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CS383C NUMERICAL ANALYSIS

Homework 04

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Part I

Exercises on Solving LLS Problems

Exercise 2.

We can prove the existence of LQ factorization for $A_{m \times n} (m < n)$ by induction on m .

- Base Case: $m = 1$

$$A = (a_0^H) = l_{00}q_0^H = \left(\begin{array}{c|c} l_{00} & 0 \end{array} \right) \left(\begin{array}{c} q_0^H \\ 0 \end{array} \right) \quad (1)$$

Let $L_L = l_{00}$ and $Q_T = q_0^H$, we have

$$A = \left(\begin{array}{c|c} L_L & 0 \end{array} \right) \left(\begin{array}{c} Q_T \\ 0 \end{array} \right) = L_L Q_T \quad (2)$$

holds for $m = 1$.

- Inductive Case: assume that existence of LQ factorization holds for $m - 1$, then we show that it also holds for m .

That is, we partition

$$A = \left(\begin{array}{c} A_0 \\ a_1^H \end{array} \right) \quad (3)$$

Assume that $A_0 = L_{00}Q_0$ holds, then we show that $\exists L_L, Q_T$, s.t. $A = L_L Q_T$ holds.

Let

$$L_L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^H & \rho_{11} \end{array} \right), \quad Q_T = \left(\begin{array}{c} Q_0 \\ q_1^H \end{array} \right) \quad (4)$$

where $l_{10}^H = a_1^H Q_0^H$ and $\rho_{11} = \|a_1^H - l_{10}^H Q_0\|_2$. Then we have

$$L_L Q_T = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^H & \rho_{11} \end{array} \right) \left(\begin{array}{c} Q_0 \\ q_1^H \end{array} \right) = \left(\begin{array}{c} L_{00}Q_0 \\ l_{10}^H Q_0 + \rho_{11} q_1^H \end{array} \right) = \left(\begin{array}{c} L_{00}Q_0 \\ a_1^H \end{array} \right) = \left(\begin{array}{c} A_0 \\ a_1^H \end{array} \right) = A \quad (5)$$

- By principle of induction, $A = L_L Q_T$ holds for arbitrary m .

Exercise 3.

$$\min_z \|Az - y\|_2 = \min_z \|LQz - y\|_2 \quad (6)$$

$$= \min_{\{z=Q^H w\}} \|LQQ^H w - y\|_2 \quad (7)$$

$$= \min_{\{z=Q^H w\}} \|Lw - y\|_2 \quad (8)$$

$$= \min_{\{z=Q^H w\}} \left\| \left(\begin{array}{c|c} L_L & 0 \end{array} \right) \left(\begin{array}{c} w_T \\ w_B \end{array} \right) - y \right\|_2 \quad (9)$$

$$= \min_{\{z=Q^H w\}} \|L_L w_T - y\|_2 \quad (10)$$

which derives solution for $w_T = L_L^{-1}y$. And then the general solution is

$$z = Q^H w = \left(\begin{array}{c|c} Q_T^H & Q_B^H \end{array} \right) \left(\begin{array}{c} w_T \\ w_B \end{array} \right) = Q_T^H w_T + Q_B^H w_B = Q_T^H L_L^{-1}y + Q_B^H w_B \quad (11)$$

where w_B can be arbitrary vector in \mathbb{C}^{n-r} .

Exercise 4.

```

%% Homework 03: LQ Factorization
% Copyright 2014 The University of Texas at Austin
%
% For licensing information see
%     http://www.cs.utexas.edu/users/flame/license.html
%
% Programmed by: Jimmy Lin
%     jimmylin@utexas.edu
function [ A_out, L_out, Q_out ] = LG_CGS_unb( A, L, Q )
    [ AT, ...
      AB ] = FLA_Part_2x1( A, ...
                          0, 'FLA.TOP' );

    [ LTL, LTR, ...
      LBL, LBR ] = FLA_Part_2x2( L, ...
                                0, 0, 'FLA.TL' );

    [ QT, ...
      QB ] = FLA_Part_2x1( Q, ...
                          0, 'FLA.TOP' );
    while ( size( AT, 1 ) < size( A, 1 ) )
        [ A0, ...
          alt, ...
          A2 ] = FLA_Repart_2x1_to_3x1( AT, ...
                                       AB, ...
                                       1, 'FLA.BOTTOM' );

        [ L00,  l01,      L02,  ...
          l10t, lambda11, l12t, ...
          L20,  l21,      L22 ] = FLA_Repart_2x2_to_3x3( LTL, LTR, ...
                                                         LBL, LBR, ...
                                                         1, 1, 'FLA.BR' );

        [ Q0, ...
          qlt, ...
          Q2 ] = FLA_Repart_2x1_to_3x1( QT, ...
                                       QB, ...
                                       1, 'FLA.BOTTOM' );

        %-----%
        l10t = alt * Q0;
        temp = alt - l10t * Q0;
        lambda11 = norm(temp, 2);
        qlt = temp / lambda11;
        %-----%

        [ AT, ...
          AB ] = FLA_Cont_with_3x1_to_2x1( A0, ...
                                       alt, ...
                                       A2, ...
                                       'FLA.TOP' );

        [ LTL, LTR, ...
          LBL, LBR ] = FLA_Cont_with_3x3_to_2x2( L00,  l01,      L02,  ...
                                                  l10t, lambda11, l12t, ...
                                                  L20,  l21,      L22, ...
                                                  'FLA.TL' );

        [ QT, ...
          QB ] = FLA_Cont_with_3x1_to_2x1( Q0, ...
                                       qlt, ...
                                       Q2, ...
                                       'FLA.TOP' );

    end
    A_out = [ AT
              AB ];
    L_out = [ LTL, LTR
              LBL, LBR ];
    Q_out = [ QT
              QB ];
return

