Vectors and Matrices (Elementary)

HU-HTAKM
Website: https://htakm.github.io/htakm_test/

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This document contains all the knowledge about vectors and matrices that you may have learned in high school mathematics.

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Chapter 1

Introduction to Vectors

In daily life, many quantities are described by **magnitude** alone, such as length, time, and temperature. These are called **scalars**. However, some quantities are described by both **magnitude** and **direction**. For example, the acceleration of an object. For simplicity, we visualize vectors in two dimensions.

Definition 1.1. A **vector** is represented by a directed line segment. The arrowhead represents the direction, while the length represents the magnitude. The vector is denoted by $\overrightarrow{AB}$ (A to B), **a**, or $\vec{a}$.

Definition 1.2. The magnitude (length) of a vector **a** is denoted by $|\vec{a}|$.

Remark 1.2.1. We say that two vectors $\vec{a}$ and $\vec{b}$ are equal if they have the same magnitude and direction.

In 2D, the length of vectors can be obtained using the distance formula. They are represented as follows:

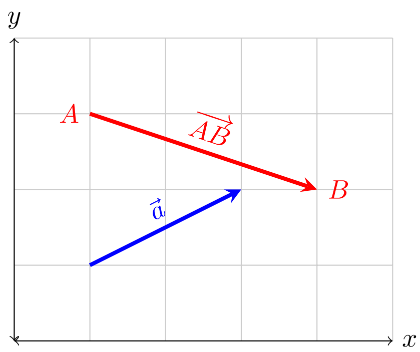


Figure 1.1: Vector

Definition 1.3. The **negative vector** of a vector $\vec{u}$ is the vector with the same magnitude but opposite direction. It is denoted by $-\vec{u}$.

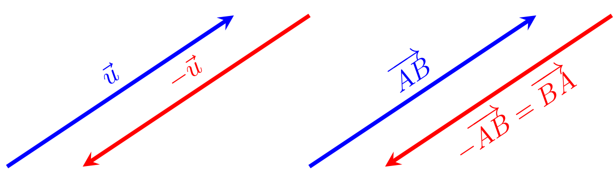


Figure 1.2: A vector and its corresponding negative vector

Definition 1.4. A **zero vector** is a vector with zero magnitude and no direction. It is denoted by $\vec{0}$.

Definition 1.5. A **unit vector** is a vector with magnitude 1. It is denoted by $\hat{u}$.

Theorem 1.6. $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$ is a unit vector with the same direction as $\vec{u}$.

Example 1.1. The most common unit vectors are the unit vectors along the axes.

1. $\hat{i}$ corresponds to the unit vector along the x-axis.
2. $\hat{j}$ corresponds to the unit vector along the y-axis.
3. $\hat{k}$ corresponds to the unit vector along the z-axis.

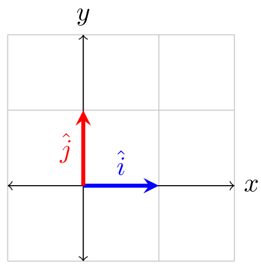
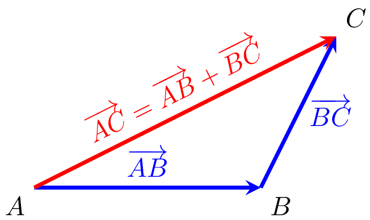
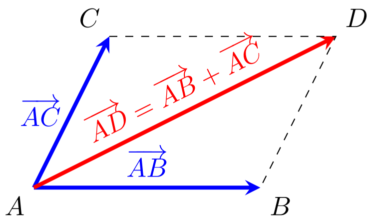


Figure 1.3: The unit vectors along the x-axis and y-axis

Definition 1.7. The addition of two vectors is also a vector and is defined as follows.



(a) Triangle Law of Addition



(b) Parallelogram Law of Addition

Definition 1.8. The subtraction of two vectors $\vec{u}$ and $\vec{v}$ is $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

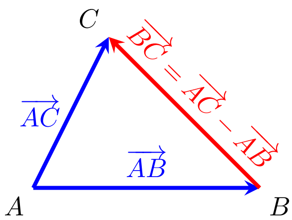


Figure 1.5: Subtraction

Definition 1.9. The scalar multiplication of a vector $\vec{u}$ is defined as follows:

1. If $\lambda > 0$, then $\lambda\vec{u}$ is a vector with magnitude $\lambda\,|u|$ and the same direction as $\vec{u}$.
2. If $\lambda < 0$, then $\lambda\vec{u}$ is a vector with magnitude $-\lambda\,|u|$ and the opposite direction to $\vec{u}$.
3. If $\lambda = 0$, then $\lambda\vec{u}$ is a zero vector.

