4. Convex optimization problems

- optimization problem in standard form

- convex optimization problems

- quasiconvex optimization

- linear optimization

- quadratic optimization

- geometric programming

- generalized inequality constraints

- semidefinite programming

- vector optimization

Optimization problem in standard form

$$\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\
& h_i(x) = 0, \quad i = 1, \dots, p
\end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable

- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function

- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, m$, are the inequality constraint functions

- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \le 0, \; i = 1, \dots, m, \; h_i(x) = 0, \; i = 1, \dots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)

- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

a feasible x is **optimal** if $f_0(x) = p^\star$; X_{opt} is the set of optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \le 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \le R \end{array}$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point

- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$

- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal

- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \ \cap \ \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem

- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints

- a problem is **unconstrained** if it has no explicit constraints $(m = p = 0)$

example:
$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal

- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \le 0, \quad i = 1, \dots, m \\
& a_i^T x = b_i, \quad i = 1, \dots, p
\end{aligned}
$$

- $f_0, f_1, \dots, f_m$ are convex; equality constraints are affine

- problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \dots, f_m$ convex)

often written as

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \le 0, \quad i = 1, \dots, m \\
& Ax = b
\end{aligned}
$$

important property: feasible set of a convex optimization problem is convex
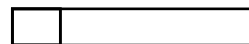
example

$$\begin{array}{ll}
\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\
& h_1(x) = (x_1 + x_2)^2 = 0
\end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex

- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine

- equivalent (but not identical) to the convex problem

$$\begin{array}{ll}
\text{minimize} & x_1^2 + x_2^2 \\
\text{subject to} & x_1 \leq 0 \\
& x_1 + x_2 = 0
\end{array}$$

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
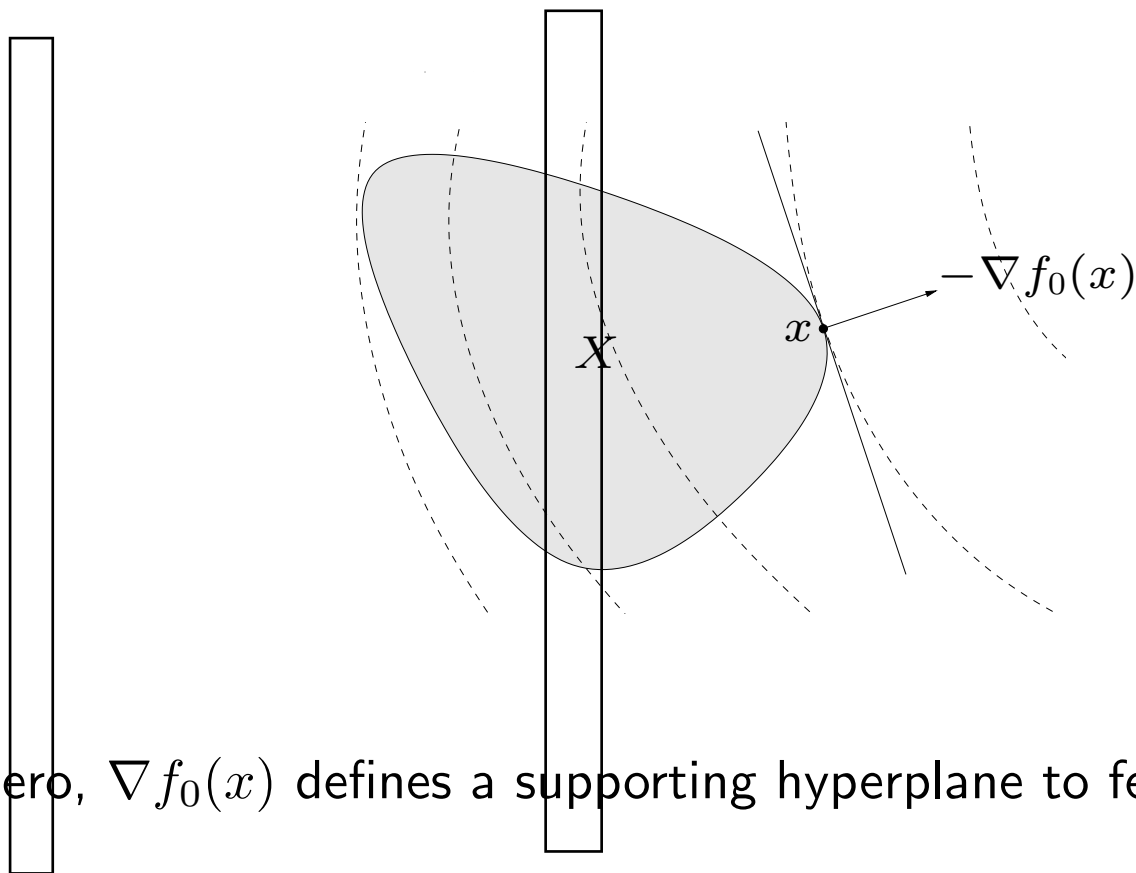- $\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$$

