Monogenicity* and Torsion

Hanson Smith

University of Connecticut

Table of contents

Background and Motivation

Monogenicity

Let $f(x) \in \mathbb{Z}[x]$ be monic irreducible polynomial of degree n. Suppose α is a root of $f(x)$ and consider $\mathbb{Q}(\alpha)$. We call the field $\mathbb{Q}(\alpha)$ a *number field*. You can think of a number field as a generalization of $\mathbb{Q}$. Number theorists love $\mathbb{Z}$, and $\mathbb{Z}$ lives in $\mathbb{Q}$, so what is the analogue of $\mathbb{Z}$ living in $\mathbb{Q}(\alpha)$? Define *the ring of integers of* $\mathbb{Q}(\alpha)$, denoted $\mathcal{O}_{\mathbb{Q}(\alpha)}$, to be the set of all solutions to monic polynomials in $\mathbb{Z}[x]$ that are in $\mathbb{Q}(\alpha)$.

Our question today will be, When is $\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathbb{Z}[\beta]$ for some $\beta \in \mathbb{Q}(\alpha)$? To be more explicit, Is there a $\beta \in \mathbb{Q}(\alpha)$ such that

$$\mathcal{O}_{\mathbb{Q}(\alpha)} = \left\{ a_0 + a_1\beta + a_2\beta^2 + \cdots a_{n-1}\beta^{n-1} : a_i \in \mathbb{Z} \right\}?$$

When this is the case we say $\mathbb{Q}(\alpha)$ is *monogenic* or $\mathcal{O}_{\mathbb{Q}(\alpha)}$ admits a *power integral basis*.

Monogenicity

Let L/K be an extension of number fields of degree n with respective rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. We say L *is monogenic over* K or $\mathcal{O}_L$ *admits a power* $\mathcal{O}_K$*-integral basis* if $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in L$. More explicitly, $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is an $\mathcal{O}_K$-basis for the $\mathcal{O}_K$-module $\mathcal{O}_L$.

Let's get our hands dirty with this definition to see why it is natural and important!

Monogenicity of Quadratics

Take $\mathbb{Q}(\sqrt{d})$, with d square-free. What is the ring of integers of this field? What are the solutions to monic polynomials in $\mathbb{Z}[x]$?

Take an arbitrary element $a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$. We compute

$$f(x) = \left(x - \left(a + b\sqrt{d}\right)\right)\left(x - \left(a - b\sqrt{d}\right)\right) = x^2 - 2ax + a^2 - b^2 d.$$

In order for $f(x)$ to be in $\mathbb{Z}[x]$, we need $2a \in \mathbb{Z}$. If $a \in \mathbb{Z}$, then $b \in \mathbb{Z}$ in order to have $f(x) \in \mathbb{Z}[x]$. But, if $a = \frac{a_0}{2}$, then a bit of arithmetic modulo 4 shows that we need $b = \frac{b_0}{2}$ and $\left(a_0^2 - b_0^2 d\right)/4 \in \mathbb{Z}$ in order to have $a^2 - b^2 d \in \mathbb{Z}$. For this case to work out, it is necessary and sufficient that $d \equiv 1 \bmod 4$. Thus the ring of integers of $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ if $d \equiv 1 \bmod 4$ and $\mathbb{Z}\left[\sqrt{d}\right]$ otherwise. In either case, $\mathbb{Q}(\sqrt{d})$ is monogenic, but $\sqrt{d}$ generates a power integral basis if and only if $d \not\equiv 1 \bmod 4$. The key here that we will generalize later on is d is not a square modulo 2^2.

Monogenicity of Cyclotomics

Let's imagine we are building algebraic number theory. We want to understand number fields and rings of integers. We've already figured out a few things about $\mathbb{Q}(\sqrt{d})$. Where do we go next? How about we look at number fields generated by really nice polynomials? What about the polynomial $x^n - 1$?

Let ζ_n be a primitive n^{th} root of unity, so $\zeta_n^n = 1$, but $\zeta_n^k \neq 1$ for $1 \leq k < n$. Let's look at $\mathbb{Q}(\zeta_n)$. It is a bit more difficult than in the quadratic case, but one can show that the ring of integers of $\mathbb{Q}(\zeta_n)$ is $\mathbb{Z}[\zeta_n]$!

Flippant Conjecture: Every number field is monogenic!

“All that glistens is not gold.”

Does this always happen? When one is learning (or discovering) algebraic number theory, they might be tempted to think every extension of $\mathbb{Q}$ is monogenic. It works for the first two families of number fields we encounter, so maybe we expect it always happens.