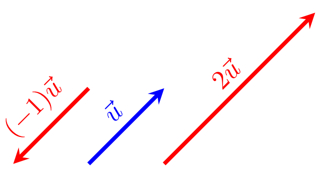


Figure 1.6: Scalar multiplication of a vector

Lemma 1.10. For any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and any real numbers λ, μ,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3. $\vec{u} + \vec{0} = \vec{u}$
4. $\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a}$
5. $\lambda(\vec{u} \pm \vec{v}) = \lambda\vec{u} \pm \lambda\vec{v}$
6. $(\lambda \pm \mu)\vec{u} = \lambda\vec{u} \pm \mu\vec{u}$

Theorem 1.11. If $\vec{u}$ and $\vec{v}$ are non-zero and not parallel, then:

1. If $\lambda\vec{u} + \mu\vec{v} = \vec{0}$, then $\lambda = \mu = 0$.
2. If $\lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$, then $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

Remark 1.11.1. Knowing these properties, we can deduce that any vector in 2D space can be represented in terms of $\hat{\imath}$ and $\hat{\jmath}$, and any vector in 3D space can be represented in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

We have now finished introducing the basics of vectors. Vectors can be used to perform various operations.

Definition 1.12. A **position vector** is a vector that starts at the origin O.

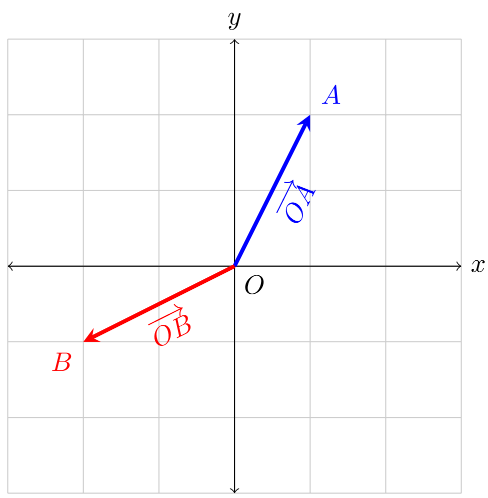


Figure 1.7: Position Vectors

Theorem 1.13. (Section formula) If P is a point on AB such that $AP : PB = r : s$, then $\vec{p} = \frac{s\vec{a}+r\vec{b}}{s+r}$, where $\vec{a}$, $\vec{b}$, and $\vec{p}$ are the position vectors of A, B, and P, respectively.

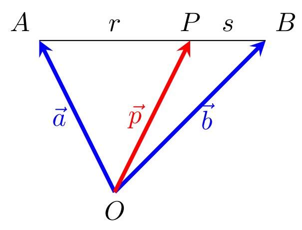


Figure 1.8: Section formula

Chapter 2

Introduction to Matrices

In real life, we often process data using a rectangular array of real numbers.

Definition 2.1. A **matrix** is a rectangular array of real numbers arranged in the form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The **dimension** of this matrix is $m \times n$. The numbers inside the matrix are called **elements** of the matrix. It is denoted by $\mathbf{A}$.

In advanced mathematics, we use vectors to define a matrix in the form:

$$\begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{pmatrix}$$

where $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ are m-dimensional vectors.

Definition 2.2. A **zero matrix** is a matrix where all the elements are zero.

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

An $m \times n$ zero matrix is denoted as $\mathbf{0}_{m\times n}$ or $\mathbf{0}$.

Definition 2.3. A **row matrix** is a matrix with only one row. A **column matrix** is a matrix with only one column.

Example 2.1. $\begin{pmatrix} 1 & 2 \end{pmatrix}$ is a row matrix.

Example 2.2. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is a column matrix.

Definition 2.4. A **square matrix** of order n is a matrix with dimension $n \times n$. The elements $a_{11}, a_{22}, \cdots, a_{nn}$ are the principal diagonal elements.