which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \text{for all feasible } y$$

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

- **unconstrained problem**: x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad x \succeq 0, \qquad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } z$$

- **introducing equality constraints**

$$
\begin{array}{ll}
\text{minimize} & f_0(A_0 x + b_0) \\
\text{subject to} & f_i(A_i x + b_i) \le 0, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\, y_i) & f_0(y_0) \\
\text{subject to} & f_i(y_i) \le 0, \quad i = 1, \ldots, m \\
& y_i = A_i x + b_i, \quad i = 0, 1, \ldots, m
\end{array}
$$

- **introducing slack variables for linear inequalities**

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\, s) & f_0(x) \\
\text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
& s_i \ge 0, \quad i = 1, \ldots m
\end{array}
$$

- **epigraph form**: standard form convex problem is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\, t) & t \\
\text{subject to} & f_0(x) - t \le 0 \\
& f_i(x) \le 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- **minimizing over some variables**

$$
\begin{array}{ll}
\text{minimize} & f_0(x_1, x_2) \\
\text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \tilde{f}_0(x_1) \\
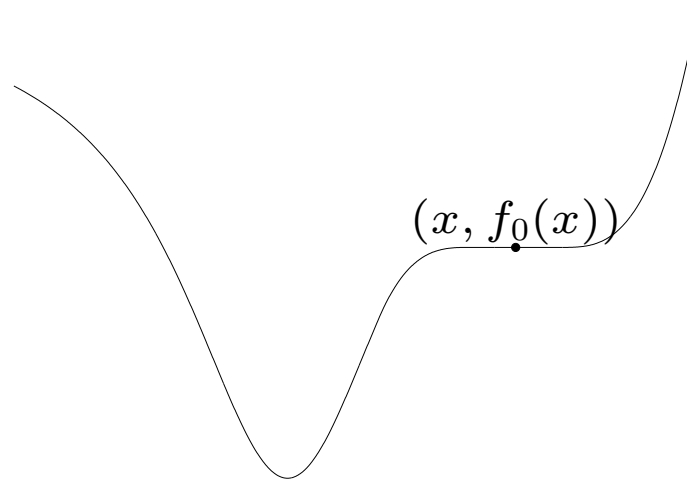\text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m
\end{array}
$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

Quasiconvex optimization

$$\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\
& Ax = b
\end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex, $f_1, \dots, f_m$ convex

can have locally optimal points that are not (globally) optimal

$(x, f_0(x))$

convex representation of sublevel sets of f_0

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(x)$ is convex in x for fixed t

- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, *i.e.*,

$$f_0(x) \le t \quad \Longleftrightarrow \quad \phi_t(x) \le 0$$

example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \ge 0$, $q(x) > 0$ on $\mathbf{dom}\, f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \ge 0$, ϕ_t convex in x

- $p(x)/q(x) \le t$ if and only if $\phi_t(x) \le 0$

quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \le 0, \qquad f_i(x) \le 0, \quad i = 1, \ldots, m, \qquad Ax = b \qquad (1)$$

- for fixed t, a convex feasibility problem in x

- if feasible, we can conclude that $t \ge p^\star$; if infeasible, $t \le p^\star$

