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CS383C Numerical Analysis

Midterm Exam

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Exercises

Exercise 1.

(a) Prove that $||\cdot||$ is a vector norm.

To prove $||\cdot||$ is a vector norm, we have to show all the following three property holds:

- If $x \neq 0$, then $||x|| > 0$.
- For arbitrary scalar α and vector x, $||\alpha x|| = |\alpha| \cdot ||x||$.
- For arbitrary vector x and y, $||x + y|| \leq ||x|| + ||y||$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i,\ \delta_i > 0$ holds. Besides, since $x \neq 0$, then $\exists i,\ x_i^2 > 0$.

$$||x|| = \sqrt{x^T D x} = \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} > 0 \qquad\qquad x \neq 0 \tag{1}$$

Therefore, we proved that if $x \neq 0$, then $||x|| > 0$. □

Proof.

$$||\alpha x|| = \sqrt{(\alpha x)^T D(\alpha x)} = \sqrt{\sum_{i=0}^{m-1} \alpha^2 \delta_i x_i^2} = \sqrt{\alpha^2 \sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \sqrt{\sum_{i=0}^{m-1} \delta_i x_i^2} = |\alpha| \cdot ||x|| \tag{2}$$

□

Proof.

$$||x + y||^2 = (\sqrt{(x + y)^T D(x + y)})^2 = x^T Dx + x^T Dy + 2x^T Dy \tag{3}$$

$$(||x|| + ||y||)^2 = (\sqrt{x^T Dx})^2 + (\sqrt{y^T Dy})^2 = x^T Dx + y^T Dy + 2\sqrt{x^T Dx}\sqrt{y^T Dy} \tag{4}$$

Now we show that $x^T Dy \leq \sqrt{x^T Dx}\sqrt{y^T Dy}$, which is equivalent to

$$\Big(\sum_{i=0}^{m-1} \delta_i x_i y_i\Big)^2 \leq \Big(\sum_{i=0}^{m-1} \delta_i x_i^2\Big)\Big(\sum_{i=0}^{m-1} \delta_i y_i^2\Big) \tag{5}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \delta_i \delta_j x_i y_i x_j y_j \leq \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \delta_i \delta_j x_i^2 y_j^2 \tag{6}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i \delta_j (2x_i y_i x_j y_j) + \sum_{i=0}^{m-1} \delta_i^2 x_i y_i x_i y_i \leq \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i \delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) + \sum_{i=0}^{m-1} \delta_i^2 x_i^2 y_i^2 \tag{7}$$

$$\iff \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i \delta_j (2x_i y_i x_j y_j) \leq \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \delta_i \delta_j (x_i^2 y_j^2 + x_j^2 y_i^2) \tag{8}$$

which obivously holds if we apply basic inequality on each (i, j) pair,

$$2x_i y_i x_j y_j = 2(x_i y_j)(x_j y_i) \leq x_i^2 y_j^2 + x_j^2 y_i^2 \tag{9}$$

Since we have proved $x^T Dy \leq \sqrt{x^T Dx}\sqrt{y^T Dy}$, then

$$||x + y||^2 \leq (||x|| + ||y||)^2 \tag{10}$$

which leads to desired triangle inequality since $||\cdot|| \geq 0$,

$$||x + y|| \leq ||x|| + ||y|| \tag{11}$$

□

Since all three property have been proved, it can be concluded that

$$||\cdot|| \text{ defined by } ||x|| = \sqrt{x^T Dx} \text{ is a norm.} \tag{12}$$

(b) Show that $||\cdot||$ is a matrix norm.

To prove $||\cdot||$ is a matrix norm, we have to show all the following three property holds:

- If $A \neq 0$, then $||A|| > 0$.
- For arbitrary scalar α and matrix A, $||\alpha A|| = |\alpha| \cdot ||A||$.
- For arbitrary matrix A and B, $||A+B|| \leq ||A|| + ||B||$.

Now we turn to prove each of them individually.

