THE UNIVERSITY OF TEXAS
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CS383C NUMERICAL ANALYSIS

HW05 Numerical Stability

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Exercises

Exercise 3.

3.1

Write 1 as floating number.

$$.1\underbrace{00\cdots0}_{t-1}\times 2^1 \tag{1}$$

3.2

Show that $\mathbf{u} = \frac{1}{2} \cdot 2^{1-t}$

Proof. Let $\chi = .\delta_0\delta_1 \cdots \delta_{t-1}\delta_t \cdots$ and $\overset{\checkmark}{\chi}$ to be the value stored in t-digit floating number with rounding mechanism. Then if $\delta_t = 0$, then $\chi = \overset{\checkmark}{\chi}$ and $|\delta\chi| = 0 \leq 2^{e-t-1}$. But if $\delta_t = 1$, due to the rounding mechanism, then $\chi < \overset{\checkmark}{\chi}$ and

$$|\delta\chi| = |\chi - \overset{\checkmark}{\chi}| = |.\delta_0\delta_1 \cdots \delta_{t-1}\delta_t \cdots \times 2^e - .\delta_0\delta_1 \cdots \delta'_{t-1} \times 2^e| \leq .\underbrace{00\cdots0}_{t}1 \times 2^e = 2^{e-t-1} \tag{2}$$

For χ, since $\delta_0 = 1$ (normalized)

$$|\chi| = |.\delta_0\delta_1 ... \times 2^e| \geq .1 \times 2^e \geq 2^{e-1} \tag{3}$$

Thus,

$$\frac{|\delta\chi|}{|\chi|} \leq \frac{2^{e-t-1}}{2^{e-1}} = \frac{1}{2} \cdot 2^{1-t} \tag{4}$$

Then,

$$|\delta\chi| \leq \frac{1}{2} \cdot 2^{1-t}|\chi| \tag{5}$$

Now, we have

$$\mathbf{u} = \frac{1}{2}2^{1-t} \tag{6}$$

□

Exercise 10.

Show that $|AB| \leq |A||B|$.

Proof. Let $C = AB$. And the (i,j) entry of $|C|$ is given by

$$|c_{i,j}| = |\sum_{p=0}^{k} a_{i,k}b_{k,j}| \leq \sum_{p=0}^{k} |a_{i,k}b_{k,j}| \leq \sum_{p=0}^{k} |a_{i,k}||b_{k,j}| \tag{7}$$

which equals (i,j) entry of $|A||B|$. Hence, we have

$$|AB| \leq |A||B| \tag{8}$$

□

Exercise 12.

12.1

Show that if $|A| \leq |B|$, then $||A||_1 \leq ||B||_1$.

Proof. Partition $A_{m\times n} = \left(\begin{array}{c|c|c|c} a_0 & a_1 & ... & a_{n-1} \end{array} \right)$ where a_j indicates the j-th column of matrix A. Similarly, we partition $B_{m\times n} = \left(\begin{array}{c|c|c|c} b_0 & b_1 & ... & b_{n-1} \end{array} \right)$ where b_j indicates the j-th column of matrix B. Then we use a_{ij} and b_{ij} to denote i-th element of a_j and b_j respectively.

$$||A||_1 = \max_{0\leq j<n} ||a_j||_1 = \max_{0\leq j<n} \sum_{i=0}^{m-1} |a_{ij}| \leq \max_{0\leq j<n} \sum_{i=0}^{m-1} |b_{ij}| = \max_{0\leq j<n} ||b_j||_1 = ||B||_1 \tag{9}$$

□

Lemma 1. *For arbitrary matrix* $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & ... & a_{n-1} \end{array} \right)$, $||A||_1 = \max_{0\leq j<n} ||a_j||_1$.

Proof. This lemma has been proved in Notes on Norms. □

12.2

Show that if $|A| \leq |B|$, then $||A||_\infty \leq ||B||_\infty$.

