The University of Texas
at Austin

EE381V Large Scale Optimization

Problem Set 4

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1 Conjugate Gradient Algorithm

1.1 M_1

Command to be executed in matlab:

```
>> load ConjugateGradient.mat
>> CGS(M1, b1, x)
```

Plot

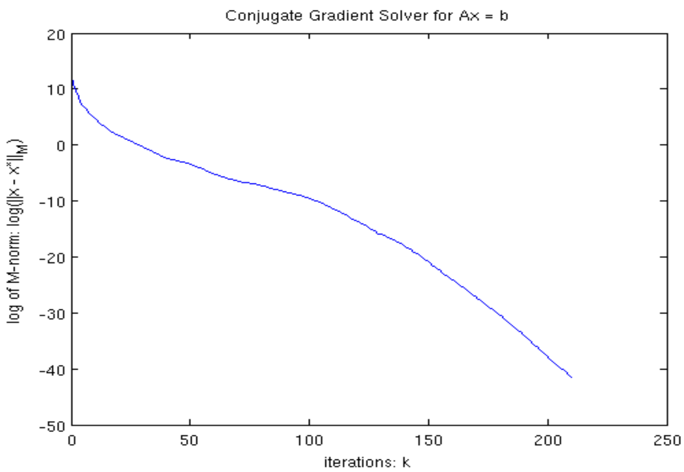


Figure 1: Conjugate Gradient Solver for M_1

1.2 M_2

Command to be executed in matlab:

```
>> load ConjugateGradient.mat
>> CGS(M2, b2, x)
```

