Monogenic Fields Arising from Trinomials

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Abstract. We call a polynomial *monogenic* if a root θ has the property that $\mathbb{Z}[\theta]$ is the full ring of integers in $\mathbb{Q}(\theta)$. Using the Montes algorithm, we find sufficient conditions for $x^n + ax + b$ and $x^n + cx^{n-1} + d$ to be monogenic (this was first studied by Jakhar, Khanduja, and Sangwan using other methods). Weaker conditions are given for $n = 5$ and $n = 6$. We also show that each of the families $x^n + bx + b$ and $x^n + cx^{n-1} + cd$ are monogenic infinitely often and give some positive densities in terms of the coefficients.

1. Introduction

Let K be a number field, and denote its ring of integers by $\mathcal{O}_K$. If $\mathcal{O}_K = \mathbb{Z}[\theta]$ for some $\theta \in \mathcal{O}_K$, we say that $\mathcal{O}_K$ admits a *power integral basis* or that K is *monogenic*. The classification of monogenic number fields is often known as Hasse's problem.

We use the term *monogenic* to refer to any polynomial f for which a root θ has the property that $\mathbb{Z}[\theta]$ is the full ring of integers in $\mathbb{Q}(\theta)$. Our work seeks to give sufficient conditions for certain polynomials to be monogenic. By elementary considerations, any polynomial having a squarefree discriminant is automatically monogenic. Both Kedlaya [14] and Boyd, Martin, and Thom [2] find families of polynomials with squarefree discriminant (hence monogenic). We study families with non-squarefree discriminant.

Our main tool in approaching Hasse's problem is the Montes algorithm (for an overview, see [17]; for in-depth treatments, see [3] or [9]). We limit ourselves to irreducible trinomials of the form $x^n + ax + b$ or $x^n + ax^{n-1} + b$, whose discriminants are strongly non-squarefree (see Theorem 2.1).

For these families, we are able to provide sufficient conditions (Theorems 3.5 and 3.6) for monogeneity, for any degree $n \geq 2$, as well as weaker requirements (Theorems 3.1, 3.2, 3.3, and 3.4) when $n = 5$ or $n = 6$. Using the Montes algorithm to treat the case $n = 4$ has already been studied in [17]. We also study a non-trinomial family in Theorem 3.7. Furthermore, we demonstrate infinite families of polynomials (Theorems 3.8 and 3.9) whose roots yield power integral bases for their associated rings of integers infinitely often, namely $x^n + bx + b$ and $x^n + cx^{n-1} + cd$; in fact, we give a density on the coefficients satisfying the sufficient conditions of Theorems 3.5 and 3.6.

The literature regarding monogenic fields is extensive. See [15] for a general survey of the results. Much of the literature focuses on a given degree or Galois

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group. Our monogenic families are of varying degree, so we only briefly survey the literature regarding families of varying degree. Classically, monogeneity is known for cyclotomic fields and the maximal real subfields thereof. Gras [7] shows that, with the exception of maximal real subfields of cyclotomic fields, abelian extensions of prime degree greater than 5 are not monogenic. Gras [6] also shows that almost all abelian extensions with degree coprime to 6 are not monogenic. Gassert [5] gives necessary and sufficient conditions for the monogeneity of extensions of the form $x^n + a$. In [4], Gassert investigates the monogeneity of extensions given by shifted Chebyshev polynomials. Jones and Phillips [12] investigate trinomials of the form $x^n + a(m, n)x + b(m, n)$ with m an indeterminate. They find infinitely many distinct monogenic fields and classify the Galois groups, which are either S_n or A_n. Although there is overlap with our family $x^n + ax + b$, the methods we employ are distinct.

As this work was in final edits for release, the authors were made aware of an overlapping recent parallel research line. Using an extension of Dedekind's index criterion, Jakhar, Khanduja, and Sangwan ([10] and [11]) established necessary and sufficient conditions for any trinomial to be monogenic. Their work is more complete than our Theorems 3.5 and 3.6, but our methods are distinct (we use the Montes Algorithm). Concurrently but independent from our work, Jones and White [13] prove infinitude and analyze the density of certain families of monogenic trinomials. In particular, they provide a more complete theorem than our Theorem 3.8 for trinomials of the form $x^n + bx + b$, but do not address the family of our Theorem 3.9.

The outline of the paper is as follows. In Section 2 we establish notation, quote some previous results we will need, and give a very brief overview of the Montes algorithm, our main tool in proving these trinomials yield monogenic fields. We will formally state our results in Section 3. With Section 4 we use the Montes algorithm to prove the roots of the trinomials we are considering yield power integral bases. Section 5 establishes the infinitude of some of our families. Finally, Section 6 contains some computational data for comparison to the sufficient conditions of Theorems 3.5 and 3.6, and the densities of Theorems 3.8 and 3.9.

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2. Notation, Definitions, and Lemmas

In Table 1 we outline some standard notation that will be in use throughout the paper.

We will need the following well-known result relating field discriminants and polynomial discriminants. Let f be a monic irreducible polynomial of degree $n > 1$ and let θ be a root. Then

$$\Delta_f = \Delta_K [\mathcal{O}_K : \mathbb{Z}[\theta]]^2. \tag{1}$$

Table 1. Notation

K	a finite degree extension of $\mathbb{Q}$
$\mathcal{O}_K$	the ring of integers of K
Δ_K	the absolute discriminant of K
f, g, ϕ	a monic polynomial in x
Δ_f	the discriminant of the polynomial f
θ	root of a polynomials
n	the degree of a polynomial
$a, b, c, \dots$	integer coefficients of a polynomial
p	a prime number
v_p	the p-adic valuation
m	a modulus
$\overline{f}$	f as viewed in $(\mathbb{Z}/m\mathbb{Z})[x]$, when m is clear

When we generalize our methods to trinomials of arbitrary degree, it is essential to know the discriminant. For this we need the following computation of Greenfield and Drucker.

