAS
AT AUSTIN

EE381V LARGE SCALE OPTIMIZATION

Problem Set 1

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Table of Contents

I	Matlab and Computational Assignment	2
1	Gradient Descent on three matrices	2
1.1	$X1, b1$	2
1.2	$X2, b2$	3
1.3	$X3, b3$	4
2	$\gamma = 1$ for the second matrix	5
II	Written Problems	6
1	Othorognal Subspace	6
(a)	Show that if U is a subspace, then so is U^\perp	6
(b)	Show that $(U^\perp)^\perp = U$	6
(c)	Show that if $U, W \subseteq V$ are subspaces of V , then $U \subseteq W \Leftrightarrow U^\perp \supseteq W^\perp$	6
(d)	Show that X^\perp makes sense, X^\perp is a subspace and $(X^\perp)^\perp \supseteq X$	7
(e)	Show that any $v \in V$ can be written uniquely as $v = u + u^\perp$	7
2	Boyd and Vandenberghe, Ex. 2.10	8
(a)	Show that if $A \in \mathbb{S}_+^n$ then the set C is convex	8
(b)	Show that C_1 is convex if there exists $\lambda \in \mathbb{R}$ such that $(A + \lambda gg^T) \in \mathbb{S}_+^n$	9
3	Boyd and Vandenberghe, Ex. 2.21	10
7	$\{x : \ x - v_1\ \leq \ x - v_2\ \} = \{x : c^T x \leq d\}$	11
8	Exists C such that $CA = B$	11
A	Codes Printout	12
(a)	Gradient Descent Routine	12
(b)	Running Script	13

List of Figures

1	Illustration for gradient descent on $X1$, staring with $b1$ by $\gamma = 0.5$ and 3.0	2
2	Illustration for gradient descent on $X2$, starting with $b2$ by $\gamma = 1.5$ and 3.0	3
3	Illustration for gradient descent on $X3$ staring with $b3$ by $\gamma = 0.005$ and 0.05	4
4	Plotting figure for gradient descent with $\gamma = 1$ on the second matrix	5

Part I

Matlab and Computational Assignment

1 Gradient Descent on three matrices

Command to get executed:

```
>> gd_run_script()
```

1.1 $X1, b1$

- Range of γ that leads to convergence: $(0, 2)$
- Range of γ that leads to divergence: $(2, +\infty)$
- Explanation: if $\gamma = 2$, the program indicates that

$$\forall k, f(x^{k+1}) = f(x^k)$$

Since the above equation is constantly true (independent of the minima), we can conclude that gradient descent with $\gamma = 2$ goes to the opposite side of that quadratic curve. Intuitively, the program will diverge if we set larger ($\gamma > 2$) and converge if we set smaller ($\gamma < 2$).

- Two illustrative examples: $\gamma = 0.5$ and $\gamma = 3.0$

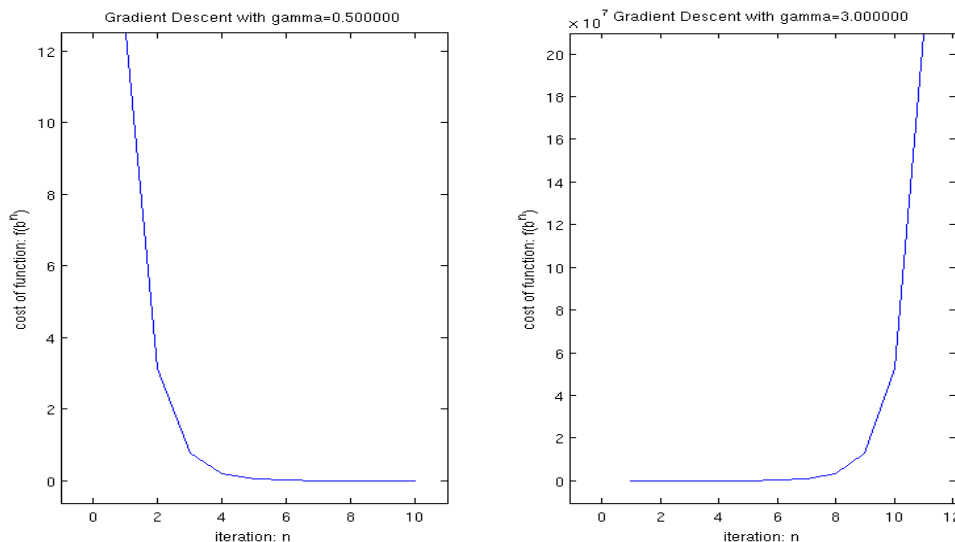


Figure 1: Illustration for gradient descent on $X1$, starting with $b1$ by $\gamma = 0.5$ and 3.0