Proof. Since D is diagonal with positive diagonal elements, then $\forall i,\ \delta_i > 0$ holds. And Since $A \neq 0$, then there is at least one column vector that is not zero vector. Let us assume it is the j-th column, a_j, that is non-zero. Then

$$||A|| = ||DA||_2 = \max_{||x||_2 \neq 0} \frac{||DAx||_2}{||x||_2} \geq \frac{||DAe_j||_2}{||e_j||_2} > 0 \tag{13}$$

Therefore, we proved that if $A \neq 0$, then $||A|| > 0$. □

Proof. According to the class notes, $||\cdot||_2$ is a norm (since it is one of p-induced-norm) and then for arbitrary scalar β and arbitrary matrix B,

$$||\alpha B||_2 = |\beta| \cdot ||B||_2 \tag{14}$$

Now we consider $||\alpha A||$

$$\begin{aligned}
||\alpha A|| &= ||D(\alpha A)||_2 && (15)\\
&= ||\alpha DA||_2 && (16)\\
&= |\alpha| \cdot ||DA||_2 && \text{natural consequence of (19)} \quad (17)\\
&= |\alpha| \cdot ||A|| && (18)
\end{aligned}$$

Thus, we proved that $||\alpha A|| = |\alpha| \cdot ||A||$. □

Proof. According to the class notes, $||\cdot||_2$ is a norm (since it is one of p-induced-norm) and then for arbitrary matrix C and arbitrary matrix D,

$$||C + D||_2 \leq ||C||_2 + ||D||_2 \tag{19}$$

Now we consider $||A+B||$ for arbitrary matrix A and B,

$$\begin{aligned}
||A+B|| &= ||D(A+B)||_2 && (20)\\
&= \max_{x\neq 0} \frac{||DAx + DBx||_2}{||x||_2} && (21)\\
&\leq \max_{x\neq 0} \frac{||DAx||_2 + ||DBx||_2}{||x||_2} && (22)\\
&= \max_{x\neq 0} \left(\frac{||DAx||_2}{||x||_2} + \frac{||DBx||_2}{||x||_2} \right) && (23)\\
&\leq \max_{x\neq 0} \frac{||DAx||_2}{||x||_2} + \max_{y\neq 0} \frac{||DBy||_2}{||y||_2} && (24)\\
&= ||DA||_2 + ||DB||_2 = ||A|| + ||B|| && (25)
\end{aligned}$$

Hence, we proved that $||A+B|| \leq ||A|| + ||B||$. □

Since all three property have been proved, it can be concluded that

$$||\cdot|| \text{ defined by } ||A|| = ||DA||_2 \text{ is a norm.} \tag{26}$$

Exercise 2.

(a) Show that $(..)^{-1} = (..)$.

Proof.

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array}\right) = \left(\begin{array}{c|c} L_{00}L_{00}^{-1} & 0 \\ \hline l_{10}^T L_{00}^{-1} - \frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} \cdot \lambda 11 & \frac{1}{\lambda_{11}} \cdot \lambda_{11} \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array}\right) = I \tag{27}$$

Multiply both side on the left with $\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^{-1}$ and then we have (assume all inversion exists)

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^{-1} = \left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -\frac{l_{10}^T L_{00}^{-1}}{\lambda_{11}} & \frac{1}{\lambda_{11}} \end{array}\right) \tag{28}$$

□

(b) Show that $H_1 H_0 = ..$

According to given formula for H_i, we have

$$H_0 = I - \frac{1}{\tau_0} u_0 u_0^T \text{ and } H_1 = I - \frac{1}{\tau_1} u_1 u_1^T \tag{29}$$

Now we manipulate $H_1 H_0$,

$$\begin{aligned} H_1 H_0 &= (I - \frac{1}{\tau_1} u_1 u_1^T)(I - \frac{1}{\tau_0} u_0 u_0^T) & (30) \\ &= I - \frac{1}{\tau_1} u_1 u_1^T - \frac{1}{\tau_0} u_0 u_0^T + \frac{1}{\tau_0 \tau_1} u_1 u_1^T u_0 u_0^T & (31) \\ &= I - (u_0|u_1)\left(\begin{array}{c|c} \frac{1}{\tau_0} & 0 \\ \hline -\frac{u_0^T u_1}{\tau_0 \tau_1} & \frac{1}{\tau_1} \end{array}\right)(u_0|u_1)^T & (32) \\ &= I - (u_0|u_1)\left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array}\right)^{-1}(u_0|u_1)^T & (33) \end{aligned}$$

where $\tau_0 = \frac{u_0^T u_0}{2}$ and $\tau_1 = \frac{u_1^T u_1}{2}$. Hence, it is concluded that (assume all inversion exists)

$$H_1 H_0 = I - (u_0|u_1)\left(\begin{array}{c|c} \tau_0 & 0 \\ \hline u_0^T u_1 & \tau_1 \end{array}\right)^{-1}(u_0|u_1)^T \tag{34}$$