Theorem 2.1. *[8, Theorem 4] Consider the trinomial $f(x) = x^n + ax^k + b$. Write N for $\dfrac{n}{\gcd(n,k)}$ and K for $\dfrac{k}{\gcd(n,k)}$. The discriminant of the trinomial is*

$$\Delta_f = (-1)^{\frac{n^2-n}{2}} b^{k-1} \left(n^N b^{N-K} - (-1)^N (n-k)^{N-K} k^K a^N\right)^{\gcd(n,k)}.$$

We now outline the notation necessary for the Montes algorithm. We mirror [17] in our exposition. We extend the standard p-adic valuation by defining the p-adic valuation of $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ to be

$$v_p(f(x)) = \min_{0 \le i \le n} (v_p(a_i)).$$

If $\phi(x), f(x) \in \mathbb{Z}[x]$ are such that $\deg \phi \le \deg f$, then we can write

$$f(x) = \sum_{i=0}^{k} a_i(x) \phi(x)^i,$$

for some k, where each $a_i(x) \in \mathbb{Z}[x]$ has degree less than $\deg \phi$. We call the above expression the *ϕ-adic development* of $f(x)$. We associate to the ϕ-adic development of f a Newton polygon by taking the lower convex hull of the integer lattice points $(i, v_p(a_i(x)))$. We call the sides of the Newton polygon with negative slope the *principal ϕ-polygon*. The number of integer lattice points (m, n) with $m, n > 0$ on or under the principal ϕ-polygon is called the *ϕ index of f* and denoted $\mathrm{ind}_\phi(f)$.

Associated to each side of the principal ϕ-polygon is a polynomial called the *residual polynomial*. To avoid technicality, we will not define the residual polynomial in general. For our purposes it suffices to note that residual polynomials attached to sides whose only integer lattice points are the initial vertex and terminal vertex are linear polynomials. Again, the interested reader is encouraged to consult [17] for a brief account of the Montes algorithm or [3] and [9] for in-depth descriptions and proofs.

Now we state the Theorem of the Index, which is our main tool in proving monogeneity.

Theorem 2.2. *[3, Theorem 1.9] Choose monic polynomials $\phi_1, \ldots, \phi_k \in \mathbb{Z}[x]$ whose reductions modulo p are exactly the distinct irreducible factors of $\overline{f(x)}$. Let θ be a root of $f(x)$. Then,*

$$v_p([\mathcal{O}_K : \mathbb{Z}[\theta]]) \geq \operatorname{ind}_{\phi_1}(f) + \cdots + \operatorname{ind}_{\phi_k}(f).$$

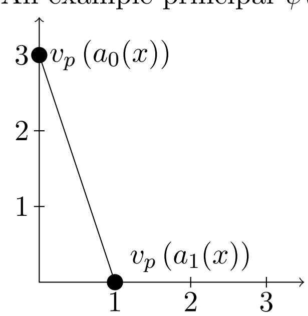
Further, equality holds if and only if, for every ϕ_i, each side of the principal ϕ_i-polygon has a separable residual polynomial.

Remark 2.3. This remark is a summary of [17, Remark 2.4]. Suppose the reduction of $f(x)$ modulo p has the form

$$\overline{f(x)} \equiv \psi(x)\gamma(x),$$

where $\psi(x)$ is an irreducible that is relatively prime to $\gamma(x)$. In this case $\psi(x)$ does not contribute to the index. This is because the $a_1(x)$ coefficient of the $\psi(x)$-adic[1] development has p-adic valuation 0. The following is an example of what the principal $\psi(x)$-polygon looks like in this case.

FIGURE 1. An example principal $\psi(x)$-polygon.



To make the example more explicit, suppose -1 is not a square modulo p and $f(x) = x^3 + px^2 + (p^3+1)x + p^3 + p$. Here $\psi(x) = x^2 + 1$. Since $f(x)$ has ψ-adic development $(p^3x + p^3) + (x+p)(x^2+1)$, we have the principal $\psi(x)$-polygon shown in the figure.

[1]We will continually commit the sin of using the same notation to identify a factor of $f(x)$ modulo p and a lift of that factor to $\mathbb{Z}[x]$, when no confusion can arise.

Lastly, in our paper 'density' refers to natural density. Let $A \subseteq \mathbb{N}$ and $a(x) := \#\{a \in A \mid a \leq x\}$. If

$$\lim_{x\to\infty} \frac{a(x)}{x} = \alpha,$$

we say that A has *natural density* α in $\mathbb{N}$.

3. Statements of Results

Consider the two families $f(x) = x^n + ax + b$ and $g(x) = x^n + cx^{n-1} + d$. The discriminants are

$$\Delta_f = (-1)^{\frac{n^2-n}{2}} \left(n^n b^{n-1} + (1-n)^{n-1} a^n\right)$$

and

$$\Delta_g = (-1)^{\frac{n^2-n}{2}} d^{n-2} \left(n^n d + (1-n)^{n-1} c^n\right).$$

We investigate the $n = 5$ and $n = 6$ cases in depth.

Theorem 3.1. *Let $f(x) = x^5 + ax + b \in \mathbb{Z}[x]$ be irreducible and let θ be a root. Suppose $\dfrac{2^8a^5 + 5^5b^4}{\gcd(2^8a^5, 5^5b^4)}$ is squarefree, and suppose for each prime $p \mid \gcd(2a, 5b)$, one of the following conditions holds:*

(1) $p \mid a$ and $p \mid b$, but $p^2 \nmid b$.
(2) $p = 2$, $2 \nmid a$, and $a + b \equiv 1 \pmod 4$.
(3) $p = 5$, $5 \nmid b$, and $b \not\equiv 1 + a, 7 + 2a, 18 + 3a, 24 + 4a \pmod{25}$.

Then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.

Theorem 3.2. *Let $f(x) = x^6 + ax + b \in \mathbb{Z}[x]$ be irreducible and let θ be a root. Suppose $\dfrac{6^6b^5 - 5^5a^6}{\gcd(6^6b^5, 5^5a^6)}$ is squarefree and that, for each prime $p \mid \gcd(6b, 5a)$, one of the following conditions holds:*

(1) $p \mid a$ and $p \mid b$, but $p^2 \nmid b$.
(2) $p = 2$, $2 \nmid b$, and $a + b \equiv 1 \pmod 4$.
*(3) $p = 3$, $3 \nmid b$, and the image of (a, b) in $(\mathbb{Z}/9\mathbb{Z})^2$ is **not** in the set*

$$\{(0,1), (0,8), (3,2), (3,5), (6,2), (6,5)\}.$$

(4) $p = 5$, $5 \nmid a$, and $a \not\equiv 1 - 4b, 7 + 3b, 18 + 3b, 24 + 4b \pmod{25}$.