```

Exercise 5.

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Homework 03: Householder LQ Transformation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

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%
% Programmed by: Jimmy Lin
%      jimmylin@utexas.edu

function [ A_out, T_out ] = HLQ_unb( A, T )

[ ATL, ATR, ...
  ABL, ABR ] = FLA_Part_2x2( A, ...
                             0, 0, 'FLA_TL' );

[ TT, ...
  TB ] = FLA_Part_2x1( T, ...
                       0, 'FLA_TOP' );

while ( size( ATL, 1 ) < size( A, 1 ) )

    [ A00, a01,      A02, ...
      a10t, alpha11, a12t, ...
      A20, a21,      A22 ] = FLA_Repart_2x2_to_3x3( ATL, ATR, ...
                                                    ABL, ABR, ...
                                                    1, 1, 'FLA_BR' );

    [ T0, ...
      t1t, ...
      T2 ] = FLA_Repart_2x1_to_3x1( TT, ...
                                    TB, ...
                                    1, 'FLA_BOTTOM' );

    %-----%

    [ alpha11, temp, t1t ] = Housev( alpha11, a12t );
    a12t = temp;
    w21 = (a21 + A22 * a12t) / t1t;
    a21 = a21 - w21;
    A22 = A22 - w21 * a12t;

    %-----%

    [ ATL, ATR, ...
      ABL, ABR ] = FLA_Cont_with_3x3_to_2x2( A00, a01,      A02, ...
                                              a10t, alpha11, a12t, ...
                                              A20, a21,      A22, ...
                                              'FLA_TL' );

    [ TT, ...
      TB ] = FLA_Cont_with_3x1_to_2x1( T0, ...
                                       t1t, ...
                                       T2, ...
                                       'FLA_TOP' );

end

A_out = [ ATL, ATR
          ABL, ABR ];
T_out = [ TT
          TB ];