Expectation is the root of all heartache.

- William Shakespeare

Dedekind-Kummer Factorization

Theorem (Dedekind building on work of Kummer)

Let $f(x)$ be a monic, irreducible polynomial in $\mathbb{Z}[x]$ with α denoting a root. If $p \in \mathbb{Z}$ is a prime that does not divide $\left[\mathcal{O}_{\mathbb{Q}(\alpha)} : \mathbb{Z}[\alpha]\right]$, then the factorization of p in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ mirrors the factorization of $f(x)$ in $\mathbb{F}_p[x]$. That is,

$$f(x) \equiv f_1(x)^{e_1} \cdots f_r(x)^{e_r} \bmod p \quad \textit{and} \quad (p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

For example, consider $\mathbb{Q}(\alpha)$ where α is a root of $x^3 - x^2 - 2x - 8$. Dedekind computed the factorization $(2) = \mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_2''$.

Thus, if this field is monogenic, there is a cubic polynomial that generates and has **three** distinct linear factors in $\mathbb{F}_2[x]$.

Common Index Divisors

One can generalize Dedekind's example: If a prime $p < n$ splits completely in an extension $K/\mathbb{Q}$ of degree n, then K is not monogenic. [Hensel, 1894] builds on these ideas to show the following.

Theorem

Fix a prime p. The prime p divides $[\mathcal{O}_K : \mathbb{Z}[\theta]]$ for every algebraic integer θ generating K over $\mathbb{Q}$ if and only if there is an integer f such that the number of prime ideal factors of $p\mathcal{O}_K$ with inertia degree f is greater than the number of monic irreducibles of degree f in $\mathbb{F}_p[x]$.

Any p satisfying this theorem is called a *common index divisor*. The terms 'essential discriminant divisor' and 'inessential discriminant divisor' also appear in the literature. The shortcomings of the English nomenclature likely come from what Neukirch [Neukirch, 1999, page 207] calls "the untranslatable German catch phrase [...] 'außerwesentliche Diskriminantenteile.'" Our nomenclature is closer to Fricke's 'ständiger Indexteiler.'

Other Obstructions

It turns out that common index divisors are not the only obstruction to monogenicity:

Example: [Narkiewicz, 2004, Chapter 2.2.6] Consider the number field given by $K = \mathbb{Q}\left(\sqrt[3]{7 \cdot 5^2}\right) = \mathbb{Q}\left(\sqrt[3]{5 \cdot 7^2}\right)$. The elements $\left\{1, \sqrt[3]{7 \cdot 5^2}, \sqrt[3]{7^2 \cdot 5}\right\}$ form an integral basis for $\mathcal{O}_K$. For any fixed prime p, one can find $\alpha \in \mathcal{O}_K$ such that $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ is not divisible by p; however, K is not monogenic.

In fact, the index $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ can be described with a form, called the *index form*. In the example above it is $5X_1^3 - 7X_2^3$. We can consider it modulo 7 to see that there are no values for which the form is ± 1.

Radicals

Radical (Pure) Motivation

Remember, the ring of integers of $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\sqrt{d}]$ if and only if d is squarefree and $d \not\equiv 1 \bmod 4$. Can we generalize?

[Westlund, 1910] computed the discriminant and an integral basis for the radical extensions $\mathbb{Q}\left(\sqrt[p]{\alpha}\right)$ over $\mathbb{Q}$, where $\alpha \in \mathbb{Z}$ and p is a prime. Westlund also identified when $\sqrt[p]{\alpha}$ yields a power integral basis for $\mathbb{Q}\left(\sqrt[p]{\alpha}\right)$. [Gassert, 2017] (see also [Jakhar et al., 2017]) gives necessary and sufficient conditions for the ring of integers of $\mathbb{Q}\left(\sqrt[n]{\alpha}\right)$ to be $\mathbb{Z}\left[\sqrt[n]{\alpha}\right]$. Having $\sqrt[n]{\alpha}$ generate a power integral basis is dependent on the congruence

$$\alpha^p \equiv \alpha \bmod p^2, \tag{1}$$

where p divides n. Loosely speaking, Congruence (1) is an obstruction to $\sqrt[n]{\alpha}$ generating a power integral basis.

The Question

Let K be a number field and let $x^n - \alpha$ be an irreducible element of $K[x]$. Consider $L = K(\sqrt[n]{\alpha})$.