1.2 $X2, b2$

- Range of γ that leads to convergence: $(0, 2)$
- Range of γ that leads to divergence: $(2, +\infty)$
- Explanation: if $\gamma = 2$, the program indicates that

$$\forall k, f(x^{k+1}) = f(x^k)$$

Since the above equation is constantly true (independent of the minima), we can conclude that gradient descent with $\gamma = 2$ goes to the opposite side of that quadratic curve. Intuitively, the program will diverge if we set larger ($\gamma > 2$) and converge if we set smaller ($\gamma < 2$).

- Two illustrative examples: $\gamma = 1.5$ and $\gamma = 3.0$

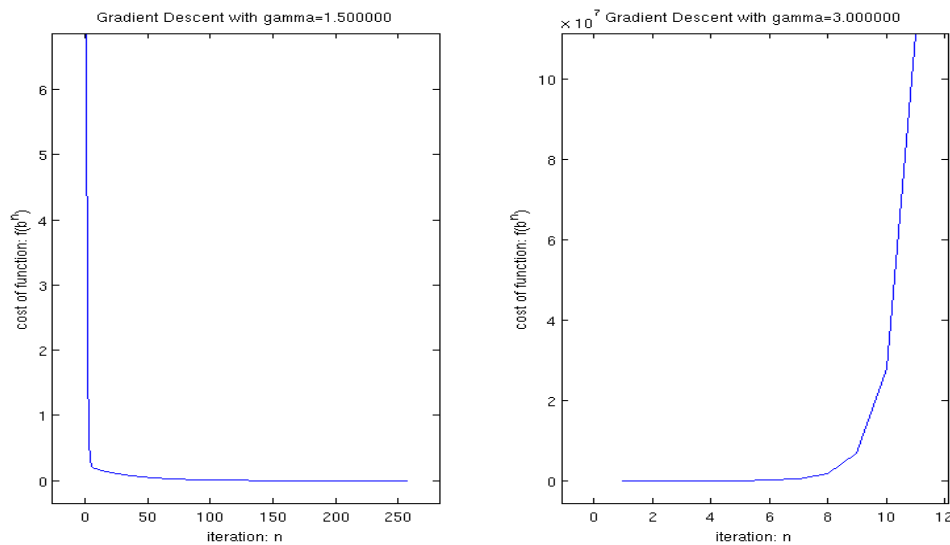


Figure 2: Illustration for gradient descent on $X2$, starting with $b2$ by $\gamma = 1.5$ and 3.0

1.3 $X3, b3$

- Range of γ that leads to convergence: $(0, 0.02)$
- Range of γ that leads to divergence: $(0.02, +\infty)$
- Explanation: if $\gamma = 0.02$, the program indicates that

$$\forall k, f(x^{k+1}) = f(x^k)$$

Since the above equation is constantly true (independent of the minima), we can conclude that gradient descent with $\gamma = 0.02$ goes to the opposite side of that quadratic curve. Intuitively, the program will diverge if we set larger ($\gamma > 0.02$) and converge if we set smaller ($\gamma < 0.02$).

- Two illustrative examples: $\gamma = 0.005$ and $\gamma = 0.05$

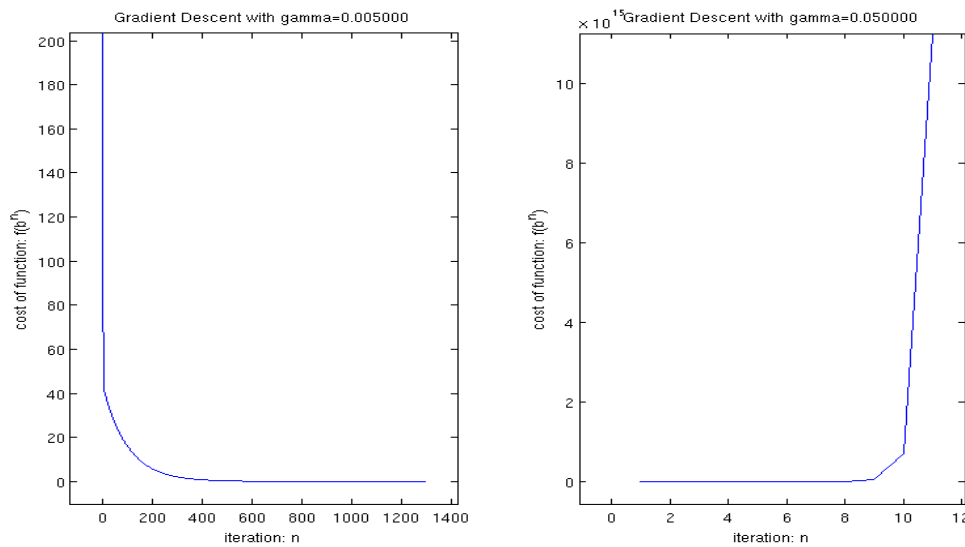


Figure 3: Illustration for gradient descent on $X3$ starting with $b3$ by $\gamma = 0.005$ and 0.05

