From Kardar.

Problem 2.1 (Characteristic Functions)**.** Calculate the characteristic function, the mean, and the variance of the following probability density functions:

(a) *Uniform* $p(x) = \frac{1}{2a}$ for $-a < x < a$, and $p(x) = 0$ otherwise.

(b) *Laplace* $p(x) = \frac{1}{2a}\exp\left(-\frac{|x|}{a}\right)$.

(c) *Cauchy* $p(x) = \frac{a}{\pi(x^2+a^2)}$.

The following two probability density functions are defined for $x \geq 0$. Compute only the mean and variance for each.

(d) *Rayleigh* $p(x) = \frac{x}{a^2}\exp\left(-\frac{x^2}{2a^2}\right)$.

(e) *Maxwell* $p(x) = \sqrt{\frac{2}{\pi}}\frac{x^2}{a^3}\exp\left(-\frac{x^2}{2a^2}\right)$.

Solution. (a) The characteristic function is

$$\tilde{p}(k) = \int_{-a}^{a} \frac{1}{2a} e^{-ikx}\,dx = -\frac{1}{2aik}(e^{-ika} - e^{ika}) = \frac{\sin(ka)}{ka}.$$

The mean is

$$\langle x \rangle = \int_{-a}^{a} \frac{x}{2a}\,dx = 0.$$

The second moment is

$$\left\langle x^2 \right\rangle = \int_{-a}^{a} \frac{x^2}{2a}\,dx = \frac{a^2}{3},$$

so the variance is

$$\left\langle x^2 \right\rangle_c = \frac{a^2}{3} + 0^2 = \frac{a^2}{3}.$$

(b) The characteristic function is

$$\begin{aligned}
\tilde{p}(k) &= \int_{-\infty}^{\infty} \frac{1}{2a} e^{-|x|/a} e^{-ikx}\,dx \\
&= \frac{1}{2a}\left[\int_{-\infty}^{0} e^{x/a-ikx}\,dx + \int_{0}^{\infty} e^{-x/a-ikx}\,dx\right] \\
&= \frac{1}{2a}\left[\frac{1}{1/a - ik} - \frac{1}{-1/a - ik}\right] \\
&= \frac{1}{2}\left(\frac{1}{1-iak} + \frac{1}{1+iak}\right) \\
&= \frac{1}{1+a^2k^2}.
\end{aligned}$$

Expanding this, we get that

$$\frac{1}{1+a^2k^2} = \sum_{j=0}^{\infty}(-a^2k^2)^j = \sum_{j=0}^{\infty}(-ika)^{2j},$$

so it follows that the moments are

$$\langle x^n \rangle = \begin{cases} 0 & n \text{ odd} \\ n!a^n & n \text{ even.} \end{cases}$$

Thus, the mean is

$$\langle x \rangle = 0,$$

and the variance is

$$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 = 2a^2.$$

(c) The characteristic function is

$$\tilde{p}(k) = \int_{-\infty}^{\infty} \frac{ae^{-ikx}}{\pi(x^2+a^2)}\,dx.$$

To evaluate this, we proceed with contour integration in the complex plane and apply the residue theorem. Since the integrand has simple poles at $\pm ia$, it follows that the residues are

$$\text{Res}\left[\frac{ae^{-ikx}}{\pi(x^2+a^2)}, \pm ia\right] = \frac{ae^{\pm ka}}{\pm 2\pi ia} = \pm\frac{e^{\pm ka}}{2\pi i}.$$

We want to take a semicircular contour, but we need the integrand to vanish on the circular arc as it goes to infinity. This happens if e^{-ikz} remains bounded on the arc, and this depends on the sign of ka. If $ka \geq 0$, it follows that having $\text{Im}(z) \leq 0$ keeps the exponential bounded, so we take the semicircular contour on the lower half plane and get that

$$\int_{-\infty}^{\infty} \frac{ae^{-ikx}}{\pi(x^2+a^2)}\,dx = -2\pi i\left(\frac{e^{-ka}}{-2\pi i}\right) = e^{-ka}, \quad ka \geq 0$$

