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CS383C NUMERICAL ANALYSIS

Midterm Exam

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Exercise 1.

(a) Prove that $\|\cdot\|$ is a vector norm.

To prove $\|\cdot\|$ is a vector norm, we have to show all the following three property holds:

- If $x \neq 0$, then $\|x\| > 0$.
- For arbitrary scalar α and vector x , $\|\alpha x\| = |\alpha| \cdot \|x\|$.
- For arbitrary vector x and y , $\|x + y\| \leq \|x\| + \|y\|$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i, \delta_i > 0$ holds. Besides, since $x \neq 0$, then $\exists i, x_i^2 > 0$.

$$\|x\| = \sqrt{x^T D x} = \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} > 0 \quad x \neq 0 \quad (1)$$

Therefore, we proved that if $x \neq 0$, then $\|x\| > 0$. \square

Proof.

$$\|\alpha x\| = \sqrt{(\alpha x)^T D (\alpha x)} = \sqrt{\sum_{i=0}^{m-1} \alpha^2 \delta_i x_i^2} = \sqrt{\alpha^2 \sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \cdot \|x\| \quad (2)$$

\square

Proof.

$$\|x + y\|^2 = (\sqrt{(x + y)^T D (x + y)})^2 = x^T D x + x^T D y + 2x^T D y \quad (3)$$

$$(\|x\| + \|y\|)^2 = (\sqrt{x^T D x})^2 + (\sqrt{y^T D y})^2 = x^T D x + y^T D y + 2\sqrt{x^T D x} \sqrt{y^T D y} \quad (4)$$

Now we show that $x^T D y \leq \sqrt{x^T D x} \sqrt{y^T D y}$, which is equivalent to

$$\left(\sum_{i=0}^{m-1} \delta_i x_i y_i \right)^2 \leq \left(\sum_{i=0}^{m-1} \delta_i x_i^2 \right) \left(\sum_{i=0}^{m-1} \delta_i y_i^2 \right) \quad (5)$$

$$\iff \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \delta_i \delta_j x_i y_i x_j y_j \leq \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \delta_i \delta_j x_i^2 y_j^2 \quad (6)$$

$$\iff \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} \delta_i \delta_j (2x_i y_i x_j y_j) + \sum_{i=0}^{m-1} \delta_i^2 x_i y_i x_i y_i \leq \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} \delta_i \delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) + \sum_{i=0}^{m-1} \delta_i^2 x_i^2 y_i^2 \quad (7)$$

$$\iff \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} \delta_i \delta_j (2x_i y_i x_j y_j) \leq \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} \delta_i \delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) \quad (8)$$

which obviously holds if we apply basic inequality on each (i, j) pair,

$$2x_i y_i x_j y_j = 2(x_i y_j)(x_j y_i) \leq x_i^2 y_j^2 + x_j^2 y_i^2 \quad (9)$$

Since we have proved $x^T D y \leq \sqrt{x^T D x} \sqrt{y^T D y}$, then

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2 \quad (10)$$

which leads to desired triangle inequality since $\|\cdot\| \geq 0$,

$$\|x + y\| \leq \|x\| + \|y\| \quad (11)$$

\square

Since all three property have been proved, it can be concluded that

$$\|\cdot\| \text{ defined by } \|x\| = \sqrt{x^T D x} \text{ is a norm.} \quad (12)$$

(b) Show that $\|\cdot\|$ is a matrix norm.

To prove $\|\cdot\|$ is a matrix norm, we have to show all the following three property holds:

- If $A \neq 0$, then $\|A\| > 0$.
- For arbitrary scalar α and matrix A , $\|\alpha A\| = |\alpha| \cdot \|A\|$.
- For arbitrary matrix A and B , $\|A + B\| \leq \|A\| + \|B\|$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i, \delta_i > 0$ holds. And Since $A \neq 0$, then there is at least one column vector that is not zero vector. Let us assume it is the j -th column, a_j , that is non-zero. Then

$$\|A\| = \|DA\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|DAx\|_2}{\|x\|_2} \geq \frac{\|DAe_j\|_2}{\|e_j\|_2} > 0 \quad (13)$$

Therefore, we proved that if $A \neq 0$, then $\|A\| > 0$. □

Proof.

$$\|\alpha A\| = \|D(\alpha A)\|_2 \quad (14)$$

$$= \|\alpha DA\|_2 \quad (15)$$

$$= |\alpha| \cdot \|DA\|_2 \quad \|\cdot\|_2 \text{ is a norm.} \quad (16)$$

$$= |\alpha| \cdot \|A\| \quad (17)$$

Thus, we proved that $\|\alpha A\| = |\alpha| \cdot \|A\|$. □

Proof.