(c) Show that $(I - \frac{1}{\tau_1} u_1 u_1^T)(I - U_0 L_{00}^T U_0^T) = I - ...$

$$\begin{aligned} (I - \frac{1}{\tau_1} u_1 u_1^T)(I - U_0 L_{00}^T U_0^T) &= I - \frac{1}{\tau_1} u_1 u_1^T - U_0 L_{00}^T U_0^T + \frac{1}{\tau_1} u_1 u_1^T U_0 L_{00}^T U_0^T & (35) \\ &= I - (U_0|u_1)\left(\begin{array}{c|c} L_{00}^{-1} & 0 \\ \hline -u_1^T U_0 L_{00}^{-1}/\tau_1 & \frac{1}{\tau_1} \end{array}\right)(U_0|u_1)^T & (36) \\ &= I - (U_0|u_1)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array}\right)^{-1}(U_0|u_1)^T & (37) \end{aligned}$$

where $\tau_1 = \frac{u_1^T u_1}{2}$. Thus, it is concluded that (assume all inversion exists)

$$(I - \frac{1}{\tau_1} u_1 u_1^T)(I - U_0 L_{00}^T U_0^T) = I - (U_0|u_1)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline u_1^T U_0 & \tau_1 \end{array}\right)^{-1}(U_0|u_1)^T \tag{38}$$

(d) Show that $H_{k-1}\cdots H_1H_0 = I - UL^{-1}U^H$

We can do mathematical induction on m. We separate the discussion of $k=1$ and $k>1$.

- Particular Case: $k=1$,

  Proof.

$$H_0 = I - \frac{1}{\tau_0}u_0u_0^T = I - \underbrace{(u_0)}_{U_{(0)}}\underbrace{(\frac{1}{\tau_0})}_{L_{(0)}^{-1}}\underbrace{(u_0)^T}_{U_{(0)}^T} \tag{39}$$

  Hence, $H_{k-1}\cdots H_1H_0 = I - UL^{-1}U^H$ holds for $k=1$. □

  Note that this case is particular because it derives matrix which is indeed scalar.

- Base Case: $k=2$, this can be proved by using result of (b).
- Inductive Case: assume that for $p \geq 2$, (I.H.)

$$H_{p-1}\cdots H_1H_0 = I - U_{(p-1)}L_{(p-1)}^{-1}U_{(p-1)}^H \tag{40}$$

  show that there exist one lower triangular matrix $L_{(p)}$ and one matrix $U_{(p)}$, such that

$$H_pH_{p-1}\cdots H_1H_0 = I - U_{(p)}L_{(p)}^{-1}U_{(p)}^H \tag{41}$$

  holds.

  Proof. With application of the result of (c), we have

$$\begin{aligned} H_p(H_{p-1}H_{p-2}\cdots H_1H_0) &= (1 - \frac{1}{\tau_p}u_pu_p^T)(I - U_{(p-1)}L_{(p-1)}^{-1}U_{(p-1)}^H) & (42)\\ &= I - \underbrace{(U_{(p-1)}|u_p)}_{U_{(p)}}\underbrace{\left(\begin{array}{c|c} L_{(p-1)} & 0 \\ \hline u_p^TU_{(p-1)} & \tau_p \end{array}\right)^{-1}}_{L_{(p)}^{-1}}\underbrace{(U_{(p-1)}|u_p)^T}_{U_{(p)}^T} & (43)\\ &= I - U_{(p)}L_{(p)}^{-1}U_{(p)}^T & (44) \end{aligned}$$

  where $U_{(p)} = (U_{(p-1)}|u_p)$, $L_{(p)} = \left(\begin{array}{c|c} L_{(p-1)} & 0 \\ \hline u_p^TU_{(p-1)} & \tau_p \end{array}\right)$ and $\tau_p = \frac{u_p^Tu_p}{2}$. □

- By the principle of mathematical induction, it can be concluded that

$$H_{k-1}\cdots H_1H_0 = I - UL^{-1}U^H \tag{45}$$

  holds for all $k > 0$.