Then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.

Theorem 3.3. *Let $g(x) = x^5 + cx^4 + d \in \mathbb{Z}[x]$ be irreducible and θ a root. Suppose d and $\dfrac{5^5d + 2^8c^5}{\gcd(5^5d, 2^8c^5)}$ are squarefree, and if $5 \mid c$, then $c + d \not\equiv 1, 7, 18, 24 \pmod{25}$. Then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.*

Theorem 3.4. *Let $g(x) = x^6 + cx^5 + d \in \mathbb{Z}[x]$ be irreducible and θ a root. Suppose d and $\dfrac{6^6d - 5^5c^6}{\gcd(6^6d, 5^5c^6)}$ are squarefree. If $2 \mid c$ and $2 \nmid d$, assume $c + d \equiv 1 \pmod 4$.*

If $3 \mid c$ and $3 \nmid d$, assume the image of (c,d) in $(\mathbb{Z}/9\mathbb{Z})^2$ is in the set

$$\{(3,1),(3,4),(3,7),(6,1),(6,4),(6,7),(0,1),(0,2),(0,4),(0,5).\}.$$

Then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.

One can generalize the proofs of Theorems 3.1, 3.2, and 3.3 to obtain the following slightly weaker, but more general theorems for f and g of arbitrary degree.

Theorem 3.5. *Let $n > 1$, let $f(x) = x^n + ax + b \in \mathbb{Z}[x]$ be irreducible, and let θ be a root of f. If $\dfrac{(1-n)^{n-1}a^n + n^n b^{n-1}}{\gcd\left((1-n)^{n-1}a^n, n^n b^{n-1}\right)}$ is squarefree and for every prime p dividing $\gcd\left((1-n)a, nb\right)$ one has $p \mid a$, $p \mid b$, and $p^2 \nmid b$, then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.*

Theorem 3.6. *Suppose $g(x) = x^n + cx^{n-1} + d \in \mathbb{Z}[x]$ is an irreducible polynomial and θ is a root. If d is squarefree and $n^n d + (1-n)^{n-1}c^n$ is squarefree away from primes dividing d, then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.*

The following is another general family that is a straight forward consequence of our methods.

Theorem 3.7. *Let $f(x) = x^n + b_{n-1}a^{e_{n-1}}x^{n-1} + \cdots + b_1 a^{e_1} x + a \in \mathbb{Z}[x]$ with $e_i \geq 1$ and $b_i \in \mathbb{Z}$ be irreducible and let θ be a root. If a is squarefree and Δ_f is squarefree away from primes dividing a, then $\mathbb{Q}(\theta)$ is monogenic and θ is a generator of the ring of integers.*

With sufficient conditions in hand, one can ask about the density of coefficients satisfying these conditions. Naturally, we would like to prove the infinitude of some of the families of monogenic fields.

Theorem 3.8. *Fix $n > 2$. Let θ be a root of $f(x) = x^n + bx + b \in \mathbb{Z}[x]$. Then, there are infinitely many b such that f is irreducible and $K = \mathbb{Q}(\theta)$ is monogenic, with θ being the generator for its ring of integers. In addition, the density of such b is at least*

$$\frac{6}{\pi^2} - \left(1 - \frac{6}{\pi^2}\prod_{p \mid (n-1)}\left(1 - \frac{1}{p^2}\right)^{-1}\right) > 21.58\%.$$

Theorem 3.9. *Fix $n > 2$. Let c be a nonzero integer such that $c \neq \pm 1$ and c is squarefree. Suppose $g(x) = x^n + cx^{n-1} + cd \in \mathbb{Z}[x]$ is irreducible and let θ be a root. Consider the quantity*

$$B = \frac{6}{\pi^2}\prod_{p \mid c}\frac{p}{p+1} - \left(1 - \frac{6}{\pi^2}\prod_{p \mid n}\frac{p^2}{p^2-1}\right).$$

Then B gives a lower bound on the density of d such that $K = \mathbb{Q}(\theta)$ is monogenic, with θ being the generator for its ring of integers. In particular, if c has exactly one prime factor, or has exactly two prime factors and is coprime to 6, then $B > 0$ and there are infinitely many d yielding monogenic fields.

Remark 3.10. The densities above are merely a biproduct of our proof methods, and appear weak compared to actual densities observed by computation. See Section 6 for these data.

Using the methods below, one can also prove the family x^5+x^4+d yields infinitely many monogenic fields as well. The relevant proofs are very similar to those below, so we exclude this case for brevity.

4. Proofs

We begin with proofs of Theorems 3.5, 3.6 and 3.7, which involve a fairly simple application of the Montes technique.

Proof of Theorem 3.5. We are considering the irreducible integral polynomial $f(x) = x^n + ax + b$ with discriminant

$$\Delta_f = (-1)^{\frac{n^2-n}{2}} \left(n^n b^{n-1} + (1-n)^{n-1} a^n\right).$$

Our hypotheses guarantee that $\dfrac{n^n b^{n-1} + (1-n)^{n-1} a^n}{\gcd\left(n^n b^{n-1}, (1-n)^{n-1} a^n\right)}$ is squarefree, and for every prime p dividing $\gcd\left(n^n b^{n-1}, (1-n)^{n-1} a^n\right)$, we have $p \mid a$, $p \mid b$, and $p^2 \nmid b$.