Question: When does $\sqrt[n]{\alpha}$ generate an $\mathcal{O}_K$-integral basis for $\mathcal{O}_L$?

† It is worth noting that even the existence of a $\mathcal{O}_K$-integral basis is not guaranteed.

A Digestible Corollary

Theorem ([Smith, 2021b])

Let p be an odd, rational prime and define $K = \mathbb{Q}(\zeta_p)$. Note $(1 - \zeta_p)$ is the unique prime ideal of $\mathcal{O}_K$ above p. Let $\alpha \in \mathcal{O}_K$ and suppose that $x^p - \alpha$ is irreducible in $\mathcal{O}_K[x]$. The ring of integers of $K(\sqrt[p]{\alpha})$ is $\mathcal{O}_K[\sqrt[p]{\alpha}]$ if and only if (α) is a square-free ideal of $\mathcal{O}_K$ and the congruence

$$\alpha^p \equiv \alpha \bmod (1 - \zeta_p)^2 \tag{2}$$

is not satisfied.

This is quite a contrast to the situation over $\mathbb{Q}$. [Gras, 1986] shows that the only abelian extensions of prime degree $p \geq 5$ are maximal real subfields of cyclotomic fields.

A Radical Result

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ dividing n, we write p for the residue characteristic and f for the residue class degree. If $\mathfrak{p}$ divides n, we factor $n = p^e m$ with $\gcd(m, p) = 1$. Let ε be congruent to e modulo f with $1 \leq \varepsilon \leq f$ and define β to be α to the power $p^{f-\varepsilon}$. By construction β is a p^e-th root of α modulo $\mathfrak{p}$. Our congruence generalizes to

$$\alpha^{p^{f-\varepsilon+e}} = \beta^{p^e} \equiv \alpha \bmod \mathfrak{p}^2. \tag{3}$$

In the case where $e \leq f$, this is simply

$$\alpha^{p^f} \equiv \alpha \bmod \mathfrak{p}^2.$$

Theorem ([Smith, 2021b]) The ring of integers of $K\left(\sqrt[n]{\alpha}\right)$ is $\mathcal{O}_K\left[\sqrt[n]{\alpha}\right]$ if and only if (α) is a square-free ideal of $\mathcal{O}_K$ and every prime $\mathfrak{p}$ dividing n does not satisfy Congruence (3).

A Very Recent Radical Result

Theorem (S.) If p is a common index divisor for $\mathbb{Q}(\sqrt[n]{a})/\mathbb{Q}$, then it is necessary that $p \mid n$.
If $n = p^m$, then for p to be a common index divisor it is necessary and sufficient that $v_p\left(a^{p^m} - a\right) > p$.

Division Field Motivation

Cyclotomic Fields ($\mathbb{G}_m$-Division Fields)

Consider $\mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^*$. The torsion points of $\mathbb{G}_m(\mathbb{Q})$ are the roots of unity, the solutions to $x^n - 1$.

One way to study $\mathbb{G}_m(\mathbb{Q})$ is to look at the n^{th} $\mathbb{G}_m(\mathbb{Q})$-division field: $\mathbb{Q}\left(\mathbb{G}_m(\mathbb{Q})[n]\right) = \mathbb{Q}(\zeta_n)$. The primes dividing n are the only ramified primes. The residue class degree of a prime p not dividing n is the least positive integer f such that $p^f \equiv 1 \bmod n$.

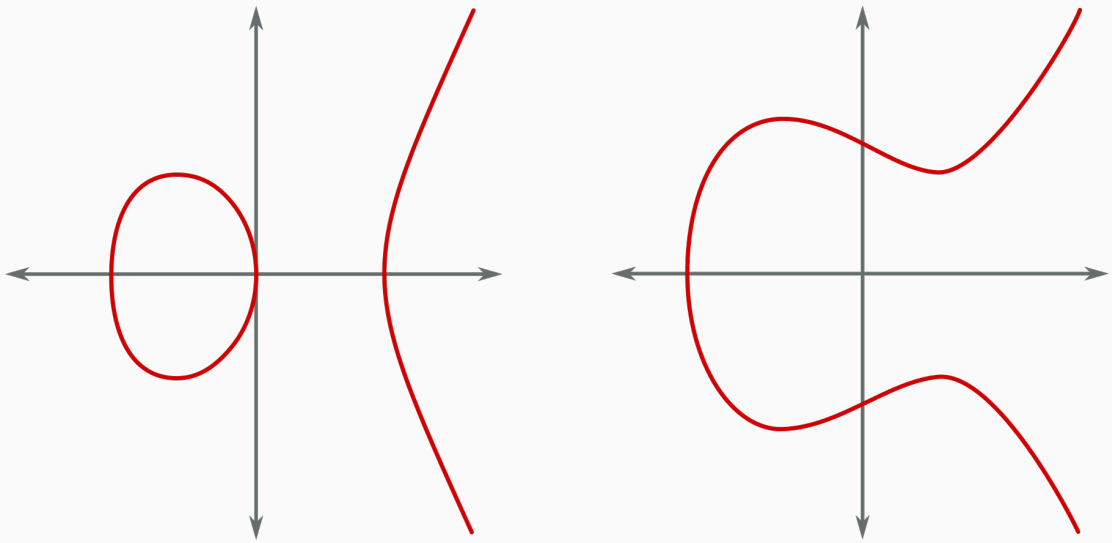
The ring of integers is $\mathbb{Z}[\zeta_n]$.