2 $\gamma = 1$ for the second matrix

Command to get executed:

```
>> gamma = 1;
>> [b2_opt, iters, all_costs] = gd (X2, b2, gamma);
```

Plotting: figure for $\gamma = 1$

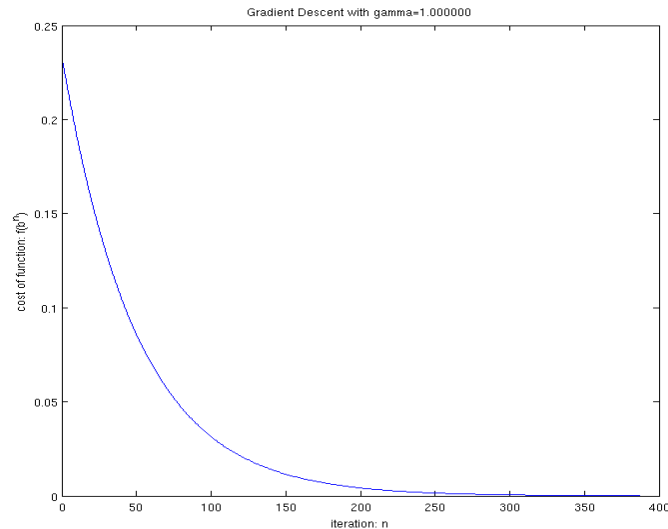


Figure 4: Plotting figure for gradient descent with $\gamma = 1$ on the second matrix

Explanation: Through the smooth plotted curve, we guess that the gradient descent method got linear convergence when $\gamma = 1$ on X_2 . Hence, we trace convergence rate $\text{conv_rate} = f(x^k)/f(x^{k-1})$ as follows:

```
Iter: 2, Cost: 2.254428e-01, Conv_Rate: 0.980100
Iter: 3, Cost: 2.209565e-01, Conv_Rate: 0.980100
Iter: 4, Cost: 2.165594e-01, Conv_Rate: 0.980100
Iter: 5, Cost: 2.122499e-01, Conv_Rate: 0.980100
Iter: 6, Cost: 2.080261e-01, Conv_Rate: 0.980100
Iter: 7, Cost: 2.038864e-01, Conv_Rate: 0.980100
...
...
Iter: 381, Cost: 1.107934e-04, Conv_Rate: 0.980100
Iter: 382, Cost: 1.085886e-04, Conv_Rate: 0.980100
Iter: 383, Cost: 1.064277e-04, Conv_Rate: 0.980100
Iter: 384, Cost: 1.043098e-04, Conv_Rate: 0.980100
Iter: 385, Cost: 1.022340e-04, Conv_Rate: 0.980100
Iter: 386, Cost: 1.001996e-04, Conv_Rate: 0.980100
Iter: 387, Cost: 9.820558e-05, Conv_Rate: 0.980100
Converged to zeros!
```

In terms of above dumps and the fact that $f(x^*) = 0$, we can conclude that when $\gamma = 1$

$$f(x^{k+1}) - f(x^*) = 0.9801 \cdot (f(x^k) - f(x^*))$$

which supports our previous guess that

Gradient Descent with $\gamma = 1$ on second matrix leads to **linear convergence**.

Part II

Written Problems

1 Othorognal Subspace

(a) Show that if U is a subspace, then so is U^\perp

Proof. Since U is a subspace, then we have U satisfying all three properties shown below:

- $\mathbf{0} \in U$
- $\forall \mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{u}_1 + \mathbf{u}_2 \in U$
- $\forall \mathbf{u} \in U, \alpha \in \mathbb{R}, \alpha \mathbf{u} \in U$

Now we show that U^\perp is also a subspace by indicating U^\perp satisfies all three properties as above U do.