If a prime p divides $[\mathcal{O}_K : \mathbb{Z}[\theta]]$, then $p^2 \mid \Delta_f$. Hence we apply the Montes technique to this finite list of primes $p^2 \mid \Delta_f$, which is exactly those p satisfying $p \mid \gcd\left((n-1)a, nb\right)$. For such a p, our hypotheses guarantee that

$$f(x) \equiv x^n \pmod{p}.$$

Hence we need only consider the ϕ-development of f for $\phi(x) = x$. This is

$$f(x) = x^n + ax + b$$

i.e. $a_0(x) = b$, $a_1(x) = a$, and $a_n(x) = 1$, with $a_i(x) = 0$ otherwise. Therefore the associated principal x-polygon originates at $(0, 1)$, since $v_p(b) = 1$ by assumption. Since $a_n(x) = 1$, this is enough to guarantee that there are no lattice points below the polygon; the case of $n = 5$ is shown in Figure 2. Note that, since all of our polynomials are monic, any principal ϕ-polygon originating at $(0, 1)$ has no positive integer lattice points on or above it. Theorem 2.2 ensures that the primes dividing $\gcd\left((n-1)a, nb\right)$ will not contribute to $[\mathcal{O}_K : \mathbb{Z}[\theta]]$. □

Proof of Theorem 3.6. We are considering the irreducible integral polynomial $g(x) = x^n + cx^{n-1} + d$ with discriminant

$$\Delta_g = (-1)^{\frac{n^2-n}{2}} d^{n-2} \left(n^n d + (1-n)^{n-1} c^n\right).$$

Our hypotheses stipulate that both d and $n^n d + (1-n)^{n-1} c^n$ are squarefree.

As in the previous proof, we need only consider primes p such that $p^2 \mid \Delta_g$. Under our hypotheses, any such prime divides d. In this case

$$g(x) \equiv x^n + cx^{n-1} \equiv x^{n-1}(x+c) \pmod{p}.$$

By Remark 2.3, we need only consider the x-adic development of g regardless of whether p divides c or not. This is

$$g(x) = x^n + cx^{n-1} + d.$$

As in the previous proof, the principal x-polygon will have only one side, and since d is squarefree, that side will originate at $(0, 1)$ and descend to $(n, 0)$ or $(n - 1, 0)$ depending on whether p divides c or not. Theorem 2.2 ensures that these primes will not contribute to $[\mathcal{O}_K : \mathbb{Z}[\theta]]$. $\square$

Proof of Theorem 3.7. As in the previous proofs, our hypotheses guarantee that we need only consider primes dividing a. For each prime p dividing a,

$$f(x) \equiv x^n \pmod{p}.$$

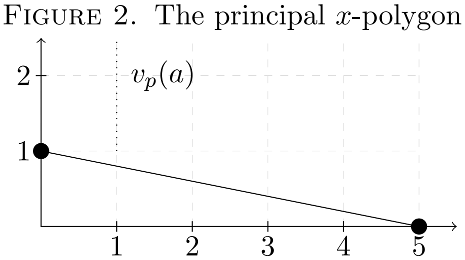
The x-adic development of f is $x^n + a$, and, since a is squarefree, the principal x-polygon is, as before, one-sided with the side originating at $(0, 1)$. Thus each prime factor of a contributes nothing to the index $[\mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta]]$. $\square$

For the remaining results, we will be particularly deliberate with our proof of Theorem 3.1. The proofs of the other theorems are very similar, so we will only highlight aspects that are distinct from the proof of Theorem 3.1.

Proof of Theorem 3.1. Recall our set-up: $f(x) = x^5+ax+b \in \mathbb{Z}[x]$ is irreducible, θ is a root, $K = \mathbb{Q}(\theta)$, and $\Delta_f = 5^5b^4 + 4^4a^5 = 3125b^4 + 256a^5$. By equation 1 and the hypothesis that $\frac{5^5b^4+2^8a^5}{\gcd(5^5b^4,2^8a^5)}$ is squarefree, the only prime factors p of $[\mathcal{O}_K : \mathbb{Z}[\theta]]$ are divisors of $\gcd(5b, 2a)$. We consider the cases given in the theorem statement.

Case 1. $p \mid a$ and $p \mid b$, and $p^2 \nmid b$. Then, $f(x) = x^5 + ax + b \equiv x^5 \pmod{p}$, and the x-adic development of $f(x)$ is again x^5+ax+b. The principal x-polygon is given in Figure 2. The integer lattice point corresponding to $(1, v_p(a))$ lies on the dotted line. The residual polynomial is linear, and thus separable. Therefore Theorem 2.2

FIGURE 2. The principal x-polygon



yields $v_p([\mathcal{O}_K : \mathbb{Z}[\theta]]) = \mathrm{ind}_x(f) = 0$.

Case 2. Suppose that $p = 2$ and $2 \nmid a$. Since $2 \mid 5b$ and $\gcd(2, 5) = 1$, we see $2 \mid b$. As a result,

$$f(x) = x^5 + ax + b \equiv x^5 + ax \equiv x(x^4 + a) \pmod{2}.$$

However, $2 \nmid a$ implies that $a \equiv 1 \pmod{2}$. Hence

$$f(x) \equiv x(x^4 + 1) \equiv x(x^4 + 1^4) \equiv x(x + 1)^4 \pmod{2}.$$

Since x is separable and $\gcd(x, (x+1)^4) = 1$, x does not contribute to the index. See Remark 2.3. Thus we only need to look at the $(x+1)$-adic development of f, which is

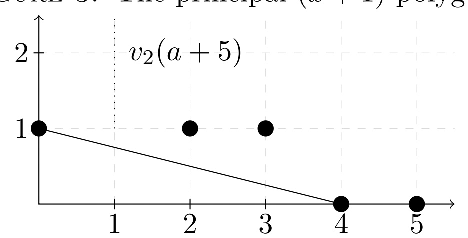
$$f(x) = (x+1)^5 - 5(x+1)^4 + 10(x+1)^3 - 10(x+1)^2 + (a+5)(x+1) + b - a - 1.$$

Note $a+5 \equiv 1+5 \equiv 0 \pmod 2$, so $v_2(a+5) \geq 1$. Thus when $v_2(b-a-1)$ is greater than 1, the point $(1,1)$ will always be on or below the principal $(x+1)$-polygon. This point contributes to $\mathrm{ind}_{x-1}(f)$, which forces $v_2([\mathcal{O}_K : \mathbb{Z}[\theta]]) > 0$. We wish to ensure that $v_2(b-a-1) = 1$. Since $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$, we examine four possibilities for the image of a, b in $\mathbb{Z}/4\mathbb{Z}$. These can seen in the table below.