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
    1. $t := (l + u)/2$.
    2. Solve the convex feasibility problem (1).
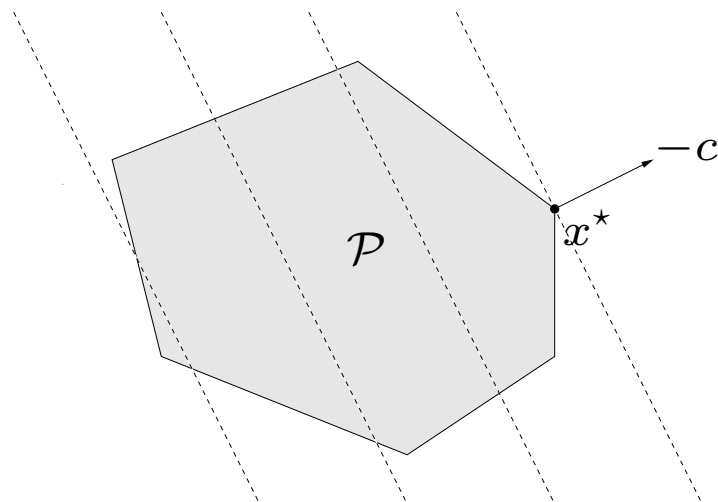   3. **if** (1) is feasible, $u := t$;   **else** $l := t$.
until $u - l \le \epsilon$.

---

requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations (where u, l are initial values)

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions

- feasible set is a polyhedron

Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \le t, \quad i = 1, \ldots, m \end{array}$$
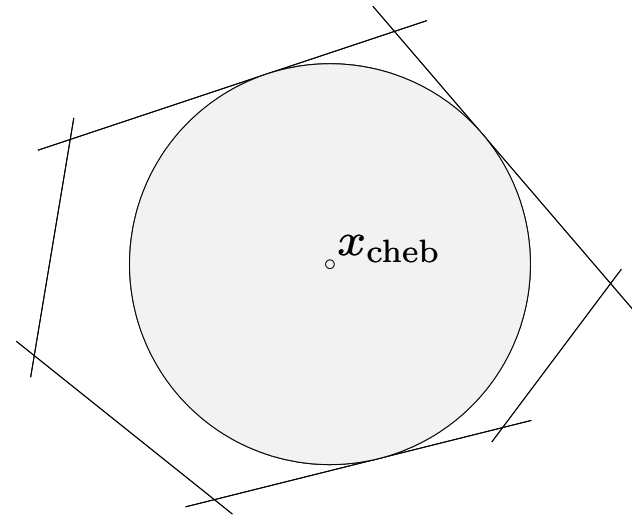
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \le b_i, \ i = 1, \dots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$$

- $a_i^T x \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \le r\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

- hence, x_c, r can be determined by solving the LP

$$
\begin{array}{ll}
\text{maximize} & r \\
\text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1, \dots, m
\end{array}
$$

Linear-fractional program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^T x + f > 0\}$$

- a quasiconvex optimization problem; can be solved by bisection

- also equivalent to the LP (variables y, z)

$$\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

generalized linear-fractional program

$$f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e_i^T x + f_i > 0,\ i = 1,\ldots,r\}$$

a quasiconvex optimization problem; can be solved by bisection

example: Von Neumann model of a growing economy

$$\begin{array}{ll}
\text{maximize (over } x,\ x^+) & \min_{i=1,\ldots,n} x_i^+/x_i \\
\text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax
\end{array}$$

- $x, x^+ \in \mathbf{R}^n$: activity levels of n sectors, in current and next period