$$\|A + B\| = \|D(A + B)\|_2 \quad (18)$$

$$= \max_{x \neq 0} \frac{\|DAx + DBx\|_2}{\|x\|_2} \quad (19)$$

$$\leq \max_{x \neq 0} \frac{\|DAx\|_2 + \|DBx\|_2}{\|x\|_2} \quad (20)$$

$$= \max_{x \neq 0} \left(\frac{\|DAx\|_2}{\|x\|_2} + \frac{\|DBx\|_2}{\|x\|_2} \right) \quad (21)$$

$$\leq \max_{x \neq 0} \frac{\|DAx\|_2}{\|x\|_2} + \max_{y \neq 0} \frac{\|DBy\|_2}{\|y\|_2} \quad (22)$$

$$= \|DA\|_2 + \|DB\|_2 = \|A\| + \|B\| \quad (23)$$

Hence, we proved that $\|A + B\| \leq \|A\| + \|B\|$. □

Since all three property have been proved, it can be concluded that

$$\|\cdot\| \text{ defined by } \|A\| = \|DA\|_2 \text{ is a norm.} \quad (24)$$

Exercise 2.**(a) Show that** $(..)^{-1} = (..)$.*Proof.*

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array} \right) \left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array} \right) = \left(\begin{array}{c|c} L_{00} L_{00}^{-1} & 0 \\ \hline l_{10}^T L_{00}^{-1} - \frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} \cdot \lambda_{11} & \frac{1}{\lambda_{11}} \cdot \lambda_{11} \end{array} \right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array} \right) = I \quad (25)$$

Hence, it is concluded that (assume all inversion exists)

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array} \right)^{-1} = \left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array} \right) \quad (26)$$

□

(b) Show that $H_1 H_0 = ..$ According to given formula for H_i , we have

$$H_0 = I - \frac{1}{\tau_0} u_0 u_0^T \quad (27)$$

$$H_1 = I - \frac{1}{\tau_1} u_1 u_1^T \quad (28)$$

Now we manipulate $H_1 H_0$,

$$H_1 H_0 = \left(I - \frac{1}{\tau_1} u_1 u_1^T \right) \left(I - \frac{1}{\tau_0} u_0 u_0^T \right) \quad (29)$$

$$= I - \frac{1}{\tau_1} u_1 u_1^T - \frac{1}{\tau_0} u_0 u_0^T + \frac{1}{\tau_0 \tau_1} u_1 u_1^T u_0 u_0^T \quad (30)$$

$$= I - (u_0 | u_1) \left(\begin{array}{c|c} \frac{1}{\tau_0} & 0 \\ \hline -\frac{u_0^T u_1}{\tau_0 \tau_1} & \frac{1}{\tau_1} \end{array} \right) (u_0 | u_1)^T \quad (31)$$

$$= I - (u_0 | u_1) \left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array} \right)^{-1} (u_0 | u_1)^T \quad (32)$$

Hence, it is concluded that (assume all inversion exists)

$$H_1 H_0 = I - (u_0 | u_1) \left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array} \right)^{-1} (u_0 | u_1)^T \quad (33)$$

(c) Show that $(1 - \frac{1}{\tau_1 u_1 u_1^T}).. = I - ...$

$$\left(I - \frac{1}{\tau_1} u_1 u_1^T \right) \left(I - U_0 L_{00}^T U_0^T \right) = I - \frac{1}{\tau_1} u_1 u_1^T - U_0 L_{00}^T U_0^T + \frac{1}{\tau_1} u_1 u_1^T U_0 L_{00}^T U_0^T \quad (34)$$

$$= I - (U_0 | u_1) \left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -u_1^T U_0 L_{00}^{-1} / \tau_1 & \frac{1}{\tau_1} \end{array} \right) (U_0 | u_1)^T \quad (35)$$

$$= I - (U_0 | u_1) \left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array} \right)^{-1} (U_0 | u_1)^T \quad (36)$$

Thus, it is concluded that (assume all inversion exists)

$$\left(I - \frac{1}{\tau_1} u_1 u_1^T \right) \left(I - U_0 L_{00}^T U_0^T \right) = I - (U_0 | u_1) \left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array} \right)^{-1} (U_0 | u_1)^T \quad (37)$$

(d) **Show that** $H_{k-1} \cdots H_1 H_0 = I - UL^{-1}U^H$

We can do mathematical induction on m . We separate the discussion of $k = 1$ and $k > 1$.

