



THE UNIVERSITY OF TEXAS
AT AUSTIN

CS383C NUMERICAL ANALYSIS

Homework 03

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Exercise 2. Show that if H is a reflector, then

2.1 $HH = I$

Since H is a reflector, we have

$$H = I - 2uu^H \quad (1)$$

where u is unit vector ($\|u\|_2^2 = 1$).

$$\begin{aligned} HH &= (I - 2uu^H)(I - 2uu^H) \\ &= I \cdot I - 2uu^H - 2uu^H + 4(uu^H)(uu^H) \\ &= I - 4uu^H + 4\|u\|_2^2 uu^H \\ &= I - 4uu^H + 4uu^H \\ &= I \end{aligned} \quad (2)$$

Lemma 1. For arbitrary vector $a \in \mathbb{C}^n$, $b \in \mathbb{C}^n$, $c \in \mathbb{C}^n$, $ab^H c = (b^H c)a$.

Proof.

$$ab^H c = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} b^H c = \begin{pmatrix} a_0 b^H c \\ a_1 b^H c \\ \vdots \\ a_{n-1} b^H c \end{pmatrix} c = \begin{pmatrix} a_0 b^H c \\ a_1 b^H c \\ \vdots \\ a_{n-1} b^H c \end{pmatrix} = \begin{pmatrix} (b^H c)a_0 \\ (b^H c)a_1 \\ \vdots \\ (b^H c)a_{n-1} \end{pmatrix} = (b^H c) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \quad (3)$$

$$= (b^H c)a \quad (4)$$

□

Lemma 2. $(uu^H)(uu^H) = \|u\|_2^2 uu^H$

Proof.

$$\begin{aligned} (uu^H)(uu^H) &= (uu^H u)u^H \\ &= ((u^H u)u)u^H \\ &= (\|u\|_2^2 u)u^H \\ &= \|u\|_2^2 uu^H \end{aligned} \quad (5)$$

□

2.2 $H = H^H$

$$\begin{aligned} H^H &= (I - 2uu^H)^H \\ &= I^H - (2uu^H)^H \\ &= I - 2(u^H)^H u^H \\ &= I - 2uu^H \\ &= H \end{aligned} \quad (6)$$

2.3 $HH^H = I$

In terms of (6), multiply both sides with H and then reuse conclusion in (2)

$$H^H H = HH = I \quad (7)$$

Exercise 4. Show that if $x \in \mathbb{R}^n$, $v = x \mp \|x\|_2 e_0$ and $\tau = v^T v / 2$, then

$$(I - \frac{1}{\tau} v v^T) x = \pm \|x\|_2 e_0$$

We start from the reflector $I - \frac{1}{\tau} v v^T$ on x ,

$$\begin{aligned}
& (I - \frac{1}{\tau} v v^T) x \\
&= \left(I - \frac{2 v v^T}{v^T v} \right) x \\
&= \left(I - \frac{2(x \mp \|x\|_2 e_0)(x \mp \|x\|_2 e_0)^T}{(x \mp \|x\|_2 e_0)^T (x \mp \|x\|_2 e_0)} \right) x \\
&= \left(I - 2 \frac{x x^T \mp \|x\|_2 (x e_0^T + e_0 x^T) + \|x\|_2^2 e_0 e_0^T}{2\|x\|_2^2 \mp 2\|x\|_2 e_0^T x} \right) x \\
&= \left(I - \frac{x x^T \mp \|x\|_2 (x e_0^T + e_0 x^T) + \|x\|_2^2 e_0 e_0^T}{\|x\|_2^2 \mp \|x\|_2 e_0^T x} \right) x \\
&= \frac{\|x\|_2^2 \mp \|x\|_2 e_0^T x - x x^T \pm \|x\|_2 (x e_0^T + e_0 x^T) - \|x\|_2^2 e_0 e_0^T}{\|x\|_2^2 \mp \|x\|_2 e_0^T x} x \\
&= \frac{\|x\|_2^2 x \mp \|x\|_2 e_0^T x x - x x^T x \pm \|x\|_2 (x e_0^T + e_0 x^T) x - \|x\|_2^2 e_0 e_0^T x}{\|x\|_2^2 \mp \|x\|_2 e_0^T x} \\
&= \frac{(\|x\|_2^2 x - x x^T x) + (\mp \|x\|_2 e_0^T x x \pm \|x\|_2 x e_0^T x) \pm \|x\|_2 e_0 x^T x - \|x\|_2^2 e_0 e_0^T x}{\|x\|_2^2 \mp \|x\|_2 e_0^T x} \\
&= \frac{\pm \|x\|_2 e_0 x^T x - \|x\|_2^2 e_0 e_0^T x}{\|x\|_2^2 \mp \|x\|_2 e_0^T x} \\
&= \frac{\pm e_0 x^T x - \|x\|_2 e_0 e_0^T x}{\|x\|_2 \mp e_0^T x} \\
&= \frac{\pm (\|x\|_2^2 e_0 - \|x\|_2 (e_0^T x) e_0)}{\|x\|_2 \mp e_0^T x} \\
&= \frac{\pm (\|x\|_2 - (e_0^T x)) \|x\|_2 e_0}{\|x\|_2 \mp e_0^T x} \\
&= \pm \|x\|_2 e_0
\end{aligned} \tag{8}$$

Note that above derivation frequently makes use of the above lemma 1 in real case.