Exercise 3.

(a) Relationship between m, n and r.

$$rank(A) = r \leq \min(m, n) \tag{46}$$

(b) Reduced SVD in terms of U, V and Σ.

- $A = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}$ and $V_r = V_L$.

$$A = U\Sigma V^T = (U_L | U_R) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) (V_L | V_R)^T = U_L \Sigma_{TL} V_L^T \tag{47}$$

- $(A^T A)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1}\Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$\begin{aligned}
(A^T A)^{-1} &= \left((U\Sigma V^T)^T U\Sigma V^T\right)^{-1} && (48)\\
&= \left(V\Sigma^T U^T U \Sigma V^T\right)^{-1} && (49)\\
&= \left(V\Sigma\Sigma V^T\right)^{-1} && (50)\\
&= \left((V_L | V_R) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) (V_L | V_R)^T \right)^{-1} && (51)\\
&= \left(V_L \Sigma_{TL}\Sigma_{TL} V_L^T\right)^{-1} && (52)\\
&= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (53)\\
&= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}(V_L^{-T})^T && (54)
\end{aligned}$$

- $(A^T A)^{-1}A^T = U_r \Sigma_r V_r^T$, where $U_r = V_L^{-T}$, $\Sigma_r = \Sigma_{TL}^{-1}$ and $V_r = U_L$.

$$\begin{aligned}
(A^T A)^{-1}A^T &= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1}V_L\Sigma_{TL}^T U_L^T && (55)\\
&= V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}\Sigma_{TL}^T U_L^T && (56)\\
&= V_L^{-T}\Sigma_{TL}^{-1}U_L^T && (57)
\end{aligned}$$

- $A(A^T A)^{-1} = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = \Sigma_{TL}^{-1}\Sigma_{TL}^{-1}$ and $V_r = V_L^{-T}$.

$$\begin{aligned}
A(A^T A)^{-1} &= U_L\Sigma_{TL}V_L^T V_L^{-T}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (58)\\
&= U_L\Sigma_{TL}\Sigma_{TL}^{-1}\Sigma_{TL}^{-1}V_L^{-1} && (59)\\
&= U_L\Sigma_{TL}^{-1}V_L^{-1} && (60)\\
&= U_L\Sigma_{TL}^{-1}(V_L^{-T})^T && (61)
\end{aligned}$$

- $A(A^T A)^{-1}A^T = U_r \Sigma_r V_r^T$, where $U_r = U_L$, $\Sigma_r = I^{r\times r}$ and $V_r = U_L$.

$$\begin{aligned}
A(A^T A)^{-1}A^T &= (U_L\Sigma_{TL}^{-1}V_L^{-1})V_L\Sigma_{TL}^T U_L^T && (62)\\
&= U_L\Sigma_{TL}^{-1}\Sigma_{TL}^T U_L^T && (63)\\
&= U_L I U_L^T && (64)
\end{aligned}$$

Exercise 4.

(a) $\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1$ where δv_{11} is small.

Proof.

$$\psi_1 = [v_{11}\check{\chi}_1]_* = \frac{v_{11}\check{\chi}_1}{1+\epsilon_*} = (1 - \frac{\epsilon_*}{1+\epsilon_*})v_{11}\check{\chi}_1 = (v_{11} - \frac{\epsilon_* v_{11}}{1+\epsilon_*})\check{\chi}_1 \equiv (v_{11} + \delta v_{11})\check{\chi}_1 \tag{65}$$

where

$$\delta v_{11} = -\frac{\epsilon_* v_{11}}{1+\epsilon_*}, \text{ where } |\epsilon_*| \le \mathbf{u} \tag{66}$$