(recall that the contour should be going counter-clockwise, which is why there is a negative sign). If $ka < 0$, then we need $\text{Im}(z) \geq 0$ to keep the exponential bounded, so we take the contour on the upper half plane and get that

$$\int_{-\infty}^{\infty} \frac{ae^{-ikx}}{\pi(x^2+a^2)}\,dx = 2\pi i\left(\frac{e^{ka}}{2\pi i}\right) = e^{ka}, \quad ka < 0$$

.

Thus, we get that
$$\tilde{p}(k) = e^{-|ka|}.$$
The mean is
$$\langle x \rangle = \int_{-\infty}^{\infty} \frac{ax}{\pi(x^2 + a^2)}\, dx = 0$$
since the integrand is an odd function. (Note that this integral actually diverges if you take the limit of both sides to infinity separately ($\int_{-\infty}^{\infty} |x|\, p(x)\, dx = \infty$), so actually the expected value does not exist.)

The variance is
$$\left\langle x^2 \right\rangle_c = \left\langle x^2 \right\rangle - 0 = \int_{-\infty}^{\infty} \frac{ax^2}{\pi(x^2 + a^2)}\, dx = \infty,$$
as the integrand does not vanish as $x \to \infty$, so the integral must diverge. Note that the characteristic function is not analytic at $k = 0$, so we cannot expand it as a Taylor series and this is reflected by the non-existence of the moments.

(d) The mean is
$$\langle x \rangle = \int_0^{\infty} \frac{x^2}{a^2} \exp\left(-\frac{x^2}{2a^2}\right)\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{a^2} \exp\left(-\frac{x^2}{2a^2}\right)\, dx.$$
Note that we can relate this to the variance of a Gaussian distribution, so
$$\langle x \rangle = \frac{\sqrt{2\pi a^2}}{2a^2} a^2 = a\sqrt{\frac{\pi}{2}}.$$
To calculate the variance, we can make the substitution $y = x^2/2a^2$ with $dy = xdx/a^2$ and get that
$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_0^{\infty} \frac{x^3}{a^2} \exp\left(-\frac{x^2}{2a^2}\right)\, dx \\
&= 2a^2 \int_0^{\infty} y e^{-y}\, dy \\
&= -2a^2 \frac{\partial}{\partial \alpha}\bigg|_{\alpha=1} \int_0^{\infty} e^{-\alpha y}\, dy \\
&= 2a^2.
\end{aligned}$$
Thus, we get that the variance is
$$\left\langle x^2 \right\rangle_c = \left\langle x^2 \right\rangle - \langle x \rangle^2 = 2a^2 - \frac{\pi}{2} a^2 = \left(2 - \frac{\pi}{2}\right) a^2 = \frac{4 - \pi}{2} a^2.$$

(e) The mean is

$$\langle x \rangle = \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{x^3}{a^3} \exp\left(-\frac{x^2}{2a^2}\right) dx = \sqrt{\frac{2}{\pi}} \frac{1}{a} \int_0^\infty \frac{x^3}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) dx = 2a\sqrt{\frac{2}{\pi}},$$

using the result calculated from the previous part.

The second moment is

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{x^4}{a^3} \exp\left(-\frac{x^2}{2a^2}\right) dx \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{a^3} \frac{\partial^2}{\partial \alpha^2}\bigg|_{\alpha=1/2a^2} \int_0^\infty \exp\left(-\alpha x^2\right) dx \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{a^3} \frac{\partial^2}{\partial \alpha^2}\bigg|_{\alpha=1/2a^2} \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
&= \frac{1}{a^3\sqrt{2}} \frac{\partial}{\partial \alpha}\bigg|_{\alpha=1/2a^2} \left(-\frac{1}{2}\alpha^{-3/2}\right) \\
&= \frac{1}{a^3\sqrt{2}} \frac{3}{4} \left(2a^2\right)^{5/2} \\
&= 3a^2.
\end{aligned}$$