a	b	$b-a-1$
3	2	2
3	0	0
1	2	0
1	0	2

When (a, b) is congruent to (3,2) or (1,0) modulo 4, i.e., $a+b \equiv 1 \pmod 4$, we have the principal $(x+1)$-polygon in Figure 3. The integer lattice point corresponding to

FIGURE 3. The principal $(x+1)$-polygon



$(1, v_2(a+5))$ is on the dotted line above the principal $(x+1)$-polygon. The residual polynomial is linear and thus separable. Therefore Theorem 2.2 yields

$$v_2[\mathcal{O}_K : \mathbb{Z}[\theta]] = \mathrm{ind}_{x-1}(f) = 0.$$

Case 3. Now, suppose that $p = 5$ and $5 \nmid b$. Since $5 \mid 2a$, we see that $5 \mid a$. Thus

$$f(x) = x^5 + ax + b \equiv x^5 + b \equiv x^5 + b^5 \equiv (x+b)^5 \pmod 5.$$

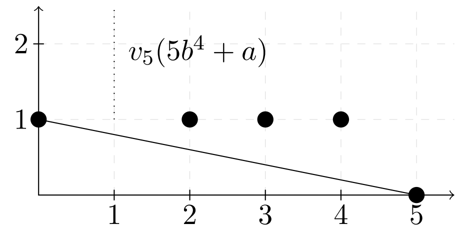
The $(x+b)$-adic development of $f(x)$ is

$$(x+b)^5 - 5b(x+b)^4 + 10b^2(x+b)^3 - 10b^3(x+b)^2 + (5b^4+a)(x+b) - b^5 - ba + b.$$

We compute $v_5(-b^5 - ba + b) \geq 1$ and $v_5(5b^4 + a) \geq 1$. Thus if $v_5(-b^5 - ba + b) > 1$, the lattice point $(1,1)$ will be on or under the principal $(x+b)$-polygon. This will contribute to $\mathrm{ind}_{x+b}(f)$. To avoid this situation, we must ensure that 25 does not divide $-b^5 - ba + b$.

Note that $b_0 = 1, 2, 3, 4$ are solutions to $-x^5 - ax + x \equiv 0 \pmod 5$. We use Hensel's lemma to obtain solutions modulo 25, and we find that the solutions (a, b) are of the form $(a, 1+a)$, $(a, 7+2a)$, $(a, 18+3a)$, and $(a, 24+4a)$. If (a, b) is not one of these pairs, then $25 \nmid \left(-b^5 - ba + b\right)$. As a consequence, we obtain the principal $(x+b)$-polygon in Figure 4. The integer lattice point $(1, v_5(5b^4 + a))$ lies above the

FIGURE 4. The principal $(x+b)$-polygon



principal $(x+b)$-polygon on the dotted line. The residual polynomial is linear and thus separable. Therefore Theorem 2.2 yields $v_5([\mathcal{O}_K : \mathbb{Z}[\theta]]) = \mathrm{ind}_{x+b}(f) = 0$.

Thus, for all primes p that could possibly divide $[\mathcal{O}_K : \mathbb{Z}[\theta]]$, we have shown $v_p([\mathcal{O}_K : \mathbb{Z}[\theta]]) = 0$. Therefore, $[\mathcal{O}_K : \mathbb{Z}[\theta]] = 1$, so K is monogenic and θ yields a power integral basis. □

Remark 4.1. In all of the cases above, we required $v_p(a_0(x))$ to be less than or equal to 1. This ensures that the principal polygons will be one-sided and there will be no positive integer lattice points below it. Figure 5 illustrates this. Just as above, in further applications of the Montes algorithm, we will ensure that $p^2 \nmid a_0(x)$ whenever $p \mid a_1(x)$, for each prime p of concern.

Proof of Theorem 3.2. Recall our set-up: $f(x) = x^6 + ax + b \in \mathbb{Z}[x]$ is irreducible, θ is a root, $K = \mathbb{Q}(\theta)$, and $\Delta_f = 6^6b^5 - 5^5a^6 = 46656b^5 - 3125a^6$. We assume that $\frac{6^6b^5 - 5^5a^6}{\gcd(6^6b^5, 5^5a^6)}$ is squarefree and consider primes p dividing $\gcd(6b, 5a)$. Our approach and case 1 are exactly analogous to the proof of Theorem 3.1.

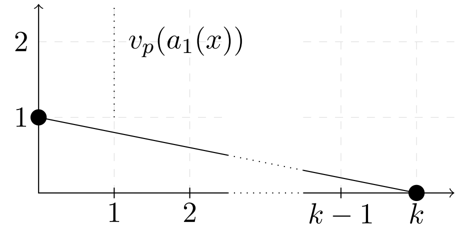
Case 2. Suppose $p = 2$ and $2 \nmid b$. We see $2 \mid a$ and as a result

$$f(x) = x^6 + ax + b = x^6 + b \equiv (x^3 + b)^2 \pmod 2.$$

Furthermore $b \equiv 1 \pmod 2$, so

$$f(x) \equiv (x^3 + 1)^2 \equiv \left[(x+1)(x^2 + x + 1)\right]^2 \pmod 2.$$

FIGURE 5. The principal $\phi(x)$-polygon



We have two irreducible factors to consider. For the irreducible factor $(x+1)$, the $(x+1)$-adic development of $f(x)$ is

$$\begin{aligned}(x+1)^6 - 6(x+1)^5 + 15(x+1)^4 - 20(x+1)^3 \\ + 15(x+1)^2 + (a-6)(x+1) - a + b + 1.\end{aligned}$$

We observe that $a_1(x) = a - 6 \equiv 0 \pmod 2$, so $v_2(a_1(x)) \geq 1$. We want to ensure $4 \nmid a_0(x) = -a + b + 1$. Thus (a, b) must be equivalent to $(2, 3)$ or $(0, 1)$ modulo 4.