Elliptic Curves

An *elliptic curve* is a smooth, projective curve of genus one with a specified point (the identity). You can think of an elliptic curve E as a donut.

Elliptic Curves

An *elliptic curve* is a smooth, projective curve of genus one with a specified point (the identity). You can think of an elliptic curve E as the solutions to an equation of the form $y^2 = x^3 + Ax + B$.



For this talk, the reason that we care about elliptic curves is the points of an elliptic curve form an abelian group.

Elliptic Curves

Mordell-Weil: If K is a number field, then $E(K) \cong \mathbb{Z}^r \times E(K)_{\text{tors}}$. The integer r is called the *rank* and $E(K)_{\text{tors}}$ is called the *torsion subgroup*.

We denote the n-torsion points of an elliptic curve E by $E[n]$. Over $\mathbb{C}$ one has $E(\mathbb{C})[n] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We will be looking at the n-division fields of an elliptic curve: $\mathbb{Q}(E[n])$. The only ramified primes in $\mathbb{Q}(E[n])$ are the primes dividing n and the primes dividing the discriminant of E.

Splitting in $\mathbb{Q}(E[n])$

Let a_p be the trace of Frobenius at p, let b_p be the index $[\mathcal{O}_K : \mathrm{End}_{\mathbb{F}_p}(E)]$, and write Δ_{End} for the discriminant of $\mathrm{End}_{\mathbb{F}_p}(E)$. Consider the matrix

$$\sigma_p = \begin{bmatrix} \dfrac{a_p + b_p\delta_{\mathrm{End}}}{2} & b_p \\ \dfrac{b_p(\Delta_{\mathrm{End}} - \delta_{\mathrm{End}})}{4} & \dfrac{a_p - b_p\delta_{\mathrm{End}}}{2} \end{bmatrix}, \tag{4}$$

where $\delta_{\mathrm{End}} = 0, 1$ according to whether $\Delta_{\mathrm{End}} \equiv 0, 1$ modulo 4.

[Duke and Tóth, 2002]: Suppose n is prime to p. When reduced modulo n, the matrix σ_p yields a global representation of the Frobenius class over p in $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$. In particular, the order of σ_p modulo n is the residue class degree of p in $\mathbb{Q}(E[n])$.

Comparing $\mathbb{Q}(\zeta_n)$ and $\mathbb{Q}(E[n])$

Motivating question: When is $\mathbb{Q}(E[n])$ *monogenic*?

Gonzáles-Jiménez and Lozano-Robledo show that $\mathbb{Q}(E[n])$ coincides with $\mathbb{Q}(\zeta_n)$ sometimes. In particular when $n = 2, 3, 4,$ and 5 this can happen.

Monogenic 2-Division Fields

Legendre form, $E_\lambda : \ y^2 = x(x-1)(x-\lambda)$, requires solving

$$g_\lambda(\lambda) = \lambda^6 - 3\lambda^5 + \left(6 - \frac{j}{256}\right)\lambda^4 + \left(\frac{j}{128} - 7\right)\lambda^3 + \left(6 - \frac{j}{256}\right)\lambda^2 - 3\lambda + 1,$$

where j is the j-invariant of E_λ. Notice that $\lambda \in \mathbb{Q}$ yields a family of elliptic curves with monogenic 2-division fields since $\mathbb{Q}(E_\lambda[2]) = \mathbb{Q}$. Less trivially, we have the following family.