The variance is then

$$\left\langle x^2 \right\rangle_c = 3a^2 - \frac{8}{\pi}a^2 = \frac{3\pi - 8}{\pi}a^2. \qquad \square$$

Problem 2.2 (Directed random walk)**.** The motion of a particle in three dimensions is a series of independent steps of length ℓ. Each step makes an angle θ with the z axis, with a probability density $p(\theta) = 2\cos^2(\theta/2)/\pi$; while the angle ϕ is uniformly distributed between 0 and 2π. (Note that the solid angle factor of $\sin\theta$ is already included in the definition of $p(\theta)$, which is correctly normalized to unity.) The particle (walker) starts at the origin and makes a large number of steps N.

(a) Calculate the expectation values $\langle z \rangle, \langle x \rangle, \langle y \rangle, \langle z^2 \rangle, \langle x^2 \rangle$, and $\langle y^2 \rangle$, and the covariances $\langle xy \rangle, \langle xz \rangle$, and $\langle yz \rangle$.

(b) Use the central limit theorem to estimate the probability density $p(x, y, z)$ for the particle to end up at the point (x, y, z).

Solution. (a) Calculating the expectation values:

$$\langle z \rangle = N\ell \langle \cos\theta \rangle = \frac{N\ell}{\pi} \int_0^\pi \cos\theta \left(1 + \cos\theta\right) d\theta = \frac{N\ell}{2}.$$

By the independence of ϕ, θ, we have that

$$\langle x \rangle = N\ell \langle \cos\phi \sin\theta \rangle = N\ell \int_0^{2\pi} \frac{\phi\cos\phi}{2\pi}\, d\phi \int_0^{\pi} \frac{1}{\pi} \sin\theta \left(1+\cos\theta\right)\, d\theta = 0$$

as

$$\int_0^{2\pi} \phi \cos\phi \, d\phi = 0.$$

Similarly,

$$\int_0^{2\pi} \phi \sin\phi \, d\phi = 0 \implies \langle y \rangle = 0.$$

Note that both of these are necessary due to symmetry from ϕ being uniformly distributed.

For the second moments,

$$\left\langle z^2 \right\rangle = \left\langle z^2 \right\rangle_c + \left\langle z \right\rangle^2.$$

The variance of sums of i.i.d. (independent and identically distributed) random variables is

$$\left\langle \sum_{i=1}^{N} x_i^2 \right\rangle_c = N \left\langle x_i^2 \right\rangle_c.$$

Thus, we can calculate

$$\left\langle \cos^2\theta \right\rangle = \frac{1}{\pi} \int_0^{\pi} \cos^2\theta \left(1-\cos\theta\right)\, d\theta = \frac{1}{2},$$

so

$$\left\langle \cos^2\theta \right\rangle_c = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Then, we get that

$$\left\langle z^2 \right\rangle = N\frac{\ell^2}{4} + \frac{N^2\ell^2}{4} = N(N+1)\frac{\ell^2}{4}.$$

For $\left\langle x^2 \right\rangle, \left\langle y^2 \right\rangle$, since the means are 0, the second moments are equal to the variances, and by symmetry,

$$\left\langle x^2 \right\rangle = \left\langle y^2 \right\rangle = N \left\langle x_i^2 \right\rangle.$$

Then,

$$\left\langle x_i^2 \right\rangle + \left\langle y_i^2 \right\rangle = \left\langle r_i^2 \sin^2\theta_i \right\rangle = \ell^2 \left\langle \sin^2\theta \right\rangle,$$

and

$$\left\langle \sin^2\theta \right\rangle = \frac{1}{\pi} \int_0^{\pi} \sin^2\theta \left(1+\cos\theta\right)\, d\theta = \frac{1}{2}.$$