We turn our attention to the other irreducible factor, x^2+x+1. The (x^2+x+1)-adic development of f is

$$(x^2 + x + 1)^3 - 3x(x^2 + x + 1)^2 + (2x - 2)(x^2 + x + 1) + ax + b + 1.$$

It is clear that $a_1(x) = 2x - 2 \equiv 0 \pmod 2$, so $v_2(a_1(x)) \geq 1$. We need to ensure that $4 \nmid a_0(x) = ax + b + 1$. Thus we need either $v_2(a) = 1$ or $b \equiv 1 \pmod 4$.

Since the conditions coming from the irreducible factor $(x+1)$ are more restrictive, we conclude that (a, b) must be equivalent to either $(2, 3)$ or $(0, 1)$ modulo 4 to ensure that $v_2([\mathcal{O}_K : \mathbb{Z}[\theta]]) = 0$.

Case 3. Suppose $p = 3$ and $3 \nmid b$. Since $3 \mid a$, we have

$$f(x) = x^6 + ax + b \equiv x^6 + b \equiv (x^2 + b)^3 \pmod 3.$$

There are two subcases.

Subcase 3.1. Suppose $b \equiv 1$ modulo 3. Then, $x^2+b \equiv x^2+1$, which is irreducible. Hence, $f(x) \equiv (x^2 + 1)^3 \pmod 3$. The $(x^2 + 1)$-adic development of f is

$$(x^2 + 1)^3 - 3(x^2 + 1)^2 + 3(x^2 + 1) + ax + b - 1.$$

Clearly, $v_3(a_1(x)) = 1$. To ensure $v_3(ax+b-1) = 1$, we require either $a \not\equiv 0$ modulo 9 or $b \not\equiv 1$ modulo 9.

Subcase 3.2. Suppose $b \equiv 2$ modulo 3. Then, $x^2 + b \equiv x^2 + 2 \equiv (x + 1)(x + 2) \pmod 3$, and

$$f(x) \equiv ((x + 1)(x + 2))^3 \pmod 3.$$

First, we examine the factor $(x + 1)$. The $(x + 1)$-adic development of f is

$$\begin{aligned}(x+1)^6 - 6(x+1)^5 + 15(x+1)^4 - 20(x+1)^3 \\ + 15(x+1)^2 + (a-6)(x+1) - a + b + 1.\end{aligned}$$

We observe that $v_3(a - 6) \geq 1$. To avoid $-a + b + 1 \equiv 0 \pmod 9$, we must ensure that $a \not\equiv b + 1 \pmod 9$.

Lastly, we look at the factor $(x + 2)$. The $(x + 2)$-adic development of f is

$$\begin{aligned}(x+2)^6 - 12(x+2)^5 + 60(x+2)^4 - 160(x+2)^3 \\ + 240(x+2)^2 + (a-192)(x+2) - 2a + b + 64.\end{aligned}$$

Since $v_3(a - 192) \geq 1$, we want $-2a + b + 64 \not\equiv 0 \pmod 9$. This happens when $b \not\equiv 2a - 64 \equiv 2a - 1 \pmod 9$.

Therefore, we see $v_3([\mathcal{O}_K : \mathbb{Z}[\theta]]) = 0$ so long as the reduction of (a, b) modulo 9 is not one of

$$\{(0, 1), (0, 8), (3, 2), (3, 5), (6, 2), (6, 5)\}.$$

Case 4. Finally, assume that $p = 5$ and $5 \nmid a$. Since $5 \mid b$, we have

$$f(x) \equiv x^6 + ax \equiv x(x+a)^5.$$

The only irreducible factor that concerns us is $x + a$. The $(x + a)$-adic development of $f(x)$ is

$$\begin{aligned}(x+a)^6 - 6a(x+a)^5 + 15a^2(x+a)^4 - 20a^3(x+a)^3\\ + 15a^4(x+a)^2 + (-6a^5+a)(x+a) + a^6 - a^2 + b.\end{aligned}$$

Proceeding in the same manner as in the proof of Theorem 3.1, we must ensure a is not of the form $1 - 4b, 7 + 3b, 18 + 3b,$ or $24 + 4b \pmod{25}$.

For all primes p which could possible divide $[\mathcal{O}_K : \mathbb{Z}[\theta]]$, we've shown $v_p([\mathcal{O}_K : \mathbb{Z}[\theta]]) = 0$. Thus $\mathcal{O}_K = \mathbb{Z}[\theta]$. □

Proof of Theorem 3.3. We consider the irreducible polynomial $g(x) = x^5 + cx^4 + d \in \mathbb{Z}[x]$ with θ denoting a root. One computes $\Delta_g = d^3\left(5^5 d + 4^4 c^5\right) = d^3\left(3125d + 256c^5\right)$. We assume that d and $\dfrac{5^5 d + 2^8 c^5}{\gcd\left(5^5 d, 2^8 c^5\right)}$ are squarefree. The possible prime divisors of the index are primes dividing d or $\gcd(5d, 2c)$. The only prime we may have to consider that isn't a divisor of d is 5.

For primes $p \mid d$ we have

$$g(x) \equiv x^5 + cx^4 \equiv x^4(x+c) \pmod p.$$

The factor $x + c$ contributes nothing to the index as Remark 2.3 illustrates. The x-adic development is again $g(x)$. Since the principal x-polygon originates at $(0, 1)$ and terminates at $(5, 0)$, it bounds no integer lattice points that contribute to the index. Theorem 2.2 ensures p contributes nothing to the index.

***p* = 5:** Suppose now that $5 \mid c$ and $5 \nmid d$. We have

$$f(x) = x^5 + cx^4 + d \equiv x^5 + d \equiv x^5 + d^5 \equiv (x+d)^5 \pmod 5.$$

The $(x + d)$-adic development of $f(x)$ is given by

$$\begin{aligned}(x+d)^5 + (c-5d)(x+d)^4 + (10d^2 - 4cd)(x+d)^3\\ + (6cd^2 - 10d^3)(x+d)^2 + (5d^4 - 4cd^3)(x+d) + cd^4 + d - d^5.\end{aligned}$$

As usual we must ensure that $25 \nmid \left(cd^4 + d - d^5\right)$. One computes that $25 \mid \left(cd^4 + d - d^5\right)$ if and only if $c + d \equiv 1, 7, 18,$ or $24 \pmod{25}$.