- $(Ax)_i$, $(Bx^+)_i$: produced, resp. consumed, amounts of good i

- x_i^+/x_i: growth rate of sector i

allocate activity to maximize growth rate of slowest growing sector

Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic

- minimize a convex quadratic function over a polyhedron
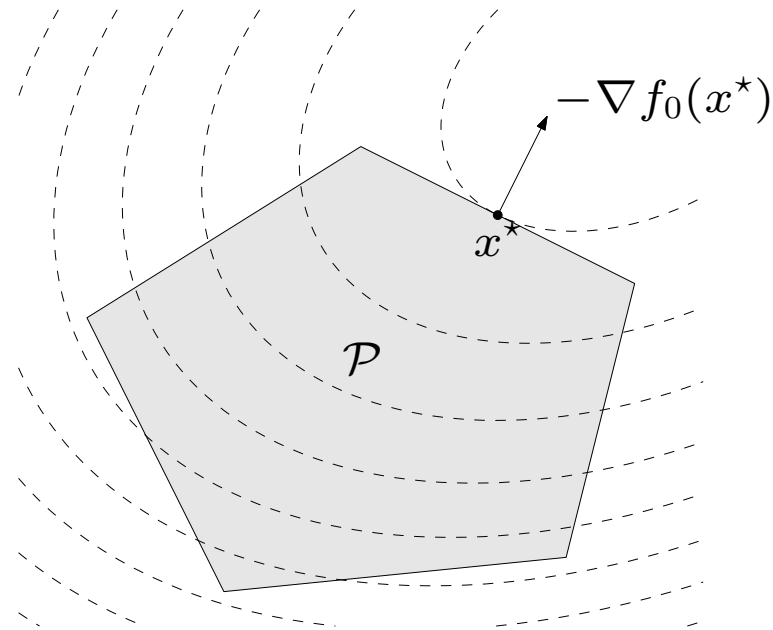
Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)

- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

linear program with random cost

$$
\begin{array}{ll}
\text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\
\text{subject to} & Gx \preceq h, \quad Ax = b
\end{array}
$$

- c is random vector with mean $\bar{c}$ and covariance Σ

- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic

- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \dots, m \\ & Fx = g \end{array}$$

$(A_i \in \mathbf{R}^{n_i \times n},\ F \in \mathbf{R}^{p \times n})$

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i + 1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP

- more general than QCQP and LP

Robust linear programming

the parameters in optimization problems are often uncertain, $e.g.$, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \dots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \dots, m, \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \dots, m \end{array}$$

deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\} \qquad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n \times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of P_i

- robust LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \le b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to the SOCP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \quad i = 1, \ldots, m
\end{array}
$$

(follows from $\sup_{\|u\|_2 \le 1}(\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)

- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\mathbf{prob}(a_i^T x \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

 where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} \, dt$ is CDF of $\mathcal{N}(0,1)$

- robust LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m,
\end{array}
$$

 with $\eta \ge 1/2$, is equivalent to the SOCP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \le b_i, \quad i = 1, \ldots, m
\end{array}
$$

Geometric programming

monomial function

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c > 0$; exponent a_i can be any real number

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

geometric program (GP)

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 1, \quad i = 1, \ldots, m \\
& h_i(x) = 1, \quad i = 1, \ldots, p
\end{array}
$$

with f_i posynomial, h_i monomial

Geometric program in convex form

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

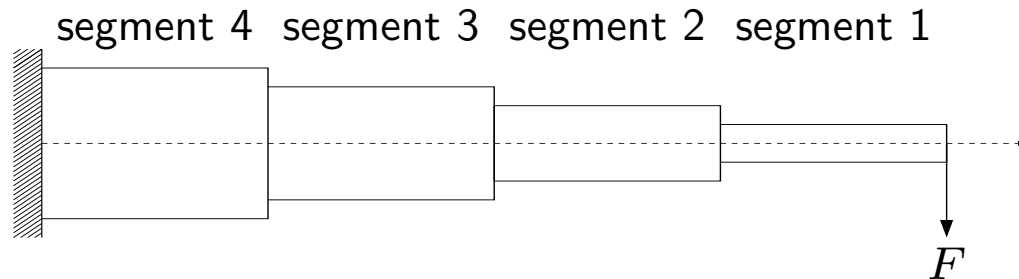$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left(\sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$
\begin{array}{ll}
\text{minimize} & \log \left(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\
\text{subject to} & \log \left(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\
& Gy + d = 0
\end{array}
$$