Theorem ([Smith, 2021c])
Suppose $g_\lambda(x)$ is irreducible. When $j \in \mathbb{Z}$ is divisible by 256 and both $\frac{j}{256}$ and $\frac{j}{64} - 27 = \frac{1}{64}(j - 1728)$ are square-free, then $\mathbb{Q}(\lambda) = \mathbb{Q}\left(E_\lambda[2]\right)$ is monogenic over $\mathbb{Q}$ with λ generating a power integral basis for the ring of integers.

Results for Division Fields of Elliptic Curves

Main Result A

There are a lot of division fields $\mathbb{Q}(E[n])$ that are not monogenic!

Algorithm/theorem statement for $p = 2$ [Smith, 2021c, Thm. 4.4]

If E is an elliptic curve over $\mathbb{Q}$ whose reduction at the prime 2 has trace of Frobenius a_2 and such that, for one of the n listed on the following slide, the Galois representation

$$\rho_{E,n} : \mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

is surjective. Then $\mathbb{Q}(E[n])$ is not monogenic. Moreover, 2 is a common index divisor of $\mathbb{Q}(E[n])$.

Results for $p = 2$

a_2	σ_2	non-monogenic n
0	$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	3, 5, 9, 11, 15, 17, 21, 27, 33, 43, 51, 57, 63, 85, 91, 93, 105, 117, 129, 171, 195, 255, 257, 273, 315, 331, 341, 381, 455, 513, 585, 657, 683, 771, 819, 993
1	$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$	11
-1	$\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$	11, 23
2	$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$	5, 13, 15, 17, 41, 51, 65, 85, 91, 105, 117, 145, 195, 205, 255, 257, 273, 315, 455, 565, 585, 771, 819
-2	$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$	5, 13, 15, 17, 41, 51, 65, 85, 91, 105, 117, 145, 195, 205, 255, 257, 273, 315, 455, 565, 585, 771, 819

Main Result B

There are a lot of division fields $\mathbb{Q}(E[n])$ that are not monogenic!

Theorem (Cor. 4.8 of [Smith, 2021c])

Let $E/\mathbb{Q}$ be an elliptic curve without CM, then for infinitely many $n > 1$ the division field $\mathbb{Q}(E[n])$ is not monogenic.

Results for Abelian Varieties of Dimension > 1

...Or How to Sound Like You Understood a Talk

If you do something for elliptic curves, you can always ask the question, "Can I do this for abelian varieties?"

Difficulties

The construction of the Frobenius in [Duke and Tóth, 2002] was very important for our work with elliptic curves. They use Deuring lifting for their construction. For an arbitrary abelian variety such a canonical lift does not necessarily exist. Canonical lifts exist if we restrict to ordinary or almost ordinary abelian varieties, but we are interested in low p-rank too.

Difficulties

Instead, we opted to generalize the approach taken by [Centeleghe, 2016] This approach relies on the fact that if A is an abelian variety over a field k with CM by a Gorenstein ring (i.e., if $\mathrm{End}_k(A)$ is a Gorenstein ring), then the Tate module $T_\ell(A)$ is free of rank one over $\mathrm{End}_k(A) \otimes \mathbb{Z}_\ell$. This is great! Now we **just** need to write down a basis for the relevant orders in an arbitrary CM field of degree $2g$, where the dimension g is greater than 1. Even if we restrict to $g = 2$ and to maximal orders, this last step is difficult and results in an overwhelming number of cases. Thus we focus on the minimal case.

The Minimal Endomorphism Ring

Suppose $|k| = p^m = q$. $\mathrm{End}_k(A)$ must contain **Frobenius** π and its dual **verschiebung** v. In fact, all orders of $\mathrm{End}_k(A) \otimes \mathbb{Q}$ containing π and v are endomorphism rings. Thus the smallest possible endomorphism ring is $\mathbb{Z}[\pi, v]$.

The characteristic polynomial of π is a *Weil q-polynomial*. We restrict to abelian varieties with irreducible Weil q-polynomials so that $\mathbb{Z}[\pi, v]$ is Gorenstein.

The Matrix Representing Frobenius

Let A/k be an abelian variety with an irreducible Weil q-polynomial and $\mathrm{End}_k(A) \cong \mathbb{Z}[\pi, v]$. First, note that $\{1, \pi, \ldots, \pi^g, v, \ldots, v^{g-1}\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}[\pi, v]$.

Write

$$f(x) = x^{2g} + a_{2g-1}x^{2g-1} + \cdots + a_1 x + a_0$$

for the Weil q-polynomial of A. The following matrix yields the action of π on $\mathbb{Z}[\pi, v]$, and hence on $T_\ell(A)$.