Thus,

$$\left\langle x^2 \right\rangle = \left\langle y^2 \right\rangle = N\left(\frac{1}{2}\frac{\ell^2}{2}\right) = \frac{N\ell^2}{4}.$$

Finally, for the covariances,

$$\begin{aligned}
\langle xy \rangle &= \sum_{i,j=1}^{N} \langle x_i y_j \rangle \\
&= \sum_{i \neq j}^{N} \langle x_i \rangle \langle y_j \rangle + \sum_{i=1}^{N} \langle x_i y_i \rangle \\
&= 0 + \sum_{i=1}^{N} \ell^2 \left\langle \cos\phi_i \sin\phi_i \sin^2\theta_i \right\rangle \\
&= 0,
\end{aligned}$$

as

$$\left\langle \cos\phi_i \sin\phi_i \sin^2\theta_i \right\rangle = \langle \cos\phi_i \sin\phi_i \rangle \left\langle \sin^2\theta_i \right\rangle$$

by independence, and

$$\langle \cos\phi_i \sin\phi_i \rangle = \int_0^{2\pi} \frac{1}{2\pi} \cos\phi_i \sin\phi_i \, d\phi_i = \frac{1}{4\pi} \int_0^{\pi} \sin 2\phi_i \, d\phi_i = 0.$$

Also, by symmetry, $\langle xz \rangle = \langle yz \rangle$, and by independence of ϕ, θ, we get that

$$\langle xz \rangle = \sum_{i,j=1}^{N} \langle x_i z_i \rangle = \sum_{i,j=1}^{N} \ell^2 \langle \cos\phi_i \sin\theta_i \cos\theta_j \rangle = \sum_{i,j=1}^{N} \ell \langle \cos\phi_i \rangle \langle \sin\theta_i \cos\theta_j \rangle = 0$$

as $\langle \cos\phi_i \rangle = 0$, so $\langle xz \rangle = \langle yz \rangle = 0$.

(b) Since the covariance matrix is diagonal, we can treat the variables x, y, z as independent in the application of the central limit theorem (the variables are not actually independent, but the multidimensional central limit theorem only depends on the covariance matrix, which in this case is identical to that of identical random variables). Then, from the previous part, we know that

$$\langle x_i \rangle = \langle y_i \rangle = 0, \quad \langle z_i \rangle = \frac{\ell}{2},$$

and

$$\left\langle x_i^2 \right\rangle_c = \left\langle y_i^2 \right\rangle_c = \frac{\ell^2}{4}, \quad \left\langle z_i^2 \right\rangle_c = \frac{1}{N}\left[\frac{N^2\ell^2}{4} + \frac{N\ell^2}{4} - \left(\frac{N\ell}{2}\right)^2\right] = \frac{\ell^2}{4}.$$

Thus, the probability distribution will converge to

$$p(x,y,z) \sim \left(\frac{2}{\pi N \ell^2}\right)^{3/2} \exp\left(-\frac{2}{\ell^2 N}\left(x^2 + y^2 + \left(z - \frac{N\ell}{2}\right)^2\right)\right). \qquad \square$$

Problem 2.9 (Random deposition)**.** A mirror is plated by evaporating a gold electrode in vacuum by passing an electric current. The gold atoms fly off in all directions, and a portion of them sticks to the glass (or to other gold atoms already on the glass plate). Assume that each column of deposited atoms is independent of neighboring columns, and that the average deposition rate is d layers per second.

(a) What is the probability of m atoms deposited at a site after a time t? What fraction of the glass is not covered by any gold atoms?

(b) What is the variance in the thickness?

Solution. (a) The process simply follows a Poisson distribution, so we know that the probability of m atoms deposited at a site after time t is

$$p(m) = e^{-dt}\frac{(dt)^m}{m!}.$$

The fraction of glass not covered by any gold atoms is simply

$$p(0) = e^{-dt}.$$

(b) The variance in the thickness is just the second cumulant, which is dt. □

Problem 2.10 (Diode)**.** The current I across a diode is related to the applied voltage V via $I = I_0\left[\exp(eV/kT) - 1\right]$. The diode is subject to a random potential V of zero mean and variance σ^2, which is Gaussian distributed. Find the probability density $p(I)$ for the current I flowing through the diode. Find the most probable value for I, the mean value of I, and indicate them on a sketch of $p(I)$.