Design of cantilever beam

F

- N segments with unit lengths, rectangular cross-sections of size $w_i \times h_i$

- given vertical force F applied at the right end

design problem

minimize   total weight
subject to   upper & lower bounds on w_i, h_i
    upper bound & lower bounds on aspect ratios h_i/w_i
    upper bound on stress in each segment
    upper bound on vertical deflection at the end of the beam

variables: w_i, h_i for $i = 1, \ldots, N$

objective and constraint functions

- total weight $w_1 h_1 + \cdots + w_N h_N$ is posynomial

- aspect ratio h_i/w_i and inverse aspect ratio w_i/h_i are monomials

- maximum stress in segment i is given by $6iF/(w_i h_i^2)$, a monomial

- the vertical deflection y_i and slope v_i of central axis at the right end of segment i are defined recursively as

$$
\begin{aligned}
v_i &= 12(i - 1/2)\frac{F}{Ew_i h_i^3} + v_{i+1} \\[2ex]
y_i &= 6(i - 1/3)\frac{F}{Ew_i h_i^3} + v_{i+1} + y_{i+1}
\end{aligned}
$$

for $i = N, N-1, \ldots, 1$, with $v_{N+1} = y_{N+1} = 0$ (E is Young's modulus)

v_i and y_i are posynomial functions of w, h

formulation as a GP

$$\text{minimize} \quad w_1 h_1 + \cdots + w_N h_N$$

$$\text{subject to} \quad w_{\max}^{-1} w_i \le 1, \quad w_{\min} w_i^{-1} \le 1, \quad i = 1, \ldots, N$$

$$h_{\max}^{-1} h_i \le 1, \quad h_{\min} h_i^{-1} \le 1, \quad i = 1, \ldots, N$$

$$S_{\max}^{-1} w_i^{-1} h_i \le 1, \quad S_{\min} w_i h_i^{-1} \le 1, \quad i = 1, \ldots, N$$

$$6 i F \sigma_{\max}^{-1} w_i^{-1} h_i^{-2} \le 1, \quad i = 1, \ldots, N$$

$$y_{\max}^{-1} y_1 \le 1$$

note

- we write $w_{\min} \le w_i \le w_{\max}$ and $h_{\min} \le h_i \le h_{\max}$

$$w_{\min}/w_i \le 1, \qquad w_i/w_{\max} \le 1, \qquad h_{\min}/h_i \le 1, \qquad h_i/h_{\max} \le 1$$

- we write $S_{\min} \le h_i/w_i \le S_{\max}$ as

$$S_{\min} w_i/h_i \le 1, \qquad h_i/(w_i S_{\max}) \le 1$$

Minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalue $\lambda_{\mathrm{pf}}(A)$

- exists for (elementwise) positive $A \in \mathbf{R}^{n \times n}$

- a real, positive eigenvalue of A, equal to spectral radius $\max_i |\lambda_i(A)|$

- determines asymptotic growth (decay) rate of A^k: $A^k \sim \lambda_{\mathrm{pf}}^k$ as $k \to \infty$

- alternative characterization: $\lambda_{\mathrm{pf}}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0\}$

minimizing spectral radius of matrix of posynomials

- minimize $\lambda_{\mathrm{pf}}(A(x))$, where the elements $A(x)_{ij}$ are posynomials of x

- equivalent geometric program:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \sum_{j=1}^{n} A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \dots, n \end{array}$$

variables λ, v, x

Generalized inequality constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $f_0 : \mathbf{R}^n \to \mathbf{R}$ convex; $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ K_i-convex w.r.t. proper cone K_i

- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

extends linear programming $(K = \mathbf{R}^m_+)$ to nonpolyhedral cones

Semidefinite program (SDP)

$$\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& Ax = b
\end{array}$$

with F_i, $G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)

- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

LP and SOCP as SDP

LP and equivalent SDP

$$\text{LP:} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array} \qquad \text{SDP:} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

$$\text{SOCP:} \quad \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

$$\text{SDP:} \quad \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m \end{array}$$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^k$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$