Then

$$|\delta v_{11}| = |-\frac{\epsilon_*}{1+\epsilon_*}v_{11}| \tag{67}$$
$$= \frac{|\epsilon_*|}{|1+\epsilon_*|}|v_{11}| \tag{68}$$
$$\le \frac{\mathbf{u}}{|1+\epsilon_*|}|v_{11}| \tag{69}$$
$$\le \frac{\mathbf{u}}{1+|\epsilon_*|}|v_{11}| \tag{70}$$
$$\le \frac{\mathbf{u}}{1+|\epsilon_*|-|\epsilon_*|-\mathbf{u}}|v_{11}| \tag{71}$$
$$= \frac{\mathbf{u}}{1-\mathbf{u}}|v_{11}| = \gamma_1|v_{11}| \tag{72}$$

Hence, we proved that

$$\psi_1 = (v_{11} - \delta v_{11})\check{\chi}_1 \text{ where } |\delta v_{11}| = \gamma_1|v_{11}| \tag{73}$$

□

(b)

Proof. Start from dot product. We directly use the corollary in notes.

$$d \stackrel{\Delta}{=} [u_{12}^T\check{x_2}] = (u_{12}^T + \delta u_{12}^T)\check{x_2} \text{ , where } |\delta u_{12}^T| \le \gamma_k|u_{12}^T| \le \gamma_{k+1}|u_{12}^T| \tag{74}$$

Then we consider the subtraction step with Standard Computational Model (SCM).

$$s \stackrel{\Delta}{=} [\psi_1 - d] = (\psi_1 - d)(1+\epsilon_-) = (\psi_1 - d) + (\psi_1 - d)\epsilon_- \equiv (\psi_1 - d) + \delta\psi_1 = (\psi_1 + \delta\psi_1) - d \tag{75}$$

where $|\delta\psi_1| = |(\psi_1 - d)\epsilon_-| \le |\psi_1\epsilon_-| = |\epsilon_-||\psi_1| \le \mathbf{u}|\psi_1|$. Then we consider the division step with Alternative Computational Model (ACM).

$$\check{\chi}_1 = [\frac{s}{v_{11}}] = \frac{s}{v_{11}}(1+\epsilon_/)^{-1} = \frac{s}{v_{11}(1+\epsilon_/)} \tag{76}$$

which can be rewritten as

$$(v_{11} + \delta v_{11})\check{\chi}_1 = s \tag{77}$$

where $|\delta v_{11}| = |\epsilon_/ v_{11}| \le |\epsilon_/| \cdot |v_{11}| \le \mathbf{u}|v_{11}|$. Then we have

$$(v_{11} + \delta v_{11})\check{\chi}_1 = (\psi_1 + \delta\psi_1) - d = (\psi_1 + \delta\psi_1) - (u_{12}^T + \delta u_{12}^T)\check{x_2} \tag{78}$$

where

$$|\delta v_{11}| \le \mathbf{u}|v_{11}|,\ |\delta\psi_1| \le \mathbf{u}|\psi_1|,\ |\delta u_{12}^T| \le \gamma_{k+1}|u_{12}^T| \tag{79}$$

□

(c) Induction for the desired result

To prove the desired result, let us do mathematical induction on k (the iteration of algorithm).

- Base Case: $k = 0$

  Proof. In this case, $U^{0\times 0}$ does not participate in computation, v_{11} is the right-bottom-most element, and no pre-computed $\check{x}_2$. Hence, it is natural that

$$(v_{11} + \delta v_{11})\check{\chi}_1 = (\psi_1 + \delta\psi_1) \tag{80}$$

  which indicates that $(U + \Delta U)\check{x} = (y + \delta y)$ holds for $k = 0$. (first iteration) $\square$

- Inductive case: Assume that at p-th iteration ($p > 0$), pre-computed $\check{x}_{2(p)}$ satisfy

$$(U_{(p)} + \Delta U_{(p)})\check{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \tag{81}$$

  where $|\Delta U_{(p)}| \leq \gamma_n |U_{(p)}|$ and $|\delta y_{2(p)}| \leq \mathbf{u}|y_{2(p)}|$, (I.H.) holds.
  Show that for the pre-computed $\check{x}_2$ at next iteration $\check{x}_{2(p+1)} = \left(\begin{array}{c}\check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)}\end{array}\right)$ and $y_{2(p+1)} = \left(\begin{array}{c}\psi_{1(p)} \\ \hline y_{2(p)}\end{array}\right)$

$$(U_{(p+1)} + \Delta U_{(p+1)})\check{x}_{2(p+1)} = (y_{(p+1)} + \Delta y_{(p+1)}) \tag{82}$$

  where $|\Delta U_{(p+1)}| \leq \gamma_n |U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u}|y_{2(p+1)}|$, holds for some upper triangle $U_{(p+1)}$.