- Since $\forall \mathbf{u} \in U, \langle \mathbf{0}, \mathbf{u} \rangle = 0$ and $\mathbf{0} \in V$ ($\mathbf{0} \in U \subseteq V$), then it turned out that $\mathbf{0} \in U^\perp$.
- Let \mathbf{u} be arbitrary vector s.t. $\mathbf{u} \in U$, and $\mathbf{x}_1, \mathbf{x}_2$ to be distinct vector s.t. $\mathbf{x}_1 \in U^\perp$ and $\mathbf{x}_2 \in U^\perp$. By definition of U^\perp , we have $\langle \mathbf{x}_1, \mathbf{u} \rangle = 0$ and $\langle \mathbf{x}_2, \mathbf{u} \rangle = 0$. Then it is obvious that $\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{u} \rangle = 0$. That is $\mathbf{x}_1 + \mathbf{x}_2 \in U^\perp$. Therefore, $\forall \mathbf{x}_1, \mathbf{x}_2 \in U^\perp, \mathbf{x}_1 + \mathbf{x}_2 \in U^\perp$ was proved.
- Let \mathbf{x} be arbitrary vector s.t. $\mathbf{x} \in U^\perp$, \mathbf{u} be arbitrary vector s.t. $\mathbf{u} \in U$ and arbitrary $\alpha \in \mathbb{R}$. By definition of U^\perp , we have $\langle \mathbf{x}_1, \mathbf{u} \rangle = 0$. Since inner product is linear operator, it is obvious that $\langle \alpha \mathbf{x}_1, \mathbf{u} \rangle = 0$. That is $\alpha \mathbf{x}_1 \in U^\perp$. Therefore, $\forall \mathbf{x} \in U^\perp, \alpha \in \mathbb{R}, \alpha \mathbf{x} \in U^\perp$ was proved.

Since U^\perp contains $\mathbf{0}$, and is closed under addition and scalar multiplication, it turned out that U^\perp is a subspace. Therefore, the statement that if U is a subspace, then so is U^\perp was proved. \square

(b) Show that $(U^\perp)^\perp = U$

Proof. By contradiction. Assume that $(U^\perp)^\perp \neq U$ and then show the contradiction. By definition of U^\perp , we have $U^\perp = \{\forall \mathbf{v} \in V, \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U\}$ and $(U^\perp)^\perp = \{\forall \mathbf{x} \in V, \langle \mathbf{x}, \mathbf{v} \rangle = 0, \forall \mathbf{v} \in U^\perp\}$. Since $(U^\perp)^\perp \neq U$, then we can say that $\exists \mathbf{x} \notin U, \langle \mathbf{x}, \mathbf{v} \rangle = 0, \forall \mathbf{v} \in U^\perp$. That is to say, $\exists \mathbf{x} \in V$ but $\notin U, \langle \mathbf{x}, \mathbf{v} \rangle = 0, \forall \mathbf{v} \in U^\perp$ s.t. $\langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U$. However, such \mathbf{x} does not exist. Hence, we reject initial assumption and conclude that $(U^\perp)^\perp = U$. \square

(c) Show that if $U, W \subseteq V$ are subspaces of V , then $U \subseteq W \Leftrightarrow U^\perp \supseteq W^\perp$

Proof of $U \subseteq W \Rightarrow U^\perp \supseteq W^\perp$. $U^\perp = \{\forall \mathbf{v} \in V, \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U\}$ and $W^\perp = \{\forall \mathbf{v} \in V, \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W\} = \{\forall \mathbf{v} \in V, \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W \text{ and } \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U\}$ (This is valid because $\forall U \subseteq W$ and then $\mathbf{u} \in W$). Now since membership of W^\perp requires one more condition, then it is obvious that $\mathbf{v} \in W^\perp \Rightarrow \mathbf{v} \in U^\perp$, and $\mathbf{v} \in U^\perp \not\Rightarrow \mathbf{v} \in W^\perp$ hold for arbitrary \mathbf{v} . Therefore, we can conclude that $U^\perp \supseteq W^\perp$. \square

Proof of $U^\perp \supseteq W^\perp \Rightarrow U \subseteq W$. By definition, we have $U^\perp = \{\mathbf{v} \in V | \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U\}$ and $W^\perp = \{\mathbf{v} \in V | \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W\}$. Since $U^\perp \supseteq W^\perp$, then we have $U^\perp \cap W^\perp = W^\perp$. Then $U^\perp \cap W^\perp = \{\mathbf{v} \in V | \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U \text{ and } \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W\} = \{\mathbf{v} \in V | \langle \mathbf{v}, \mathbf{x} \rangle = 0, \forall \mathbf{x} \in W \cup U\} = W^\perp$. Then we have $W \cup U = W$, which naturally derives $U \subseteq W$. \square