- follows from

$$\lambda_{\max}(A) \leq t \quad \Longleftrightarrow \quad A \preceq tI$$

Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\max}(A(x)^T A(x))\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{R}^{p \times q}$)

equivalent SDP

$$\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0
\end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$

- constraint follows from

$$\|A\|_2 \le t \iff A^T A \preceq t^2 I, \quad t \ge 0$$
$$\iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0$$

Vector optimization

general vector optimization problem

$$\begin{array}{ll}
\text{minimize (w.r.t. } K) & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \in \mathbf{R}^q$

convex vector optimization problem

$$\begin{array}{ll}
\text{minimize (w.r.t. } K) & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}$$

with f_0 K-convex, $f_1, \ldots, f_m$ convex

Optimal and Pareto optimal points

set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x is **optimal** if $f_0(x)$ is the minimum value of $\mathcal{O}$

- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$

$x^\star$ is optimal

x^{po} is Pareto optimal
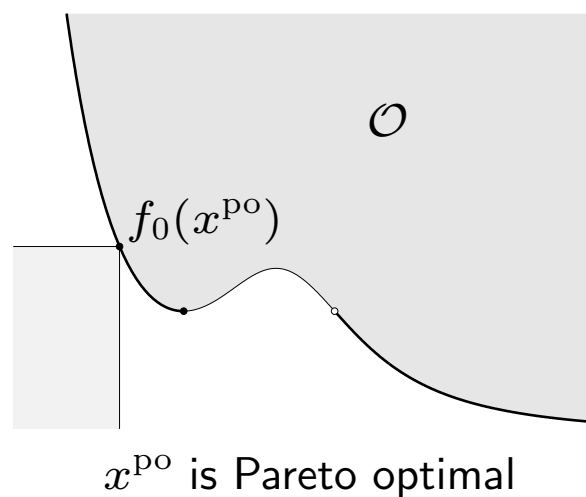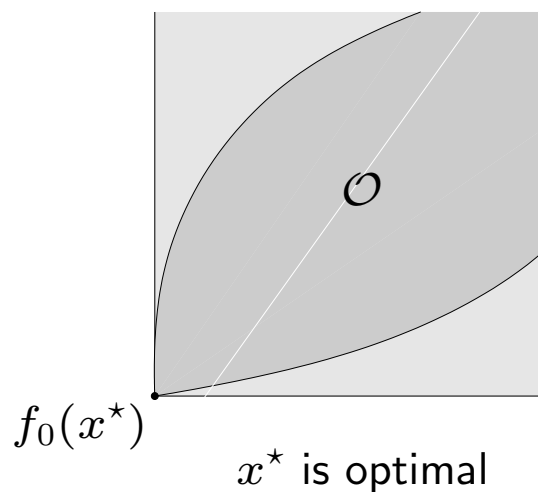
Multicriterion optimization

vector optimization problem with $K = \mathbf{R}_+^q$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- q different objectives F_i; roughly speaking we want all F_i's to be small

- feasible $x^\star$ is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^\star) \preceq f_0(y)$$

  if there exists an optimal point, the objectives are noncompeting

- feasible x^{po} is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\mathrm{po}}) \quad \Longrightarrow \quad f_0(x^{\mathrm{po}}) = f_0(y)$$

  if there are multiple Pareto optimal values, there is a trade-off between the objectives

Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$
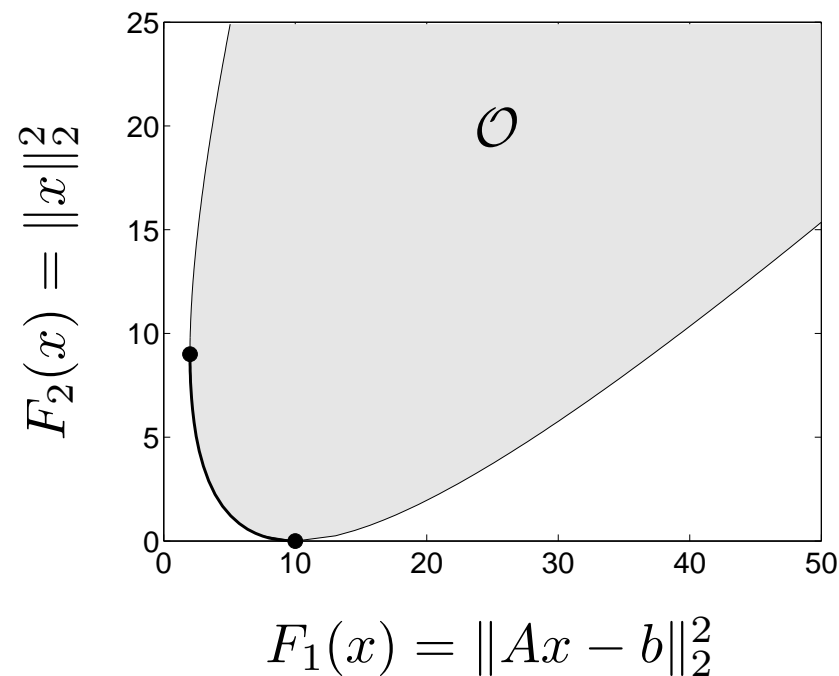

example for $A \in \mathbf{R}^{100 \times 10}$; heavy line is formed by Pareto optimal points

Risk return trade-off in portfolio optimization

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{R}_+^2) & (-\bar{p}^T x, x^T \Sigma x) \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

- $x \in \mathbf{R}^n$ is investment portfolio; x_i is fraction invested in asset i

- $p \in \mathbf{R}^n$ is vector of relative asset price changes; modeled as a random variable with mean $\bar{p}$, covariance Σ

- $\bar{p}^T x = \mathbf{E}\, r$ is expected return; $x^T \Sigma x = \mathbf{var}\, r$ is return variance
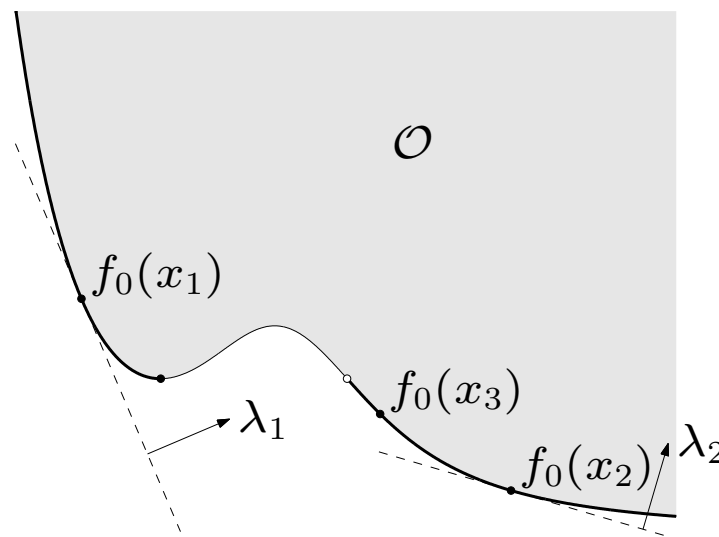
example
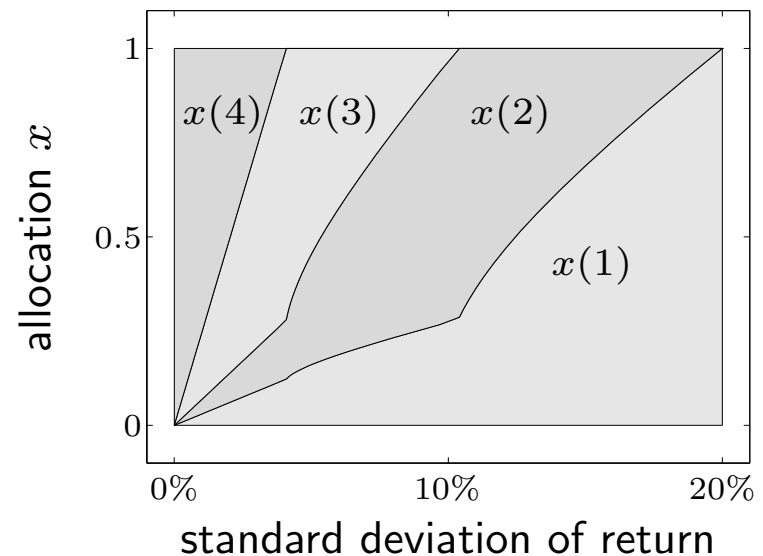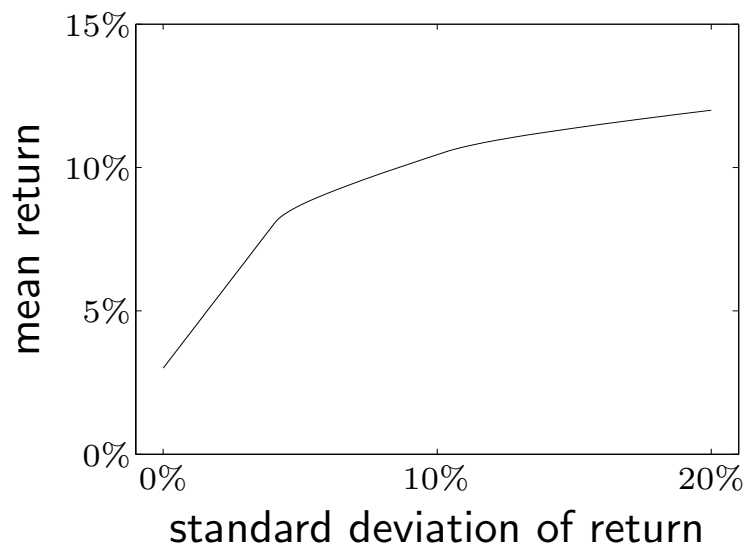
Scalarization

