THE UNIVERSITY OF TEXAS
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CS331 ALGORITHM

Assignment 06

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Contents

1 Exercise 1

Let i, j, and k be distinct integers in $[n]$ such that j lies between i and k, i.e., either $i < j < k$ and $k < j < i$. Assume that bids $\alpha(i)$ and $\alpha(j)$ are linear. Prove that if $gap(p, i, j) = 0$ and $g(j, k) = 0$, then $g(i, j) = 0$.

Based on the known condition as follows,

$$gap(p, i, j) = 0 \tag{1}$$
$$gap(p, j, k) = 0 \tag{2}$$

And the assumed condition that

$$i, j, k \in [n] \tag{3}$$
$$\alpha(j) \text{ is linear} \tag{4}$$
$$\alpha(i) \text{ is linear} \tag{5}$$

We have

$$max(0, s(i) \cdot g(i, j) - f(p, i, j)) = 0 \tag{6}$$
$$max(0, s(j) \cdot g(j, k) - f(p, j, k)) = 0 \tag{7}$$

That is,

$$s(i) \cdot g(i, j) - f(p, i, j) \leq 0 \tag{8}$$
$$s(j) \cdot g(j, k) - f(p, j, k) \leq 0 \tag{9}$$

We sum up (8) and (9), then get

$$s(i) \cdot g(i, j) - f(p, i, j) + s(j) \cdot g(j, k) - f(p, j, k) \leq 0 \tag{10}$$

This can be simplified to be

$$s(i) \cdot g(i, k) - f(p, i, k) \leq g(j, k) \cdot (s(i) - s(j)) \tag{11}$$

According to the known condition, we have

$$i < j < k \text{ or } k < j < i \tag{12}$$

In the case of $i < j < k$, we have $g(j, k) = q_{\beta(k)} - q_{\beta(j)} \geq 0$ and $s(i) - s(j) \leq 0$ since $\alpha(i)$ and $\alpha(j)$ are matched to βi and βj in MWMCM M, respectively. Similarly, in the case of $k < j < i$, we have $g(i, j) = q_{\beta(j)} - q_{\beta(i)} \leq 0$ and $s(i) - s(j) \geq 0$. Therefore, it can be concluded that in either case,

$$g(j, k) \cdot (s(i) - s(j)) \leq 0 \tag{13}$$

Hence, we have

$$s(i) \cdot g(i, k) - f(p, i, k) \leq g(j, k) \leq 0 \tag{14}$$

Since $\alpha(i)$ is linear bid and $i \in [n]$

$$gap(p, i, k) = 0 \tag{15}$$

2 Exercise 2

Let p be a price vector for G such that $L(p,n) \wedge R(p,0)$ holds. Prove that $gap(p,i,j) = 0$ for all integers i and j in $[n]$. Hint: Make use of Exercise 1.

Case 1. In the case of $i \in \{-1, n\}$ or in the case of $i \in [n]$ and $\alpha(i)$ is single-item bid, it can be easy to conclude that $\forall i, j,\ gap(p,i,j) = 0$ by definition of *gap* function.

□

Case 2. In the case of $i \in [n]$ and $\alpha(i)$ is linear bid, things are more complicated. There are two subcases, (1) $i > j$, (2) $j > i$. We provide detailed proof as follows.

Based on give condition $L(p,n) \wedge R(p,0)$, we have both of the followings hold.

$$L(p,n) \tag{16}$$
$$R(p,0) \tag{17}$$

For $i > j$, we can iteratively employ (17) and the result of exercise 1 to reach the conclusion of $\forall i, j,\ gap(p,i,j) = 0$. Specifically, apply the (17) repeatedly and since $\alpha(i)$ is linear bid, it will eventually reach $\alpha(i)$.

$$gap(p, \tau(j), j) = 0 \tag{18}$$
$$gap(p, \tau(\tau(j)), \tau(j)) = 0 \tag{19}$$
$$\vdots$$
$$gap(p, \tau_s(j), \tau_{s-1}(j)) = 0 \tag{20}$$
$$gap(p, i, \tau_s(j)) = 0 \tag{21}$$

Note that s denotes the number of times we apply rule of $R(p,0)$ and $\tau_s(j)$ denotes the integer derived by apply s times of $R(p,0)$ on original integer j.

Then we can make use of the result of exercise 1 to address the series of equations (18) - (21) and conclude that

$$\forall i, j\ gap(p,i,j) = 0 \tag{22}$$

Note that the above proof can be formalized by doing mathematical induction on the number of times we employ $R(p,0)$, which is s.

For $i < j$, we can iteratively employ (16) and the result of exercise 1 to reach the conclusion of $\forall i, j,\ gap(p,i,j) = 0$. The detailed proof follows the same routine as the case of $i > j$ in the above.

In terms of the dicussion over two subcases, we have $\forall i, j,\ gap(p,i,j) = 0$ in the case of $i \in [n]$ and $\alpha(i)$ is linear bid.

□

According to the dicussion over *case 1* and *case 2*, it can be concluded that *generally*

$$\forall i, j,\ gap(p,i,j) = 0 \tag{23}$$