



THE UNIVERSITY OF TEXAS  
AT AUSTIN

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CS331 ALGORITHM

**Assignment 06**

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## 1 Exercise 2

Let  $i, j$ , and  $k$  be distinct integers in  $[n]$  such that  $j$  lies between  $i$  and  $k$ , i.e., either  $i < j < k$  and  $k < j < i$ . Assume that bids  $\alpha(i)$  and  $\alpha(j)$  are linear. Prove that if  $gap(p, i, j) = 0$  and  $g(j, k) = 0$ , then  $g(i, j) = 0$ .

Based on the known condition as follows,

$$gap(p, i, j) = 0 \quad (1)$$

$$gap(p, j, k) = 0 \quad (2)$$

And the assumed condition that

$$i, j, k \in [n] \quad (3)$$

$$\alpha(j) \text{ is linear} \quad (4)$$

$$\alpha(i) \text{ is linear} \quad (5)$$

We have

$$max(0, s(i) \cdot g(i, j) - f(p, i, j)) = 0 \quad (6)$$

$$max(0, s(j) \cdot g(j, k) - f(p, j, k)) = 0 \quad (7)$$

That is,

$$s(i) \cdot g(i, j) - f(p, i, j) \leq 0 \quad (8)$$

$$s(j) \cdot g(j, k) - f(p, j, k) \leq 0 \quad (9)$$

We sum up (8) and (9), then get

$$s(i) \cdot g(i, j) - f(p, i, j) + s(j) \cdot g(j, k) - f(p, j, k) \leq 0 \quad (10)$$

This can be simplified to be

$$s(i) \cdot g(i, k) - f(p, i, k) \leq g(j, k) \cdot (s(i) - s(j)) \quad (11)$$

According to the known condition, we have

$$i < j < k \text{ or } k < j < i \quad (12)$$

In the case of  $i < j < k$ , we have  $g(j, k) = q_{\beta(k)} - q_{\beta(j)} \geq 0$  and  $s(i) - s(j) \leq 0$  since  $\alpha(i)$  and  $\alpha(j)$  are matched to  $\beta i$  and  $\beta j$  in MWMCM  $M$ , respectively. Similarly, in the case of  $k < j < i$ , we have  $g(i, j) = q_{\beta(j)} - q_{\beta(i)} \leq 0$  and  $s(i) - s(j) \geq 0$ . Therefore, it can be concluded that in either case,

$$g(j, k) \cdot (s(i) - s(j)) \leq 0 \quad (13)$$

Hence, we have

$$s(i) \cdot g(i, k) - f(p, i, k) \leq g(j, k) \leq 0 \quad (14)$$

Since  $\alpha(i)$  is linear bid and  $i \in [n]$

$$gap(p, i, k) = 0 \quad (15)$$

## 2 Exercise 1

Let  $p$  be a price vector for  $G$  such that  $L(p, n) \wedge R(p, 0)$  holds. Prove that  $gap(p, i, j) = 0$  for all integers  $i$  and  $j$  in  $[n]$ . Hint: Make use of Exercise 1.

*Case 1.* In the case of  $i \in \{-1, n\}$  or in the case of  $i \in [n]$  and  $\alpha(i)$  is single-item bid, it can be easy to conclude that  $\forall i, j, gap(p, i, j) = 0$  by definition of  $gap$  function. □

*Case 2.* In the case of  $i \in [n]$  and  $\alpha(i)$  is linear bid, things are more complicated. There are two subcases, (1)  $i > j$ , (2)  $j > i$ . We provide detailed proof as follows.

Based on give condition  $L(p, n) \wedge R(p, 0)$ , we have both of the followings hold.

$$L(p, n) \tag{16}$$

$$R(p, 0) \tag{17}$$

For  $i > j$ , we can iteratively employ (17) and the result of exercise 1 to reach the conclusion of  $\forall i, j, gap(p, i, j) = 0$ . Specifically, apply the (17) repeatedly and since  $\alpha(i)$  is linear bid, it will eventually reach  $\alpha(i)$ .

$$gap(p, \tau(j), j) = 0 \tag{18}$$

$$gap(p, \tau(\tau(j)), \tau(j)) = 0 \tag{19}$$

$$\vdots$$

$$gap(p, \tau_s(j), \tau_{s-1}(j)) = 0 \tag{20}$$

$$gap(p, i, \tau_s(j)) = 0 \tag{21}$$

Note that  $s$  denotes the number of times we apply rule of  $R(p, 0)$  and  $\tau_s(j)$  denotes the integer derived by apply  $s$  times of  $R(p, 0)$  on original integer  $j$ .

Then we can make use of the result of exercise 1 to address the series of equations (18) - (21) and conclude that

$$\forall i, j, gap(p, i, j) = 0 \tag{22}$$

Note that the above proof can be formalized by doing mathematical induction on the number of times we employ  $R(p, 0)$ , which is  $s$ .

For  $i < j$ , we can iteratively employ (16) and the result of exercise 1 to reach the conclusion of  $\forall i, j, gap(p, i, j) = 0$ . The detailed proof follows the same routine as the case of  $i > j$  in the above.

In terms of the dicussion over two subcases, we have  $\forall i, j, gap(p, i, j) = 0$  in the case of  $i \in [n]$  and  $\alpha(i)$  is linear bid. □

According to the dicussion over *case 1* and *case 2*, it can be concluded that *generally*

$$\forall i, j, gap(p, i, j) = 0 \tag{23}$$