Plot

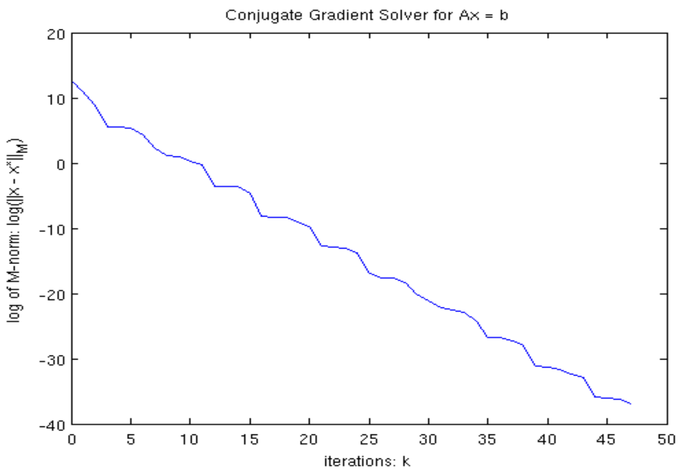


Figure 2: Conjugate Gradient Solver for M_2

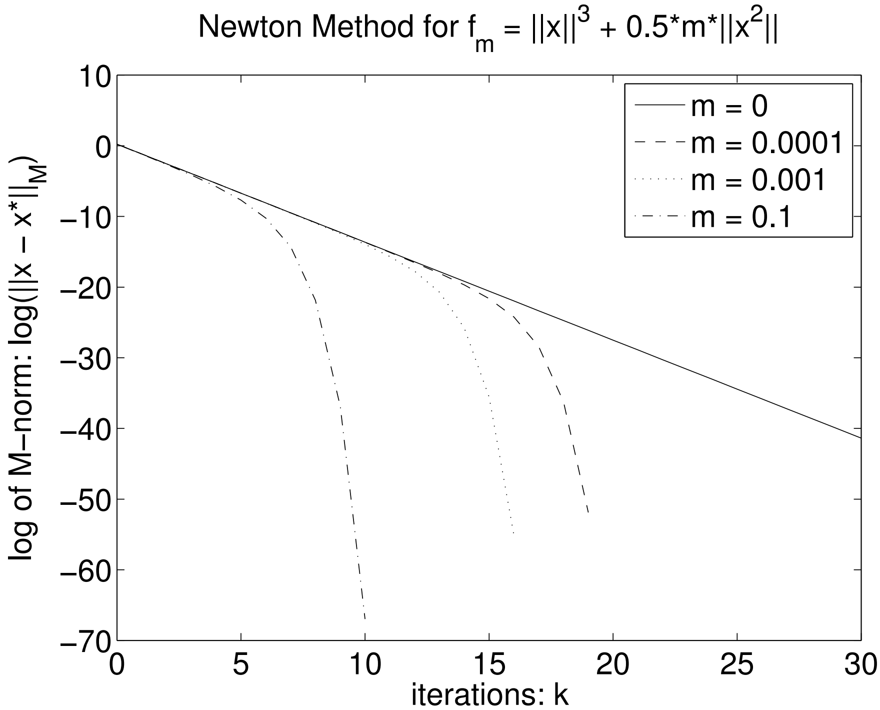
2 Newtons Method

2.1 plots for various m

Command to run:

```
>> Newton(0, 'k-')
>> hold on
>> Newton(0.0001, 'k--')
>> Newton(0.001, 'k:')
>> Newton(0.1, 'k-.')
>> Newton(0, 'k-')
```

Plots Note that the initial point here is $0.5 * ones(5, 1)$.



2.2 Explanation

From the figure, we can see that the $f_m(\cdot)$ goes to quadratic convergence phrase with fewer number of iterations if m is greater. The intuitive explanation is that the larger m will cause quadratic convergence condition $||\nabla f(x)|| < \frac{m^2}{L}$ easier to satisfy. Note that one extreme is when $m = 0$, our newton method solver never goes into quadratic convergence phrase (only linear convergence phrase).

3 Central Path

3.1 Find a function F

$$F = -\sum_{i=1}^{4} log(x_i) - log(4 - x_1 - 3x_2 - x_4) - log(3 - 2x_1 - x_2) - log(3 - x_2 - 4x_3 - x_4) \quad (1)$$

3.2 Find analytic center x_F^*

$$x_F^* = arg \min_{x \in domF} F(x) \quad (2)$$
$$= (0.5488, 0.3091, 0.2543, 0.6485) \quad (3)$$

3.3 Generate a central path

The commands we used for generating a central path with various α

```
>> t_init = 5; alpha1=0.01; alpha2=0.1; alpha3=0.5; alpha4=1e-3;
>> CP(t_init, alpha1, 'k-');
>> hold on
>> CP(t_init, alpha2, 'k--');
>> CP(t_init, alpha3, 'k:');
>> CP(t_init, alpha4, 'k-.');
```

3.4 Plot the error $log(||x^{(k)} - x^*||)$ w.r.t k

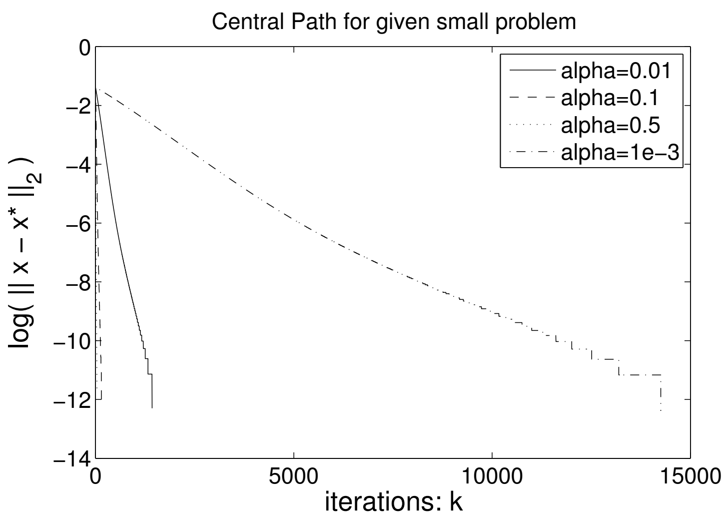
In order to plot the errors, we first employ CVX to obtain the optimal feasible point x^* by

```
>> CVX_solve_CP
```