- (d) Show that X^\perp makes sense, X^\perp is a subspace and $(X^\perp)^\perp \supseteq X$
- (e) Show that any $v \in V$ can be written uniquely as $v = u + u^\perp$

2 Boyd and Vandenberghe, Ex. 2.10

(a) Show that if $A \in \mathbb{S}_+^n$ then the set C is convex

Proof. Assume that $A \in \mathbb{S}_+^n$. Let $\mathbf{x}_1, \mathbf{x}_2$ to be arbitrary vector such that $\mathbf{x}_1 \in C, \mathbf{x}_2 \in C$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. Then we show arbitrary linear combination $\mathbf{x}_* = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2, \lambda \in [0, 1]$ also belongs to the set C . According to the definition of set C , we have

$$\mathbf{x}_1^T A \mathbf{x}_1 + b^T \mathbf{x}_1 + c \leq 0 \quad (1)$$

$$\mathbf{x}_2^T A \mathbf{x}_2 + b^T \mathbf{x}_2 + c \leq 0 \quad (2)$$

By positive semidefiniteness of A , we have

$$\mathbf{x}^T A \mathbf{x} \geq 0, \forall \mathbf{x} \quad (3)$$

That is

$$\mathbf{x}_1^T A \mathbf{x}_1 \geq 0 \quad (4)$$

$$\mathbf{x}_2^T A \mathbf{x}_2 \geq 0 \quad (5)$$

$$(\mathbf{x}_2 - \mathbf{x}_1)^T A (\mathbf{x}_2 - \mathbf{x}_1) \geq 0 \quad (6)$$

Besides,

$$\lambda \geq 0 \quad (7)$$

$$1 - \lambda \geq 0 \quad (8)$$

$$\lambda - 1 \leq 0 \quad (9)$$

Then we investigate the property of $\mathbf{x}_*^T A \mathbf{x}_* + b^T \mathbf{x}_* + c$.

$$\mathbf{x}_*^T A \mathbf{x}_* + b^T \mathbf{x}_* + c \quad (10)$$

$$= (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2)^T A (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) + b^T (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) + c \quad (11)$$

$$= \lambda^2 \mathbf{x}_1^T A \mathbf{x}_1 + (1 - \lambda)^2 \mathbf{x}_2^T A \mathbf{x}_2 + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) + \lambda b^T \mathbf{x}_1 + (1 - \lambda) b^T \mathbf{x}_2 + c \quad (12)$$

$$= \lambda^2 \mathbf{x}_1^T A \mathbf{x}_1 - \lambda \mathbf{x}_1^T A \mathbf{x}_1 + (1 - \lambda)^2 \mathbf{x}_2^T A \mathbf{x}_2 - (1 - \lambda) \mathbf{x}_2^T A \mathbf{x}_2 + \lambda(\mathbf{x}_1^T A \mathbf{x}_1 + b^T \mathbf{x}_1 + c) + (1 - \lambda)(\mathbf{x}_2^T A \mathbf{x}_2 + b^T \mathbf{x}_2 + c) + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) \quad (13)$$

$$\leq \lambda(\lambda - 1)(\mathbf{x}_1^T A \mathbf{x}_1 + \mathbf{x}_2^T A \mathbf{x}_2) + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) \quad (14)$$

$$= \lambda(\lambda - 1)((\mathbf{x}_2 - \mathbf{x}_1)^T A (\mathbf{x}_2 - \mathbf{x}_1)) \quad (15)$$

$$\leq 0 \quad (16)$$

Then, it is obvious that

$$\mathbf{x}_* \in C \quad (17)$$

Since \mathbf{x}_* is arbitrary convex combination of \mathbf{x}_1 and \mathbf{x}_2 , then

$$C \text{ is convex set.} \quad (18)$$

Hence, it is proved that

$$A \in \mathbb{S}_+^n \Rightarrow C \text{ is convex.} \quad (19)$$

□

(b) Show that C_1 is convex if there exists $\lambda \in \mathbb{R}$ such that $(A + \lambda gg^T) \in \mathbb{S}_+^n$

Proof. Assume that there exists $\lambda \in \mathbb{R}$ such that $(A + \lambda gg^T) \in \mathbb{S}_+^n$. Let λ^* be the λ that satisfies $(A + \lambda gg^T) \in \mathbb{S}_+^n$. Now we show that

$$C_1 = C \cap \{\mathbf{x} \in \mathbb{R}^n : g^T \mathbf{x} + h = 0\} \quad (20)$$

$$= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T A \mathbf{x} + b^T \mathbf{x} + c \leq 0 \text{ and } g^T \mathbf{x} + h = 0\} \quad (21)$$

is convex.