Definition 2.5. A **diagonal matrix** is a square matrix where elements that are not principal diagonal elements are zero.

Example 2.3. $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix.

Definition 2.6. An **identity matrix** is a diagonal matrix where the principal diagonal elements are all 1.

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

An $n \times n$ identity matrix is denoted as $\mathbf{I}_n$ or $\mathbf{I}$.

Definition 2.7. For two $m \times n$ matrices $\mathbf{A}$ with elements a_{ij} and $\mathbf{B}$ with elements b_{ij}, the addition and subtraction of $\mathbf{A}$ and $\mathbf{B}$ are defined as:

$$\mathbf{A} \pm \mathbf{B} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Definition 2.8. For an $m \times n$ matrix $\mathbf{A}$ with elements a_{ij} and a scalar k, scalar multiplication is defined as:

$$k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

Lemma 2.9. For $m \times n$ matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and scalars λ, μ:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
3. $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
4. $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
5. $(\lambda \pm \mu)\mathbf{A} = \lambda\mathbf{A} \pm \mu\mathbf{A}$
6. $\lambda(\mathbf{A} \pm \mathbf{B}) = \lambda\mathbf{A} \pm \lambda\mathbf{B}$
7. $\lambda(\mu\mathbf{A}) = (\lambda\mu)\mathbf{A}$
8. $0\mathbf{A} = \mathbf{0}$
9. $\lambda\mathbf{0} = \mathbf{0}$

There is an important scalar that can be computed from square matrices. It is called the determinant.

Definition 2.10. For any square matrix $\mathbf{A}$, the **determinant** of $\mathbf{A}$ is a real number denoted by $|\mathbf{A}|$ or $\det \mathbf{A}$.

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

How do we calculate the determinant?

Definition 2.11. For a 2×2 matrix:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Definition 2.12. For a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Definition 2.13. Given a matrix $\mathbf{A}$ with elements a_{ij}:

1. The **minor** of the element a_{ij}, denoted by M_{ij}, is the determinant of order $(n-1)$ obtained by removing the i-th row and j-th column of $|\mathbf{A}|$.
2. The **cofactor** of the element a_{ij}, denoted by A_{ij}, is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Example 2.4. Given a matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$:

$$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}, \qquad A_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}, \qquad M_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, \qquad A_{33} = +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Theorem 2.14. For any identity matrix $\mathbf{I}$, $|\mathbf{I}| = 1$.

Theorem 2.15. (**Cofactor Expansion**) We can expand the determinant using cofactors along the i-th row or j-th column:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Calculating determinants directly can sometimes be tedious. Fortunately, we can perform row and column operations on the matrix to simplify the process.

Definition 2.16. By definition, we can perform addition and scalar multiplication on determinants:

$$\begin{vmatrix} a_{11} & a_{12}+b_1 & \dots & a_{1n} \\ a_{21} & a_{22}+b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2}+b_n & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21}+b_1 & a_{22}+b_2 & \dots & a_{2n}+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & \lambda a_{12} & \dots & a_{1n} \\ a_{21} & \lambda a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \lambda a_{n2} & \dots & a_{nn} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \mu a_{21} & \mu a_{22} & \dots & \mu a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \mu \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

We can also change the order:

$$-\begin{vmatrix} a_{12} & a_{11} & \dots & a_{1n} \\ a_{22} & a_{21} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Theorem 2.17. We can add multiples of rows and columns to other rows and columns, respectively.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} - \lambda a_{1j} & a_{12} & \dots & a_{1n} \\ a_{21} - \lambda a_{2j} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - \lambda a_{nj} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} - \mu a_{i1} & a_{12} - \mu a_{i2} & \dots & a_{1n} - \mu a_{in} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Example 2.5.

$$\begin{vmatrix} 2 & 5 & 10 \\ 10 & 3 & 4 \\ 2 & 5 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 10 \\ 10 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Chapter 3

Product of Vectors and Matrices

When calculating the products of vectors and matrices, the operations differ significantly from scalar multiplication. In this chapter, we will discuss these operations and their applications. Let us start with vectors.

Definition 3.1. Given two vectors $\vec{u}$ and $\vec{v}$, the **dot product** (scalar product) of $\vec{u}$ and $\vec{v}$ is defined as:

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$

where θ is the angle between the two vectors.

