Problem Definition

Given two line segments **S1** and **S2** in 3D space.
Segment **S1** is defined by endpoints **A1** and **B1**.
Segment **S2** is defined by endpoints **A2** and **B2**.
The goal is to find the shortest distance **R** between any point on **S1** and any point on **S2**.

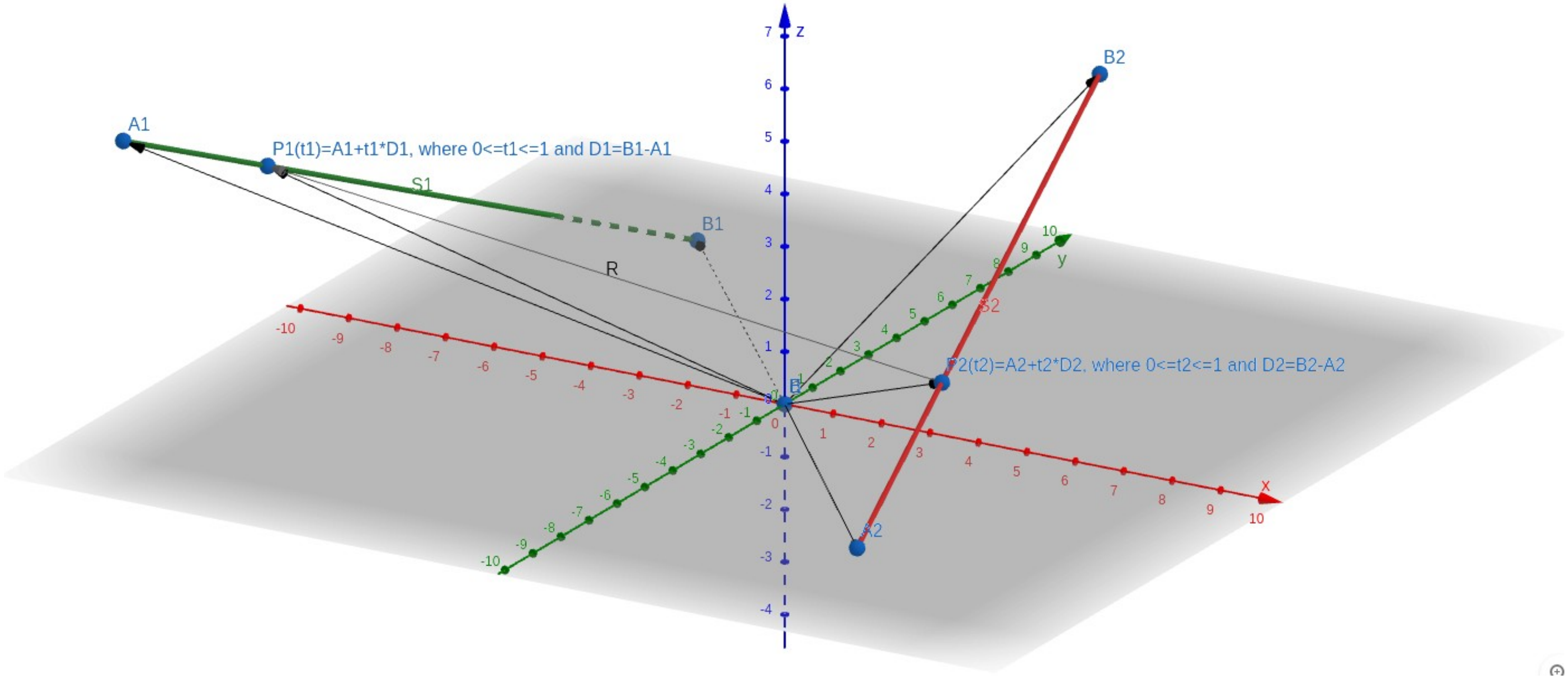

Figure 1: Solution Visualization

Parametric Representations

Let represent any point on **S1** and **S2** in **parametric** form:

$\overrightarrow{P1}(t1)=\overrightarrow{A1}+t1*\overrightarrow{D1}$, where $t1\in[0.0,1.0]$ and $\overrightarrow{D1}=\overrightarrow{B1}-\overrightarrow{A1}$ is **direction** of **S1**
$\overrightarrow{P2}(t2)=\overrightarrow{A2}+t2*\overrightarrow{D2}$, where $t2\in[0.0,1.0]$ and $\overrightarrow{D2}=\overrightarrow{B2}-\overrightarrow{A2}$ is **direction** of **S2**
$\vec{R}(t1,t2)=\overrightarrow{P2}(t2)-\overrightarrow{P1}(t1)$, where $\vec{R}(t1,t2)$ is a vector between some points on **S1** and **S2**
after **substitution**:
$\vec{R}(t1,t2)=(\overrightarrow{A2}+t2*\overrightarrow{D2})-(\overrightarrow{A1}+t1*\overrightarrow{D1})$
$\vec{R}(t1,t2)=(\overrightarrow{A2}-\overrightarrow{A1})+t2*\overrightarrow{D2}-t1*\overrightarrow{D1}$

Closest Point Criteria

We need **to minimize** the distance between the points on **S1** and **S2** - length of the vector $\vec{R}(t1,t2)$.
For the closest points on the lines, the vector connecting the points on the two lines must be **orthogonal** to both direction vectors $\overrightarrow{D1}$ and $\overrightarrow{D2}$.

