The University of Texas
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CS331 Algorithm

Assignment 04

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1 Exercise 1

Exercise 1. Let $G = (U, V, E)$ be a configuration. Prove that if p is a stable price vector for G and v is an item in V,then p_v is at least the start price of v.

Overview. We provide the proof by contradiction. We first assume that there exist a item v for which, the p_v can be less than the start price of item v. That is,

$$\exists v \in V, p_v < p_v^* \tag{1}$$

where p_v^* is the start price of item v. And there are two cases we should consider as follows:

- for item v, there is no unit-demand bid wins it
- for item v, there is at least one unit-demand bid wins it

We provide dicussion over two cases above, and then show the contradiction in each case. □

Notational explanation:

- p_v^R is the reserve price for item v
- p_v^* is the start price for item v
- $w(u, v)$ is the unit-demand bid u 's bidding price to item v

No unit-demand bid wins it. Since unit-demand bid u does not win item v, we have $w(u, v) \leq p_v^R$ There are two subcases when if there is no unit-demand bid wins it.

- $\neg\exists(u, v),\ w(u, v) \geq p_v^R$, in this case, the $p_v = p_v^*$, contradicts to (1)
- $\exists(u, v),\ w(u, v) \geq p_v^R$, $p_v = max(w(u, v), w(u', v'), ..) \geq p_v^*$, contradicts to (1)

□

At least one unit-demand bid wins it. Only consider the first unit-demand bid u winning item v. Since u wins the item v, we have

$$w(u, v) \geq p_v^R \geq p_v^* \tag{2}$$

At this moment, the $p_v = p_v^R \geq p_v^*$. And the later unit-demand bid u' which wins item v must satisfy

$$w(u', v) \geq w(u, v) \tag{3}$$

And at that time, the p_v in price vector will be updated to be $w(u, v)$, which is larger than previous price. Hence, we can conclude that

$$p_v \geq p_v^* \tag{4}$$

This contradicts to (1). An formal alternative to prove this is to have induction on the number of unit-demand bid ever winning item v and show a persistent contradiction. But here we just provide brief proof. □

Since in all possibilities indicates a contradiction to (1), we can negate the initial assumption to have

$$\neg\exists v \in V, p_v < p_v^* \tag{5}$$

That is to conclude,

$$\forall v \in V, p_v \geq p_v^* \tag{6}$$

2 Exercise 2

Exercise 2. Let $G = (U, V, E)$ be a configuration, let (M, p) and (M', p) be stable solutions for G, let (u, v) be an edge in M, and assume that $w(u, v) > p_v$. Prove that u is matched to some item v' in M', and $w(u, v) - p_v = w(u, v') - p'_v$.

Since (M, p) is stable solution, and $(u, v) \in M$, we can instantiate the second property of stable solution and have

$$w(u, v) - p_v \geq w(u, v') - p_{v'} \tag{7}$$

Since we assume that $w(u, v) > p_v$, it can be avoided that two bids have the same bidding highest price for item v (p_v is the second highest bidding price). Hence, the other MWMCM is not made by the instability as the same highest bidding price.

Similarly for stable solution (M', p), we have

$$w(u, v') - p_{v'} \geq w(u, v) - p_v \tag{8}$$

Therefore, we have

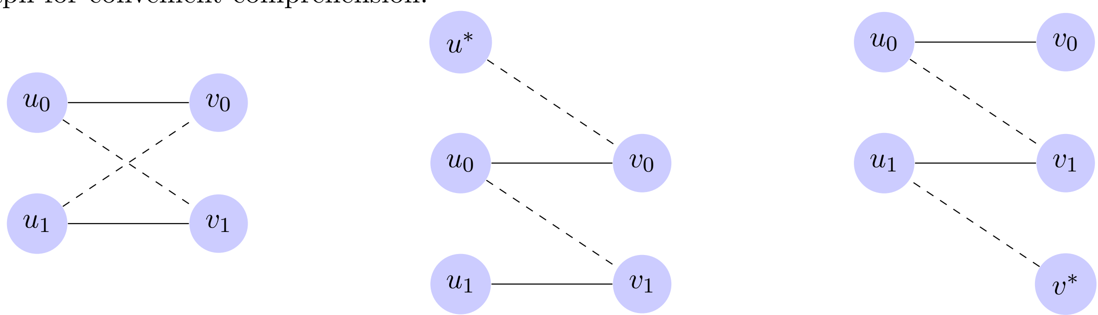
$$w(u, v) - p_v = w(u, v') - p_{v'} \tag{9}$$