to find Pareto optimal points: choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

if x is optimal for scalar problem,
then it is Pareto-optimal for vector
optimization problem

for convex vector optimization problems, can find (almost) all Pareto
optimal points by varying $\lambda \succ_{K^*} 0$

Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

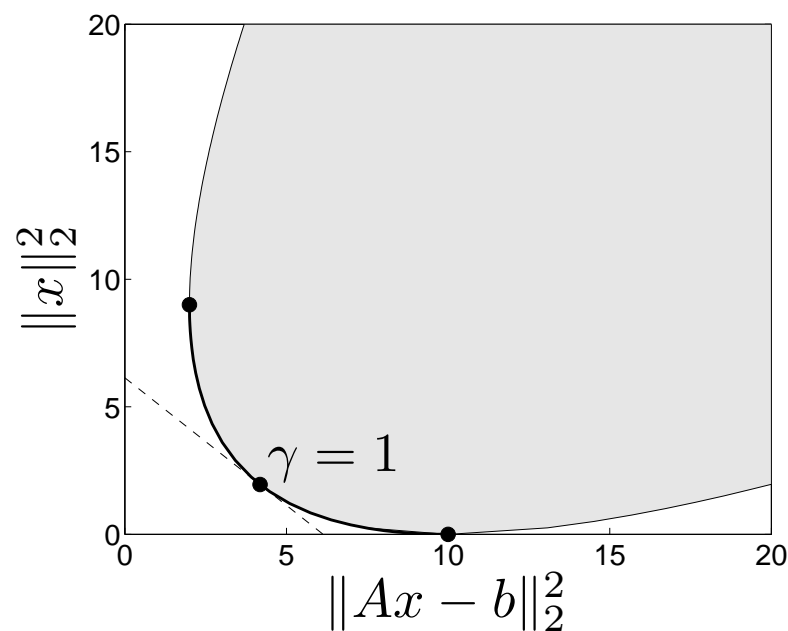$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

examples

- regularized least-squares problem of page 4–43

take $\lambda = (1, \gamma)$ with $\gamma > 0$

minimize $\quad \|Ax - b\|_2^2 + \gamma\|x\|_2^2$

for fixed γ, a LS problem

- risk-return trade-off of page 4–44

$$\begin{array}{ll} \text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

for fixed $\gamma > 0$, a quadratic program