The University of Texas
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CS383C Numerical Analysis

Homework 03

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Exercise 2. Show that if H is a reflector, then

2.1 $HH = I$

Since H is a reflector, we have

$$H = I - 2uu^H \tag{1}$$

where u is unit vector ($||u||_2^2 = 1$).

$$\begin{aligned} HH &= (I - 2uu^H)(I - 2uu^H) \\ &= I \cdot I - 2uu^H - 2uu^H + 4(uu^H)(uu^H) \\ &= I - 4uu^H + 4||u||_2^2 uu^H \\ &= I - 4uu^H + 4uu^H \\ &= I \end{aligned} \tag{2}$$

Lemma 1. *For arbitrary vector $a \in \mathbb{C}^n$, $b \in \mathbb{C}^n$, $c \in \mathbb{C}^n$, $ab^H c = (b^H c)a$.*

Proof.

$$ab^H c = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} b^H c = \begin{pmatrix} a_0 b^H \\ a_1 b^H \\ \vdots \\ a_{n-1} b^H \end{pmatrix} c = \begin{pmatrix} a_0 b^H c \\ a_1 b^H c \\ \vdots \\ a_{n-1} b^H c \end{pmatrix} = \begin{pmatrix} (b^H c)a_0 \\ (b^H c)a_1 \\ \vdots \\ (b^H c)a_{n-1} \end{pmatrix} = (b^H c) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \tag{3}$$

$$= (b^H c)a \tag{4}$$

□

Lemma 2. $(uu^H)(uu^H) = ||u||_2^2 uu^H$

Proof.

$$\begin{aligned} (uu^H)(uu^H) &= (uu^H u)u^H \\ &= ((u^H u)u)u^H \\ &= (||u||_2^2 u)u^H \\ &= ||u||_2^2 uu^H \end{aligned} \tag{5}$$

□

2.2 $H = H^H$

$$\begin{aligned} H^H &= (I - 2uu^H)^H \\ &= I^H - (2uu^H)^H \\ &= I - 2(u^H)^H u^H \\ &= I - 2uu^H \\ &= H \end{aligned} \tag{6}$$

2.3 $HH^H = I$

In terms of (6), multiply both sides with H and then reuse conclusion in (2)

$$H^H H = HH = I \tag{7}$$

Exercise 4. Show that if $x \in \mathbb{R}^n$, $v = x \mp ||x||_2 e_0$ and $\tau = v^T v/2$, then $(I - \frac{1}{\tau} vv^T)x = \pm||x||_2 e_0$

We start from the reflector $I - \frac{1}{\tau}vv^T$ on x,

$$
\begin{aligned}
& (I - \frac{1}{\tau}vv^T)x \\
= & \left(I - \frac{2vv^T}{v^Tv}\right)x \\
= & \left(I - \frac{2(x \mp ||x||_2e_0)(x \mp ||x||_2e_0)^T}{(x \mp ||x||_2e_0)^T(x \mp ||x||_2e_0)}\right)x \\
= & \left(I - 2\frac{xx^T \mp ||x||_2(xe_0^T + e_0x^T) + ||x||_2^2e_0e_0^T}{2||x||_2^2 \mp 2||x||_2e_0^Tx}\right)x \\
= & \left(I - \frac{xx^T \mp ||x||_2(xe_0^T + e_0x^T) + ||x||_2^2e_0e_0^T}{||x||_2^2 \mp ||x||_2e_0^Tx}\right)x \\
= & \frac{||x||_2^2 \mp ||x||_2e_0^Tx - xx^T \pm ||x||_2(xe_0^T + e_0x^T) - ||x||_2^2e_0e_0^T}{||x||_2^2 \mp ||x||_2e_0^Tx}x \\
= & \frac{||x||_2^2x \mp ||x||_2e_0^Txx - xx^Tx \pm ||x||_2(xe_0^T + e_0x^T)x - ||x||_2^2e_0e_0^Tx}{||x||_2^2 \mp ||x||_2e_0^Tx} \\
= & \frac{(||x||_2^2x - xx^Tx) + (\mp||x||_2e_0^Txx \pm ||x||_2xe_0^Tx) \pm ||x||_2e_0x^Tx - ||x||_2^2e_0e_0^Tx}{||x||_2^2 \mp ||x||_2e_0^Tx} \\
= & \frac{\pm||x||_2e_0x^Tx - ||x||_2^2e_0e_0^Tx}{||x||_2^2 \mp ||x||_2e_0^Tx} \\
= & \frac{\pm e_0x^Tx - ||x||_2e_0e_0^Tx}{||x||_2 \mp e_0^Tx} \\
= & \frac{\pm||x||_2^2e_0 - ||x||_2(e_0^Tx)e_0}{||x||_2 \mp e_0^Tx} \\
= & \frac{\pm(||x||_2 - (e_0^Tx))||x||_2e_0}{||x||_2 \mp e_0^Tx} \\
= & \pm||x||_2e_0
\end{aligned} \tag{8}
$$

Note that above derivation frequently makes use of the above lemma 1 in real case.