For every possible prime p that could divide the index, we find $v_p([\mathcal{O}_K : \mathbb{Z}[\theta]]) = 0$. Therefore $\mathcal{O}_K = \mathbb{Z}[\theta]$. □

Proof of Theorem 3.4. We are considering $g(x) = x^6 + cx^5 + d$. We have $\Delta_g = -d^4(6^6 d - 5^5 c^6)$ and by hypothesis d and $\frac{6^6 d - 5^5 c^6}{\gcd(6^6 d, 5^5 c^6)}$ are squarefree. We consider primes p dividing d or $\gcd(6d, 5c)$. The only primes we may have to consider than are not divisors of d are 2 and 3.

If $p \mid d$, a routine argument shows that p cannot contribute to the index.

***p* = 2:** Suppose $2 \mid c$ and $2 \nmid d$. Reducing yields

$$g(x) = x^6 + cx^5 + d \equiv (x+1)^2(x^2+x+1)^2 \pmod 2.$$

The $(x+1)$-adic development is

$$g(x) = (x+1)^6 + (-6+c)(x+1)^5 + (15-5c)(x+1)^4 + (-20+10c)(x+1)^3 \\ + (15-10c)(x+1)^2 + (-6+5c)(x+1) - c + d + 1.$$

Thus we require (c,d) reduces to either $(0,1)$ or $(2,3)$ in $(\mathbb{Z}/4\mathbb{Z})^2$.

Continuing, the (x^2+x+1)-adic development is

$$g(x) = (x^2+x+1)^3 + (-3x+cx-2c)(x^2+x+1)^2 \\ + (2x+cx+3c-2)(x^2+x+1) - cx - c + d + 1.$$

We are concerned with $v_2(-cx-c+d+1) = \min(v_2(-c), v_2(-c+d+1))$. Thus we require $c \equiv 2 \pmod 4$ or $d \equiv 1 \pmod 4$. The $(x+1)$-adic development is more restrictive, so we insist that (c,d) reduces to either $(0,1)$ or $(2,3)$ in $(\mathbb{Z}/4\mathbb{Z})^2$.

***p* = 3:** Suppose $3 \mid c$ and $3 \nmid d$. If $d \equiv 1 \pmod 3$, then $g(x) \equiv (x^2+1)^3 \pmod 3$. The (x^2+1)-adic development is

$$g(x) = (x^2+1)^3 + (cx-2c-3)(x^2+x+1)^2 \\ + (6x+cx+3c)(x^2+x+1) - cx - c + d + 2.$$

Thus we require either $c \equiv 3, 6 \pmod 9$ or $d \equiv 1, 4 \pmod 9$.

If $d \equiv 2$ modulo 3, then $g(x) \equiv (x+1)^3(x-1)^3$ modulo 3. The $(x+1)$-adic development is above. The $(x-1)$-adic development is

$$g(x) = (x-1)^6 + (6+c)(x-1)^5 + (15+5c)(x-1)^4 + (20+10c)(x-1)^3 \\ (15+10c)(x-1)^2 + (6+5c)(x-1) + 1 + c + d.$$

Combining this with the conditions coming from the $(x+1)$-adic development we require $v_3(d-c+1) = v_3(d+c+1) = 1$. Therefore the image of (c,d) in $(\mathbb{Z}/9\mathbb{Z})^2$ must be either $(0,2)$ or $(0,5)$.

$\square$

5. Infinitude of the Families

In this section we will restrict some of our families and find that they are monogenic infinitely often. To do this, we will actually prove that the coefficients yielding monogenic fields have positive density in $\mathbb{Z}$. This requires considering the density of squarefree values of parts of the discriminant. In general, showing a polynomial takes on many squarefree values can be difficult: for example, it is not known whether there is a single quartic polynomial that is squarefree infinitely often [1]. In our case, we need only some results on linear polynomials.

The first is a result from Prachar [16] about the density of squarefree integers congruent to m modulo k. Let $S(x; m, k)$ denote the number of squarefree integers not exceeding x that are congruent to m modulo k.

Theorem 5.1. *If $(m,k)=1$ and $k \le x^{\frac{2}{3}-\epsilon}$, then*

$$S(x;m,k) \sim \frac{6x}{\pi^2 k}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1} \qquad (x\to\infty).$$

We will also need to know the number of integers not exceeding x that are squarefree and coprime to k. Denote this quantity by $T(x;k)$. The following is a straight forward corollary of Theorem 5.1 if one notes there are $\phi(k) = k\prod_{p\mid k}\left(\frac{p-1}{p}\right)$ distinct congruence classes modulo k that are relatively prime to k.

Corollary 5.2. *With the notation as above*

$$T(x;k) \sim \frac{6x}{\pi^2}\prod_{p\mid k}\frac{p}{p+1} \qquad (x\to\infty).$$

Proof of Theorem 3.8. We are considering the polynomial $f(x)=x^n+bx+b$. First note that if b is squarefree, then f is irreducible by Eisenstein's Criterion. Recall that the density of squarefree b is $\frac{6}{\pi^2}$. We've computed $\Delta_f = \pm b^{n-1}(n^n + (1-n)^{n-1}b)$, and Theorem 3.5 tells us that if $n^n+(1-n)^{n-1}b$ is squarefree, the number field generated by f is monogenic. By Theorem 5.1, the density of b so that $n^n+(1-n)^{n-1}b$ is squarefree is

$$1-\frac{6}{\pi^2}\prod_{p\mid(n-1)}\left(1-\frac{1}{p^2}\right)^{-1} < 1-\frac{6}{\pi^2}.$$

For the purposes of a lower bound on the density of

$$\{b\in\mathbb{N} : b \text{ and } n^n+(1-n)^{n-1}b \text{ are squarefree}\},$$

the worst case scenario is that all these non-squarefree numbers occur as $n^n+(1-n)^{n-1}b$ for b squarefree. Thus the density of squarefree b with $n^n+(1-n)^{n-1}b$ also squarefree is at least

$$\frac{6}{\pi^2}-\left(1-\frac{6}{\pi^2}\prod_{p\mid(n-1)}\left(1-\frac{1}{p^2}\right)^{-1}\right) > \frac{6}{\pi^2}-\left(1-\frac{6}{\pi^2}\right)\approx 21.58\%.$$

□

Proof of Theorem 3.9. Consider $g(x)=x^n+cx^{n-1}+cd$ with n and c fixed, c squarefree, $c\neq\pm1$, and $\gcd(c,n)=1$. We consider the density of d for which g is monogenic. Without affecting density, we may assume $\gcd(c,d)=1$. Since $c\neq\pm1$, Eisenstein's criterion shows g is irreducible.