Exercise 5. Complex

It is easy to show that the conclusion in (8) can extend to complex space. That is,

$$(I - \frac{1}{\tau}vv^H)x = \oplus \|x\|_2 e_0 \quad (9)$$

where $x \in \mathbb{C}^n$, $v = x \oplus \|x\|_2 e_0$ and $\tau = v^H v / 2$.

Let

$$v = \begin{pmatrix} 1 \\ u_2 \end{pmatrix}, \quad x = \begin{pmatrix} \chi \\ x_2 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (10)$$

Then have

$$\left(I - \frac{1}{\tau} \begin{pmatrix} 1 \\ u_2 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \right) \cdot \begin{pmatrix} \chi \\ x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix} \quad (11)$$

where $\tau = v^H v / 2 = (1 + u_2^H u_2) / 2$ and $\rho = \oplus \|x\|_2$.

Lemma 3. If $x \in \mathbb{C}^n$, $v = x \oplus \|x\|_2 e_0$ and $\tau = v^H v / 2$, then $(I - \frac{1}{\tau}vv^H)x = \oplus \|x\|_2 e_0$.

Proof. We start from the reflector $I - \frac{1}{\tau}vv^H$ on x ,

$$\begin{aligned} & (I - \frac{1}{\tau}vv^H)x \\ &= \left(I - \frac{2vv^H}{v^H v} \right) x \\ &= \left(I - \frac{2(x \oplus \|x\|_2 e_0)(x \oplus \|x\|_2 e_0)^H}{(x \oplus \|x\|_2 e_0)^H (x \oplus \|x\|_2 e_0)} \right) x \\ &= \left(I - 2 \frac{xx^H \oplus \|x\|_2 (xe_0^H + e_0 x^H) + \|x\|_2^2 e_0 e_0^H}{2\|x\|_2^2 \oplus 2\|x\|_2 e_0^H x} \right) x \\ &= \left(I - \frac{xx^H \oplus \|x\|_2 (xe_0^H + e_0 x^H) + \|x\|_2^2 e_0 e_0^H}{\|x\|_2^2 \oplus \|x\|_2 e_0^H x} \right) x \\ &= \frac{\|x\|_2^2 \oplus \|x\|_2 e_0^H x - xx^H \oplus \|x\|_2 (xe_0^H + e_0 x^H) - \|x\|_2^2 e_0 e_0^H}{\|x\|_2^2 \oplus \|x\|_2 e_0^H x} x \\ &= \frac{\|x\|_2^2 x \oplus \|x\|_2 e_0^H xx - xx^H x \oplus \|x\|_2 (xe_0^H + e_0 x^H)x - \|x\|_2^2 e_0 e_0^H x}{\|x\|_2^2 \oplus \|x\|_2 e_0^H x} \\ &= \frac{(\|x\|_2^2 x - xx^H x) + (\oplus \|x\|_2 e_0^H xx \oplus \|x\|_2 xe_0^H x) \oplus \|x\|_2 e_0 x^H x - \|x\|_2^2 e_0 e_0^H x}{\|x\|_2^2 \oplus \|x\|_2 e_0^H x} \\ &= \frac{\oplus \|x\|_2 e_0 x^H x - \|x\|_2^2 e_0 e_0^H x}{\|x\|_2^2 \oplus \|x\|_2 e_0^H x} \\ &= \frac{\oplus e_0 x^H x - \|x\|_2 e_0 e_0^H x}{\|x\|_2 \oplus e_0^H x} \\ &= \frac{\oplus \|x\|_2^2 e_0 - \|x\|_2 (e_0^H x) e_0}{\|x\|_2 \oplus e_0^H x} \\ &= \frac{\oplus (\|x\|_2 - (e_0^H x)) \|x\|_2 e_0}{\|x\|_2 \oplus e_0^H x} \\ &= \oplus \|x\|_2 e_0 \end{aligned} \quad (12)$$

□

Exercise 6. Matrix Equivalence

We start from right hand side

$$\begin{aligned}
 RHS &= I - \frac{1}{\tau_1} \begin{pmatrix} 0 \\ 1 \\ u_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ u_2 \end{pmatrix}^H \\
 &= \left(I - \frac{1}{\tau_1} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{pmatrix} 1 \\ u_2 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \right) \\
 &= \left(I - \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{\tau_1} \begin{pmatrix} 1 \\ u_2 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \right) \\
 &= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I - \frac{1}{\tau_1} \begin{pmatrix} 1 \\ u_2 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \\
 &= LHS
 \end{aligned} \tag{13}$$

Exercise 11. Expensive Algorithm

As indicated by **Theorem 10** in the note, we have cost of the algorithm in Figure 6 for $A \in \mathbb{C}^{m \times n}$

$$C_{FormQ}(m, n) = 2mn^2 - \frac{2}{3}n^3 \tag{14}$$

For $m = n$, the cost can be simplified as

$$C_{FormQ}(A) = \frac{4}{3}n^3 = \mathcal{O}(n^3) \tag{15}$$

However, if we accumulate Q by using n householder transformation with

$$Q = (\dots((IH_0)H_1)\dots H_{n-1}) \tag{16}$$

Then the cost we have is at least

$$C_{accumulation}(A) = n^3 \cdot (n - 1) = \mathcal{O}(n^4) \tag{17}$$

where n^3 comes from each one matrix multiplication, and $n - 1$ comes from the total number of householder matrix H_i ($i = 0, 1, 2, \dots, n - 1$).

Comparing formula (15) and (17), it is obvious that the accumulation method is much more expensive than algorithm in Figure 6.