3 Exercise 3

Exercise 3. Let $G = (U, V, E)$ be a configuration. Prove that if (M, p) is a stable solution for G, then M is an MWMCM of G. Hint: Let M be an MWMCM of G, and make use of the fact that the bipartite graph $G = (U, V, M \oplus M)$ consists of a collection of disjoint paths and cycles, where each path and cycle in the collection is of even length.

Overview. Let M' be an MWMCM of G, and it is easy to see the fact that the bipartite graph $G = (U, V, M)$ consists of a collection of disjoint paths and cycles, where each path and cycle in the collection is of even length. Hence, there are **three cases** we need to consider for G', **(1) cycle (2) path with unit-demand bid as endpoint (3) path with item as endpoint**. The intuitive idea of our proof is that in all cases mentioned above, the total weight of edges in stable solution M is no less than the total weight of edges in the MWMCM M' and since M' is MWMCM (total weight should be no less than any MCM), the only possible scenario is that M is also a MWMCM. □

Before diving into the detailed proof for all three cases, we first domonstrate you the representative graph for convenient comprehension.



Note that the case shown above from left to right are (1), (2) and (3) respectively. And for every edge shown above, if it is full line, then it is in the stable solution MCM M, while the dashed line represents pairs in MWMCM M'.

Next, we are going to provide proof for three general cases. For specific notation reference, you can refer to the representative graph.

Cycle. The proof is simple in this case. Since every bid node is in a cycle, we have that in G', each bid node has at most two edges connecting to it, one is in M and the other in M'. The same property also applies to the item node. Otherwise, it cannot form a cycle. According to the second property of stable solution, we have

$$\forall u, (u, v) \in M,\ \ (u, v') \in M', s.t.\ w(u, v) - p_v \geq w(u, v') - p_{v'} \tag{10}$$

Sum up all inequalities, and then we will have all item price removed on both side of the resulted inequality,

$$\sum_u (w(u, v) - p_v) \geq \sum_u (w(u, v') - p_{v'}) \tag{11}$$

In one cycle, it can be easily observed that

$$\sum_u p_v = \sum_u p_{v'} \tag{12}$$

Note that in one cycle, the set of bid nodes can only be matched to the same set of item nodes, regardless of the MCM we talks about. By (11) and (12), we have

$$\sum_u w(u, v) \geq \sum_u w(u, v') \tag{13}$$

Thus, in one cycle subgraph, the total weight of edges in M is no less than the total weight of edges in M'.

$\square$

Unit-demand bid as endpoint. In case of a path with unit-demand bid as its endpoint, things are a little more complicated. Since u^* is unmatched in stable solution MCM M and according to the third property of stable solution , we have

$$\forall v,\ w(u^*, v) \le p_v \tag{14}$$

Instantiate it, we have

$$-w(u^*, v_1) \ge -p_{v_1} \tag{15}$$

For every immediate unit-demand bid u, it is easy to see that they all have one edge in M and one in M'. In terms of the second property of stable solution,

$$\forall u',\ degree(u') = 2,\ (u', v') \in M,\ (u', v'') \in M',\ s.t. \tag{16}$$

$$w(u', v') - p_{v'} \ge w(u', v'') - p_{v''} \tag{17}$$

For the other end edge (u_n, v_n) of this path, we can have

$$p_{v_n} \le w(u_n, v_n) \tag{18}$$

$$-p_{v_n} \ge -w(u_n, v_n) \tag{19}$$

Sum up (16), we have

$$\sum_{u'} (w(u', v') - p_{v'}) \ge \sum_{u'} (w(u', v'') - p_{v''}) \tag{20}$$

$$\sum_{u'} (w(u', v')) - p_{v_1} \ge \sum_{u'} (w(u', v'')) - p_{v_n} \tag{21}$$

According to the (15), we have

$$\sum_{u'} (w(u', v')) - w(u^*, v_1) \ge \sum_{u'} (w(u', v')) - p_{v_1} \tag{22}$$