We've computed $\Delta_g = \pm(cd)^{n-2}\left(n^n(cd)+(1-n)^{n-1}c^n\right)$. Since $\gcd(c,d)=1$, the product cd is squarefree. If $cdn^n+(1-n)^{n-1}c^n$ is squarefree, then by Theorem 3.6 g is monogenic. The factor of c is extraneous, so we will investigate when $dn^n+(1-n)^{n-1}c^{n-1}$ is squarefree.

From Corollary 5.2, the proportion of squarefree d that are coprime to c is given by

$$\frac{6}{\pi^2}\prod_{p\mid c}\frac{p}{p+1}.$$

In addition, Theorem 5.1 tells us the density of squarefree integers congruent to $(1-n)^{n-1}c^{n-1}$ modulo n^n is

$$\frac{6}{\pi^2}\prod_{p\mid n}\frac{p^2}{p^2-1}.$$

Thus (much as in the last proof) a lower bound on the density of squarefree d coprime to c such that $(1-n)^{n-1}c^{n-1}+dn^n$ is squarefree is,

$$B=\frac{6}{\pi^2}\prod_{p\mid c}\frac{p}{p+1}-\left(1-\frac{6}{\pi^2}\prod_{p\mid n}\frac{p^2}{p^2-1}\right). \tag{2}$$

This is non-trivial if this value is positive.

If c has exactly one prime factor, then

$$\frac{6}{\pi^2}\prod_{p\mid c}\frac{p}{p+1}\geq\frac{6\cdot 2}{\pi^2\cdot 3}.$$

Hence

$$B>\frac{6\cdot 2}{\pi^2\cdot 3}-\left(1-\frac{6}{\pi^2}\right)\approx 0.013.$$

On the other hand, if c has at most two prime factors and is coprime to 6, then we have

$$\frac{6}{\pi^2}\prod_{p\mid c}\frac{p}{p+1}\geq\frac{6\cdot 5\cdot 7}{\pi^2\cdot 6\cdot 8}.$$

Hence

$$B>\frac{6\cdot 5\cdot 7}{\pi^2\cdot 6\cdot 8}-\left(1-\frac{6}{\pi^2}\right)\approx 0.051.$$

□

In both proofs, the densities would be improved if we could heuristically assume that squarefreeness of the two relevant quantities was independent. Thus, heuristically, we should expect a lower bound for the family of Theorem 3.8 of at least

$$B_1:=\frac{36^2}{\pi^4}\prod_{p\mid(n-1)}\left(1-\frac{1}{p^2}\right)^{-1}$$

and for the family of Theorem 3.9 of at least

$$B_2:=\frac{6^2}{\pi^4}\prod_{p\mid c}\frac{p}{p+1}\prod_{p\mid n}\frac{p^2}{p^2-1}.$$

Table 2. Monogenic Percentages for Degrees 5 and 6

Family	% monogenic	% with θ a generator	% satisfying hypotheses of relevant Theorem
$x^5 + bx + b$	52.46	50.50	50.50
$x^6 + bx + b$	58.49	57.71	57.71
$x^5 + cx^4 + c$	44.84	43.10	35.92
$x^6 + cx^5 + c$	58.68	58.00	29.00
$x^5 + ax + b$	61.17	60.86	60.86
$x^6 + ax + b$	61.10	60.90	60.90
$x^5 + cx^4 + d$	55.78	51.80	51.80
$x^6 + cx^5 + d$	45.43	44.66	26.00
$x^5 + 2x^4 + 2d$	36.88	33.67	33.67
$x^5 + 3x^4 + 3d$	43.96	43.29	43.29
$x^5 + 4x^4 + 4d$	65.97	0.00	0.00
$x^5 + 5x^4 + 5d$	42.19	42.08	42.08
$x^5 + 6x^4 + 6d$	32.05	28.85	28.85
$x^5 + 7x^4 + 7d$	45.18	45.13	45.13
$x^5 + 8x^4 + 8d$	13.08	0.00	0.00
$x^6 + 2x^5 + 2d$	40.29	38.48	38.48
$x^6 + 3x^5 + 3d$	43.53	43.28	14.43
$x^6 + 4x^5 + 4d$	2.50	0.00	0.00
$x^6 + 5x^5 + 5d$	48.11	48.09	16.03
$x^6 + 6x^5 + 6d$	30.38	28.86	28.84
$x^6 + 7x^5 + 7d$	51.57	51.57	17.19
$x^6 + 8x^5 + 8d$	4.91	0.00	0.00
$\sum_{i=0}^{5} ax^i$	31.11	29.50	29.50
$\sum_{i=0}^{6} ax^i$	57.27	56.52	56.52

For families with a single parameter a, b, c, or d, the values tested were $[-500000, 500000]$. For families with two parameters the values tested were $[-500, 500]$. The percentages are rounded to the nearest hundredth.

Note that, using $\prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \zeta(2)$,

$$0.28 \approx 27/\pi^4 \leq B_1 \leq 6/\pi^2 \approx 0.61.$$

Similarly, we have

$$0 \leq B_2 \leq \frac{6^2}{\pi^4} \prod_p \left(1 + \frac{1}{p^2 - 1}\right) \leq 72/\pi^4 \approx 0.74.$$

6. Computational Data

Table 2 is for comparison to Theorems 3.8 and 3.9. We can see it is rare that a trinomial yields a monogenic field for which it is not a generator. It is also noteworthy that our theorems for the monogeneity of the trinomials f and g don't capture all instances in which f and g yield monogenic fields. Specifically, there are instances when the relevant factors of Δ_f and Δ_g are not square-free, but those square factors don't contribute to the index. It does not appear the machinery we use is adequate to understand these square factors.

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