Scalar product

Scalar product(Inner product , Dot product) of vector $\vec{A}$ and vector $\vec{B}$:
1. In geometric interpretation: $\vec{A}\cdot\vec{B}=|\vec{A}|*|\vec{B}|*\cos(\widehat{AB})$;
2. In algebraic interpretation: $\vec{A}\cdot\vec{B}=XA*XB+YA*YB+ZA*ZB$.

Vector $\vec{A}$ is **orthogonal** to vector $\vec{B}$, if:
1. In geometric interpretation $|\vec{A}|*|\vec{B}|*\cos(\widehat{AB})=0$ ($\cos(PI/2)=0$) ;

2. In algebraic interpretation $\vec{A}\cdot\vec{B}=XA*XB+YA*YB+ZA*ZB=0$.

System of equations

Let formulate our problem as system of equations.

$\overrightarrow{D1} \cdot \vec{R}(t1, t2) = 0$

$\overrightarrow{D2} \cdot \vec{R}(t1, t2) = 0$

Substitute expressions

Let $\vec{A} = \overrightarrow{A2} - \overrightarrow{A1}$

$\overrightarrow{D1} \cdot (\vec{A} + t2 * \overrightarrow{D2} - t1 * \overrightarrow{D1}) = 0$

$\overrightarrow{D2} \cdot (\vec{A} + t2 * \overrightarrow{D2} - t1 * \overrightarrow{D1}) = 0$

Transform expressions

$t1 * (\overrightarrow{D1} \cdot \overrightarrow{D1}) - t2 * (\overrightarrow{D1} \cdot \overrightarrow{D2}) = \overrightarrow{D1} \cdot \vec{A}$

$t2 * (\overrightarrow{D2} \cdot \overrightarrow{D2}) - t1 * (\overrightarrow{D2} \cdot \overrightarrow{D1}) = -\overrightarrow{D2} \cdot \vec{A}$

Simplify expressions

$t1 * G11 - t2 * G12 = C1$	**Where:**	$G11 = \overrightarrow{D1} \cdot \overrightarrow{D1}$ magnitude squared of D1
$t2 * G22 - t1 * G12 = C2$		$G12 = \overrightarrow{D1} \cdot \overrightarrow{D2}$ dot product of D1 and D2
		$G22 = \overrightarrow{D2} \cdot \overrightarrow{D2}$ magnitude squared of D2
		$C1 = \overrightarrow{D1} \cdot \vec{A}$
		$C2 = -\overrightarrow{D2} \cdot \vec{A}$

Represent system of equations in matrix form

$$\begin{bmatrix} G11 & -G12 \\ -G12 & G22 \end{bmatrix} \begin{bmatrix} t1 \\ t2 \end{bmatrix} = \begin{bmatrix} C1 \\ C2 \end{bmatrix}$$

Solving the System of Equations

Let find **t1n** and **t2n** parameters, which specifies **the closest** to each other points on **S1** and **S2**.

$$\begin{bmatrix} t1n \\ t2n \end{bmatrix} = \begin{bmatrix} G11 & -G12 \\ -G12 & G22 \end{bmatrix}^{-1} \begin{bmatrix} C1 \\ C2 \end{bmatrix}$$

Determinant

$$det(G) = G11 * G22 - G12^2$$

Inverse Matrix

$$G^{-1} = \frac{1}{G11 * G22 - G12^2} \begin{bmatrix} G22 & G12 \\ G12 & G11 \end{bmatrix}$$

Substituting

$$\begin{bmatrix} t1n \\ t2n \end{bmatrix} = \frac{1}{G11 * G22 - G12^2} \begin{bmatrix} G22 & G12 \\ G12 & G11 \end{bmatrix} \begin{bmatrix} C1 \\ C2 \end{bmatrix}$$

Final analytical expressions for t1n and t2n

$$t1n=\frac{G22*C1+G12*C2}{G11*G22-G12^2}$$

$$t2n=\frac{G12*C1+G11*C2}{G11*G22-G12^2}$$

Return to coordinates formalism

Define Points

A1 = [XA1, YA1, ZA1]

B1 = [XB1, YB1, ZB1]

A2 = [XA2, YA2, ZA2]

B2 = [XB2, YB2, ZB2]

Evaluate Direction Vectors

$\vec{D1}=\vec{B1}-\vec{A1}$ = [XB1 – XA1, YB1 – YA1, ZB1 – ZA1] = [XD1, YD1, ZD1]

$\vec{D2}=\vec{B2}-\vec{A2}$ = [XB2 – XA2, YB2 – YA2, ZB2 – ZA2] = [XD2, YD2, ZD2]