Let \mathbf{x}_1 and \mathbf{x}_2 be arbitrary vector that $\mathbf{x}_1 \in C_1$, $\mathbf{x}_2 \in C_1$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. Let \mathbf{x}_* to be the arbitrary convex combination of \mathbf{x}_1 and \mathbf{x}_2 as $\mathbf{x}_* = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2$, and next we show that $\mathbf{x}_* \in C_1$.

By the definition of C_1 (See. (21)), we have

$$\mathbf{x}_1^T A \mathbf{x}_1 + b^T \mathbf{x}_1 + c \leq 0 \text{ and } g^T \mathbf{x}_1 + h = 0 \quad (22)$$

$$\mathbf{x}_2^T A \mathbf{x}_2 + b^T \mathbf{x}_2 + c \leq 0 \text{ and } g^T \mathbf{x}_2 + h = 0 \quad (23)$$

Then we have

$$g^T \mathbf{x}_2 + h - (g^T \mathbf{x}_1 + h) = 0 \quad (24)$$

$$g^T (\mathbf{x}_2 - \mathbf{x}_1) = 0 \quad (25)$$

Derivation for $g^T \mathbf{x}_* + h = 0$ is as follows:

$$g^T \mathbf{x}_* + h \quad (26)$$

$$= g^T (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) + h \quad (27)$$

$$= \lambda g^T \mathbf{x}_1 + (1 - \lambda) g^T \mathbf{x}_2 + h \quad (28)$$

$$= \lambda (g^T \mathbf{x}_1 + h) + (1 - \lambda) (g^T \mathbf{x}_2 + h) \quad (29)$$

$$= 0 \quad (30)$$

Derivation for $\mathbf{x}_*^T A \mathbf{x}_* + b^T \mathbf{x}_* + c \leq 0$ is as follows:

$$\mathbf{x}_*^T A \mathbf{x}_* + b^T \mathbf{x}_* + c \quad (31)$$

$$= (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2)^T A (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) + b^T (\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) + c \quad (32)$$

$$= \lambda^2 \mathbf{x}_1^T A \mathbf{x}_1 + (1 - \lambda)^2 \mathbf{x}_2^T A \mathbf{x}_2 + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) + \lambda b^T \mathbf{x}_1 + (1 - \lambda) b^T \mathbf{x}_2 + c \quad (33)$$

$$= \lambda^2 \mathbf{x}_1^T A \mathbf{x}_1 - \lambda \mathbf{x}_1^T A \mathbf{x}_1 + (1 - \lambda)^2 \mathbf{x}_2^T A \mathbf{x}_2 - (1 - \lambda) \mathbf{x}_2^T A \mathbf{x}_2 \quad (34)$$

$$+ \lambda(\mathbf{x}_1^T A \mathbf{x}_1 + b^T \mathbf{x}_1 + c) + (1 - \lambda)(\mathbf{x}_2^T A \mathbf{x}_2 + b^T \mathbf{x}_2 + c) + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) \quad (35)$$

$$\leq \lambda(\lambda - 1)(\mathbf{x}_1^T A \mathbf{x}_1 + \mathbf{x}_2^T A \mathbf{x}_2) + \lambda(1 - \lambda)(\mathbf{x}_1^T A \mathbf{x}_2 + \mathbf{x}_2^T A \mathbf{x}_1) \quad (36)$$

$$= \lambda(\lambda - 1)((\mathbf{x}_2 - \mathbf{x}_1)^T A (\mathbf{x}_2 - \mathbf{x}_1)) \quad (37)$$

$$= \lambda(\lambda - 1) \left((\mathbf{x}_2 - \mathbf{x}_1)^T (A + \lambda^* g g^T) (\mathbf{x}_2 - \mathbf{x}_1) - (\mathbf{x}_2 - \mathbf{x}_1)^T (\lambda^* g g^T) (\mathbf{x}_2 - \mathbf{x}_1) \right) \quad (38)$$

$$\leq \lambda(1 - \lambda)(\mathbf{x}_2 - \mathbf{x}_1)^T (\lambda^* g g^T) (\mathbf{x}_2 - \mathbf{x}_1) \quad (39)$$

$$= 0$$

Then we have

$$\mathbf{x}_*^T A \mathbf{x}_* + b^T \mathbf{x}_* + c \leq 0 \text{ and } g^T \mathbf{x}_* + h = 0 \quad (40)$$