For implementation details of this program, please go to appendix.
Then we have optimal feasible point for plotting $log(||x^{(k)} - x^*||)$

$$x^* = (0, 0, 0, 0) \quad (4)$$

Plots



4 Larger Linear Program

4.1 Find a function F

For given $A^{m \times n}$

$$F = -\sum_{i=1}^{N} log(x_i) - \sum_{i=1}^{m} log(b - a_i^T x) \tag{5}$$

4.2 Find analytic center x_F^*

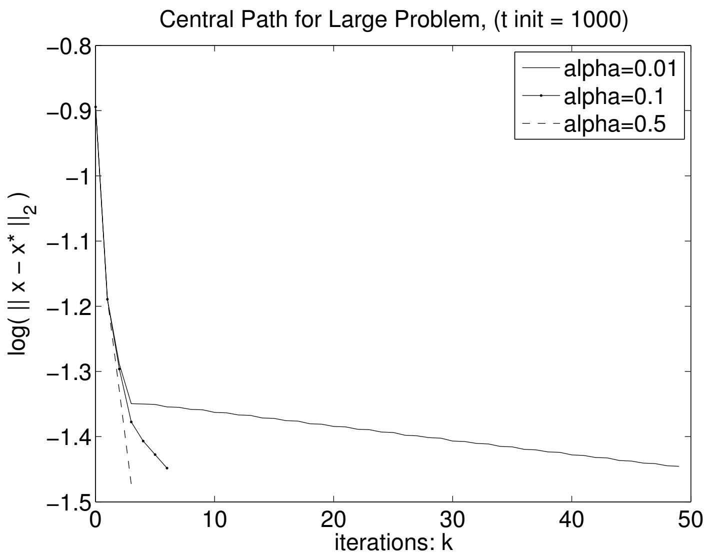
Newton's method start from $x_0 = 0.01 * ones(100, 1)$. We first employ CVX to compute a feasible point and then use newton_solver to figure out the analytic center. Since it has 50 elements, we are not going to post it here.

4.3 Generate a central path

Using similar commands with ones in response to previous question.