$\vec{A}=\vec{A2}-\vec{A1}$ = [XA2 – XA1, YA2 – YA1, ZA2 – ZA1] = [XA, YA, ZA]

Evaluate Coefficients

G11 $= XD1*XD1 + YD1*YD1 + ZD1*ZD1 = (XB1 - XA1)*(XB1 - XA1) + (YB1 - YA1)*(YB1 - YA1) + (ZB1 - ZA1)*(ZB1 - ZA1)$

G12 $= XD1*XD2 + YD1*YD2 + ZD1*ZD2 = (XB1 - XA1)*(XB2 - XA2) + (YB1 - YA1)*(YB2 - YA2) + (ZB1 - ZA1)*(ZB2 - ZA2)$

G22 $= XD2*XD2 + YD2*YD2 + ZD2*ZD2 = (XB2 - XA2)*(XB2 - XA2) + (YB2 - YA2)*(YB2 - YA2) + (ZB2 - ZA2)*(ZB2 - ZA2)$

C1 $= XD1*XA + YD1*YA + ZD1*ZA = (XB1 - XA1)*(XA2 - XA1) + (YB1 - YA1)*(YA2 - YA1) + (ZB1 - ZA1)*(ZA2 - ZA1)$

C2 $= XD2*XA + YD2*YA + ZD2*ZA = (XB2 - XA2)*(XA2 - XA1) + (YB2 - YA2)*(YA2 - YA1) + (ZB2 - ZA2)*(ZA2 - ZA1)$

Evaluate t1

$$t1n = \frac{G22*C1+G12*C2}{G11*G22-G12^2}$$

t1n =

(((XB2 – XA2)*(XB2 – XA2) + (YB2 – YA2)*(YB2 – YA2) + (ZB2 – ZA2)*(ZB2 – ZA2)) *

((XB1 – XA1)*(XA2 – XA1) + (YB1 – YA1)*(YA2 – YA1) + (ZB1 – ZA1)*(ZA2 – ZA1))) +

(((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2)) *

((XB2 – XA2)*(XA2 – XA1) + (YB2 – YA2)*(YA2 – YA1) + (ZB2 – ZA2)*(ZA2 – ZA1)))) /

(((XB1 – XA1)*(XB1 – XA1) + (YB1 – YA1)*(YB1 – YA1) + (ZB1 – ZA1)*(ZB1 – ZA1)) *

((XB2 – XA2)*(XB2 – XA2) + (YB2 – YA2)*(YB2 – YA2) + (ZB2 – ZA2)*(ZB2 – ZA2))) -

(((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2)) *

((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2))))

Evaluate t2

$$t2n = \frac{G12 * C1 + G11 * C2}{G11 * G22 - G12^2}$$

t2n =

(((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2)) *

((XB1 – XA1)*(XA2 – XA1) + (YB1 – YA1)*(YA2 – YA1) + (ZB1 – ZA1)*(ZA2 – ZA1)) +

(((XB1 – XA1)*(XB1 – XA1) + (YB1 – YA1)*(YB1 – YA1) + (ZB1 – ZA1)*(ZB1 – ZA1)) *

((XB2 – XA2)*(XA2 – XA1) + (YB2 – YA2)*(YA2 – YA1) + (ZB2 – ZA2)*(ZA2 – ZA1))) /

(((XB1 – XA1)*(XB1 – XA1) + (YB1 – YA1)*(YB1 – YA1) + (ZB1 – ZA1)*(ZB1 – ZA1)) *

((XB2 – XA2)*(XB2 – XA2) + (YB2 – YA2)*(YB2 – YA2) + (ZB2 – ZA2)*(ZB2 – ZA2)) -

(((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2)) *

((XB1 – XA1)*(XB2 – XA2) + (YB1 – YA1)*(YB2 – YA2) + (ZB1 – ZA1)*(ZB2 – ZA2)))

Closest Points on Each Segment

Parametric representation of the closest points on **S1** and **S2**.

Evaluated in the coordinates of endpoints of segments:

P1N(t1n) = [XA1 + t1n*XD1, YA1 + t1n*YD1, ZA1 + t1n*ZD1] = [XP1, YP1, ZP1]

P2N(t2n) = [XA2 + t2n*XD2, YA2 + t2n*YD2, ZA2 + t2n*ZD2] = [XP2, YP2, ZP2]

Where **P1N**, **P2N –** the closest points.

Distance between P1N and P2N

Let calculate the Euclidean distance between the closest points:

$$Distance = \| \overrightarrow{P2N}(t2n) - \overrightarrow{P1N}(t1n) \|$$

$$Distance = \sqrt{(XP2 - XP1)^2 + (YP2 - YP1)^2 + (ZP2 - ZP1)^2}$$

DISTANCE_S1toS2 = SQRT(((XA2 + t2n*XD2) - (XA1 + t1n*XD1))^2 + ((YA2 + t2n*YD2) - (YA1 + t1n*YD1))^2 + ((ZA2 + t2n*ZD2) - (ZA1 + t1n*ZD1))^2);