That is to say, for arbitrary convex combination $\mathbf{x}_* \in C_1$. Then

$$C_1 \text{ is convex set.} \quad (41)$$

Hence, it is proved that

$$\exists \lambda \in \mathbb{R}, (A + \lambda g g^T) \in \mathbb{S}_+^n \Rightarrow C_1 \text{ is convex.} \quad (42)$$

□

3 Boyd and Vandenberghe, Ex. 2.21

Proof. Let $(a_1, b_1) \in S$, $(a_2, b_2) \in S$ and $(a_1, b_1) \neq (a_2, b_2)$, then show that arbitrary convex combination $(a_*, b_*) = (\lambda a_1 + (1 - \lambda)a_2, \lambda b_1 + (1 - \lambda)b_2) \in S$.

By definition of set S , we have

$$a_1^T x \leq b_1 \quad \forall x \in C, \text{ and } a_1^T x \geq b_1 \quad \forall x \in D \quad (43)$$

$$a_2^T x \leq b_2 \quad \forall x \in C, \text{ and } a_2^T x \geq b_2 \quad \forall x \in D \quad (44)$$

Then we turn to discuss $a_*^T x_c$ as follows: let x_c be arbitrary vector in set C ,

$$a_*^T x_c = (\lambda a_1 + (1 - \lambda)a_2)^T x_c \quad (45)$$

$$= \lambda a_1^T x_c + (1 - \lambda)a_2^T x_c \quad (46)$$

$$\leq \lambda b_1 + (1 - \lambda)b_2 \quad (47)$$

$$= b_* \quad (48)$$

and now let x_d be arbitrary vector in set D ,

$$a_*^T x_d = (\lambda a_1 + (1 - \lambda)a_2)^T x_d \quad (49)$$

$$= \lambda a_1^T x_d + (1 - \lambda)a_2^T x_d \quad (50)$$

$$\geq \lambda b_1 + (1 - \lambda)b_2 \quad (51)$$

$$= b_* \quad (52)$$

Now since we have $a_*^T x_c \leq b_* \quad \forall x_c \in C$ and $a_*^T x_d \geq b_* \quad \forall x_d \in D$, then we can conclude that

$$(a_*, b_*) \in S \quad (53)$$

Hence, it is proved that

$$S \text{ is convex.} \quad (54)$$

□

$$\mathbf{7} \quad \{x : \|x - v_1\| \leq \|x - v_2\|\} = \{x : c^T x \leq d\}$$

Proof.

$$\|x - v_1\| \leq \|x - v_2\| \quad (55)$$

$$\Leftrightarrow \|x - v_1\|^2 \leq \|x - v_2\|^2 \quad (56)$$

$$\Leftrightarrow (x - v_1)^T (x - v_1) \leq (x - v_2)^T (x - v_2) \quad (57)$$

$$\Leftrightarrow x^T x - x^T v_1 - v_1^T x + v_1^T v_1 \leq x^T x - x^T v_2 - v_2^T x + v_2^T v_2 \quad (58)$$

$$\Leftrightarrow -v_1^T x - v_1^T x + v_1^T v_1 \leq -v_2^T x - v_2^T x + v_2^T v_2 \quad (59)$$

$$\Leftrightarrow 2(v_2 - v_1)^T x \leq v_2^T v_2 - v_1^T v_1 \quad (60)$$

Let $c = 2(v_2 - v_1)$ and $d = v_2^T v_2 - v_1^T v_1 = \|v_2\|^2 - \|v_1\|^2$, then

$$c^T x \leq d \quad (61)$$

where $c = 2(v_2 - v_1)$ and $d = \|v_2\|^2 - \|v_1\|^2$.

Hence, we can conclude that

$$\{x : \|x - v_1\| \leq \|x - v_2\|\} = \{x : c^T x \leq d\} \quad (62)$$

□

8 Exists C such that $CA = B$

Proof. By Contradiction. Assume that there does not exist such C that $CA = B$. This means

$$\forall C, \nexists \mathbf{x}, CA\mathbf{x} = B\mathbf{x} \quad (63)$$

Let $\mathbf{x}_* \in \mathbb{R}^m$ such that $A\mathbf{x}_* = 0$. Since we already have

$$A\mathbf{x} = 0 \Rightarrow B\mathbf{x} = 0 \quad (64)$$

Then we have

$$B\mathbf{x}_* = 0 \quad (65)$$

Therefore, we find a \mathbf{x}_* such that $CA\mathbf{x}_* = B\mathbf{x}_* = 0, \forall C$, which contradict the statement (63). Hence, the initial assumption should be rejected and then it is proved that

$$\exists C \text{ such that } CA = B \quad (66)$$

□

A Codes Printout

(a) Gradient Descent Routine

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem set 1: Standard Gradient Descent with fixed step size
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usage:
%   [b, iter, all_costs] = gd (X, b_init, gamma)
% Parameters:
%   X: matrix in quadratic optimization
%   b_init: starting vector of variable b
%   gamma: fixed step size
% Note that stopping criteria is set by absolute eps = 10e-5.