4.4 Plot the error $log(||x^{(k)} - x^*||)$ w.r.t k

Plot



5 Gradient and Newton

Command to be executed:

```
>> Rosenbrock
```

Plot

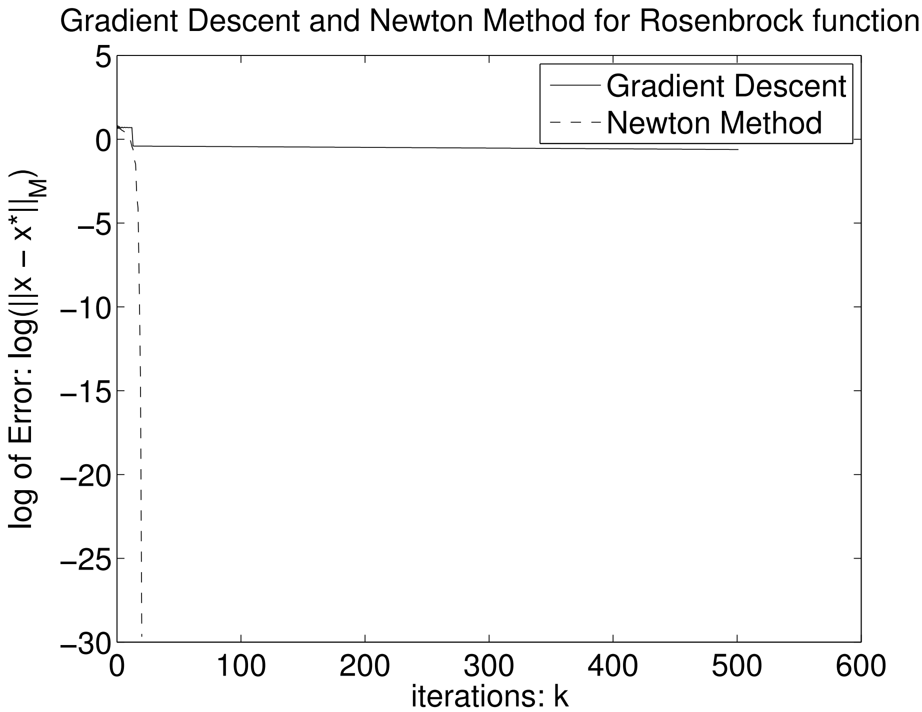


Figure 3: Gradient Descent and Newton Method on Rosenbrock function

Part II
Written Problems

6 $\alpha-$holder

Now we derive the rate of convergence with step size $t = 1$,

$$
\begin{aligned}
||\nabla f(x^+)||_2 &= ||\nabla f(x + \Delta x_{nt}) - \nabla f(x) - \nabla^2 f(x)\Delta x_{nt}||_2 && (6)\\
&= \left|\left| \int_0^1 \big(\nabla^2 f(x + t\Delta x_{nt}) - \nabla^2 f(x)\big)\Delta x_{nt} dt \right|\right|_2 && (7)\\
&\leq \int_0^1 ||\big(\nabla^2 f(x + t\Delta x_{nt}) - \nabla^2 f(x)\big)||_2 \cdot ||\Delta x_{nt}||_2 \cdot dt && (8)\\
&\leq \int_0^1 H||t\Delta x_{nt}||_2^\alpha \cdot ||\Delta x_{nt}||_2 \cdot dt && (9)\\
&= H||\Delta x_{nt}||_2^{1+\alpha} \cdot \frac{1}{1+\alpha} t^{1+\alpha}\Big|_0^1 && (10)\\
&= \frac{H}{1+\alpha}||\Delta x_{nt}||_2^{1+\alpha} && (11)\\
&= \frac{H}{1+\alpha}||\nabla^2 f(x)^{-1}\nabla f(x)||_2^{1+\alpha} && (12)\\
&\leq \frac{H}{(1+\alpha)m^{1+\alpha}}||\nabla f(x)||_2^{1+\alpha} && (13)\\
& && (14)
\end{aligned}
$$

Thus, we can easily find a constant C, such that

$$C \cdot ||\nabla f(x^+)||_2 \leq \Big(C \cdot ||\nabla f(x)||_2\Big)^{1+\alpha} \quad (15)$$

where $C = (\frac{H}{(1+\alpha)m^{1+\alpha}})^{-\alpha}$.
Recursively apply (15) and simulate the same inference of (9.34) and (9.35) in Boyd's book, we can conclude that once $||\nabla f(x^{(k)})||$ is small enough, then the convergence will go into convergence phrase with order at $1 + \alpha$.

A Codes Printout

A.1 Conjugate Gradient Algorithm

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 1. Conjugate Gradient Algorithm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function CGS(M, b, x_opt)
    [R, C] = size(M);
    assert (R == C, 'M should be square matrix.');
    assert (R == size(b,1), 'Dim of M and b should be consistent.');

    EPSILON = 10e-10; % how close solution do we need
    x_0 = zeros(size(x_opt)); % initial point

    listK = [];
    listlogMdiff = [];

    k = 0;
    x = x_0;
    r = b - M * x; % residual
    p = r;
    while 1,
        diff = x - x_opt;
        logMdiff = log(diff' * M * diff);

        fprintf ('iteration: %d, log(||x - x*||_M) = %f \n', k, logMdiff)
        listK = [listK k];
        listlogMdiff = [listlogMdiff logMdiff];

        alpha = (r' * r) / (p' * M * p);
        x = x + alpha * p;
        r_new = r - alpha * M * p;
        if r_new < EPSILON,
            break
        end
        beta = (r_new' * r_new) / (r' * r);
        p = r_new + beta * p;
        r = r_new;
        k = k + 1;
    end
    plot (listK, listlogMdiff)
    title('Conjugate Gradient Solver for Ax = b')
    xlabel('iterations: k')
    ylabel('log of M-norm: log(||x - x*||_M)')
end
```

A.2 Newtons Method

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 2. Newton Method
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function Newton(m, marker)
    x_opt = zeros(5, 1); % initial point
    t = 1; % step size fixed at 1
    x_0 = [0.5 0.5 0.5 0.5 0.5]';

    listK = [];
    listlogMdiff = [];

    k = 0;
    x = x_0;
    while 1,
        diff = x - x_opt;
        logMdiff = log(norm(diff,2)^2);

        fprintf ('iteration: %d, log(||x - x*||_2^2) = %f \n', k, logMdiff)
        listK = [listK k];
        listlogMdiff = [listlogMdiff logMdiff];

        grad = grad_func(x, m);
        hess = hess_func(x, m);
        x = x - t * inv(hess) * grad;

        k = k + 1;
        %if grad' * hess * grad <= eps,
        if logMdiff < -40,
            break
        end
    end
    plot (listK, listlogMdiff, marker)
    title('Newton Method for f_m', 'fontsize', 18)
    xlabel('iterations: k', 'fontsize', 18)
    ylabel('log of M-norm: log(||x - x*||_M)', 'fontsize', 18)
    set(gca, 'fontsize', 18)
end

function val = func(x, m)
    normx = norm(x, 2);
    val = normx^3 + 0.5 * m * normx^2;
end

function val = grad_func(x, m)
    val = 3 * norm(x, 2) * x + m * x;
end

function val = hess_func(x, m)
    len = size(x,1);
    normx = norm(x, 2);
    val = (3 / normx) * x * x' + ( 3* normx + m) * eye(len, len);
end
```