  Proof. In terms of result of (b), we have

$$(v_{11(p)} + \delta v_{11(p)})\check{\chi}_{1(p)} + (u_{12(p)}^T + \delta u_{12(p)}^T)\check{x}_{2(p)} = (\psi_{1(p)} + \delta\psi_{1(p)}) \tag{83}$$

  where $\delta v_{11(p)} \leq \mathbf{u}|v_{11(p)}|$, $\delta\psi_{1(p)} \leq \mathbf{u}|\psi_{1(p)}|$, and $\delta u_{12(p)}^T \leq \gamma_{p+1}|u_{12(p)}^T|$.
  Combined with (I.H.)

$$(U_{(p)} + \Delta U_{(p)})\check{x}_{2(p)} = (y_{2(p)} + \Delta y_{2(p)}) \tag{84}$$

  we have

$$\left(\begin{array}{c|c} v_{11(p)} + \delta v_{11(p)} & u_{12(p)}^T + \delta u_{12(p)}^T \\ \hline 0 & U_{(p)} + \Delta U_{(p)} \end{array}\right)\left(\begin{array}{c}\check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)}\end{array}\right) = \left(\begin{array}{c}\psi_{1(p)} + \delta\psi_{1(p)} \\ \hline y_{2(p)} + \Delta y_{2(p)}\end{array}\right) \tag{85}$$

  where $|\Delta U_{(p)}| \leq \gamma_n|U_{(p)}|$, $|\delta v_{11(p)}| \leq \mathbf{u}|v_{11(p)}| \leq \gamma_n|v_{11(p)}|$, $|\delta u_{12(p)}^T| \leq \gamma_{p+1}|u_{12(p)}^T| \leq \gamma_n|u_{12(p)}^T|$ and $\delta\psi_{1(p)} \leq \mathbf{u}|\psi_{1(p)}|$, $|\delta y_{2(p)}| \leq \mathbf{u}|y_{2(p)}|$. (Note that $0 \leq p \leq n-1$)
  which is equivalent to

$$\left(\underbrace{\left(\begin{array}{c|c} v_{11(p)} & u_{12(p)}^T \\ \hline 0 & U_{(p)}\end{array}\right)}_{U_{(p+1)}} + \underbrace{\left(\begin{array}{c|c} \delta v_{11(p)} & \delta u_{12(p)}^T \\ \hline 0 & \Delta U_{(p)}\end{array}\right)}_{\Delta U_{(p+1)}}\right)\underbrace{\left(\begin{array}{c}\check{\chi}_{1(p)} \\ \hline \check{x}_{2(p)}\end{array}\right)}_{\check{x}_{2(p+1)}} = \left(\underbrace{\left(\begin{array}{c}\psi_{1(p)} \\ \hline y_{2(p)}\end{array}\right)}_{y_{2(p+1)}} + \underbrace{\left(\begin{array}{c}\delta\psi_{1(p)} \\ \hline \Delta y_{2(p)}\end{array}\right)}_{\delta y_{2(p+1)}}\right) \tag{86}$$

$$(U_{(p+1)} + \Delta U_{(p+1)})\check{x}_{2(p+1)} = (y_{2(p+1)} + \delta y_{2(p+1)}) \tag{87}$$

  where $|\Delta U_{(p+1)}| \leq \gamma_n|U_{(p+1)}|$ and $|\delta y_{2(p+1)}| \leq \mathbf{u}|y_{2(p+1)}|$.
  Hence, we sucessfully proved (82). $\square$

- By principle of mathematical induction, it is proved w.l.o.g. that

$$(U + \Delta U)\check{x} = (y + \delta y) \text{ , where } |\Delta U| \leq \gamma_n|U| \text{ and } |\delta y| \leq \mathbf{u}|y| \tag{88}$$