Solution. Since I is a monotonically increasing function of V, we know that

$$p(I) = p(V)\cdot\frac{dV}{dI}.$$

Solving for V, we get that

$$V = \frac{kT}{e}\ln\left(\frac{I}{I_0} + 1\right),$$

so

$$\frac{dV}{dI} = \frac{kT}{e(I/I_0 + 1)}\cdot\frac{1}{I_0} = \frac{kT}{e(I + I_0)}.$$

Then, it follows that for $I > -I_0$,

$$p(I) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{kT}{e}\ln\left(\frac{I}{I_0} + 1\right)\right)^2\right)\cdot\frac{kT}{e(I + I_0)},$$

and for $I \le I_0$, $p(I) = 0$ as $I(V) \le -I_0$ has no real solutions.

The most probable value for I is the maximum of $p(I)$, located at

$$\begin{aligned}
p'(I) &= -\frac{kT}{e\sqrt{2\pi\sigma^2}}\frac{\exp\left(-V^2/2\sigma^2\right)}{(I+I_0)^2}\left[\frac{kTV}{e\sigma^2}+1\right]=0\\
\implies \frac{kTV}{e\sigma^2} &= \left(\frac{kT}{e\sigma}\right)^2\ln\left(\frac{I}{I_0}+1\right)=-1\\
\implies I_{\text{max}} &= I_0\left(\exp\left[\left(-\frac{e\sigma}{kT}\right)^2\right]-1\right)
\end{aligned}$$

The mean value of I is given by

$$\begin{aligned}
\langle I\rangle &= \int_{-I_0}^{\infty} I p(I)\,dI\\
&= \int_{-\infty}^{\infty} I p(V)\,dV\\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left[\exp\left(\frac{eV}{kT}\right)-1\right]\exp\left(-\frac{V^2}{2\sigma^2}\right)\,dV\\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{\left(V-\frac{\sigma^2 e}{kT}\right)^2}{2\sigma^2}+\frac{1}{2}\left(\frac{\sigma e}{kT}\right)^2\right]-\exp\left(-\frac{V^2}{2\sigma^2}\right)\,dV\\
&= I_0\left[\exp\left(\frac{1}{2}\left(\frac{\sigma e}{kT}\right)^2\right)-1\right].
\end{aligned}$$

Graphically, this looks like

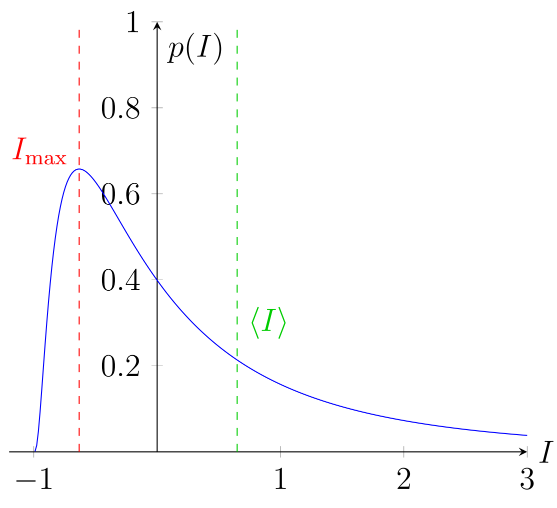
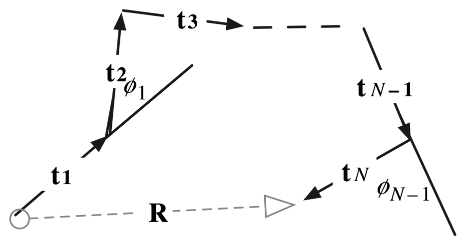


Figure 1: $p(I)$ with I_{max} and $\langle I\rangle$.