Lemma 3.2. For any vectors $\vec{u}, \vec{v}, \vec{w}$ and any scalar λ:

1. $\vec{u} \cdot \vec{0} = \vec{0} \cdot \vec{u} = 0$
2. $\vec{u} \cdot \vec{u} = |\vec{u}|^2 \geq 0$
3. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
4. $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$
5. $\lambda(\vec{u} \cdot \vec{v}) = (\lambda\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\lambda\vec{v})$
6. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
7. $|\vec{u} \cdot \vec{v}| \leq |\vec{u}|\,|\vec{v}|$
8. $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2(\vec{u} \cdot \vec{v})$

Theorem 3.3. For any two non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u} \perp \vec{v}$.

Lemma 3.4. In the 3D coordinate system:

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1, \qquad \hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0.$$

Given two vectors $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, the dot product of $\vec{u}$ and $\vec{v}$ is defined as:

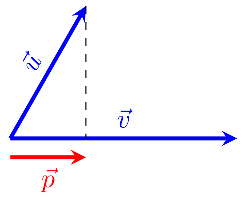
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Interestingly, from the definition, we can project one vector onto another.

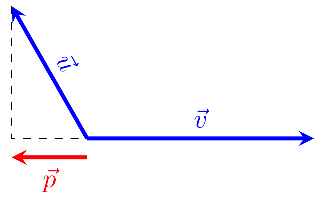
Definition 3.5. Given two non-zero vectors $\vec{u}$ and $\vec{v}$, the **projection** of $\vec{u}$ onto $\vec{v}$ is a vector $\vec{p}$ defined as:

$$\vec{p} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

The magnitude of $\vec{p}$ is $|\vec{p}| = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$.



(a) Projection with equal direction



(b) Projection with opposite direction

Another type of product for vectors is the cross product.

Definition 3.6. Given two vectors $\vec{u}$ and $\vec{v}$, the **cross product** (vector product) of $\vec{u}$ and $\vec{v}$ is defined as:

$$\vec{u} \times \vec{v} = (|\vec{u}|\,|\vec{v}| \sin\theta)\hat{n},$$

where θ is the angle between $\vec{u}$ and $\vec{v}$, and $\hat{n}$ is the unit vector perpendicular to both $\vec{u}$ and $\vec{v}$, with its direction determined by the right-hand rule.

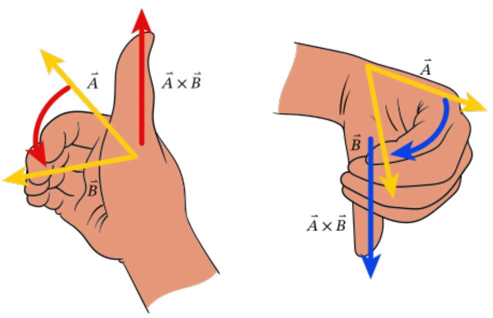


Figure 3.2: Right-Hand Rule

Lemma 3.7. For any vectors $\vec{u}, \vec{v}, \vec{w}$ and any scalar λ:

1. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
2. $\vec{u} \times \vec{u} = \vec{0}$
3. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
4. $(\lambda\vec{u}) \times \vec{v} = \vec{u} \times (\lambda\vec{v}) = \lambda(\vec{u} \times \vec{v})$
5. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
6. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
7. $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2\,|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$

We can use the determinant to calculate the cross product.

Theorem 3.8. In the 3D coordinate system:

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0, \qquad \hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}.$$

Given two vectors $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, the cross product of $\vec{u}$ and $\vec{v}$ is defined as:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Now that we have introduced both types of vector products, what are their applications? One simple application is calculating areas and volumes.

Theorem 3.9. The area of the parallelogram formed by two vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.

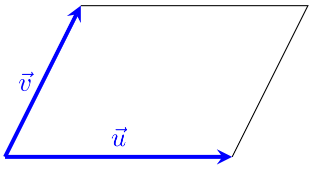


Figure 3.3: Parallelogram formed by two vectors

Theorem 3.10. The area of the triangle formed by two vectors $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}|\vec{u} \times \vec{v}|$.

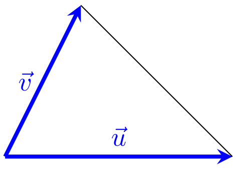


Figure 3.4: Triangle formed by two vectors

Theorem 3.11. The volume of the parallelepiped formed by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is $|\vec{u} \cdot (\vec{v} \times \vec{w})|$.