Proof. Partition $A_{m\times n} = \left(\begin{array}{c} a_0 \\ \hline a_1 \\ \hline \vdots \\ \hline a_{m-1} \end{array} \right)$, where a_i indicates the i-th row of matrix A. Similarly, we partition $B_{m\times n} = \left(\begin{array}{c} b_0 \\ \hline b_1 \\ \hline \vdots \\ \hline b_{m-1} \end{array} \right)$, where b_i indicates the i-th row of matrix B. Then we use a_{ij} and b_{ij} to denote j-th element of a_i and b_i respectively.

$$||A||_\infty = \max_{0\leq i<m} ||a_i||_1 = \max_{0\leq i<m} \sum_{j=0}^{n-1} |a_{ij}| \leq \max_{0\leq i<m} \sum_{j=}^{n-1} |b_{ij}| = \max_{0\leq i<m} ||b_i||_1 = ||B||_\infty \tag{10}$$

□

Lemma 2. *For arbitrary matrix* $A = \left(\begin{array}{c} a_0 \\ \hline a_1 \\ \hline \vdots \\ \hline a_{m-1} \end{array} \right)$, $||A||_\infty = \max_{0\leq i<m} ||a_i||_1$.

Proof. This lemma has been proved in Notes on Norms. □

12.3

Show that if $|A| \leq |B|$, then $||A||_F \leq ||B||_F$. Let $A, B \in \mathbb{R}^{m\times n}$ and a_{ij}, b_{ij} be (i, j) entry of A, B respectively.

Proof.

$$||A||_F = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |a_{ij}|^2 \leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |b_{ij}|^2 = ||B||_F \tag{11}$$

□

Exercise 13.

$$\begin{aligned}
\check{\kappa} &= [(\chi_0\psi_0 + \chi_1\psi_1) + \chi_2\psi_2] && (12)\\
&= [[\chi_0\psi_0 + \chi_1\psi_1] + [\chi_2\psi_2]] && (13)\\
&= [[[\chi_0\psi_0] + [\chi_1\psi_1]] + [\chi_2\psi_2]] && (14)\\
&= [[\chi_0\psi_0(1+\epsilon_*^{(0)}) + \chi_1\psi_1(1+\epsilon_*^{(1)})] + \chi_2\psi_2(1+\epsilon_*^{(2)})] && (15)\\
&= [\big(\chi_0\psi_0(1+\epsilon_*^{(0)}) + \chi_1\psi_1(1+\epsilon_*^{(1)})\big)(1+\epsilon_+^{(1)}) + \chi_2\psi_2(1+\epsilon_*^{(2)})] && (16)\\
&= \Big(\big(\chi_0\psi_0(1+\epsilon_*^{(0)}) + \chi_1\psi_1(1+\epsilon_*^{(1)})\big)(1+\epsilon_+^{(1)}) + \chi_2\psi_2(1+\epsilon_*^{(2)})\Big)(1+\epsilon_+^{(2)}) && (17)\\
&= \chi_0\psi_0(1+\epsilon_*^{(0)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)}) + \chi_1\psi_1(1+\epsilon_*^{(1)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)}) + \chi_2\psi_2(1+\epsilon_*^{(2)})(1+\epsilon_+^{(2)}) && (18)\\
&= \begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}^T \begin{pmatrix}\epsilon_0(1+\epsilon_*^{(0)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)})\\ \epsilon_1(1+\epsilon_*^{(1)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)})\\ \epsilon_2(1+\epsilon_*^{(2)})(1+\epsilon_+^{(2)})\end{pmatrix} && (19)\\
&= \begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}^T \left(\begin{array}{c|c|c}(1+\epsilon_*^{(0)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)}) & 0 & 0\\ 0 & (1+\epsilon_*^{(1)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)}) & 0\\ 0 & 0 & (1+\epsilon_*^{(2)})(1+\epsilon_+^{(2)})\end{array}\right)\begin{pmatrix}\epsilon_0\\ \epsilon_1\\ \epsilon_2\end{pmatrix} && (20)\\
&= \begin{pmatrix}\chi_0(1+\epsilon_*^{(0)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)})\\ \chi_1(1+\epsilon_*^{(1)})(1+\epsilon_+^{(1)})(1+\epsilon_+^{(2)})\\ \chi_2(1+\epsilon_*^{(2)})(1+\epsilon_+^{(2)})\end{pmatrix}^T \begin{pmatrix}\epsilon_0\\ \epsilon_1\\ \epsilon_2\end{pmatrix} && (21)
\end{aligned}$$

Exercise 15.