□

Problem 2.12 (Semi-flexible polymer in two dimensions (a,b,c))**.** Configurations of a model polymer can be described by either a set of vectors $\{\mathbf{t}_i\}$ of length a in two dimensions (for $i = 1, \ldots, N$), or alternatively by the angles $\{\phi_i\}$ between successive vectors, as indicated in the figure below. The polymer is at a temperature T, and



subject to an energy

$$\mathcal{H} = -\kappa \sum_{i=1}^{N-1} \mathbf{t}_i \cdot \mathbf{t}_{i+1} = -\kappa a^2 \sum_{i=1}^{N-1} \cos \phi_i,$$

where κ is related to the bending rigidity, such that the probability of any configuration is proportional to $\exp(-\mathcal{H}/k_B T)$.

(a) Show that $\langle \mathbf{t}_m \cdot \mathbf{t}_n \rangle \propto \exp(-|n-m|/\xi)$, and obtain an expression for the *persistence length* $\ell_p = a\xi$. (You can leave the answer as the ratio of simple integrals.)

(b) Consider the end-to-end distance $\mathbf{R}$ as illustrated in the figure. Obtain an expression for $\langle R^2 \rangle$ in the limit of $N \gg 1$.

(c) Find the probability $p(\mathbf{R})$ in the limit of $N \gg 1$.

Solution. (a) Without loss of generality, assume $n > m$. We can then write

$$\langle \mathbf{t}_m \cdot \mathbf{t}_n \rangle = a^2 \left\langle \cos\left(\sum_{k=m}^{n-1} \phi_k \right) \right\rangle .$$

Notice that

$$\left\langle \cos\left(\sum_{k=m}^{n-1} \phi_k \right) \right\rangle = \left\langle \cos\left(\sum_{k=m}^{n-2} \phi_k \right) \right\rangle \langle \cos \phi_{n-1} \rangle - \left\langle \sin\left(\sum_{k=m}^{n-2} \phi_k \right) \right\rangle \langle \sin \phi_{n-1} \rangle ,$$

where we have used the independence of $\{\phi_i\}$. By symmetry, $\langle \sin \phi_i \rangle = 0$, so

$$\left\langle \cos\left(\sum_{k=m}^{n-1} \phi_k \right) \right\rangle = \left\langle \cos\left(\sum_{k=m}^{n-2} \phi_k \right) \right\rangle \langle \cos \phi_{n-1} \rangle .$$

Since $\{\phi_i\}$ are i.i.d., it follows by induction that

$$\left\langle \cos\left(\sum_{k=m}^{n-1} \phi_k\right)\right\rangle = \langle \cos\phi_i\rangle^{|n-m|}.$$

Thus, it follows that

$$\langle \mathbf{t}_m \cdot \mathbf{t}_n\rangle = a^2 \exp\left[-|n-m|\left(-\ln\langle\cos\phi_i\rangle\right)\right] = a^2\exp\left[-\frac{|n-m|}{\xi}\right],$$

where

$$\xi = -\frac{1}{\ln\langle\cos\phi_i\rangle}.$$

Now, again by the independence of $\{\phi_j\}$

$$\begin{aligned}\langle\cos\phi_i\rangle &= \frac{\int \cos\phi_i \exp\left[\frac{\kappa a^2}{k_BT}\sum_{j=1}^{N-1}\cos\phi_j\right]\,d\{\phi_j\}}{\int \exp\left[\frac{\kappa a^2}{k_BT}\sum_{j=1}^{N-1}\cos\phi_j\right]\,d\{\phi_j\}}\\ &= \frac{\int_0^{2\pi}\cos\phi_i\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}{\int_0^{2\pi}\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}.\end{aligned}$$