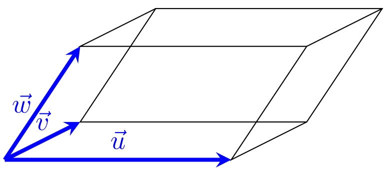


Figure 3.5: Parallelepiped formed by three vectors

Theorem 3.12. The volume of the tetrahedron formed by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is $\frac{1}{6}|\vec{u} \cdot (\vec{v} \times \vec{w})|$.

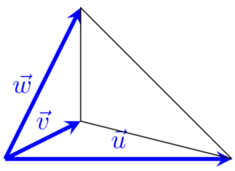
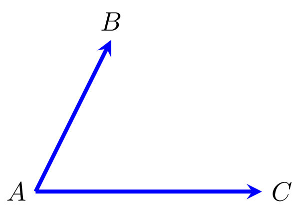


Figure 3.6: Tetrahedron formed by three vectors

We can also use the cross product to determine if three points are collinear.

Theorem 3.13. Given three points A, B, and C, they are collinear if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$.

(a) Collinear



(b) Not collinear

What about multiplying matrices?

Definition 3.14. Given an $m \times n$ matrix $\mathbf{A}$ with elements a_{ij} and an $n \times p$ matrix $\mathbf{B}$ with elements b_{ij}, the product of $\mathbf{A}$ and $\mathbf{B}$ is an $m \times p$ matrix $\mathbf{C} = \mathbf{AB}$ with each element c_{ij} obtained by:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example 3.1. Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$, then:

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 39 & 54 & 69 \\ 49 & 68 & 87 \\ 59 & 82 & 105 \end{pmatrix}.$$

Lemma 3.15. For any scalars λ and μ, and matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ that allow for matrix multiplications between them:

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
3. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
4. $(\lambda\mathbf{A})(\mu\mathbf{B}) = \lambda\mu(\mathbf{AB})$.
5. $\mathbf{0A} = \mathbf{A0} = \mathbf{0}$.
6. $\mathbf{I}_m\mathbf{A} = \mathbf{AI}_n = \mathbf{A}$ if $\mathbf{A}$ is an $m \times n$ matrix.
7. $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

Definition 3.16. Let $\mathbf{A}$ be a square matrix and n be a positive integer. Then:

$$\mathbf{A}^n = \prod_{i=1}^{n} \mathbf{A}.$$

Some matrices have a multiplicative inverse, just like scalars.

Definition 3.17. Given an $n \times n$ matrix $\mathbf{A}$, it is **invertible** (non-singular) if there exists another $n \times n$ matrix $\mathbf{B}$ such that:

$$\mathbf{AB} = \mathbf{I}_n.$$

$\mathbf{B}$ is the **inverse matrix** of $\mathbf{A}$, and is denoted by $\mathbf{A}^{-1}$.

Theorem 3.18. For any square matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{AB} = \mathbf{I}$, then $\mathbf{BA} = \mathbf{I}$ and $\mathbf{B}$ is the inverse of $\mathbf{A}$.

Theorem 3.19. For any invertible matrix $\mathbf{A}$, $\left|\mathbf{A}^{-1}\right| = |\mathbf{A}|^{-1}$.

Theorem 3.20. (**Uniqueness**) Let $\mathbf{A}$ be an invertible matrix. If both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, then $\mathbf{B} = \mathbf{C}$.

Theorem 3.21. Let $\mathbf{A}$ and $\mathbf{B}$ be two invertible $n \times n$ matrices, and λ be a non-zero scalar.

1. $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. $\lambda\mathbf{A}$ is invertible and $(\lambda\mathbf{A})^{-1} = \frac{1}{\lambda}\mathbf{A}^{-1}$.
3. $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Theorem 3.22. If $\mathbf{A}$ is invertible and k is a positive integer, then $\mathbf{A}^k$ is invertible and $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$.

Based on the definition, it seems we may need to find the inverse by brute force. Luckily, there is a way to easily find the inverse. Before that, we should introduce the transpose.