Exercise 5. Complex

It is easy to show that the conclusion in (8) can extend to complex space. That is,

$$(I - \frac{1}{\tau}vv^H)x = \oplus||x||_2 e_0 \tag{9}$$

where $x \in \mathbb{C}^n,\ v = x \oplus ||x||_2 e_0$ and $\tau = v^H v/2$.
Let

$$v = \begin{pmatrix} 1 \\ \hline u_2 \end{pmatrix},\ x = \begin{pmatrix} \chi \\ \hline x_2 \end{pmatrix},\ e = \begin{pmatrix} 1 \\ \hline 0 \end{pmatrix} \tag{10}$$

Then have

$$\left(I - \frac{1}{\tau}\begin{pmatrix} 1 \\ \hline u_2 \end{pmatrix}\begin{pmatrix} 1 \\ \hline u_2 \end{pmatrix}^H\right) \cdot \begin{pmatrix} \chi \\ \hline x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ \hline 0 \end{pmatrix} \tag{11}$$

where $\tau = v^H v/2 = (1 + u_2^H u_2)/2$ and $\rho = \oplus||x||_2$.

Lemma 3. *If $x \in \mathbb{C}^n,\ v = x \oplus ||x||_2 e_0$ and $\tau = v^H v/2$, then $(I - \frac{1}{\tau}vv^H)x = \oplus||x||_2 e_0$.*

Proof. We start from the reflector $I - \frac{1}{\tau}vv^H$ on x,

$$\begin{aligned}
& (I - \frac{1}{\tau}vv^H)x \\
= & \left(I - \frac{2vv^H}{v^H v}\right)x \\
= & \left(I - \frac{2(x \oplus ||x||_2 e_0)(x \oplus ||x||_2 e_0)^H}{(x \oplus ||x||_2 e_0)^H (x \oplus ||x||_2 e_0)}\right)x \\
= & \left(I - 2\frac{xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) + ||x||_2^2 e_0 e_0^H}{2||x||_2^2 \oplus 2||x||_2 e_0^H x}\right)x \\
= & \left(I - \frac{xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) + ||x||_2^2 e_0 e_0^H}{||x||_2^2 \oplus ||x||_2 e_0^H x}\right)x \\
= & \frac{||x||_2^2 \oplus ||x||_2 e_0^H x - xx^H \oplus ||x||_2(xe_0^H + e_0 x^H) - ||x||_2^2 e_0 e_0^H}{||x||_2^2 \oplus ||x||_2 e_0^H x}x \\
= & \frac{||x||_2^2 x \oplus ||x||_2 e_0^H xx - xx^H x \oplus ||x||_2(xe_0^H + e_0 x^H)x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{(||x||_2^2 x - xx^H x) + (\oplus||x||_2 e_0^H xx \oplus ||x||_2 xe_0^H x) \oplus ||x||_2 e_0 x^H x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{\oplus||x||_2 e_0 x^H x - ||x||_2^2 e_0 e_0^H x}{||x||_2^2 \oplus ||x||_2 e_0^H x} \\
= & \frac{\oplus e_0 x^H x - ||x||_2 e_0 e_0^H x}{||x||_2 \oplus e_0^H x} \\
= & \frac{\oplus||x||_2^2 e_0 - ||x||_2(e_0^H x)e_0}{||x||_2 \oplus e_0^H x} \\
= & \frac{\oplus(||x||_2 - (e_0^H x))||x||_2 e_0}{||x||_2 \oplus e_0^H x} \\
= & \oplus||x||_2 e_0
\end{aligned} \tag{12}$$

□

Exercise 6. Matrix Equivalence

We start from right hand side

$$
\begin{aligned}
RHS &= I - \frac{1}{\tau_1}\begin{pmatrix} 0 \\ \hline 1 \\ \hline u_2 \end{pmatrix}\begin{pmatrix} 0 \\ \hline 1 \\ \hline u_2 \end{pmatrix}^H \\
&= \left(I - \frac{1}{\tau_1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{pmatrix} 1 \\ u_2 \end{pmatrix}\begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \right) \\
&= \left(I - \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{\tau_1}\begin{pmatrix} 1 \\ u_2 \end{pmatrix}\begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \right) \\
&= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I - \frac{1}{\tau_1}\begin{pmatrix} 1 \\ u_2 \end{pmatrix}\begin{pmatrix} 1 \\ u_2 \end{pmatrix}^H \end{array} \right) \\
&= LHS
\end{aligned} \tag{13}
$$

Exercise 11. Expensive Algorithm

As indicatd by **Theorem 10** in the note, we have cost of the algorithm in Figure 6 for $A \in \mathbb{C}^{m\times n}$

$$C_{FormQ}(m,n) = 2mn^2 - \frac{2}{3}n^3 \tag{14}$$

For $m = n$, the cost can be simplified as

$$C_{FormQ}(A) = \frac{4}{3}n^3 = \mathcal{O}(n^3) \tag{15}$$

However, if we accumulate Q by using n householder transformation with

$$Q = (\ldots((IH_0)H_1)\ldots H_{n-1}) \tag{16}$$

Then the cost we have is at least

$$C_{accumulation}(A) = n^3 \cdot (n-1) = \mathcal{O}(n^4) \tag{17}$$

where n^3 comes from each one matrix multiplication, and $n-1$ comes from the total number of householder matrix H_i ($i \in \mathbb{Z}, i \in [0, n-1]$).
Comparing formula (15) and (17), it is obvious that the accumulation method is much more expensive than algorithm in Figure 6.