Thus,

$$\xi = \left[\ln\left(\frac{\int_0^{2\pi}\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}{\int_0^{2\pi}\cos\phi_i\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}\right)\right]^{-1},$$

so

$$\ell_p = a\left[\ln\left(\frac{\int_0^{2\pi}\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}{\int_0^{2\pi}\cos\phi_i\exp\left(\frac{\kappa a^2}{k_BT}\cos\phi_i\right)\,d\phi_i}\right)\right]^{-1}$$

(b) Note that

$$\langle R^2\rangle = \langle \mathbf{R}\cdot\mathbf{R}\rangle = \sum_{i,j=1}^{N}\langle \mathbf{t}_i\cdot\mathbf{t}_j\rangle.$$

From the previous part, we know that

$$\langle \mathbf{t}_i\cdot\mathbf{t}_j\rangle = a^2\exp\left(-\frac{|i-j|}{\xi}\right).$$

Notice now that we can write

$$\begin{aligned}
\sum_{i,j=1}^{N} \langle \mathbf{t}_i \cdot \mathbf{t}_j \rangle &= \sum_{i=1}^{N} \langle t_i^2 \rangle + 2 \sum_{i>j}^{N} \langle \mathbf{t}_i \cdot \mathbf{t}_j \rangle \\
&= Na^2 + 2a^2 \sum_{i>j}^{N} \exp\left(-\frac{i-j}{\xi}\right) \\
&= Na^2 + 2a^2 \sum_{k=1}^{N-1} k \exp\left(-\frac{N-k}{\xi}\right) \\
&\approx Na^2 + 2a^2 \alpha e^{-N/\xi} \int_1^{N-1} k e^{k/\xi}\, dk \\
&= Na^2 + 2a^2 \alpha e^{-N/\xi} \left(\xi e^{k/\xi}(k-\xi)\right) \Big|_{k=1}^{N-1} \\
&= Na^2 + 2a^2 \alpha \xi e^{-N/\xi} \left[e^{(N-1)/\xi}(N-1-\xi) - e^{1/\xi}(1-\xi)\right] \\
&= Na^2 + 2a^2 \alpha \xi \left(e^{-1/\xi}(N-1-\xi) - e^{-(N-1)/\xi}(1-\xi)\right),
\end{aligned}$$

where we can change the sum over k into an integral over k times a constant α for $N \gg 1$. To find α, we calculate

$$\alpha = \lim_{m\to\infty} \frac{\int_{m-1}^{m} x e^{k/\xi}\, dx}{m e^{m/\xi}} = (1 - e^{-1/\xi})\xi.$$

Thus, we get that

$$\langle R^2 \rangle \sim a^2 \left(N + 2(1-e^{-1/\xi})\xi^2 \left[e^{-1/\xi}(N-1-\xi) - e^{-(N-1)/\xi}(1-\xi)\right]\right).$$

(c) By symmetry, we know that $\langle \mathbf{R} \rangle = \langle x \rangle = \langle y \rangle = 0$, so $\langle R^2 \rangle_c = \langle R^2 \rangle$. Also by symmetry, we must have $\langle xy \rangle = 0$, so we can treat x, y as independent in the limit $N \gg 1$ for the central limit theorem. Since $\langle x^2 \rangle = \langle y^2 \rangle$ by symmetry, we know that

$$\langle x^2 \rangle + \langle y^2 \rangle = \langle R^2 \rangle \implies \langle x^2 \rangle = \langle y^2 \rangle = \frac{1}{2} \langle R^2 \rangle.$$

Then, by the central limit theorem, we get that for $N \gg 1$

$$p(\mathbf{R}) \sim \frac{1}{2\pi\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} \exp\left(-\frac{x^2}{2\langle x \rangle^2} - \frac{y^2}{2\langle y^2 \rangle}\right) = \frac{1}{\pi \langle R^2 \rangle} \exp\left(-\frac{R^2}{\langle R^2 \rangle}\right). \quad \square$$