Definition 3.23. Given an $m \times n$ matrix $\mathbf{A}$, the **transpose** of matrix $\mathbf{A}$, denoted by $\mathbf{A}^T$, is an $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

Lemma 3.24. Let λ be a scalar, and matrices $\mathbf{A}, \mathbf{B}$ that allow for matrix multiplications.

1. $\mathbf{I}^T = \mathbf{I}$.
2. $(\mathbf{A}^T)^T = \mathbf{A}$.
3. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.
4. $(\lambda\mathbf{A})^T = \lambda\mathbf{A}^T$.
5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
6. If $\mathbf{A}$ is invertible, then $\mathbf{A}^T$ is invertible and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

We can now introduce the adjoint matrix.

Definition 3.25. For any $n \times n$ square matrix $\mathbf{A}$, the **adjoint matrix** is defined by:

$$\operatorname{adj}\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T,$$

where A_{ij} is the cofactor of a_{ij}.

Using this matrix, we can obtain a matrix that is a multiple of the identity matrix using cofactor expansion.

Lemma 3.26. Let $\mathbf{A}$ be a square matrix. Then:

$$\mathbf{A}(\operatorname{adj}\mathbf{A}) = \begin{pmatrix} |\mathbf{A}| & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & |\mathbf{A}| \end{pmatrix} = |\mathbf{A}|\,\mathbf{I}.$$

As we can see, we have obtained a formula for the inverse matrix.

Theorem 3.27. Let $\mathbf{A}$ be a square matrix.

1. If $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\operatorname{adj}\mathbf{A}$.
2. $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$.

Example 3.2. Given $\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix}$, we can find that:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 0 & 3 \\ -1 & 2 & 0 \end{vmatrix} = 7.$$

Therefore, we can see that $\mathbf{A}$ is invertible. Then:

$$\operatorname{adj}\mathbf{A} = \begin{pmatrix} \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} -6 & 8 & 3 \\ -3 & 4 & 5 \\ 4 & -3 & -2 \end{pmatrix},$$

$$\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} -6 & 8 & 3 \\ -3 & 4 & 5 \\ 4 & -3 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{7} & \frac{8}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{4}{7} & \frac{5}{7} \\ \frac{4}{7} & -\frac{3}{7} & -\frac{2}{7} \end{pmatrix}.$$

Chapter 4

System of Linear Equations

One of the most important applications of matrices is solving a system of multiple linear equations.

Definition 4.1. A system of m linear equations in n unknowns is in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

It is **consistent** if it has solutions. It is **inconsistent** if it has no solutions.

For simplicity, we focus on a system of n linear equations in n unknowns. One obvious way is to use an invertible matrix.

Lemma 4.2. We can turn the system of linear equations into the form:

$$\mathbf{AX} = \mathbf{B}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

If $\mathbf{A}$ is a square matrix, $\mathbf{A}$ is invertible if and only if $\mathbf{AX} = \mathbf{B}$ has a unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Theorem 4.3. Given a system of n linear equations in n unknowns $\mathbf{AX} = \mathbf{B}$:

1. $\mathbf{AX} = \mathbf{B}$ has a unique solution if and only if $|\mathbf{A}| \neq 0$.
2. $\mathbf{AX} = \mathbf{B}$ has either no solutions or infinitely many solutions if and only if $|\mathbf{A}| = 0$.

Finding an invertible matrix to solve the system of linear equations can be tedious. We introduce Cramer's Rule, which makes use of determinants.

Theorem 4.4. (**Cramer's Rule**) For a system of two linear equations in two unknowns, we let:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}, \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \Delta_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \qquad \Delta_y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

If $\Delta \neq 0$, then $x = \frac{\Delta_x}{\Delta}$ and $y = \frac{\Delta_y}{\Delta}$ is the unique solution of the system.
For a system of three linear equations in three unknowns, we let:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}, \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

If $\Delta \neq 0$, then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, and $z = \frac{\Delta_z}{\Delta}$ is the unique solution of the system.

Interestingly, Cramer's Rule can help us determine whether a system of linear equations has no solutions or infinitely many solutions.

Theorem 4.5. For a system of two linear equations in two unknowns:

1. If $\Delta \neq 0$, then the system has a unique solution.
2. If $\Delta = 0$ and either $\Delta_x \neq 0$ or $\Delta_y \neq 0$, then the system has no solutions.
3. If $\Delta = 0$ and $\Delta_x = \Delta_y = 0$, then the system has infinitely many solutions.