- Particular Case: $k = 1$,

Proof.

$$H_0 = I - \frac{1}{\tau_0} u_0 u_0^T = I - \underbrace{(u_0)}_{U_{(0)}} \underbrace{\left(\frac{1}{\tau_0}\right)}_{L_{(0)}^{-1}} \underbrace{(u_0)^T}_{U_{(0)}^T} \quad (38)$$

Hence, $H_{k-1} \cdots H_1 H_0 = I - UL^{-1}U^H$ holds for $k = 1$. \square

Note that this case is particular because it derives matrix which is indeed scalar.

- Base Case: $k = 2$, this can be proved by using result of (b).
- Inductive Case: assume that (I.H.)

$$H_{k-1} \cdots H_1 H_0 = I - U_{(k-1)} L_{(k-1)}^{-1} U_{(k-1)}^H \quad (39)$$

show that

$$H_k H_{k-1} \cdots H_1 H_0 = I - U_{(k)} L_{(k)}^{-1} U_{(k)}^H \quad (40)$$

holds for lower triangular matrix $L_{(k)}$.

Proof. With application of the result of (c), we have

$$H_k(H_{k-1} H_{k-2} \cdots H_1 H_0) = \left(1 - \frac{1}{\tau_k} u_k u_k^T\right) (I - U_{(k-1)} L_{(k-1)}^{-1} U_{(k-1)}^H) \quad (41)$$

$$= I - \underbrace{(U_{(k-1)} | u_k)}_{U_{(k)}} \underbrace{\left(\begin{array}{c|c} L_{(k-1)} & 0 \\ \hline u_k^T U_{(k-1)} & \tau_k \end{array} \right)^{-1}}_{L_{(k)}^{-1}} \underbrace{(U_{(k-1)} | u_k)^T}_{U_{(k)}^T} \quad (42)$$

$$= I - U_{(k)} L_{(k)}^{-1} U_{(k)}^T \quad (43)$$

\square

- By the principle of mathematical induction, it can be concluded that

$$H_{k-1} \cdots H_1 H_0 = I - UL^{-1}U^H \quad (44)$$

holds for all $k > 0$.

Exercise 3.**(a) Relationship between m , n and r .**

$$\text{rank}(A) = r \leq \min(m, n) \quad (45)$$

(b) Reduced SVD in terms of U , V and Σ .

- $A = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}$ and $V_r = V_L$.

$$A = U \Sigma V^T = (U_L | U_R) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) (V_L | V_R)^T = U_L \Sigma_{TL} V_L^T \quad (46)$$

- $(A^T A)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1} \Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$(A^T A)^{-1} = ((U \Sigma V^T)^T U \Sigma V^T)^{-1} \quad (47)$$

$$= (V \Sigma^T U^T U \Sigma V^T)^{-1} \quad (48)$$

$$= (V \Sigma \Sigma V^T)^{-1} \quad (49)$$

$$= \left((V_L | V_R) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) (V_L | V_R)^T \right)^{-1} \quad (50)$$

$$= (V_L \Sigma_{TL} \Sigma_{TL}^T V_L^T)^{-1} \quad (51)$$

$$= V_L^{-T} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} V_L^{-1} \quad (52)$$

- $(A^T A)^{-1} A^T = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1}$ and $V_r = U_L$.

$$(A^T A)^{-1} A^T = V_L^{-T} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} V_L^{-1} V_L \Sigma_{TL}^T U_L^T \quad (53)$$

$$= V_L^{-T} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} \Sigma_{TL}^T U_L^T \quad (54)$$

$$= V_L^{-T} \Sigma_{TL}^{-1} U_L^T \quad (55)$$

- $A(A^T A)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}^{-1} \Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$A(A^T A)^{-1} = U_L \Sigma_{TL} V_L^T V_L^{-T} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} V_L^{-1} \quad (56)$$

$$= U_L \Sigma_{TL} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} V_L^{-1} \quad (57)$$

$$= U_L \Sigma_{TL}^{-1} V_L^{-1} \quad (58)$$

- $A(A^T A)^{-1} A^T = U_r \Sigma_r V_r^T$, where $U_r = I^{m \times r}$, $\Sigma_r = I^{r \times r}$ and $V_r = I^{n \times r}$.