Now we complete the missing part for the Inductive Step Case 1 of Lemma 14.

Proof. Case 1: $\prod_{i=0}^{n}(1+\epsilon)^{\pm1} = \prod_{i=0}^{n-1}(1+\epsilon)^{\pm1}(1+\epsilon_n)$.
By the inductive hypothesis, there exists a θ_n such that

$$(1+\theta_n) = \prod_{i=0}^{n-1}(1+\epsilon_i)^{\pm1} \text{ and } |\theta_n| \le n\mathbf{u}/(1-n\mathbf{u}) \qquad (22)$$

Then

$$\prod_{i=0}^{n}(1+\epsilon)^{\pm1} = \left(\prod_{i=0}^{n-1}(1+\epsilon)^{\pm1}\right)(1+\epsilon_n) = (1+\theta_n)(1+\epsilon_n) = 1 + \underbrace{\theta_n + \epsilon_n + \theta_n\cdot\epsilon_n}_{\theta_{n+1}} \qquad (23)$$

which tells us how to pick up θ_{n+1}. Then

$$\begin{aligned}
|\theta_{n+1}| &= |\theta_n + \epsilon_n + \theta_n\cdot\epsilon_n| && (24)\\
&\le |\theta_n| + |\epsilon_n| + |\theta_n|\cdot|\epsilon_n| && (25)\\
&= \frac{n\mathbf{u}}{1-n\mathbf{u}} + \mathbf{u} + \frac{n\mathbf{u}}{1-n\mathbf{u}}\cdot\mathbf{u} && (26)\\
&= \frac{n\mathbf{u} + \mathbf{u} - n\mathbf{u}^2 + n\mathbf{u}^2}{1-n\mathbf{u}} && (27)\\
&= \frac{(n+1)\mathbf{u}}{1-n\mathbf{u}} && (28)\\
&\le \frac{(n+1)\mathbf{u}}{1-(n+1)\mathbf{u}} && (29)
\end{aligned}$$

□

Exercise 18.

18.1

Show that if $n, b \geq 1$, then $\gamma_n \leq \gamma_{n+b}$.

Proof.

$$\gamma_n = \frac{n\mathbf{u}}{1-n\mathbf{u}} \leq \frac{n\mathbf{u}}{1-(n+b)\mathbf{u}} \leq \frac{(n+b)\mathbf{u}}{1-(n+b)\mathbf{u}} = \gamma_{n+b} \tag{30}$$

Note that since $\mathbf{u}$ is extremely small, then $1 - n\mathbf{u} > 0$ and $1 - (n+b)\mathbf{u} > 0$. □

18.2

Show that if $n, b \geq 1$, then $\gamma_n + \gamma_b + \gamma_n\gamma_b \leq \gamma_{n+b}$.

Proof.

$$\begin{aligned}
&\gamma_n + \gamma_b + \gamma_n\gamma_b && (31)\\
=&\frac{n\mathbf{u}}{1-n\mathbf{u}} + \frac{b\mathbf{u}}{1-b\mathbf{u}} + \frac{n\mathbf{u}}{1-n\mathbf{u}} \cdot \frac{b\mathbf{u}}{1-b\mathbf{u}} && (32)\\
=&\frac{n\mathbf{u} - nb\mathbf{u}^2 + b\mathbf{u} - nb\mathbf{u}^2 + nb\mathbf{u}^2}{(1-n\mathbf{u})(1-b\mathbf{u})} && (33)\\
=&\frac{n\mathbf{u} - nb\mathbf{u}^2 + b\mathbf{u}}{1 - n\mathbf{u} - b\mathbf{u} + nb\mathbf{u}^2} && (34)\\
\leq&\frac{n\mathbf{u} + b\mathbf{u}}{1 - n\mathbf{u} - b\mathbf{u} + nb\mathbf{u}^2} && (35)\\
\leq&\frac{n\mathbf{u} + b\mathbf{u}}{1 - n\mathbf{u} - b\mathbf{u}} && (36)\\
=&\frac{(n+b)\mathbf{u}}{1-(n+b)\mathbf{u}} && (37)\\
=&\gamma_{n+b} && (38)
\end{aligned}$$

Note that since $\mathbf{u}$ is extremely small, then $1 - n\mathbf{u} > 0$ and $1 - (n+b)\mathbf{u} > 0$. Also, note that $nb\mathbf{u}^2 \geq 0$. □

Exercise 19.