For a system of three linear equations in three unknowns:

1. If $\Delta \neq 0$, then the system has a unique solution.
2. If $\Delta = 0$ and either $\Delta_x \neq 0$, $\Delta_y \neq 0$, or $\Delta_z \neq 0$, then the system has no solutions.
3. If $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has infinitely many solutions.

We now introduce a third method of solving a system of linear equations, which is Gaussian elimination. Before that, we introduce the row echelon form.

Definition 4.6. An **augmented matrix** is a matrix that is obtained by appending an original matrix.

Example 4.1. Assume that we want to represent:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

We can use the following augmented matrix to represent this system of linear equations:

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right).$$

Remark 4.6.1. What's special about a system of linear equations is that we can perform row operations on the augmented matrix.

Definition 4.7. An augmented matrix is in **row echelon form** if:

1. The first non-zero element in each row is 1.
2. The first non-zero element in each row occurs in a column to the right of the first non-zero element in the preceding row.
3. Any zero rows are placed at the bottom of the matrix.

Example 4.2. The following are some augmented matrices that are in row echelon form:

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 3 \end{array}\right), \qquad \left(\begin{array}{ccc|c} 1 & 3 & 6 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right), \qquad \left(\begin{array}{ccc|c} 1 & 7 & 6 & 5 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Once we turn an augmented matrix into row echelon form, we can obtain the solution by back substitution.

Example 4.3. We want to find the solution of this system of linear equations:

$$\begin{cases} x + 3y + 4z = 19 \\ 2x + 5y + 3z = 21 \\ 3x + 7y + 3z = 26 \end{cases}$$

We turn this system into an augmented matrix, then perform row operations:

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 2 & 5 & 3 & 21 \\ 3 & 7 & 3 & 26 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & -1 & -5 & -17 \\ 0 & -2 & -9 & -31 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & 1 & 5 & 17 \\ 0 & -2 & -9 & -31 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

From this, we can turn the augmented matrix into linear equations again and get the solution:

$$\begin{cases} x + 3y + 4z = 19 \\ y + 5z = 17 \\ z = 3 \end{cases}, \qquad \begin{cases} x = 1 \\ y = 2 \\ z = 3. \end{cases}$$

Example 4.4. We want to find the solution of this system of linear equations:

$$\begin{cases} x + 3y + 4z = 19 \\ 2x + 5y + 3z = 21 \\ 3x + 7y + 2z = 23 \end{cases}$$

We turn this system into an augmented matrix, then perform row operations:

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 2 & 5 & 3 & 21 \\ 3 & 7 & 2 & 23 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & -1 & -5 & -17 \\ 0 & -2 & -10 & -34 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & 1 & 5 & 17 \\ 0 & -2 & -10 & -34 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 19 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From this, we can turn the augmented matrix into linear equations again and get the solution. Let t be any real number:

$$\begin{cases} x + 3y + 4z = 19 \\ y + 5z = 17 \end{cases}, \qquad \begin{cases} x = 11t - 32 \\ y = -5t + 17 \\ z = t. \end{cases}$$

There is a special kind of system of linear equations called the system of homogeneous linear equations. In this case, it is guaranteed to have solutions.

Definition 4.8. A system of m homogeneous linear equations in n unknowns is in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Theorem 4.9. For a system of n homogeneous linear equations with n unknowns $\mathbf{AX} = \mathbf{0}$:

1. $\mathbf{AX} = \mathbf{0}$ has the trivial solution (all zeros) as its only solution if and only if $|\mathbf{A}| \neq 0$.
2. $\mathbf{AX} = \mathbf{0}$ has the trivial solution and infinitely many non-trivial solutions if and only if $|\mathbf{A}| = 0$.

There is a special use for the augmented matrix. Note that when we represent a system of linear equations in $\mathbf{AX} = \mathbf{B}$:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $|\mathbf{A}| \neq 0$, by performing row operations, we get:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & \dots & a'_{1n} \\ a'_{21} & a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

We know that $\mathbf{AX} = \mathbf{IB}$ can become $\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$. It just so happens that we have obtained the inverse of $\mathbf{A}$! Therefore:

Theorem 4.10. Given an invertible matrix $\mathbf{A}$, by performing row operations, we can get:

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right) \sim \left(\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array} \right).$$