$$A(A^T A)^{-1} A^T = (U_L \Sigma_{TL}^{-1} V_L^{-1}) V_L \Sigma_{TL}^T U_L^T \quad (59)$$

$$= U_L \Sigma_{TL}^{-1} \Sigma_{TL}^T U_L^T \quad (60)$$

$$= U_L U_L^T \quad (61)$$

$$= I \quad (62)$$

Exercise 4.**(a)** $\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1$ **where** δv_{11} **is small.***Proof.*

$$\psi_1 = [v_{11}\check{\chi}_1]_* = \frac{v_{11}\check{\chi}_1}{1 + \epsilon_*} = (1 - \frac{\epsilon_*}{1 + \epsilon_*})v_{11}\check{\chi}_1 = (v_{11} - \frac{\epsilon_* v_{11}}{1 + \epsilon_*})\check{\chi}_1 \equiv (v_{11} + \delta v_{11})\check{\chi}_1 \quad (63)$$

where

$$\delta v_{11} = -\frac{\epsilon_* v_{11}}{1 + \epsilon_*}, \text{ where } |\epsilon_*| \leq \mathbf{u} \quad (64)$$

Then

$$|\delta v_{11}| = |-\frac{\epsilon_*}{1 + \epsilon_*}v_{11}| \quad (65)$$

$$= \frac{|\epsilon_*|}{|1 + \epsilon_*|}|v_{11}| \quad (66)$$

$$\leq \frac{\mathbf{u}}{|1 + \epsilon_*|}|v_{11}| \quad (67)$$

$$\leq \frac{\mathbf{u}}{1 + |\epsilon_*|}|v_{11}| \quad (68)$$

$$\leq \frac{\mathbf{u}}{1 + |\epsilon_*| - |\epsilon_*| - \mathbf{u}}|v_{11}| \quad (69)$$

$$= \frac{\mathbf{u}}{1 - \mathbf{u}}|v_{11}| = \gamma_1|v_{11}| \quad (70)$$

Hence, we proved that

$$\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1 \text{ where } |\delta v_{11}| = \gamma_1|v_{11}| \quad (71)$$

□

(b)*Proof.* Start from dot product. We directly use the corollary in notes.

$$d \triangleq [u_{12}^T \check{x}_2] = (u_{12}^T + \delta u_{12}^T)\check{x}_2, \text{ where } \delta u_{12}^T \leq \gamma_k|u_{12}^T| \leq \gamma_{k+1}|u_{12}^T| \quad (72)$$

Then we consider the subtraction step with Standard Computational Model (SCM).

$$s \triangleq [\psi_1 - d] = (\psi_1 - d)(1 + \epsilon_-) = (\psi_1 - d) + (\psi_1 - d)\epsilon_- \equiv (\psi_1 - d) + \delta\psi_1 = (\psi_1 + \delta\psi_1) - d \quad (73)$$

where $|\delta\psi_1| = |(\psi_1 - d)\epsilon_-| \leq |\psi_1\epsilon_-| = |\epsilon_-||\psi_1| \leq \mathbf{u}|\psi_1|$. Then we consider the division step with Alternative Computational Model (ACM).

$$\check{\chi}_1 = [\frac{s}{v_{11}}] = \frac{s}{v_{11}}(1 + \epsilon_+)^{-1} = \frac{s}{v_{11}(1 + \epsilon_+)} \quad (74)$$

which can be rewritten as

$$(v_{11} + \delta v_{11})\check{\chi}_1 = s \quad (75)$$

where $|\delta v_{11}| = |\epsilon_+ v_{11}| \leq |\epsilon_+| \cdot |v_{11}| \leq \mathbf{u}|v_{11}|$. Then we have

$$(v_{11} + \delta v_{11})\check{\chi}_1 = (\psi_1 + \delta\psi_1) - d = (\psi_1 + \delta\psi_1) - (u_{12}^T + \delta u_{12}^T)\check{x}_2 \quad (76)$$

where

$$|\delta v_{11}| \leq \mathbf{u}|v_{11}|, \quad |\delta\psi_1| \leq \mathbf{u}|\psi_1|, \quad |\delta u_{12}^T| \leq \gamma_{k+1}|u_{12}^T| \quad (77)$$

□

(c) Induction for the desired result

To prove the desired result, let us do mathematical induction on k (the iteration of algorithm).