function [b, iter, all_costs] = gd (X, b_init, gamma)
eps = 10e-5;
b = b_init;
last_cost = 0.5 * b' * X * b;
iter = 1;
all_costs = [];
while true,
    %% compute essential numerics and do gradient descent
    gradient = X * b;
    b = b - gamma * gradient;
    cost = 0.5 * b' * X * b;
    rate = (cost / last_cost);
    all_costs = [all_costs cost];
    %% output numeric information of this iteration
    disp(sprintf('Iter: %d, Cost: %e, Conv.Rate: %f', iter, cost, rate));
    %% quadratic optimization converges to zero
    if cost < eps,
        disp('Converged to zeros!')
        break
    end
    %% quadratic optimization diverges
    if cost >= last_cost && iter > 10,
        disp('Problem diverges!')
        break
    end
    %% prepare for next iteration
    last_cost = cost;
    iter = iter + 1;
end

%% uncomment following code for plotting individual gradient descent run
%% plot f(b^(n)) with regard to n
%% plot (1:iter, all_costs)
%% title (sprintf ('Gradient Descent with gamma=%f', gamma))
%% xlabel ('iteration: n')
%% ylabel ('cost of function: f(b^n)')

end

```

(b) Running Script

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Running scripts for applying gradient descent
%% on three given dataset
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% for X1, b1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
gamma1_one = 0.5; gamma2_two = 3;
[b1_opt_one, iter1_one, costs1_one] = Gradient_Descent(X1, b1, gamma1_one);
[b1_opt_two, iter1_two, costs1_two] = Gradient_Descent(X1, b1, gamma2_two);
subplot (1, 2, 1)
plot (1:iter1_one, costs1_one)
axis ([-0.1*iter1_one 1.1*iter1_one -0.05*max(costs1_one) max(costs1_one)])
title (sprintf ('Gradient Descent with gamma=%f', gamma1_one))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')
subplot (1, 2, 2)
plot (1:iter1_two, costs1_two)
axis ([-0.1*iter1_two 1.1*iter1_two -0.05*max(costs1_two) max(costs1_two)])
title (sprintf ('Gradient Descent with gamma=%f', gamma2_two))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% for X2, b2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
gamma2_one = 1.5; gamma2_two = 3;
[b2_opt_one, iter2_one, costs2_one] = Gradient_Descent(X2, b2, gamma2_one);
[b2_opt_two, iter2_two, costs2_two] = Gradient_Descent(X2, b2, gamma2_two);
figure()
subplot (1, 2, 1)
plot (1:iter2_one, costs2_one)
axis ([-0.1*iter2_one 1.1*iter2_one -0.05*max(costs2_one) max(costs2_one)])
title (sprintf ('Gradient Descent with gamma=%f', gamma2_one))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')
subplot (1, 2, 2)
plot (1:iter2_two, costs2_two)
axis ([-0.1*iter2_two 1.1*iter2_two -0.05*max(costs2_two) max(costs2_two)])
title (sprintf ('Gradient Descent with gamma=%f', gamma2_two))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% for X3, b3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
gamma3_one = 0.005; gamma3_two = 0.05;
[b3_opt_one, iter3_one, costs3_one] = Gradient_Descent(X3, b3, gamma3_one);
[b3_opt_two, iter3_two, costs3_two] = Gradient_Descent(X3, b3, gamma3_two);
figure()
subplot (1, 2, 1)
plot (1:iter3_one, costs3_one)
axis ([-0.1*iter3_one 1.1*iter3_one -0.05*max(costs3_one) max(costs3_one)])
title (sprintf ('Gradient Descent with gamma=%f', gamma3_one))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')
subplot (1, 2, 2)
plot (1:iter3_two, costs3_two)
axis ([-0.1*iter3_two 1.1*iter3_two -0.05*max(costs3_two) max(costs3_two)])
title (sprintf ('Gradient Descent with gamma=%f', gamma3_two))
xlabel ('iteration: n')
ylabel ('cost of function: f(b^n)')

```