A.3 Newton Solver For Central Path Generation

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 3. General Newton's method solver for Central Path problem
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function x_opt = Newton_Solver (b, A, x_0, t, c)
    eta = 0.1;
    k = 0;
    x = x_0;
    x_opt = x;
    beta = 0.8;
    while k < 100,
        grad = grad_func(b, A, x, t, c);
        hess = hess_func(b, A, x, t, c);
        delta_x = - inv(hess) * grad;
        x = x + eta * delta_x;
        if norm(x-x_opt, 2)/norm(x, 2) <= 1e-3 ,
            break;
        else
            k = k + 1;
            x_opt = x;
        end
    end
    x_opt = x;
end
function val = func (b, A, x, t, c)
    N = size(x, 1); % number of variables
    C = size(A, 1); % number of constraint
    val = 0.0;
    for i = 1:N,
        val = val - log(x(i));
    end
    for i = 1:C,
        val = val - log(b(i) - A(i,:)*x);
    end
    val = val + t* c' * x;
end
function val = grad_func (b, A, x, t, c)
    N = size(x, 1);
    C = size(A, 1); % number of constraint
    val = zeros(N, 1);
    val = val - 1 ./ x;
    for i = 1:C,
        val = val - (1 / (b(i) - A(i,:)*x)) * (-A(i,:)');
    end
    val = val + t * c;
end
function val = hess_func (b, A, x, t, c)
    N = size(x, 1);
    C = size(A, 1); % number of constraint
    val = zeros(N, N);
    for i = 1:N,
        val(i, i) = val(i, i)+ (1/x(i))^2;
    end
    for i = 1:C,
        val = val + (1 / (b(i) - A(i,:)*x))^2 * (A(i,:)*A(i,:)');
    end
end
```

A.4 Small Linear Program

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 3. Central Path
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function x = CP (t_init, alpha, marker)
    % acquired global optima from cvx command
    x_opt = zeros(4, 1);
    % create barrier system of equation
    A = [1 3 0 1; 2 1 0 0; 0 1 4 1];
    b = [4 3 3]';
    c = [2 4 1 1]';
    % find analytical center
    x_F_init = [0.2, 0.2, 0.2, 0.2]';
    t = 0;
    x_F = Newton_Solver (b, A, x_F_init, t, c);
    % generate a central path
    listK = [0];
    listLogNorm2 = [log(norm(x_F-x_opt, 2))];

    k = 1;
    t = t_init;
    x = x_F;
    while 1,
        x = Newton_Solver (b, A, x, t, c);

        logNorm2 = log(norm(x - x_opt, 2));
        listK = [listK k];
        listLogNorm2 = [listLogNorm2 logNorm2];

        if norm(x - x_opt, 2) < 1e-5,
            break
        end
        t = t * (1 + alpha);
        k = k + 1;
    end
    semilogx (listK, listLogNorm2, marker)
    xlabel('iterations: k', 'fontsize', 18)
    ylabel('log( || x - x* ||_2 )', 'fontsize', 18)
    title('Central Path for the large problem', 'fontsize', 18)
    set(gca, 'fontsize', 18)
end
```

A.5 Larger Linear Program

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 3. Central Path for the large linear program
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function x = CP_large (c, A, b, t_init, alpha, marker)
    % acquired global optima from cvx command
    [M, N] = size(A);
    x_opt = CVX_solve_CP_large (c, A, b)