The Matrix Representing Frobenius

$$\sigma_{\mathfrak{p}} = \begin{array}{c} \begin{array}{cccccccccccc} 1 & \pi & \pi^2 & \pi^{g-2} & \pi^{g-1} & \pi^g & v & v^2 & v^3 & & v^{g-1} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -qa_{g+1} & q & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & -a_g & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & -a_{g+1} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots & -a_{g+i-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & -a_{2g-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & -a_{2g-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -qa_2 & 0 & q & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -qa_3 & 0 & 0 & q & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -qa_{i+1} & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & -qa_{g-1} & 0 & 0 & 0 & \dots & q \\ 0 & 0 & \dots & 0 & 0 & -q & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \pi \\ \pi^2 \\ \pi^i \\ \pi^{g-1} \\ \pi^g \\ v \\ v^2 \\ \\ v^i \\ v^{g-2} \\ v^{g-1} \end{array}$$

Non-monogenic Division Fields of Abelian Surfaces

Algorithm/theorem statement for $p = 2$ ([Smith, 2021a])

Let $A/\mathbb{F}_2$ be an abelian surface and write the Weil 2-polynomial of A as

$$x^4 + a_3x^3 + a_2x^2 + 2a_3x + 2^2.$$

Suppose the Weil 2-polynomial is irreducible, $\mathrm{End}_{\mathbb{F}_2}(A)$ is minimal, and

$$\rho_{\hat{A},n} : \mathrm{Gal}(\mathbb{Q}(\hat{A}[n])/\mathbb{Q}) \to \mathrm{GSp}_4(\mathbb{Z}/n\mathbb{Z})$$

is surjective for some $\hat{A}/\mathbb{Q}$ that reduces to A modulo 2. Then Table 1 shows the odd $n < 500$ for which the prime 2 is a common index divisor of $\mathbb{Q}(\hat{A}[n])$ over $\mathbb{Q}$.

Non-monogenic Division Fields of Abelian Surfaces

a_3	a_2	p-rank	non-monogenic n
-3	5	2	3, 19, 31, 57, 61, 93, 171, 183
-2	2	0	5, 7, 9, 13, 15, 21, 35, 37, 39, 45, 51, 61, 63, 65, 85, 91, 105, 109, 111, 117, 119, 133, 135, 153, 171, 185, 189, 195, 205, 219, 221, 241, 247, 255, 259, 273, 285, 305, 315, 325, 327, 333, 351, 357, 365, 377, 399, 455, 481, 485
-2	3	2	7, 47
-1	-1	2	5, 9, 11, 15, 23, 37, 43, 45, 67, 111, 127, 135, 151, 185, 203, 301, 333
-1	0	1	47
-1	1	2	3, 9, 103, 127
-1	3	2	5, 15, 59
0	-3	2	3, 5, 9, 11, 15, 23, 29, 33, 37, 45, 53, 87, 111, 135, 137, 185, 203, 233, 281, 301, 333
0	-2	0	3, 5, 7, 9, 11, 13, 15, 19, 21, 27, 33, 35, 39, 43, 45, 51, 57, 63, 65, 67, 73, 77, 81, 85, 91, 93, 99, 105, 109, 111, 117, 119, 129, 133, 135, 151, 153, 171, 185, 189, 195, 201, 217, 219, 221, 231, 241, 247, 255, 259, 273, 279, 285, 301, 315, 327, 331, 333, 337, 341, 351, 357, 365, 381, 387, 399, 441, 453, 455, 481, 485

Table 1: Odd $n < 500$ where 2 is a common index divisor in $\mathbb{Q}(\hat{A}[n])$

Non-monogenic Division Fields of Abelian Surfaces

a_3	a_2	p-rank	non-monogenic n
0	-1	2	3, 17, 19, 23, 31, 57, 61, 93, 171, 183, 229
0	1	2	3, 9, 17, 19, 23, 47, 57, 61, 69, 93, 171, 183, 229
0	2	0	3, 5, 7, 9, 13, 15, 19, 21, 27, 31, 35, 39, 45, 49, 51, 57, 63, 65, 73, 77, 85, 89, 91, 93, 99, 105, 109, 111, 117, 119, 127, 133, 135, 151, 153, 161, 171, 185, 189, 195, 217, 219, 221, 231, 241, 247, 255, 259, 273, 279, 285, 301, 315, 327, 331, 333, 337, 341, 351, 357, 365, 381, 387, 399, 441, 453, 455, 481, 485
1	-1	2	5, 7, 9, 11, 15, 37, 43, 45, 67, 79, 111, 135, 185, 203, 301, 333
1	0	1	47
1	1	2	3, 9
1	3	2	5, 15, 59
2	2	0	5, 7, 9, 13, 15, 21, 35, 37, 39, 45, 51, 61, 63, 65, 85, 91, 105, 109, 111, 117, 119, 133, 135, 153, 171, 185, 189, 195, 205, 219, 221, 241, 247, 255, 259, 273, 285, 305, 315, 325, 327, 333, 351, 357, 365, 377, 399, 455, 481, 485
2	3	2	7, 47
3	5	2	3, 19, 31, 57, 61, 93, 171, 183

