Habib University
CS-113 Discrete Mathematics
Spring 2018
HW 6 Solutions

Released: 17th April, 2018

1. Prove that a graph of n vertices and $n-1$ edges has a vertex of degree 1 or an isolated vertex.

> **Solution:** Suppose, for the sake of absurdity, that all vertices of the graph are of degree at least 2. We know, by the **Handshake Theorem**, that the total number of edges in this graph should be half the number of total degrees, i.e., at least $\frac{2n}{2} = n$. But the graph has $n-1$ edges only. Therefore, our initial assumption must be wrong.

2. Prove that a graph G is 2-connected **iff** for every triple (x, y, z) of distinct vertices, G has an (x, z) path through y.

   (A graph is 2-connected if removing at least 2 vertices (and associated edges) disconnects it or produces a graph with a single vertex)

> **Solution:**
>
> To prove: *If for every triple (x,y,z) of distinct vertices, G has an (x,z) path through y, then G is 2-connected.*
>
> Suppose, for sake of absurdity, that G is 1-connected. Remove any cut vertex a from G to get $G' = G - \{a\}$. Since G' has at least one more component than G, it must have at least one pair of non-connected vertices b and c (there is no path from b to c) which were connected in G via a. By our initial assumption, there exists a path from a to c via b. Since this path cannot go through a (simple path), b and c must be connected through another vertex. Therefore, our assumption that G is 1-connected must be wrong.
>
> To prove: *If G is 2-connected, then for every triple(x,y,z) of distinct vertices, G has an (x,z) path through y.*
>
> We prove the contrapositive: *If for some triple (x,y,z), G does not have a (x,z) path through y, then G is at most 1-connected.*
>
> Consider the triple (x, y, z) for which there is no (x, z) path through y. Removing x from the graph guarantess that there is no path from y to z. Therefore, the graph must be at most 1-connected.

3. Are the following graphs isomorphic? Provide a mapping if they are, or explain why they are not.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution: Let $a_i, i \in [1,6]$ be the row-wise vertices of the graph on the left, and $b_i, i \in [1,6]$ be the row-wise vertices of the graph on the right. Here is a mapping between the two sets of vertices:

$a_1 \to b_1, a_2 \to b_5, a_3 \to b_3, a_4 \to b_6, a_5 \to b_4, a_6 \to b_2$

[There are 7 other correct mappings]

4. The complement of a graph $G = (V, E)$ is the graph

$$(V, \{\{x, y\} \subset V, x \neq y\} \setminus E)$$

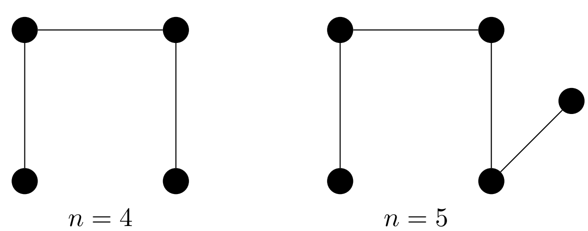
A graph is *self-complementary* if it is isomorphic to its complement.

(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such simple graphs. G

Solution: Consider a graph G and its complement G'. If G is isomorphic to G' then they must have the same number of edges. Therefore, the identity $\binom{n}{2} - e = e \to \binom{n}{2} = 2e$ where n is the number of vertices and e the number of edges of G, must hold. But neither $\binom{2}{2}$ nor $\binom{3}{2}$ is even. Therefore, the identity can not hold for $n = 2, 3$.

(b) Give one example each of self-complementary simple graphs with 4 and 5 vertices.

Solution:



5. Consider a graph G with two sets of vertices M and N and a set of undirected edges E such that $\forall m \in M, n \in N(\{m, n\} \in E)$. Determine (and prove) all $|M|, |N| \in \mathbb{N}$ for which G is

(a) Eulerian

Solution: An Euler cycle exists iff $|M|, |N| \geq 2$ are both even. This is because an Euler cycle exists iff all vertices are of even degree.

(b) Hamiltonian

Solution: A Hamilton Cycle exists iff $|M| = |N| \geq 2$. Here are the proofs of the if and only if directions:

if: Any bipartite graph with vertex sets of unequal size $|A|, |B|$ must observe the inequality $|A| < |B|$. Suppose, for sake of absurdity, that a Hamilton Cycle exists. Then, each vertex in A must contribute 2 edges to the Hamilton cycle or a total of $2|A|$ edges. Similarly, each vertex in B must contribute 2 edges to the cycle or a total of $2|B|$ edges. But $2|A| \neq 2|B|$ which is a contradiction to our assumption. Therefore, a Hamilton Cycle does not exist.

only if: It is quite easy to observe that a complete bipartite graph where $|M| = |N|$ is Hamiltonian. It is left to the reader to verify this.