- Base Case: $k = 0$

Proof. In this case, $U^{0 \times 0}$ does not participate in computation, v_{11} is the right-bottom-most element, and no pre-computed \tilde{x}_2 . Hence, it is natural that

$$(v_{11} + \delta v_{11})\tilde{\chi}_1 = (\psi_1 + \delta\psi_1) \quad (78)$$

which indicates that $(U + \Delta U)\tilde{x} = (y + \delta y)$ holds for $k = 0$. (first iteration) \square

- Inductive case: Assume that at p -th iteration ($p > 0$), pre-computed $\tilde{x}_{2(p)}$ satisfy

$$(U_{(p)} + \Delta U_{(p)})\tilde{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \quad (79)$$

where $|\Delta U_{(p)}| \leq \gamma_n |U_{(p)}|$ and $|\delta y_{2(p)}| \leq \mathbf{u} |y_{2(p)}|$, (I.H.) holds.

Show that for the pre-computed \tilde{x}_2 at next iteration $\tilde{x}_{2(p+1)} = \left(\frac{\tilde{\chi}_{1(p)}}{\tilde{x}_{2(p)}} \right)$ and $y_{2(p+1)} = \left(\frac{\psi_{1(p)}}{y_{2(p)}} \right)$

$$(U_{(p+1)} + \Delta U_{(p+1)})\tilde{x}_{2(p+1)} = (y_{(p+1)} + \Delta y_{(p+1)}) \quad (80)$$

where $|\Delta U_{(p+1)}| \leq \gamma_n |U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u} |y_{2(p+1)}|$, holds for some upper triangle $U_{(p+1)}$.

Proof. In terms of result of (b), we have

$$(v_{11(p)} + \delta v_{11(p)})\tilde{\chi}_{1(p)} + (u_{12(p)}^T + \delta u_{12(p)}^T)\tilde{x}_{2(p)} = (\psi_{1(p)} + \delta\psi_{1(p)}) \quad (81)$$

where $\delta v_{11(p)} \leq \mathbf{u} |v_{11(p)}|$, $\delta\psi_{1(p)} \leq \mathbf{u} |\psi_{1(p)}|$, and $\delta u_{12(p)}^T \leq \gamma_{p+1} |u_{12(p)}^T|$.
Combined with (I.H.)

$$(U_{(p)} + \Delta U_{(p)})\tilde{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \quad (82)$$

we have

$$\left(\frac{v_{11(p)} + \delta v_{11(p)}}{0} \middle| \frac{u_{12(p)}^T + \delta u_{12(p)}^T}{U_{(p)} + \Delta U_{(p)}} \right) \left(\frac{\tilde{\chi}_{1(p)}}{\tilde{x}_{2(p)}} \right) = \left(\frac{\psi_{1(p)} + \delta\psi_{1(p)}}{y_{2(p)} + \Delta y_{2(p)}} \right) \quad (83)$$

where $|\Delta U_{(p)}| \leq \gamma_n |U_{(p)}|$, $\delta v_{11(p)} \leq \mathbf{u} |v_{11(p)}|$, $\delta u_{12(p)}^T \leq \gamma_{p+1} |u_{12(p)}^T|$

and $\delta\psi_{1(p)} \leq \mathbf{u} |\psi_{1(p)}|$, $|\delta y_{2(p)}| \leq \mathbf{u} |y_{2(p)}|$.

which is equivalent to

$$\left(\underbrace{\left(\frac{v_{11(p)}}{0} \middle| \frac{u_{12(p)}^T}{U_{(p)}} \right)}_{U_{(p+1)}} + \underbrace{\left(\frac{\delta v_{11(p)}}{0} \middle| \frac{\delta u_{12(p)}^T}{\Delta U_{(p)}} \right)}_{\Delta U_{(p+1)}} \right) \underbrace{\left(\frac{\tilde{\chi}_{1(p)}}{\tilde{x}_{2(p)}} \right)}_{\tilde{x}_{2(p+1)}} = \underbrace{\left(\frac{\psi_{1(p)}}{y_{2(p)}} \right)}_{y_{2(p+1)}} + \underbrace{\left(\frac{\delta\psi_{1(p)}}{\Delta y_{2(p)}} \right)}_{\delta y_{2(p+1)}} \quad (84)$$

$$(U_{(p+1)} + \Delta U_{(p+1)})\tilde{x}_{2(p+1)} = (y_{2(p+1)} + \delta y_{2(p+1)}) \quad (85)$$

where $|\Delta U_{(p+1)}| \leq \gamma_n |U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u} |y_{2(p+1)}|$.

Hence, we successfully proved (80). \square

- By principle of mathematical induction, it is proved w.l.o.g. that

$$(U + \Delta U)\tilde{x} = (y + \delta y), \text{ where } |\Delta U| \leq \gamma_n |U| \text{ and } |\delta y| \leq \mathbf{u} |y| \quad (86)$$