Table 2: Odd $n < 500$ where 2 is a common index divisor in $\mathbb{Q}(\hat{A}[n])$

Non-monogenic Division Fields of Abelian Varieties

General Algorithm ([Smith, 2021a]) *Let $A/\mathbb{F}_p$ be an abelian variety and write the Weil p-polynomial of A as*

$$x^{2g} + a_{2g-1}x^{2g-1} + \cdots + p^{g-1}a_{2g-1}x + p^g.$$

Suppose the Weil p-polynomial is irreducible, $\mathrm{End}_{\mathbb{F}_p}(A)$ *is minimal, and*

$$\rho_{\hat{A},n} : \mathrm{Gal}(\mathbb{Q}(\hat{A}[n])/\mathbb{Q}) \to \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$$

is surjective for some $\hat{A}/\mathbb{Q}$ that reduces to A modulo p. If $\gcd(n,p) = 1$*, then n satisfies* (5) *if and only if p is a common index divisor of* $\mathbb{Q}(\hat{A}[n])/\mathbb{Q}$*.*

$$\begin{aligned}\prod_{\ell \mid n,\ \ell \text{ prime}} \left((\ell-1)\ell^{g^2}\prod_{i=1}^{g}(\ell^{2i}-1)\right)\cdot\left(\ell^{2g^2+g+1}\right)^{v_\ell(n)-1} \\ > \sum_{d \mid \mathrm{ord}_n(\sigma_p)} p^d \mu\left(\frac{\mathrm{ord}_n(\sigma_p)}{d}\right).\end{aligned} \tag{5}$$

Non-monogenic Division Fields of Abelian Threefolds ($p = 2$)

a_5	a_4	a_3	p-rank	non-monogenic n	a_5	a_4	a_3	p-rank	non-monogenic n
-4	9	-15	3	7, 11, 23, 29, 43, 71, 87, 113, 127	0	1	-3	3	3, 9
-3	2	1	3	7, 11, 29, 43, 71, 87, 113, 127	0	1	-1	3	
-3	6	-9	3	3, 9, 27, 153	0	1	3	3	3, 9
-2	0	3	3	107, 149	0	2	-2	0	
-2	1	0	2	3, 5, 11, 55, 83	0	2	-1	3	7
-2	3	-5	3	3, 9, 27, 59, 63	1	-1	-5	3	3, 9
-2	3	-3	3	5, 83, 131	1	-1	-4	2	3, 7, 49
-2	5	-7	3	3, 7	1	0	-3	3	7, 77, 103
-1	-1	5	3	3, 9	1	0	1	3	3
-1	0	-1	3	3	1	1	0	2	3, 7
0	0	-3	3	3, 7, 9, 13, 15, 21, 27, 29, 31, 35, 39, 45, 63, 65, 87, 91, 93, 105, 117, 123, 141, 151, 195	2	4	6	0	3
0	0	-2	0	3, 7, 11, 15, 23, 29, 37, 45, 67, 71, 79	2	5	7	3	3, 7
0	0	-1	3	3, 5, 7, 15, 19, 21, 25, 35, 45, 63, 71, 75, 95, 97, 105, 123, 133	3	2	-1	3	7, 11, 23, 29, 43, 71, 87, 113, 127
0	0	1	3	3, 5, 7, 15, 19, 21, 25, 35, 45, 47, 49, 63, 75, 95, 97, 105, 123, 133	3	5	7	3	7
0	0	2	0	3, 7, 11, 15, 23, 29, 37, 45, 67	3	6	9	3	3, 9, 27, 153
0	0	3	3	3, 7, 9, 13, 15, 21, 27, 29, 31, 35, 39, 45, 47, 63, 65, 71, 87, 91, 93, 105, 117, 123, 141, 151, 195	4	9	15	3	7, 11, 29, 43, 71, 87, 113, 127

Thank You!