    % find feasible starting point
    %x_F_init = feasible(b, A, c, ones(50, 1)');
    x_F_init = 10e-9 * ones(50, 1);
    % find analytical center
    x_F = Newton_Solver (b, A, x_F_init, 0, c')
    % generate a central path
    listK = [0];
    listLogNorm2 = [log(norm(x_F-x_opt, 2))];

    k = 1;
    t = t_init;
    x = x_F;
    while k < 50,
        x = Newton_Solver (b, A, x, t, c');

        logNorm2 = log(norm(x - x_opt, 2));
        listK = [listK k];
        listLogNorm2 = [listLogNorm2 logNorm2];

        if norm(x - x_opt, 2) < 1e-5,
            break
        end
        t = t * (1 + alpha);
        k = k + 1;
    end
    plot (listK, listLogNorm2, marker)
    xlabel('iterations: k', 'fontsize', 18)
    ylabel('log( || x - x* ||_2 )', 'fontsize', 18)
    title('Central Path for the large problem', 'fontsize', 18)
    set(gca, 'fontsize', 18)
end
```

A.6 Gradient and Newton

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 5. Gradient descent and Newton on Rosenbrock function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function Rosenbrock ()
    EPSILON = 10e-10; % how close solution do we need
    alpha = 0.1;
    beta = 0.95;

    x_0 = [-1.2, 1]'; % initial point
    x_opt = [1 1]';

    [listK_GD, listError_GD] = gradient_descent (@func, @grad_func, ...
                                             x_0, x_opt, alpha, beta);
    [listK_NM, listError_NM] = newton_method (@func, @grad_func, @hess_func, ...
                                             x_0, x_opt, alpha, beta);

    plot (listK_GD, listError_GD, 'k-', listK_NM, listError_NM, 'k--')
    set(gca, 'fontsize', 18)
    legend ('Gradient Descent', 'Newton Method')
    title('Gradient Descent and Newton Method for Rosenbrock function')
    xlabel('iterations: k')
    ylabel('log of Error: log(||x - x*||_M)')
end

%% gradient descent with BTLS
function [listK, listError] = gradient_descent (f, grad_f, x_0, x_opt, alpha, beta)
    listK = [];
    listError = [];
    x = x_0;  % initial point
    k = 0;
    while 1,
        Error = log(norm(x_opt-x, 2));

        fprintf ('iteration: %d, Error = %f \n', k, Error)
        listK = [listK k];
        listError = [listError Error];
        if k > 500,
            break
        end

        gradient = grad_f(x);
        delta_x = -1.0 * gradient;
        t = 1.0;
        while f(x+t*delta_x) > f(x) + alpha*t*gradient'*delta_x,
           t = beta * t;
        end
        x = x + t * delta_x;
        k = k + 1;
    end
end
%% newton method with BTLS
function [listK, listError] = newton_method (f, grad_f, hess_f, x_0, x_opt, alpha, beta)
    listK = [];
    listError = [];
    x = x_0;  % initial point
    k = 0;
    while 1,
        % Error = f(x);
        Error = log(norm(x_opt-x, 2));

        fprintf ('iteration: %d, Error = %f \n', k, Error)
        listK = [listK k];
        listError = [listError Error];
```

```
        if k > 50,
            break
        end

        gradient = grad_f(x);
        hessian = hess_f(x);
        delta_x = -1.0 * inv(hessian) * gradient;
        t = 1.0;
        while f(x+t*delta_x) > f(x) + alpha*t*gradient'*delta_x,
           t = beta * t;
        end
        x = x + t * delta_x;
        k = k + 1;
    end
end

function [x_1, x_2] = parse (x)
    [m, n] = size(x);
    assert (m == 2 && n == 1);
    x_1 = x(1);
    x_2 = x(2);
end
function y = func (x)
    [x_1, x_2] = parse(x);
    y = 100 * (x_2 - x_1^2)^2 + (1-x_1)^2;
end
function y = grad_func (x)
    [x_1, x_2] = parse(x);
    y = zeros(2,1);
    y(1) = -400 * x_1 * (x_2-x_1^2) - 2 * (1-x_1);
    y(2) = 200 * (x_2 - x_1^2);
end
function y = hess_func (x)
    [x_1, x_2] = parse(x);
    y = zeros(2,2);
    y(1,1) = -400 * (x_2 - 3*x_1^2 ) + 2;
    y(1,2) = -400 * x_1;
    y(2,1) = y (1,2);
    y(2,2) = 200;
end
```