19.1 $k = 0$

Show that $\left(\begin{array}{c|c} I+\Sigma^{(k)} & 0 \\ \hline 0 & (1+\epsilon_1) \end{array}\right)(1+\epsilon_2) = I + \Sigma^{(k+1)}$

Proof. Given that if $k = 0$, then $\epsilon_1 = 0$ and Σ^0 is 0×0 matrix, we have

$$(1+0)\cdot(1+\epsilon_2) = \underbrace{1}_{I} + \underbrace{\epsilon_2}_{\Sigma^{(1)}} \tag{39}$$

Thus, $\left(\begin{array}{c|c} I+\Sigma^{(k)} & 0 \\ \hline 0 & (1+\epsilon_1) \end{array}\right)(1+\epsilon_2) = I + \Sigma^{(k+1)}$ holds for $k = 0$. □

19.2 $k > 0$

Proof. For arbitrary $k > 0$,

$$\left(\begin{array}{c|c} I+\Sigma^{(k)} & 0 \\ \hline 0 & (1+\epsilon_1) \end{array}\right)(1+\epsilon_2) \tag{40}$$

$$= \left(\begin{array}{c|c} (I+\Sigma^{(k)})(1+\epsilon_2) & 0 \\ \hline 0 & (1+\epsilon_1)(1+\epsilon_2) \end{array}\right) \tag{41}$$

$$= \left(\begin{array}{c|c} I+\epsilon_2 I+\Sigma^{(k)}+\epsilon_2\Sigma^{(k)} & 0 \\ \hline 0 & 1+\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2 \end{array}\right) \tag{42}$$

$$= \underbrace{\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array}\right)}_{I} + \underbrace{\left(\begin{array}{c|c} \epsilon_2 I+\Sigma^{(k)}+\epsilon_2\Sigma^{(k)} & 0 \\ \hline 0 & \epsilon_1+\epsilon_2+\epsilon_1\epsilon_2 \end{array}\right)}_{\Sigma^{(k+1)}} \tag{43}$$

which tells us that $\left(\begin{array}{c|c} I+\Sigma^{(k)} & 0 \\ \hline 0 & (1+\epsilon_1) \end{array}\right)(1+\epsilon_2) = I + \Sigma^{(k+1)}$ holds for $k > 0$. □

Exercise 23.

23.1

Show that $\check{\kappa} = (x+\delta x)^T y$, where $|\delta x| \leq \gamma_n |x|$.

Proof. Let $\delta x = \Sigma^{(n)} x$, where $\Sigma^{(n)}$ is as in Theorem 20.

$$|\delta x| = |\Sigma^{(n)}x| = \begin{pmatrix} |\theta_n\chi_0| \\ |\theta_n\chi_1| \\ \vdots \\ |\theta_2\chi_{n-1}| \end{pmatrix} \leq \begin{pmatrix} |\theta_n||\chi_0| \\ |\theta_n||\chi_1| \\ \vdots \\ |\theta_2||\chi_{n-1}| \end{pmatrix} \leq |\theta_n| \begin{pmatrix} |\chi_0| \\ |\chi_1| \\ \vdots \\ |\chi_{n-1}| \end{pmatrix} \leq \gamma_n \begin{pmatrix} |\chi_0| \\ |\chi_1| \\ \vdots \\ |\chi_{n-1}| \end{pmatrix} = \gamma_n |x| \tag{44}$$

Thus, it can be concluded for the backward analysis that

$$|\delta x| \leq \gamma_n |x| \tag{45}$$

□

23.2

Show that $\check{\kappa} = x^T(y+\delta y)$, where $|\delta y| \leq \gamma_n |y|$.

Proof. The proof for perturbation on input y is the same as that of perturbation on input x. □

Exercise 25.

Exercise 27.