References i

Centeleghe, T. G. (2016).

Integral Tate modules and splitting of primes in torsion fields of elliptic curves.

Int. J. Number Theory, 12(1):237–248.

Duke, W. and Tóth, A. (2002).

The splitting of primes in division fields of elliptic curves.

Experiment. Math., 11(4):555–565 (2003).

Gassert, T. A. (2017).

A note on the monogeneity of power maps.

Albanian J. Math., 11(1):3–12.

References ii

Gras, M.-N. (1986).

Non monogénéité de l'anneau des entiers des extensions cycliques de Q de degré premier $l \geq 5$.

J. Number Theory, 23(3):347–353.

Hensel, K. (1894).

Arithmetische Untersuchungen über die gemeinsamen ausserwesentlichen Discriminantentheiler einer Gattung.

J. Reine Angew. Math., 113:128–160.

Jakhar, A., Khanduja, S. K., and Sangwan, N. (2017).

Characterization of primes dividing the index of a trinomial.

Int. J. Number Theory, 13(10):2505–2514.

References iii

Narkiewicz, W. (2004).

Elementary and analytic theory of algebraic numbers.

Springer Monographs in Mathematics. Springer-Verlag, Berlin, third edition.

Neukirch, J. (1999).

Algebraic number theory, volume 322 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].

Springer-Verlag, Berlin.

Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

References iv


Smith, H. (2021a).

Frobenius finds non-monogenic division fields of abelian varieties.


Smith, H. (2021b).

The monogeneity of radical extensions.

Acta Arith., 198(3):313–327.


Smith, H. (2021c).

Non-monogenic division fields of elliptic curves.

J. Number Theory, 228:174–187.

References v

Westlund, J. (1910).

On the fundamental number of the algebraic number-field $k(\sqrt[p]{m})$.

Trans. Amer. Math. Soc., 11(4):388–392.

An Example with an Ordinary Elliptic Curve

Suppose E is an elliptic curve with $a_2 = 1$. The characteristic polynomial of Frobenius is $x^2 - x + 2$ and this has discriminant -7. Letting π denote the Frobenius endomorphism of E over $\mathbb{F}_2$, we have $\mathrm{End}_{\mathbb{F}_2}(E) \cong \mathbb{Z}[\pi] = \mathcal{O}_{\mathbb{Q}(\pi)}$.

Combining all this information, we see Duke and Tóth's matrix representing π is

$$\sigma_2 = \begin{bmatrix} 8/2 & (-7 \cdot 8)/4 \\ 1 & -6/2 \end{bmatrix} = \begin{bmatrix} 4 & -14 \\ 1 & -3 \end{bmatrix}.$$

Denote the order of σ_2 modulo n by $\mathrm{ord}\,(\sigma_2, n)$. This is the residue class degree of 2 in $\mathbb{Q}(E[n])$.

An Example with an Ordinary Elliptic Curve

Generically, we expect the degree of $\mathbb{Q}(E[n])$ over $\mathbb{Q}$ to be $|\operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z})|$. Thus 2 will split into $\frac{|\operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z})|}{\operatorname{ord}(\sigma_2,n)}$ primes in $\mathbb{Q}(E[n])$.

The number of irreducible polynomials of degree m in $\mathbb{F}_p[x]$ is $\frac{1}{m}\sum_{d|m} p^d \mu\left(\frac{m}{d}\right)$. With Dedekind's factorization theorem in mind, we compare $\frac{|\operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z})|}{\operatorname{ord}(\sigma_2,n)}$ and $\frac{1}{\operatorname{ord}(\sigma_2,n)} \sum_{d|\operatorname{ord}(\sigma_2,n)} 2^d \mu\left(\frac{\operatorname{ord}(\sigma_2,n)}{d}\right)$.

If the number of irreducible polynomial of degree $\operatorname{ord}(\sigma_2, n)$ in $\mathbb{F}_2[x]$ is less than $\frac{|\operatorname{GL}_2(\mathbb{Z}/n\mathbb{Z})|}{\operatorname{ord}(\sigma_2, n)}$, then 2 must divide the index of any monogenic order in $\mathcal{O}_{\mathbb{Q}(E[n])}$. We find that σ_2 has order 10 modulo 11, so that 2 splits into 1320 primes in $\mathbb{Q}(E[11])$. There are only 99 irreducible polynomials of degree 10 in $\mathbb{F}_2[x]$. Thus 2 is a common index divisor of $\mathbb{Q}(E[11])$ over $\mathbb{Q}$.