The University of Texas at Austin

CS331 Algorithm

Assignment 05

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1 Exercise 2

Consider an iteration of Algorithm A on a configuration $G = (U, V, E)$ that increments the price of an item due to a violation of Stability Condition 2. Assume that the MCM M of G associated with this execution of Algorithm A is an MWMCM of G. Let $p(resp., p_0)$ denote the price vector of G maintained by Algorithm A just before (resp., after) this iteration. Prove that if $P_0(G, p)$ holds, then so does $P_0(G, p_0)$. Hint: Make use of Lemma 1 of Assignment 4.

In order to prove that if $P_0(G, p)$ holds, then $P_0(G, p')$ holds after one iteration of algorithm A, we should prove it for all three possible violation cases of algorithm A. Let p be the price vector that violates certain stability condition and q be one arbitrary stable price vector.

Stability Condition 1 is violated. In this case, the algorithm A will halts "unsuccessfully". Hence, there is no change for price vector after the iteration. That is, we have $p = p'$. And obviously, it is true that if $P_0(G, p)$ holds, then $P_0(G, p')$ holds.

□

Stability Condition 2 is violated. Since the violation of stability condition 2 causes the iteration of algorithm A, it is true that

$$\exists u,\ (u, v) \in M,\ (u, v^*) \in E,\ s.t.\ w(u, v) - p_v < w(u, v^*) - p_{v^*} \tag{1}$$

Let us instantiate the v to v_0 and v^* to v_1, and have

$$w(u, v_0) - p_{v_0} < w(u, v_1) - p_{v_1} \tag{2}$$

In this case, the p'_{v_1} will increments by 1, it is true that

$$w(u, v_0) - p_{v_0} \leq w(u, v_1) - p'_{v_1} \tag{3}$$

That is,

$$p'_{v_1} \leq w(u, v_1) - w(u, v_0) + p_{v_0} \tag{4}$$

For the stable price vector q, according the stability condition 2, we have

$$\forall v^*,\ w(u, v^*) - q_{v^*} \leq w(u, v_0) - q_{v_0} \tag{5}$$

Let us instantiate the v^* to v_1 and have

$$w(u, v_1) - q_{v_1} \leq w(u, v_0) - q_{v_0} \tag{6}$$

That is,

$$w(u, v_1) - w(u, v_0) + q_{v_0} \leq q_{v_1} \tag{7}$$

Since $p_{v_0} \leq q_{v_0}$ holds, we have

$$w(u, v_1) - w(u, v_0) + p_{v_0} \leq w(u, v_1) - w(u, v_0) + q_{v_0} \tag{8}$$

And then we have

$$w(u, v_1) - w(u, v_0) + p_{v_0} \leq q_{v_1} \tag{9}$$

Combined with (4), we reach the destiniation as follows

$$p'_{v_1} \leq q_{v_1} \tag{10}$$

Since p'_{v_1} is the only item whose price is changed in this iteration, $\forall v \neq v_1,\ p'_v \leq q_{v_1}$ holds. Hence, we can conclude that after the iteration of algorithm A with the cause of stability violation 2, Predicate 0 still holds. That is, $P_0(G, p) \Rightarrow P_0(G, p')$ holds if one iteration is caused by stability violation 2.

$\square$

Stability Condition 3 is violated. In the violation of stability condition 3, we have

$$\exists v^*,\ (u^*, v^*) \in E,\ u^* \text{ is unmatched in } M,\ p_{v^*} < w(u^*, v^*) \tag{11}$$

Let u_0 to be a unmatched bid in M, and $(u_0, v_0) \in E$. Hence, we can instantiate v^* to v_0, and u^* to u_0

$$(u_0, v_0) \in E,\ p_{v_0} < w(u_0, v_0) \tag{12}$$

Since the algorithm A in this case will increment p_{v_0} by one and other price component remains unchanged, we have

$$p_{v_0} \leq w(u, v_0) \tag{13}$$

Since M is already MWMCM, and q is stable price vector, then (M, q) is stable solution. According to the third property of stable solution, we have

$$\forall u^* \in U,\ u^* \text{ is unmatched in } M,\ w(u^*, v_0) \leq q_{v_0} \tag{14}$$

We instantiate u^* to u_0, which is reasonable because u_0 is unmatched in M. Then we have

$$w(u_0, v_0) \leq q_{v_0} \tag{15}$$

Combined with (13), it is true that

$$p_{v_0} \leq q_{v_0} \tag{16}$$

Since other price component does not vary at that iteration, we can conclude that $p \leq q$ holds in this case. That is, $P_0(G, p) \Rightarrow P_0(G, p')$ holds if one iteration is caused by stability violation 3.

$\square$

Since one iteration in algorithm A will only be caused by one of three cases, and in all of these three cases, $P_0(G, p) \Rightarrow P_0(G, p')$ holds for that iteration, it can be concluded that if $P_1(G, M, p)$ holds, then so does $P_1(G, M, p')$.

2 Exercise 5

Consider an execution of Algorithm A on a configuration $G = (U, V, E)$. Assume that the associated MCM M of G is an MWMCM of G, and that the initial price vector p for G is such that $P(G, M, p)$ holds. Prove that Algorithm A is guaranteed to halt successfully within a finite number of iterations. Hint: Make use of Lemma 3.

Proof by Contradiction. First assume that Algorithm A is **not necessarily** guaranteed to halt successfully within a finite number of iterations. That is to say, it is possible for Algorithm A to unsuccessfully halt at certain iteration. Let us say the halting iteration to be iteration k. Since the algorithm A halts at iteration k, the only possibility comes from the violation of stability condition 1 for price vector p. That is,

$$\exists (u, v) \in M,\ w(u, v) < p_v \tag{17}$$

Let us see how the above formula contradicts the known condition $P(G, M, p)$.
For an stable price vector q, and an MWMCM M, we have

$$\forall v,\ q_v \leq w(u, v) \tag{18}$$

Combined with (17), it can be easily seen that

$$\exists (u, v) \in M,\ q_v \leq w(u, v) < p_v \tag{19}$$

However, we already know that $P(G, M, p)$ holds and if $P(G, M, p)$, then $P_1(G, M, p)$ must hold. Hence, $P_1(G, M, p)$ holds. That is,

$$\forall v,\ p_v < q_v \tag{20}$$

where q can be arbitrary stable price vector, but here we just instantiate it to be the same stable price vector in (18) for convenience.

Now we can see that there is a contradiction between (19) and (20). Hence, we should negate the initial assumption and conclude that Algorithm A is guaranteed to halt successfully within a finite number of iterations.

□