return

```

Part II

Exercises on Conditioning

Exercise 1.

Show that, for a consistent matrix norm, $\kappa(A) \geq 1$.

Proof.

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = 1 \quad (12)$$

Note that the above $\|\cdot\|$ was for arbitrary induced matrix norm. □

Lemma 1. For arbitrary matrix A and B , $\|AB\| \leq \|A\| \cdot \|B\|$.

Proof.

$$\|AB\| = \sup_{x \neq 0} \frac{\|ABx\|}{\|x\|} = \sup_{x \neq 0} \frac{\|A(Bx)\|}{\|x\|} \quad (13)$$

$$\leq \sup_{x \neq 0} \frac{\|A\| \cdot \|Bx\|}{\|x\|} \quad (14)$$

$$\leq \sup_{x \neq 0} \frac{\|A\| \cdot \|B\| \cdot \|x\|}{\|x\|} \quad (15)$$

$$= \|A\| \cdot \|B\| \quad (16)$$

Hence, it is concluded that $\|AB\| \leq \|A\| \cdot \|B\|$. □

Lemma 2. For arbitrary norm $\|\cdot\|$ and identity matrix I , $\|I\| = 1$.

Proof.

$$\|I\| = \sup_{x \neq 0} \frac{\|I \cdot x\|}{\|x\|} = \sup_{x \neq 0} \frac{\|x\|}{\|x\|} = 1 \quad (17)$$

□

Exercise 2.

If A has linearly independent columns, show that $\|(A^H A)^{-1} A^H\|_2 = \frac{1}{\sigma_{n-1}}$, where σ_{n-1} equals the smallest singular value of A .

Proof. Let U , Σ and V be singular value decomposition of A , such that $A = U\Sigma V^H$.

$$\|(A^H A)^{-1} A^H\|_2 = \|((U\Sigma V^H)^H U \Sigma V^H)^{-1} (U\Sigma V^H)^H\|_2 \quad (18)$$

$$= \|(V\Sigma^H U^H U \Sigma V^H)^{-1} V\Sigma^H U^H\|_2 \quad (19)$$

$$= \|(V\Sigma^H \Sigma V^H)^{-1} V\Sigma^H U^H\|_2 \quad (20)$$

$$= \|V^{-H} \Sigma^{-1} \Sigma^{-H} V^{-1} V\Sigma^H U^H\|_2 \quad (21)$$

$$= \|V^{-H} \Sigma^{-1} \Sigma^{-H} \Sigma^H U^H\|_2 \quad (22)$$

$$= \|V^{-H} \Sigma^{-1} U^H\|_2 \quad (23)$$

$$= \|V \Sigma^{-1} U^H\|_2 \quad (24)$$

$$= \|\Sigma^{-1}\|_2 \quad (25)$$

$$= \frac{1}{\sigma_{n-1}} \quad (26)$$

□

Lemma 3. (Unitary Invariance) For arbitrary unitary matrix U ,

$$\|UA\|_2 = \|AU\|_2 = \|A\|_2 \quad (27)$$

Lemma 4. For arbitrary diagonal matrix Σ ,

$$\|\Sigma^{-1}\|_2 = \frac{1}{\sigma_{n-1}} \quad (28)$$

where, σ_{n-1} is the least entry of Σ .

Note that above two lemmas have been proven in exercises of previous notes.

Exercise 3.

Let A have linearly independent columns. Show that $\kappa_2(A^H A) = \kappa_2(A)^2$.

Proof. We achieve the proof by employing SVD over A . Let unitary matrix U , diagonal matrix Σ and unitary matrix V be singular value decomposition of A , such that $A = U\Sigma V^H$. We start from the definition of condition number $\kappa_2(\cdot)$.

$$\kappa_2(A^H A) = \|A^H A\|_2 \cdot \|(A^H A)^{-1}\|_2 \quad (29)$$

Then we discuss the term $\|A^H A\|_2$ and $\|(A^H A)^{-1}\|_2$ respectively.

$$\|A^H A\|_2 = \|(U\Sigma V^H)^H U\Sigma V^H\|_2 \quad (30)$$

$$= \|V\Sigma^H U^H U\Sigma V^H\|_2 \quad (31)$$

$$= \|V\Sigma^H \Sigma V^H\|_2 \quad (32)$$

$$= \|\Sigma^H \Sigma\|_2 \quad (33)$$

$$= \sigma_0^2 \quad (34)$$

$$= \|A\|_2^2 \quad (35)$$

Note that σ_0 is the largest singular value of matrix A and also the largest entry of Σ .

$$\|(A^H A)^{-1}\|_2 = \|((U\Sigma V^H)^H U\Sigma V^H)^{-1}\|_2 \quad (36)$$

$$= \|(V\Sigma^H U^H U\Sigma V^H)^{-1}\|_2 \quad (37)$$

$$= \|(V\Sigma^H \Sigma V^H)^{-1}\|_2 \quad (38)$$

$$= \|V^{-H} \Sigma^{-1} \Sigma^{-H} V^{-1}\|_2 \quad (39)$$

$$= \|\Sigma^{-1} \Sigma^{-H}\|_2 \quad (40)$$

$$= \|\Sigma^{-1} \Sigma^{-1}\|_2 \quad (41)$$

$$= \frac{1}{\sigma_{n-1}^2} \quad (42)$$

$$= \|A^{-1}\|_2^2 \quad (43)$$

Now we have

$$\|A^H A\|_2 = \|A\|_2^2 \quad (44)$$

$$\|(A^H A)^{-1}\|_2 = \|A^{-1}\|_2^2 \quad (45)$$

Then

$$\kappa_2(A^H A) = \|A^H A\|_2 \cdot \|(A^H A)^{-1}\|_2 \quad (46)$$

$$= \|A\|_2^2 \cdot \|A^{-1}\|_2^2 \quad (47)$$

$$= (\|A\|_2 \cdot \|A^{-1}\|_2)^2 \quad (48)$$

$$= \kappa_2(A)^2 \quad (49)$$

Hence, it can be concluded that

$$\kappa_2(A^H A) = \kappa_2(A)^2 \quad (50)$$

□

Exercise 4.

Exercise 5.

Let $U \in \mathbb{C}^{n \times n}$ be unitary. Show that $\kappa_2(U) = 1$.

Proof.

$$\kappa_2(U) = \|U\|_2 \|U^{-1}\|_2 \quad (51)$$

$$= \sup_{x \neq 0} \frac{\|Ux\|_2}{\|x\|_2} \cdot \sup_{y \neq 0} \frac{\|U^{-1}y\|_2}{\|y\|_2} \quad (52)$$

$$= \sup_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} \cdot \sup_{y \neq 0} \frac{\|y\|_2}{\|y\|_2} \quad (53)$$

$$= 1 \cdot 1 \quad (54)$$

$$= 1 \quad (55)$$

□

Lemma 5. *For arbitrary unitary matrix U , its inverse U^{-1} is still unitary.*