Habib University
CS-113 Discrete Mathematics
Spring 2018
HW 5 Solutions

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1. Prove, by double counting:
   (a) $\Sigma_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

   Solution:
   R.H.S.: The number of pairs $(x, subset)$ where $x \in [n]$ and $subset \subseteq \{[n]\backslash x\}$ is $n2^{n-1}$.
   L.H.S.: Alternatively, to create pairs, first choose a subset A of size k from n, and then create pairs $(x, subset)$, where $x \in A$ and $subset = A\backslash x$. Since the first step can be done in $\binom{n}{k}$ ways, the pairs can be created in $k\binom{n}{k}$ ways, where $k = 1, 2, ..., n$. In total, there are $\Sigma_{k=1}^{n} k\binom{n}{k}$ pairs.

   (b) $\Sigma_{k=-m}^{n} \binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$

   Solution:
   R.H.S.: The number of ways to choose subsets of size $r + 1 + s$ from $[m + n + 1]$.
   L.H.S.: Alternatively, let A be a subset of size $r + 1 + s$ chosen from $[m + n + 1]$ and let $x = 1, 2, ..., (m+n+1)$, be the $r+1^{th}$ element in ordered A. Then the number of such subsets is

   $$\Sigma_{k=0}^{n+m} \binom{k}{r} 1 \binom{n+m-k}{s}$$

   where we choose r objects from the first k, and then s objects from the last $n + m - k$. The 1 in the middle of the summation is symbolic of the fixed $r + 1^{th}$ element. To complete the proof, we observe that

   $$\Sigma_{k=0}^{n+m} \binom{k}{r} 1 \binom{n+m-k}{s} = \Sigma_{k=-m}^{n} \binom{m+k}{r}\binom{n-k}{s}$$

2. You and a friend play a game with a pile of N gold coins. On each turn, a player can remove 1, 3, or 6 coins from the pile. The winner is the one who takes the last coin. For $0 < N < 1000$, how many starting positions are winning positions for the first player?

   Solution: 666 (I will add the complete derivation soon).

3. Six friends, who each bought a gift, distribute them amongst each other such that everyone receives a gift. How many ways are there to distribute the gifts such that no friend gets their own gift?

Solution: 265 (I will add the complete derivation soon).

4. $\boxed{\text{0 points}}$ Find the number of non-negative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = y \qquad \text{where } x_1 < x_2 < \cdots < x_n$$

5. Your friend's been working hard on developing the covert *Kabootar Messaging System (KMS).* Their technique transforms each bit string into a unique bit string before sending over an insecure network. Once on the other side, the original bit string is fully reconstructed. Your friend claims that, on average, the *KMS* decreases the length of strings before transmission, and in any case, does not increase the length of a string. Prove that they are wrong.

Solution: For sake of absurdity, suppose that KMS does what it promises, i.e., generates a bijection $A \to B$, where A is the set of all strings over some alphabet Σ and B the set of encrypted strings over Σ, such that on average, strings in A are mapped to smaller strings in B and none to larger strings. Let $A_n \subset A$ be the first n strings in lexicographic order in A. Similarly let $B_n \subset B$ be the first n strings in lexicographic order in B. According to KMS, there exists a bijection from $A_n \to B_n$. But since the average length of strings in B_n is smaller than A_n, and no string in B_n is larger (in lexicographic terms) than any string in A_n, B_n must contain less strings than A_n. Therefore, $|B_n| < n$, and the original claim is absurd.