According to the (19), we have

$$\sum_{u'} (w(u', v'')) - p_{v_n} \ge \sum_{u'} (w(u', v'')) - w(u_n, v_n) \tag{23}$$

Based on (21), (22) and (23), we have

$$\sum_{u'} (w(u', v')) - w(u^*, v_1) \ge \sum_{u'} (w(u', v'')) - w(u_n, v_n) \tag{24}$$

$$\sum_{u'} (w(u', v')) + w(u_n, v_n) \ge \sum_{u'} (w(u', v'')) + w(u^*, v_1) \tag{25}$$

Since (u_n, v_n) is in stable solution MCM M, and (u^*, v_1) is in MWMCM M', we can conclude that in one path with unit-demand bid as endpoint, the total weight of edges in M is no less than the total weight of edges in M'

$\square$

Item as endpoint. This case is more complicated than previous one. Since if one item is not matched by any unit-demand bid, it will be matched by dummy bid, say reserve bid.

Similarly to previous, in graph G', the unit-demand bid has the following property.

$$\begin{aligned} \forall u',\ degree(u') = 2,\ (u', v') \in M,\ (u', v'') \in M',\ s.t. \\ w(u', v') - p_{v'} \geq w(u', v'') - p_{v''} \end{aligned} \tag{26}$$

Sum up all inequalities, we have

$$\sum_{u'}(w(u', v')) - p_{v_0} \geq \sum_{u'}(w(u', v'')) - p_{v^*} \tag{27}$$

$$\sum_{u'}(w(u', v')) + p_{v^*} \geq \sum_{u'}(w(u', v'')) + p_{v_0} \tag{28}$$

Since the p_{v_0} is the reserve price of item v_0, which is counted in MWMCM, and p_{v^*} is the start price of item v^*, which is counted in stable solution, it can be concluded that the total weight of edges in M is no less than the total weight of edges in M'.

□

Epilogue. Since G' consists of a collection of cycles and path, and we have proven that the total weight of edges in M is no less than the total weight of edges in M' for all cases, we can conclude that

$$w(M) \geq w(M') \tag{29}$$

Since M' is MWMCM, we have

$$w(M) \leq w(M') \tag{30}$$

Hence, the only possibility can be

$$w(M) = w(M') \tag{31}$$

Therefore, the MCM in stable solution (M, p) is also MWMCM. □

4 Exercise 4

Exercise 4. Let G be a configuration, let M be an MWMCM of G, and let p be a price vector for G. Prove that digraph (G, M, p) does not contain a directed cycle of positive weight.

Proof by Contradiction. Let us assume that $digraph(G, M, p)$ does contain a directed cycle of positive weight. Since in the cycle C, all nodes are matched in G, we only need to refer to the first and second rule of provided G' formation. And here we represent the cycle C which has positive weight, by indexing involved bid set $u_0, .., u_n$, and item set $v_0, .., v_n$, such that

$$\forall i \in [0, n], (u_i, v_i) \in M$$

According to the edge formation of E' of $G'(V', E')$, in cycle C, each node must have degree of size 2, one incoming edge and one outgoing edge. For each item node indexed by i, the outgoing edges can be explained mathematically as follows,

$$(u_i, v_i) \in M \tag{32}$$

For each bid node indexed by i, whose outgoing edge is (u_i, v_j) $(j \neq j)$, we can treat it as

$$w(u_i, v_i) - p_{v_i} \leq w(u_i, v_j) - p_{v_j} \tag{33}$$

Since all the corresponding edges in (32) on E' has weight 0, there must be **at least one inequality in (33) having non-equal relationship. Otherwise, the cycle C does have positive weight.**

By summing up all inequalities in (33) and cancel out all terms about certain component of price vector, we have

$$\sum_i \left(w(u_i, v_i)\right) < \sum_i \left(w(u_i, v_j)\right) \tag{34}$$

The derived inequality indicates that for each u_i, the matching $(u_i, v_j) \in M$ rather than (u_i, v_j). Apparently, this contradicts to our encoding scheme of MWMCM M at the very beginning.

Thus, there does no exist a cycle of positive weight in digraph (G, M, p) $\square$